Dynamic Optimisation

1 Optimal Control

1.1

If left undisturbed, a population of s fish in a lake grows at the rate

$$\dot{s} = as - bs^2.$$

Fish can be consumed at a rate x which yields instantaneous utility to the local community of $\ln x$ and reduces the fish growth rate accordingly:

$$\dot{s} = as - bs^2 - x.$$

Future utility is discounted at the constant rate ρ, where $\rho < a$. Through the use of a phase diagram (or otherwise) describe the optimal consumption plan starting from the initial stock of fish $s(0) = a/b$. What happens if $\rho > a$?

The Hamiltonian for this problem is:

$$\mathcal{H} = u(x_t) + \lambda_t(as_t - bs_t^2 - x_t),$$

where we know $\rho < a$ and $s_0 = a/b$. Our first order conditions (FOCs) for this problem are the above Hamiltonian derived wrt the control variable, x, and state variable, s:

$$\frac{\partial \mathcal{H}}{\partial x_t} = u'(x_t) - \lambda_t = 0, \tag{1}$$

$$\frac{\partial \mathcal{H}}{\partial s_t} = a\lambda_t - 2b\lambda_t s_t = \rho\lambda_t - \dot{\lambda}, \tag{2}$$

where the dot notation denotes a variable's derivative wrt time. i.e. $\dot{\lambda}_t = d\lambda_t/dt$. From our FOCs, we take the expression for λ_t from (1) and take its derivative wrt time:

$$\lambda_t = u'(x_t),$$
$$\implies \dot{\lambda} = u''(x_t)\dot{x}.$$

We then use the above expression to get an expression for $\dot{x}_t$, and substitute our value for $\dot{\lambda}$ from 2:

$$\begin{aligned}
\dot{x} &= \frac{\dot{\lambda}}{u''(x_t)} \\
&= \frac{\lambda_t\rho - \lambda_t a + 2\lambda_t b s_t}{u''(x_t)} = \frac{\lambda_t(\rho - a + 2bs_t)}{u''(x_t)} \\
\dot{x} &= \frac{u'(x_t)(\rho - a + 2bs_t)}{u''(x_t)}.
\end{aligned}$$

This gives us two transition equations:

$$\dot{x} = \frac{u'(x_t)}{u''(x_t)}(\rho - a + 2bs_t) = 0,$$
$$\dot{s} = as_t - bs_t^2 - x_t = 0.$$

Since $u(x_t) = \ln x_t$, our transition equations are:

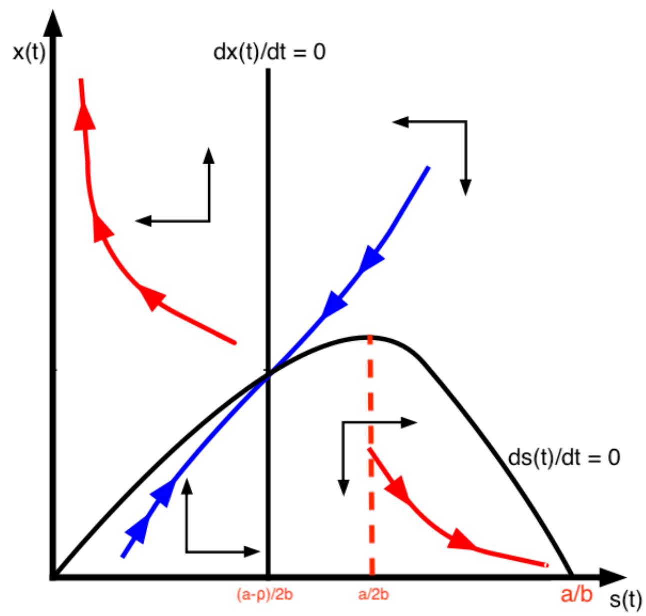
$$\dot{x} = x_t(a - \rho - 2bs_t) = 0,$$
$$\dot{s} = as_t - bs_t^2 - x_t = 0.$$

