Macroeconomics*

David Murakami†

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†St Cross College, University of Oxford. Email: david.murakami@economics.ox.ac.uk

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Part I

Growth and Business Cycles

1 Introduction and Simple General Equilibrium Models

At the heart of modern macroeconomic models is the belief that growth and "business cycles" should be explained by making explicit assumptions regarding the "deep" structural parameters of the economy, namely:

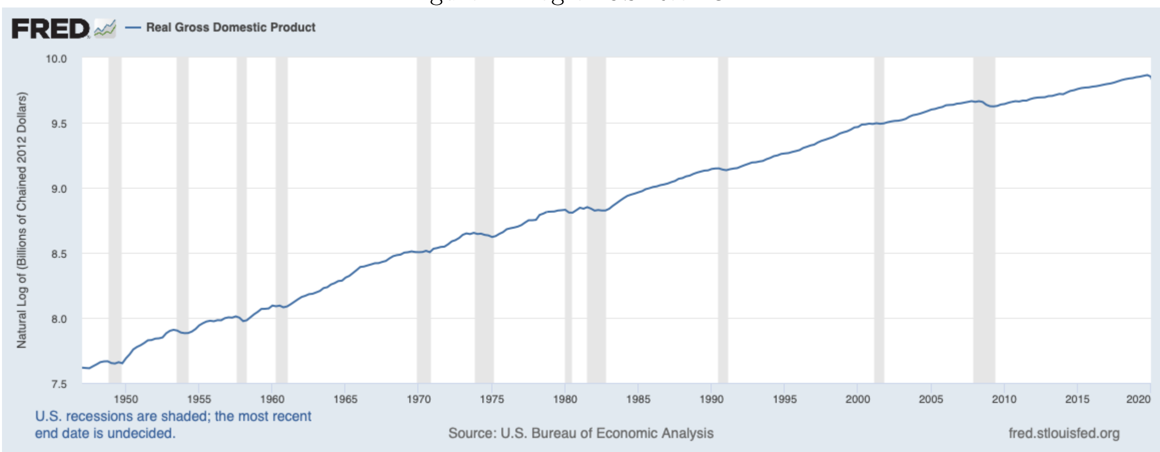
- tastes and preferences of agents;
- production technology; and
- market structure.

In this section we will focus on how to represent agents in a simple economy, define the business cycle, and talk about stylised facts of economic growth. Rounding out the section will be a focus on the consumption Euler equation, a key equation which we will revisit time and time again.

1.1 Economic growth verses the business cycle

Macroeconomists conduct "business cycle analysis" by breaking down a data series, such as GDP, into a "non-stationary" long-run trend and a "stationary" cyclical component. Consider the plot of US real GDP in Figure 1.

Figure 1: Log of US Real GDP



Let's use the simplest tool, log-linear trend, to try and break down the cyclical components of the real GDP time series to estimate the following regression

$$\ln Y_t = y_t = \alpha + gt + \epsilon_t, \tag{1}$$

where Y_t is real GDP, the trend component is $\alpha + gt$, ϵ_t is a zero-mean stationary cycle component. We can define the log difference in real GDP, Δy_t, as having two components: constant trend growth g and the change in cycle component $\Delta\epsilon_t$. We thus have:

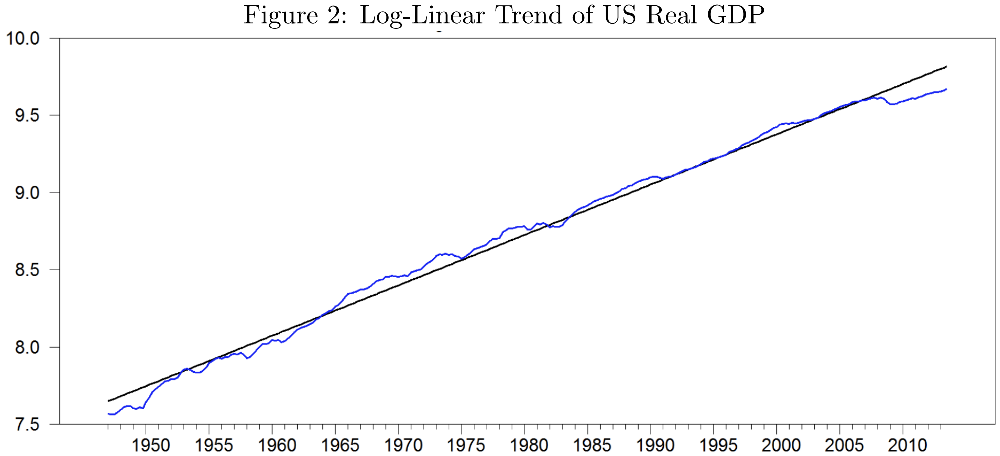
$$\begin{aligned}
\Delta y_t &= y_t - y_{t-1} \\
&= \alpha + gt + \epsilon_t - \alpha - g(t-1) - \epsilon_{t-1} \\
&= g + \epsilon_t - \epsilon_{t-1} \\
&= g - \Delta\epsilon_t.
\end{aligned}$$