Given $u(x_t) = \ln x_t$, we can clearly see that the Hamiltonian is concave in both x_t and s_t. Moreover,

$$\frac{\partial \mathcal{H}}{\partial x_t} = \frac{1}{x_t} - \lambda_t, \quad \frac{\partial^2 \mathcal{H}}{\partial x_t^2} = -\frac{1}{x_t^2},$$

so the value of x_t that maximises $\mathcal{H}$ is $1/\lambda_t$. We can see the transition dynamics of the two transition equations by plotting a phase diagram:

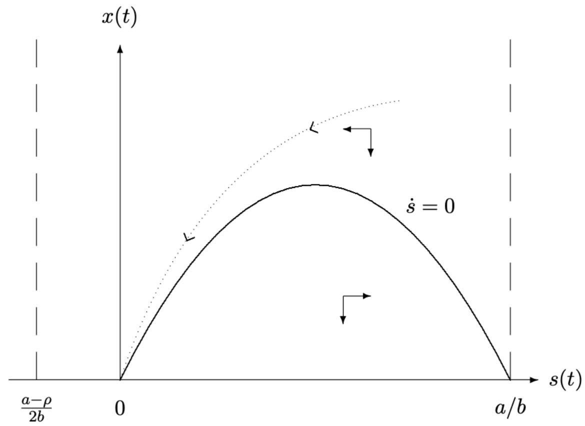
Figure 1: Phase diagram for Question 1



Given that we start at $s_0 = a/b$, our consumption of the fish stock must be sufficiently high so that we hit the saddle path given by the blue arrows. If our initial consumption of fish is insufficient, then we drift back to the starting point of $s_t = a/b$ and $x_t = 0$. Suppose that we do arrive onto the saddle path, then our steady state consumption of fish is given by $x_t^* = \frac{a-\rho}{2b}$, which is attained by solving for $\dot{x}_t = 0$. We can see that this is less than the maximum value of the fish stock due to the fact that we discount by discount factor ρ. For curiosity's sake, differentiating $\dot{s}_t$ and solving gives the max point as $s_t = \frac{a}{2b}$.

If $\rho > a$, then our impatience will lead us to consume the entire stock of fish such that the steady state level of fish consumption and fish stock per period will be at the origin.

Figure 2: Phase diagram for Question 1 Part 2



1.2

The rate at which natural gas is pumped from a deposit is given by

$$\dot{s} = -\alpha s x,$$

where $s(t)$ is the amount of gas remaining in the deposit, $x(t)$ is the input of pumping energy, and α is a constant. Let P denote the (constant) price of gas and $c(x)$ be a strictly convex function denoting the cost of pumping. Find the energy input $x(t)$ that maximises

$$\int_0^T \{p\alpha s(t)x(t) - c(x(t))\}\, dt,$$

subject to $s(0) = s_0$, $s(T) = s_T$, and $\dot{s} = -\alpha s x$.
(Hint: First show that $\dot{x} = 0$)

The problem is to choose $x(t)$ to

$$\max \int_0^T \{P\alpha s(t)x(t) - c(x(t)) + \lambda_t[-\alpha s(t)x(t) - \dot{s}]\}\, dt$$

or, equivalently

$$\max \int_0^T \left\{P\alpha s(t)x(t) - c(x(t)) + \lambda_t[-\alpha s(t)x(t)] + s(t)\dot{\lambda}\right\} dt.$$

Note: writing the Hamiltonian as

$$\mathcal{H} = P\alpha s(t)x(t) - c(x(t)) + \lambda_t(\alpha s(t)x(t)),$$

yields the exact same results as those derived below.
Taking the derivatives wrt x and s yields our FOCs:

$$\text{FOC}_x : P\alpha s(t) - c'(x(t)) - \lambda_t \alpha s(t) = 0 \tag{3}$$

$$\text{FOC}_s : P\alpha x(t) - \lambda \alpha x(t) + \dot{\lambda} = 0 \tag{4}$$

From (3) we have

$$c'(x(t)) = (P - \lambda_t)\alpha s(t),$$

and taking the derivative wrt t yields

$$\begin{aligned} c''(x(t))\dot{x} &= (P - \lambda_t)\alpha \dot{s} - \dot{\lambda}\alpha s(t) \\ \implies c''(x(t))\dot{x} &= \frac{c'(x(t))\dot{s}}{s(t)} - \dot{\lambda}\alpha s(t). \end{aligned} \tag{5}$$

From (4) we have

$$\begin{aligned} -\dot{\lambda} &= P\alpha x(t) - \lambda_t \alpha x(t) \\ &= (P - \lambda_t)\alpha x(t) \\ \implies -\dot{\lambda} &= \frac{c'(x(t))x(t)}{s(t)}. \end{aligned} \tag{6}$$

We can then substitute (6) into (5) (be careful with the '$-$' sign):

$$\begin{aligned} c''(x(t))\dot{x} &= \frac{c'(x(t))\dot{s}}{s(t)} + \frac{c'(x(t))x(t)}{s(t)}\alpha s(t) \\ &= \frac{c'(x(t))\dot{s}}{s(t)} + \alpha c'(x(t))x(t) \\ &= c'(x(t))\left[\frac{\dot{s}}{s(t)} + \alpha x(t)\right]. \end{aligned}$$

We can then use the law of motion

$$\dot{s} = -\alpha s(t)x(t),$$

to get

$$\begin{aligned} c''(x(t))\dot{x} &= c'(x(t))\left[\frac{-\alpha s(t)x(t)}{s(t)} + \alpha x(t)\right] \\ \implies \dot{x} &= \frac{c'(x(t))}{c''(x(t))}\left[\alpha x(t) - \alpha x(t)\right] = 0. \end{aligned}$$

Since $\dot{x} = 0$, this implies that $x(t)$ is some constant in each period, say k. So

$$\begin{aligned}
\dot{s} &= -\alpha s(t) k \\
s(t) &= s_0 \exp(-\alpha k t) \\
s_T &= s_0 \exp(-\alpha k T) \\
\implies \frac{s_0}{s_T} &= \exp(\alpha k T) \\
\ln\left(\frac{s_0}{s_T}\right) &= \alpha k T \\
\implies k &= \frac{1}{\alpha T} \ln\left(\frac{s_0}{s_T}\right).
\end{aligned}$$

With our initial and terminal conditions we can see that when $t = 0$, $s(t) = s_0$, and when $t = T$, $s(t) = s_T \approx 0$ if T is sufficiently large.

1.3

The price of a mineral resource is constant and equal to P. The instantaneous cost of extraction is $c(x) = Ax^2$ where $x(t)$ is the rate of extraction. The initial stock is s_0. The owner of the resource has a discount rate ρ and wishes to maximise

$$\int_0^\infty \exp(-\rho t) \left\{ Px(t) - Ax(t)^2 \right\} dt,$$

subject to $s(0) = s_0$, $s(t) \geq 0$, and $\dot{s} = -x(t)$.

The problem is to choose $x(t)$ to

$$\max \int_0^\infty \exp(-\rho t) \left\{ Px(t) - Ax(t)^2 + \lambda_t(-x(t) - \dot{s}) \right\} dt,$$

or equivalently,

$$\max \int_0^\infty \exp(-\rho t) \left\{ Px(t) - Ax(t)^2 + \lambda_t(-x(t)) + s(t)(\dot{\lambda} - \rho \lambda_t) \right\} dt.$$

and the first order conditions are:

$$\text{FOC}_x : P - 2Ax(t) - \lambda_t = 0 \tag{7}$$

$$\text{FOC}_s : \dot{\lambda} - \rho \lambda_t = 0. \tag{8}$$

From 7:

$$\begin{aligned}
\lambda_t &= P - 2Ax(t) \\
\implies \frac{d\lambda_t}{dt} &= \dot{\lambda} = -2A\dot{x}
\end{aligned}$$

and from 8:

$$\begin{aligned}
\dot{\lambda} &= \rho \lambda_t \\
\implies -2A\dot{x} &= \rho \lambda_t \\
-2A\dot{x} &= \rho(P - 2Ax(t)).
\end{aligned}$$