Plotting this log-linear fit gives us the plots that we see in Figures 2 and 3. But drawing these straight lines to detrend a series can provide misleading results. For example, suppose that the correct model is

$$y_t = g + y_{t-1} + \epsilon_t, \tag{2}$$

where growth has a constant component g and a random bit ϵ_t.[1] Cycles here are just an accumulation of all the random shocks that have affected Δy_t over time. There is no tendency to revert to the trend, as the expected growth rate is always g no matter what happened in the past. In this case, Δy_t is stationary: first differencing gets rid of the unit-root (non-stationary stochastic trend component) of the series. In this simple example, if we fit a model like (1) to a series like (2), there might appear to be mean-reverting cyclical component when there actually is not. The simple takeaway is that detrending a time series – to understand the underlying trend, business cycle component, seasonality, and any other purely random fluctuations – is not as simple as fitting in a straight line.[2]

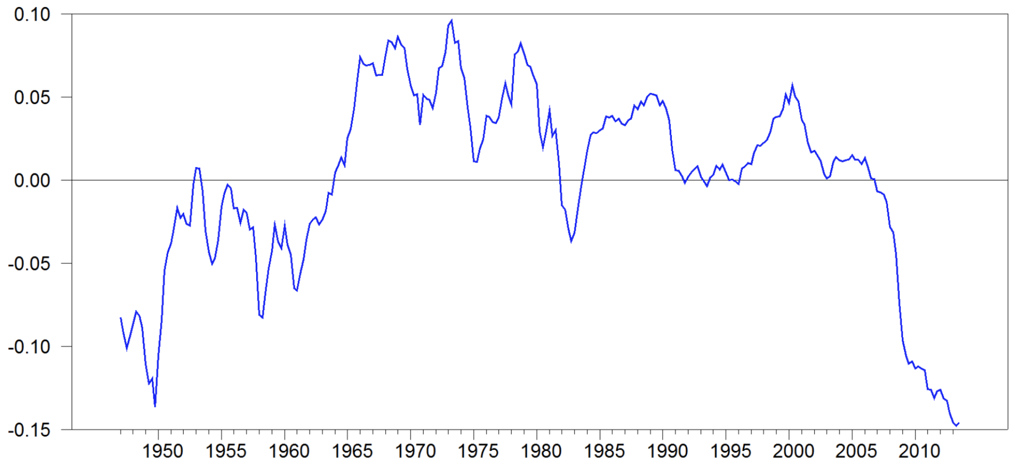
Figure 2: Log-Linear Trend of US Real GDP



[1] i.e., The data is generated by a random walk with drift.

[2] Johannes Pfeifer – the Dynare extraordinaire – has a fantastic set of notes, "A Guide to Specifying Observation Equations for the Estimation of DSGE Models", that discusses these topics in great detail.

Figure 3: Cycles from a Log-Linear Trend Model



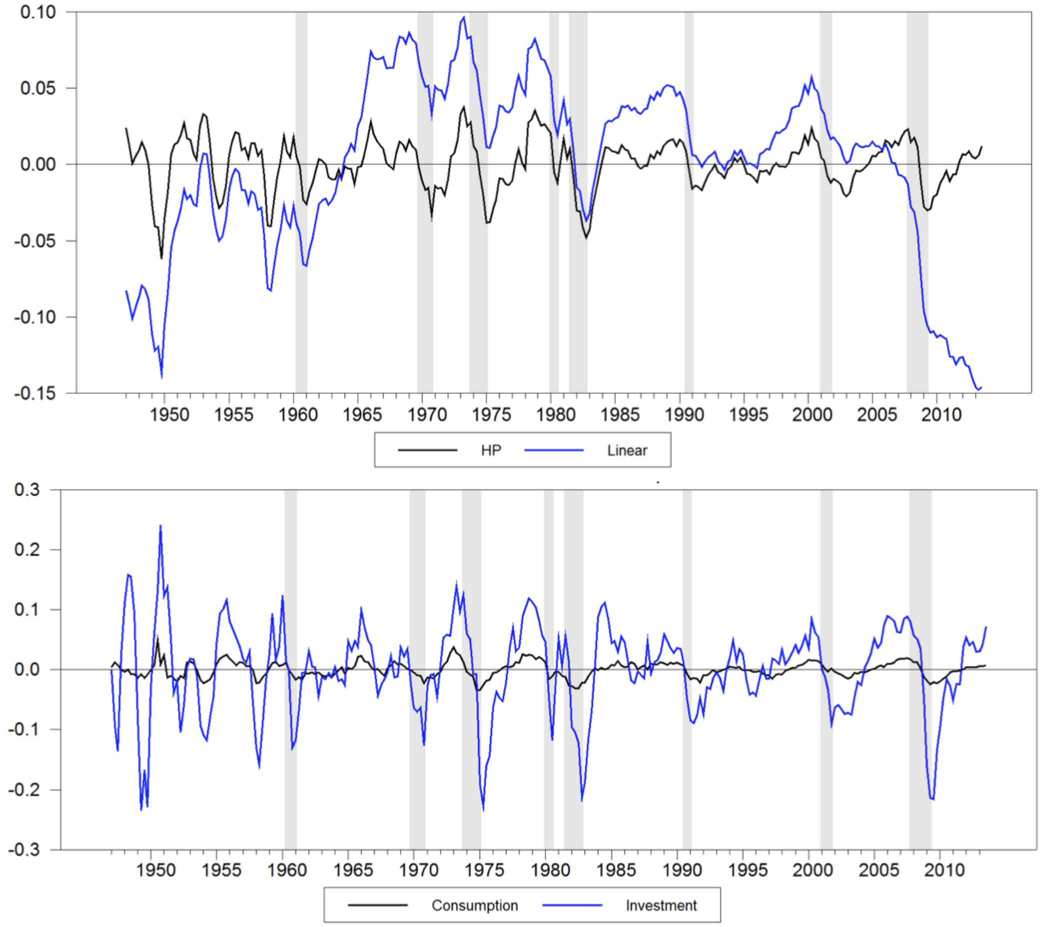
So what can we do? Well, we can use a filter, such as the Hodrick-Prescott (HP) filter,[3] to try and break down a series into its various components. The idea behind the HP filter is that the trend must be a smooth time series, rather than a typical zero-mean white noise process. This means that we would accept that the growth rate of the trend probably varies a bit over time, leaving a cycle that moves up and down over time. Hodrick and Prescott suggest choosing the time-varying trend Y_t^* so as to minimise the following

$$\min_{Y_t^*} \sum_{t=1}^{N} \left[(Y_t - Y_t^*)^2 + \lambda \left(\Delta Y_t^* - \Delta Y_{t-1}^* \right) \right]. \tag{3}$$

This method tries to minimise the sum of squared deviations between output and its trend, $(Y_t - Y_t^*)^2$, but also contains a term that emphasises minimising the change in the trend growth rate, $\lambda(\Delta Y_t^* - \Delta Y_{t-1}^*)$. λ is a parameter that we have to set, and typically this is set to 1600 for quarterly data. The larger the value of λ, the smoother the changes in the growth of the trend. Figures 4 and 5 show HP-filtered US real GDP cycles, consumption, investment and NBER-defined recessions, and US GNP with various HP filters, respectively.

[3]For more info see "Postwar U.S. Business Cycles: An Empirical Investigation" by Hodrick and Prescott (1997). This paper, and the HP filter, was actually first drafted in 1981. But it wasn't published until 1997.

Figure 4: HP-Filtered Cycles and NBER Recessions



As you can see, the HP-filter does seem to fit the quarterly data quite well, and that is probably one of the reasons why it has become the industry standard technique. However, there is also widespread concern about its use. Mainly:

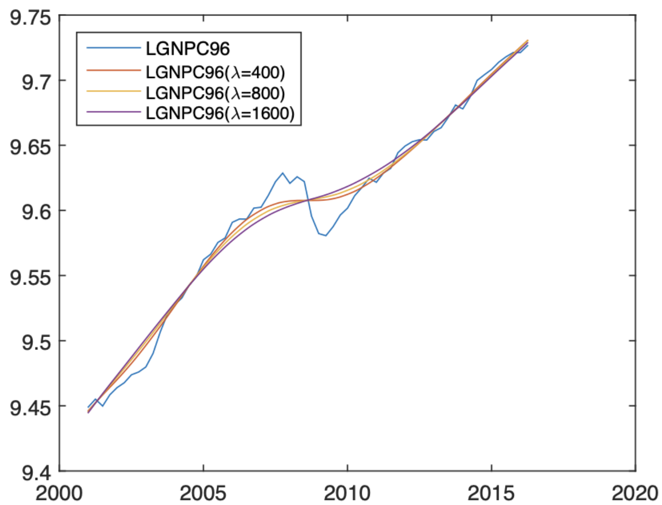
1. Business cycle facts are not invariant to the detrending filter used.

2. Other filters may be more optimal. A little bit of thought will reveal that if variables have different stochastic properties then a different detrending filter should be applied.

3. The HP filter may produce spurious cycles.

A well known result in the econometrics literature is by Nelson and Kang (1981), showing that if a linear time trend is fitted to a series which follows a random walk then the detrended data will display spurious cycles. In other words, if a researcher mistakenly thinks the trend is deterministic, then the cycles derived will be misspecified. Incorrect assumptions about the stochastic behaviour of a variable similarly mean the HP filter will exaggerate the pattern of long term growth cycles at cyclical frequencies and depress the influence of cycles at other frequencies. The result is that the HP filter may overstate the importance of business cycles.