To get an expression for $x(t)$ without $\dot{x}$ we must rearrange and integrate:

$$\begin{aligned}
\frac{2A\dot{x}}{P - 2Ax(t)} &= -\rho \\
\int \frac{2A\dot{x}}{P - 2Ax(t)} dt &= \int -\rho \; dt \\
2A \int \frac{dx}{P - 2Ax(t)} &= -\rho \int dt \\
-\ln(P - 2Ax(t)) &= -\rho t - C \\
\ln(P - 2Ax(t)) &= \rho t + C,
\end{aligned}$$

take anti-logs:

$$\begin{aligned}
P - 2Ax(t) &= K \exp(\rho t) \\
-2Ax(t) &= K \exp(\rho t) - P \\
x(t) &= \frac{1}{2A}(P - K \exp(\rho t)).
\end{aligned}$$

1.3.1

Show that the optimal policy is to extract the resource completely in some finite time T.

Since K is positive there must be a time T such that $P - K\exp(\rho t) = 0$. i.e. $x(T) = 0$; further, since $x(t)$ cannot be negative, it must be the case that $x(t) = 0 \; \forall t > T$. it cannot be optimal to abandon extraction when $s(t) > 0$, so $s(T) = 0$.

1.3.2

Obtain an explicit expression for $x(t)$ in terms of this time T (rather than s_0).

$K \exp(\rho t) = P$ implies that $K = P \exp(-\rho T)$, and so

$$x(t) = \frac{P}{2A} \left[1 - \exp(-\rho(T - t))\right].$$

1.3.3

Show that T satisfies

$$\frac{P}{2A} \left[T - \frac{1}{\rho}(1 - \exp(-\rho T)\right] = s_0.$$

What happens to the extraction rate as $s_0 \to \infty$?

Using the LOM that $\dot{s} = -x(t)$:

$$\begin{aligned}
\dot{s} &= -\frac{P}{2A} \left[1 - \exp(-\rho(T - t))\right] \\
\int \dot{s} \; dt &= -\int \frac{P}{2A} \left[1 - \exp(-\rho(T - t))\right] \\
s(t) &= -\frac{P}{2A} \left[t - \frac{1}{\rho} \exp(-\rho(T - t))\right] + K',
\end{aligned}$$

then suppose

$$\begin{aligned} 0 = s(T) &= -\frac{P}{2A}\left[t - \frac{1}{\rho}\exp(-\rho(T-t))\right] + K' \\ 0 &= -\frac{P}{2A}\left[T - \frac{1}{\rho}\right] + K' \\ \implies K' &= \frac{P}{2A}\left[T - \frac{1}{\rho}\right], \end{aligned}$$

substituting this back into $s(t)$:

$$\begin{aligned} s(t) &= -\frac{P}{2A}\left[t - \frac{1}{\rho}\exp(-\rho(T-t))\right] + \frac{P}{2A}\left[T - \frac{1}{\rho}\right] \\ &= \frac{P}{2A}\left[T - \frac{1}{\rho} - t + \frac{1}{\rho}\exp(-\rho(T-t))\right], \end{aligned}$$

and so

$$s_0 = s(0) = \frac{P}{2A}\left[T - \frac{1}{\rho} + \frac{1}{\rho}\exp(-\rho T)\right].$$

Now, take derivatives

$$\frac{ds_0}{dT} = \frac{P}{2A}\left[1 - \exp(-\rho T)\right] > 0,$$

and

$$\frac{dx(t)}{dT} = \frac{P}{2A}\rho\exp(-\rho(T-t)) > 0,$$

so $s_0 \uparrow \implies T \uparrow$, and $T \uparrow \implies x(t) \uparrow$ – the extraction rate is approximately linear.