Figure 5: US GNP and HP Trends



Even more strikingly, in the context of the Frisch-Slutsky paradigm, the HP filter can be dramatically misleading. Observed stylised facts about the business cycle reflect three factors: (i) an impulse; (ii) an propagation mechanism; and (iii) the data being trended by the HP filter and the certain statistics reported. It can be shown that for a typical macroeconomic model (ii) is unnecessary – merely assuming a process for the shock and applying the HP filter will be enough to generate business cycle patterns even if they are not there in the model. In other words, so called “stylised facts” are nothing more than artefacts. This is why some call the HP filter the “Hocus Pocus” filter – it can create business cycles from nothing.[4]

[4]Hamilton (2018) provides a lengthy explanation of the HP filter’s flaws, and provides an alternative filtering technique

1.1.1 The Lucas calculation

But should we care about business cycles? How important are fluctuations away from trend growth compared to the importance of the actual growth rate g? After all, if fluctuations are of minor importance compared to growth, then dedicating complex statistical and mathematical techniques to the explanation of shocks is a waste of time. Lucas considered a simple formulation to try and answer this question by looking at the "welfare cost" of business cycles. Suppose there are three economies: A, B, and C. Economy A grows at rate g but has business cycles, economy B grows at rate g too but does not have business cycles, and lastly, economy C grows at rate $g' > g$ but has business cycle fluctuations. So to summarise,

$$
\begin{aligned}
c_t^A &= \begin{cases} c_0(1+g)^t(1+f) & w.p.\ 0.5, \\ c_0(1+g)^t(1-f) & w.p.\ 0.5, \end{cases} \\
c_t^B &= c_0(1+g)^t, \\
c_t^C &= \begin{cases} c_0(1+g')^t(1+f) & w.p.\ 0.5, \\ c_0(1+g')^t(1-f) & w.p.\ 0.5. \end{cases}
\end{aligned}
$$

Clearly, economy B and C are better off than economy A, but the question is by how much? Suppose that the representative agent household in these economies had the following utility function

$$
u(c) = \frac{c^{1-\sigma}}{1-\sigma},
$$

where lifetime utility in economy A depends on three things: the initial level of consumption c_0 which affects every period thereafter in the same proportion, the rate of economic growth g, and the size of

in his piece "Why You Should Never Use the Hodrick-Prescott Filter". In summary, Hamilton's reasoning is: (i) the HP filter produces spurious cycles; (ii) filtered values at the end of the sample are very different from those in the middle; (iii) industry standard values for the smoothing parameter λ are statistically inaccurate; and (iv) there's a better alternative: Regress the variable at date $t+h$ on the four most recent values as of date t. Hamilton shows that his method achieves all the objectives sought by the HP filter but with none of its drawbacks.

fluctuations f. We can compute the welfare for economy A as follows:

$$\begin{aligned}
W^A(c_0, g, f) &= \mathbb{E}\left[\sum_{t=0}^{\infty} \beta^t u\left(c_t^A\right)\right] \\
&= \sum_{t=0}^{\infty}\left(\beta^t \frac{1}{2}\frac{1}{1-\sigma}\left[c_0(1+g)^t(1+f)\right]^{1-\sigma}\right) + \sum_{t=0}^{\infty}\left(\beta^t \frac{1}{2}\frac{1}{1-\sigma}\left[c_0(1+g)^t(1-f)\right]^{1-\sigma}\right) \\
&= \sum_{t=0}^{\infty} \beta^t \frac{1}{2}\frac{1}{1-\sigma}\left(c_0(1+g)^t\right)^{1-\sigma}\left((1+f)+(1-f)\right)^{1-\sigma} \\
&= \sum_{t=0}^{\infty} \beta^t \frac{1}{2}\frac{1}{1-\sigma} c_0^{1-\sigma}\left((1+g)^t\right)^{1-\sigma}\left((1+f)+(1-f)\right)^{1-\sigma} \\
&= \frac{1}{2}\frac{c_0^{1-\sigma}}{1-\sigma}\left(\frac{(1+f)^{1-\sigma}+(1-f)^{1-\sigma}}{1-\beta(1+g)^{1-\sigma}}\right). \qquad (4)
\end{aligned}$$

Now, how do we compare this welfare to economies B and C? Rather, what fraction of their consumption every year would the households in economy A be prepared to give up in order to have the features of economies B or C? For economy B, this would mean we solve for some proportion λ^B in the following equation:

$$W^A(c_0, g, f) = W^B(\lambda^B c_0, g, f).$$

So, we have from (4):

$$\begin{aligned}
\frac{1}{2}\frac{c_0^{1-\sigma}}{1-\sigma}\left(\frac{(1+f)^{1-\sigma}+(1-f)^{1-\sigma}}{1-\beta(1+g)^{1-\sigma}}\right) &= \frac{\left(\lambda^B c_0\right)^{1-\sigma}}{1-\sigma}\frac{1}{1-\beta(1+g)^{1-\sigma}} \\
\implies \lambda^B &= \left(\frac{1}{2}\left[(1+f)^{1-\sigma}+(1-f)^{1-\sigma}\right]\right)^{\frac{1}{1-\sigma}}. \qquad (5)
\end{aligned}$$

When we parameterise $f = 0.02$ and $\sigma = 2$, we get a value of $\lambda^B = 0.9996$. What does this mean? Households in economy A would be willing to give up just 0.04% of initial consumption to eliminate fluctuations. What about when we compare A to C? We get

$$\begin{aligned}
\frac{c_0^{1-\sigma}}{1-\sigma}\frac{(1+f)^{1-\sigma}+(1-f)^{1-\sigma}}{1-\beta(1+g)^{1-\sigma}} &= \frac{(\lambda^C c_0)^{1-\sigma}}{1-\sigma}\frac{(1+f)^{1-\sigma}+(1-f)^{1-\sigma}}{1-\beta(1+g')^{1-\sigma}} \\
\implies \lambda^C &= 0.826,
\end{aligned}$$

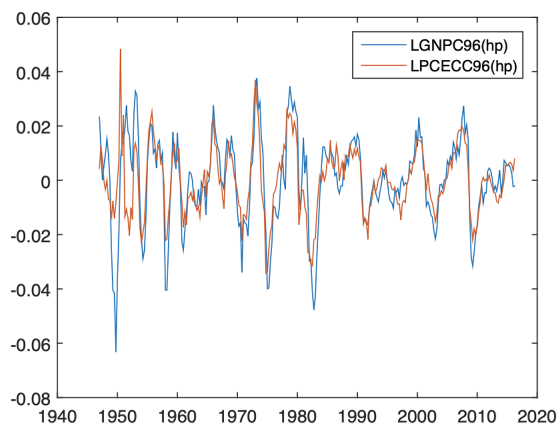
when $\beta = 0.97, \sigma = 2, g = 0.015$, and $g' = 0.025$. With this parameterisation $\lambda^C = 0.826$, which means that households in A would be willing to give up 17.4% of initial consumption to raise the rate of economic growth from 1.5% to 2.5% per year while also keeping fluctuations.

So what does this simple exercise show? It seems to suggest that growth matters a lot more than business cycle fluctuations, which probably explains why Lucas has chosen to focus on long-term growth rather than business cycle research. But there are some things that this hasn't addressed: distributional consequences of business cycles, other values of risk aversion, utility functions say nothing about unemployment, and other social consequences of recessions (e.g. political instability and crime). So while Lucas' napkin math seems to suggest that business cycles aren't as relevant compared to growth, we could say that there are factors associated with business cycles which we want to minimise, and which are not captured by this simple mathematical exercise.

1.2 Stylised facts of the business cycle

We've gone on for a bit without formally defining what a "business cycle" is, although I suspect many have got a decent understanding of it by now. A business cycle is made of an expansion (boom) and a contraction (recession). During the expansion all good things (GDP, employment, productivity, and so on) tend to go up, or grow faster than "normal", and bad things (e.g. unemployment) tend to fall. During the contraction good things go down and bad things go up.

Figure 6: Detrended GNP and Non-Durable Consumption



Using some of the techniques previously mentioned (while carefully noting caveats of the HP filter), we can extract the cyclical component (the business cycle) from raw macroeconomic time-series. Figure 6 plots detrended real US GNP alongside non-durable consumption (all variables are in logs). We can see a strong positive relationship between the two variables, with consumption leading GDP by a quarter or two.

Table 1 gives a more complete description of the volatilities and cross correlations of consumption and labour market variables. We quote results from the US because most of the theoretical models we shall examine have been constructed with this data in mind. However, surprisingly, the UK exhibits very similar properties as the US (with a bit more even split between hours and unemployment).

Table 1: Cyclical Behaviour of the US Economy (1954Q1-1991Q2)

Variable	Sd%	Cross-correlation of output with:								
		t-4	t-3	t-2	t-1	t	t+1	t+2	t+3	t+4
GNP	1.72	0.16	0.38	0.63	0.85	1.00	0.85	0.63	0.38	0.16
CND	0.86	0.40	0.55	0.68	0.78	0.77	0.64	0.47	0.27	0.06
CD	4.96	0.37	0.49	0.65	0.75	0.78	0.61	0.38	0.11	-0.13
H	1.59	0.09	0.30	0.53	0.74	0.86	0.82	0.69	0.52	0.32
Ave H	0.63	0.16	0.34	0.48	0.63	0.62	0.52	0.37	0.23	0.09
L	1.14	0.04	0.23	0.46	0.69	0.85	0.86	0.76	0.59	0.40
GNP/L	0.90	0.14	0.20	0.30	0.33	0.41	0.19	0.00	-0.18	-0.25
Ave W	0.55	0.25	0.21	0.14	0.09	0.03	-0.07	-0.09	-0.09	-0.09

Source: *Frontiers of Business Cycle Research* (Cooley and Prescott 1995).
Sd% denotes standard deviations, $t-j$ denotes the correlation between GNP at time t and the variable denoted by the first column at time $t-j$. CND stands for non-durable consumption, CD for durable consumption, H for total hours worked, Ave H is average hours worked per employee, L is employment, GNP/L is productivity, Ave W is average hourly wage based on national accounts. All unemployment data is based on household surveys.