2 Dynamic Programming

2.1 Consumption-investment with logarithmic utility & Cobb-Douglas production

Consider an agent with utility function $u(x) = \ln x$, and a capital stock s. When she consumes x this period she invests the remained $s - x$ in production and at the start of the next period this has become $f(s - x)$ where $f(y) = Ay^{\alpha}$ with $0 < \alpha \leq 1$. The Bellman equation for this problem is

$$V(s) = \max_{0 \leq x \leq s} \left\{ u(x) + \delta V(f(s - x)) \right\}.$$

Show that $V(s) = B \ln s + C$ satisfies the Bellman equation for some constants B and C. Show that the associated plan is to consume a fixed fraction of the stock each period.
Note: Setting $A = 1 + r$, the gross interest rate, and $\alpha = 1$ gives us a consumption-savings problem.

Given that $u(x) = \ln x$ and we have capital stock s which grows by the following law of motion:

$$s' = A(s - x)^{\alpha}.$$

The value function for our problem is then:

$$V(s) = \max \left\{ u(x) + \delta V(A(s - x)^{\alpha}) \right\}.$$

Since our utility is a log function, we can guess a form for $V(s)$ as $B \ln(A(s - x)^{\alpha}) + C$. So our value function becomes:

$$\max \left\{ \ln x + \delta B \ln A + \delta B \alpha \ln(s - x) + \delta C \right\}.$$

Differentiating our value function gives the following FOC:

$$\frac{1}{x} - \frac{\delta B \alpha}{s - x} = 0,$$

which then gives us

$$\begin{aligned} s - x &= \delta B \alpha x \\ \implies x &= \frac{s}{1 + \delta B \alpha}. \end{aligned}$$

So the Bellman equation for this problem is:

$$\begin{aligned}
V(s) &= \max_s \left\{ \ln \frac{s}{1+\delta B\alpha} + \delta B\alpha \ln (s-x) + \delta C \right\} \\
&= \max \left\{ \ln \frac{s}{1+\delta B\alpha} + \delta B\alpha \ln (\delta B\alpha x) + \delta C \right\} \\
&= \max \left\{ \ln \frac{s}{1+\delta B\alpha} + \delta B\alpha \ln \left(\frac{\delta B\alpha s}{1+\delta B\alpha} \right) + \delta C \right\} \\
&= \max \left\{ \ln \frac{s}{1+\delta B\alpha} + \delta B\alpha \ln \left(\frac{\delta B\alpha s}{1+\delta B\alpha} \right) + \delta C \right\} \\
&= \max \left\{ \ln s - \ln(1+\delta B\alpha) + \delta B\alpha \ln \left(\frac{\delta B\alpha}{1+\delta B\alpha} \right) + \delta B\alpha \ln s + \delta C \right\} \\
&= \max \left\{ \underbrace{[1+\delta B\alpha]}_{B} \ln s + \underbrace{\delta B\alpha \ln \left(\frac{\delta B\alpha}{1+\delta B\alpha} \right) - \ln(1-\delta B\alpha) + \delta C}_{C} \right\}.
\end{aligned}$$

So, our solution is of the form $B \ln s + C$. We know that

$$x = \frac{s}{1+\delta B\alpha},$$

which implies that

$$B = 1 + \delta B\alpha \implies B = \frac{1}{1-\delta\alpha},$$

so

$$x = \frac{s}{1+\delta\alpha B} = \frac{s}{B} = (1-\delta\alpha)s.$$

Thus, our optimal consumption per period is $x^* = (1-\delta\alpha)s$, and since $s' = s - x \implies s' = \delta\alpha s$.

2.2 'Cake-eating' with CRRA utility

Consider an agent with utility function $u(x) = \frac{x^{1-R}-1}{1-R}$, and a capital stock s. When she consumes x this period she saves the remainder $s-x$ and has this at the start of next period. Show that $V(s) = \frac{Bs^{1-R}-C}{1-R}$ satisfies the Bellman equation for some constants B and C. Show that the associated plan is again to consume a fixed fraction of the stock each period.
Note: Simultaneously setting $R = 1$ in the above exercise and $A = 1$, $\alpha = 1$ in the previous one yields identical problems – check if the solutions coincide.