There are six main stylised facts which emerge from Table 1:

1. Consumption is smoother than output.

2. Volatility in GNP is similar in magnitude to volatility in total hours.

3. Volatility in employment is greater than volatility in average hours. Therefore most labour market adjustments operate on the extensive rather than intensive margin.

4. Productivity is slightly pro-cyclical.

5. Wages are less variable than productivity.

6. There is no correlation between wages and output (nor with employment for that matter).

In terms of the neoclassical model's performance we will show that the model is relatively successful at explaining why consumption is smoother than output (at least for the US). Fact 2 shows how important labour market fluctuations are to the business cycle. The sections on unemployment later in the notes examine a number of models which try to account for fact 3, but this represents a significant problem for the basic neoclassical model. Facts 4-6 are also very problematic for the neoclassical model. The findings by Prescott and Cooley are also verified by the findings of King and Rebelo (1999), which we summarise in Table 2.

Table 2: Business Cycle Statistics for the US Economy

	Standard Deviation	Relative Standard Deviation	First Order Auto-correlation	Contemporaneous Correlation with Output
Y	1.81	1.00	0.84	1.00
C	1.35	0.74	0.80	0.88
I	5.30	2.93	0.87	0.80
N	1.79	0.99	0.88	0.88
Y/N	1.02	0.56	0.74	0.55
w	0.68	0.38	0.66	0.12
r	0.30	0.16	0.60	-0.35
A	0.98	0.54	0.74	0.78

Source: "Resuscitating Real Business Cycles" (King and Rebelo 1999).
All variables are in logarithms (with the exception of the real interest rate) and have been detrended with the HP filter. Data sources are described in Stock and Watson (1999), who created the real rate using VAR inflation expectations. Y is per capita output, C is per capita consumption, I is per capita investment, N is per capita hours, w is the real wage (compensation per hour), r is the real interest rate, and A is total factor productivity.

Some facts that emerge from the King and Rebelo (1999) study are:

1. Consumption of non-durables is less volatile than output.
2. Consumer durables are more volatile than output.
3. Investment is three times more volatile than output.
4. Government expenditures are less volatile than output.
5. Total hours worked are about the same volatility as output.
6. Capital is much less volatile than output.
7. Employment is as volatile as output, while hours per worker are much less volatile than output.
8. Labour productivity is less volatile than output
9. The real wage is much less volatile than output.

Clearly, most macroeconomic series are pro-cyclical, exhibiting a positive contemporaneous correlation with output, and are very persistent with an autocorrelation order of roughly 0.8 to 0.9. There are three acyclical series: wages, government expenditures, and the capital stock. So, any model that we build will have to account and explain these facts, which we will soon find is quite a challenge.

1.2.1 Technical aside: The AR(1) model and impulse responses

Cyclical components are positively autocorrelated (i.e., positively correlated with their own lagged values) and also exhibit random-looking fluctuations. One simple model that captures these features is the autoregressive of order 1 (AR(1)) model:

$$y_t = \rho y_{t-1} + \epsilon_t. \tag{6}$$

Suppose an AR(1) series starts out at zero. Then there is a unit shock, $\epsilon_t = 1$, then all shocks are zero afterwards. In period t we have $y_t = 1$, in period $t+1$ we have $y_{t+1} = \rho$, in period $t+n$ we have

$y_{t+n} = \rho^n$, and so on. The shock fades away gradually. How fast it does so depends on the size of ρ. The time path of y after this hypothetical shock is known as the impulse response function (IRF).

We can think of the IRF as the path followed from t onwards when shocks are $(\epsilon_t + 1, \epsilon_{t+1}, \epsilon_{t+2}, ...)$ instead of $(\epsilon_t, \epsilon_{t+1}, \epsilon_{t+2}, ...)$. i.e., The incremental effect in all future periods of a unit shock today. IRF graphs are commonly used to illustrate dynamic properties of macro data.