Given the utility function

$$u(x) = \frac{x^{1-R}-1}{1-R},$$

and the law of motion for our stock of cake is:

$$s' = s - x,$$

our value function is the following:

$$V(s) = \max_x \left\{ \frac{x^{1-R} - 1}{1-R} + \delta V(s-x) \right\}.$$

We wish to check the utility function is of CRRA form, so we try a solution to our value function of the following:

$$V(s) = \frac{Bs^{1-R} - C}{1-R},$$

so our functional equation becomes:

$$\underbrace{\frac{Bs^{1-R} - C}{1-R}}_{V(s)} = \max \left\{ \underbrace{\frac{(s-s')^{1-R} - 1}{1-R}}_{u(x)} + \delta \underbrace{\left[\frac{Bs'^{1-R} - C}{1-R} \right]}_{V(s')} \right\}. \tag{9}$$

We know[1] from the Euler equation that:

$$V'(s) = u'(x) = \beta V'(s'),$$

which implies

$$s'^R = \frac{\delta s^R}{B}$$
$$\therefore s' = \left(\frac{\beta}{B}\right)^{\frac{1}{R}} s.$$

We can substitute this back into (9) to get:

$$\frac{Bs^{1-R} - C}{1-R} = \frac{\left(s - \left(\frac{\delta}{B}\right)^{1/R} s\right)^{1-R} - 1}{1-R} + \delta \left[\frac{B\left(\left(\frac{\delta}{B}\right)^{1/R} s\right)^{1-R} - C}{1-R} \right]$$

$$\frac{Bs^{1-R} - C}{1-R} = \frac{s^{1-R} - \left(\frac{\delta}{B}\right)^{1/R} s^{1-R} - 1}{1-R} + \frac{\delta B\left(\left(\frac{\delta}{B}\right)^{1/R} s\right)^{1-R} - \delta C}{1-R}$$

$$\frac{Bs^{1-R} - s^{1-R} + \left(\frac{\delta}{B}\right)^{\frac{1-R}{R}} s^{1-R} - \delta B \left(\frac{\delta}{B}\right)^{\frac{1-R}{R}} s^{1-R}}{1-R} = \frac{C}{1-R} - \frac{1}{1-R} - \frac{\delta C}{1-R}$$

$$\underbrace{\left[B - 1 + \left(\frac{\delta}{B}\right)^{\frac{1-R}{R}} - \delta B \left(\frac{\delta}{B}\right)^{\frac{1-R}{R}} \right]}_{B} \frac{s^{1-R}}{1-R} = \underbrace{\frac{C - 1 - \delta C}{1-R}}_{C}.$$

Thus verifying that our solution to the problem is of the form $B\frac{s^{1-R}}{1-R} + C$.

[1] See 2.4.

We can get an expression for our ideal consumption of x since e know from our Euler equation conditions that

$$\frac{B}{s^R} = \frac{1}{x^R}$$
$$\implies x^* = \left(\frac{1}{B}\right)^{1/R} s.$$

2.3 'Cake-eating' with linear utility

What are the value function and the optimal plan for the cake-eating problem for an agent with linear utility function $u(x) = x$? What if $u(x) = x - 1$? Does the latter solution coincide with that from the previous problem when $R \to 0$?

When our utility function is linear such that $u(x) = x$, our lifetime discounted utility is given as:

$$\max_x \sum_{t=1}^{\infty} \beta^{1-t} x_t.$$

We can clearly see that if $0 < \beta < 1$, then the consumer will choose to consume the entirety of her cake in the first period, as the utility she gets from consumption will be discounted by β in the next period. If $\beta = 1$, then it doesn't matter when she consumes her cake, nor does it matter how much of the cake she consumes. The value function is thus:

$$V(s) = s.$$

If the utility function is of the form $u(x) = x - 1$, then the same conditions apply except for when $\beta = 1$ where she will consume all of her cake in the first period. She does so because she loses 1 util per period for which cake remains. The value function is thus:

$$V(s) = s - \frac{1}{1-\beta}.$$

2.4 Consumption-savings with CRRA utility & a stochastic interest rate

Consider an agent with utility function $u(x) = \frac{x^{1-R}-1}{1-R}$, and savings s. When she consumes x this period she saves the remainder $s-x$ and starts next period with $(1+r)(s-x)$ where r is a random variable, IID over time. Show that the solution to the Bellman equation and the associated plan are qualitatively the same as in the deterministic case, but the effect of the uncertainty makes the agent bring consumption forward if $0 < R < 1$, and defer it if $R > 1$.

Hint: For the last part, Jensen's inequality might be useful.

The value function for this problem is:

$$V(s) = \max_{0 \leq x < s} \left\{ u(x) + \beta E_t \left[V(s') \right] \right\},$$

subject to

$$s' = (1 + r')(s - x).$$

Therefore, taking the derivative of $V(s)$ wrt x yields the following FOC:

$$u'(x) = \beta E\left[V'(s')(1+r')\right].$$

Then, take the total derivative of $V(s)$:

$$\begin{aligned} V'(s)ds &= u'(x)dx + \beta E\left[V'(s')(1+r')(ds-dx)\right] \\ &= \underbrace{\left(u'(x)dx - \beta E\left[V'(s')(1+r')\right]\right)dx}_{=0 \text{ from FOC}} + \beta E\left[V'(s')(1+r')\right]ds \\ \therefore V'(s) &= \beta E\left[V'(s')(1+r')\right] = u'(x). \end{aligned}$$

Which gives the consumption Euler equation:

$$\begin{aligned} u'(x) &= \beta E\left[u'(x')(1+r')\right] \\ \implies \frac{u'(x)}{E[u'(x')]} &= \beta E[1+r']. \end{aligned}$$

Since we have CRRA utility:

$$\begin{aligned} u(x) &= \frac{x^{1-R}-1}{1-R} \\ \implies u'(x) &= \frac{1}{x^R}. \end{aligned}$$

Thus we have the following for our Euler equation:

$$\frac{E\left[x'^R\right]}{x^R} = \beta E[1+r'].$$

For the deterministic case, most of our analysis is analogous to the stochastic case. Start with our value function:

$$V(s) = \max_{0\leq x<s}\left\{u(x) + \beta E[V(s')]\right\},$$

subject to

$$s' = (1+R)(s-x).$$

Taking the derivative of the value function wrt x yields the following FOC:

$$u'(x) = \beta(1+r).$$

Then, take the total derivative of the value function:

$$\begin{aligned} V'(s)ds &= u'(x)dx + \beta E\left[V'(s')(1+r)(ds-dx)\right] \\ &= u'(x)dx - \beta E\left[V'(s')(1+r)\right]dx + \beta E\left[V'(s')(1+r)\right]ds \\ &= \underbrace{u'(x)dx - \beta(1+r)dx}_{=0} + \beta(1+r)^2(s-x)ds \\ \therefore V'(s) &= \beta(1+r)(s-x) \\ \implies V'(s) &= u'(x). \end{aligned}$$

This gives our consumption Euler equation:

$$\frac{u'(x)}{u'(x')} = \beta(1+r)$$
$$\implies \left(\frac{x'}{x}\right)^R = \beta(1+r),$$

as we have CRRA utility.

For the stochastic case, we have per period consumption as

$$x = \left[\frac{E[x'^R]}{\beta E[1+r']}\right]^{1/R},$$

and for the deterministic case as

$$x = \frac{x'}{(\beta(1+r))^{1/R}}.$$

These expressions are qualitatively identical. However since the numerator on the RHS of the stochastic expression is an expected utility, we can see that for $0 < R < 1$, the individual will choose to bring her consumption forward. If $R > 1$, then she will choose to smooth her consumption into the future.

Note: Jensen's equality states:

$$E[g(r)] \leq g(E[r]),$$

thus

$$E[x'^R] \leq E[x']^R$$