Now consider the model in (6), and suppose that the variance of ϵ_t is σ^2. The long-run variance of y_t is the same as the long-run variance of y_{t-1}, and (remembering that ϵ_t is independent of y_{t-1}) this is given by

$$\sigma_y^2 = \rho^2 \sigma_y^2 + \sigma_\epsilon^2,$$

and this simplifies to

$$\sigma_y^2 = \frac{\sigma_\epsilon^2}{1 - \rho^2}.$$

The variance of output depends positively on both shock variance, σ_ϵ^2, and also on the persistence parameter, ρ. So, the volatility of the series is partly due to the size of shocks but also due to the strength of the propagation mechanism.

1.3 Stylised facts of economic growth

Statistical properties of long-term economic growth were first summarised by Kaldor (1957). These "remarkable historical constancies revealed by recent empirical investigations" quickly become known as the "Kaldor stylised facts." While initially derived from US and UK data, the Kaldor stylised facts were later found to hold for many other countries too. These stylised facts can be summarised as follows:

1. Output per worker grows at a roughly constant rate that does not diminish over time. $\left(\frac{Y}{L}\right) \uparrow$

2. Capital per worker grows over time. $\left(\frac{K}{L}\right) \uparrow$

3. The capital/output ratio is roughly constant. $\overline{K/Y}$

4. The rate of return to capital is constant. $\bar{r}^K$

5. The share of capital and labour in net income are nearly constant. $\bar{\alpha}$

6. Real wages grow over time. $w \uparrow$

7. Constant ratios of consumption to GDP and investment to GDP. $\overline{C/Y}, \overline{I/Y}$

The idea of Kaldor's stylised facts is not that these hold every period, rather that they hold when averaging data over long periods of time. This is exactly what the HP trend is designed to do, so if Kaldor is right we would expect to see fairly constant trend output per worker growth, and so on. Let's see how these stylised facts stack up:

Figure 7: GDP Trend Growth

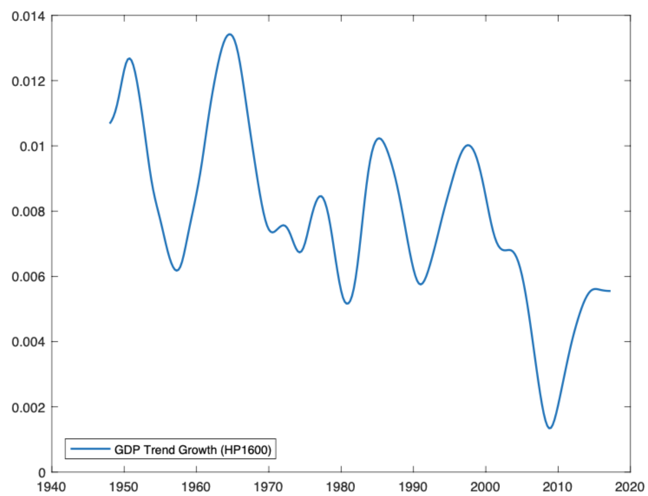
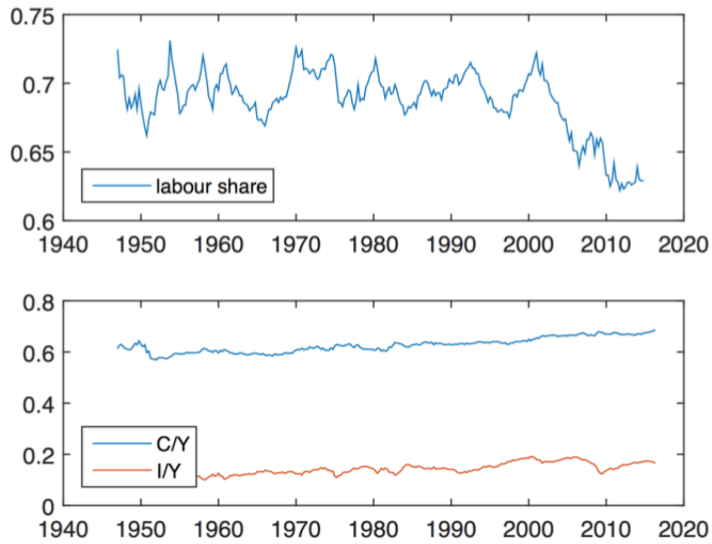


Figure 8: Labour Share and the Great Ratios in the US



Despite some recent declines in the labour share of output, the Kaldor stylised facts look pretty good. Our business cycle stylised facts and long-term growth stylised facts are looking pretty reliable. The only thing left to do now is to build models that can explain and replicate these facts.

1.3.1 A note on capital in the twenty-first century

Though not covered in most macroeconomic courses, it is worth talking a bit about Piketty's 2014 piece, *Capital in the Twenty-First Century*, to see how it relates to the Kaldor stylised facts . Piketty states there are "Two Fundamental Laws of Capitalism":

1. $\alpha = r\beta$, where r is the net rate of profit, and β is defined by the 2nd Law of Capitalism.

2. $\beta = s/g$, where β is the "ratio of wealth to income" and s is the savings rate.

It should be noted that the "2nd Law of Capitalism" is basically from conventional macroeconomic growth research: $sY = gK \implies K/Y = s/g = \beta$. Piketty argues that s has been broadly fixed throughout history, however g has decreased, leading to an increase in the ratio of wealth to income, β. This has lead to the profit share of capital rising in most developed, capitalist economies (which also means that labour's share, $(1-\alpha)$, is decreasing. Certainly, observing the data in Figure 8, we do see a decline in the labour share of income in the twenty-first century.

Some macroeconomists have stated that Piketty should have considered depreciation in his equation, $sY = (g+\delta)K$, so as to attain $\beta = s/(g+\delta)$. It should be noted that adjusting for depreciation does not invalidate the findings of Piketty. Ton Van Schaik has a good piece in VOXEU-CEPR on this, and it's a good read for those interested.[5] I also highly recommend Robert Solow's review of Piketty's book too.[6]

1.4 The consumption Euler equation and a general equilibrium model

It's now time to begin building some models that can explain the stylised facts we've observed. We will begin with very simple neoclassical models which feature only households and firms. These include the Solow-Swan model, the Ramsey model, the overlapping generations (OLG) model, and models of endogenous growth such as the AK model. These models are all rudimentary, but provide some key insights, particularly when it comes to explaining long term growth.

As a preview to where we're heading, after neoclassical growth models we will discuss vector autoregression (VAR) models and stochastic difference equations, and then move onto the real business cycle (RBC) model. The RBC model takes the Ramsey model as its foundation, and then builds in mechanisms to account for business cycle fluctuations. We will see that even the baseline RBC model can go a long way in explaining a lot of the business cycle moments we found. The baseline RBC model can be tweaked and enhanced to improve its performance, but ultimately those efforts will lead to a dead end. There are simply too many factors that the RBC model cannot account for without loosening some of the strict assumptions that keep the model together. While the RBC model will struggle to explain some dynamics in the data, it will serve as a "best case scenario" benchmark. We will then add money to the RBC model, a monetary authority/government, imperfect competition, and sticky prices to then get the New Keynesian (NK) model. The NK model will serve as our workhorse model[7] to explain macroeconomic shocks and optimal policy. But, again, that's all way ahead; for

[5] https://voxeu.org/article/piketty-s-two-laws
[6] https://newrepublic.com/article/117429/capital-twenty-first-century-thomas-piketty-reviewed
[7] With some additional tweaks and modifications, of course. This is macroeconomics after all.

now, let's build some neoclassical models. We'll start with a model that is quite similar to a Robinson Crusoe economy with fixed labour.[8]

We assume the existence of a utility function $u(c_t^i)$ where c_t^i is consumption of individual i. Notice that utility depends only on current consumption – that is, preferences are intertemporally separable. This implies that previous consumption choices do not influence marginal utility in this period. Clearly, previous values of c_t^i will influence the current choice of consumption through budget constraint effects but they do not directly influence the utility function. A number of recent studies have stressed the importance of not having intertemporally separable preferences, but for the sake of our simple model we won't pay attention to that. Households have to make two decisions: (i) how much to spend, and (ii) how much to save. Households receive a gross interest rate, $R_t = (1 + r_t)$, on any savings, and receive an endowment y_t^i in period t. Both R_t and y_t^i are treated as beyond the household's control and are known with certainty into the infinite future. Assume that the household wishes to maximise the present value of the discounted stream of utility. That is

$$\max_{\{c_{t+s}^i, a_{t+s}^i\}} \sum_{s=0}^{\infty} \beta^s u(c_{t+s}^i), \tag{7}$$

subject to the following constraints:

$$c_{t+s}^i + a_{t+s}^i = y_{t+s}^i + R_{t+s-1} a_{t+s-1}^i, \tag{8}$$

$$\lim_{T\to\infty} \frac{a_T^i}{\prod_{s=t+1}^{T-1} R_s} = 0, \tag{9}$$

where a_t^i denotes the household's asset holdings and $\beta \in (0, 1)$ is the stochastic discount factor. The second constraint is the no-Ponzi condition that rules out consumption plans based on ever-increasing levels of debt. It serves as the transversality condition to uniquely pin down the optimal path for consumption. Note that if we define $\tilde{R}_{t+1} = R_0 R_1 R_2 ... R_{t+1}$ for $t > 0$ then we can solve the period

[8]See McCandless (2008, pp. 33-48) for a good treatment of neoclassical growth models (as well as RBC models).

budget constraint (8) forward to get a present value budget constraint:

$$\sum_{t=0}^{\infty} \frac{c_t^i}{\tilde{R}_t} = a_0^i + \sum_{t=0}^{\infty} \frac{y_t^i}{\tilde{R}_t}, \tag{10}$$

which states that the present discounted value of consumption must equal initial assets plus the present discounted value of the endowment stream. To see this, begin by writing (8) as

$$y_t^i + R_{t-1} a_{t-1}^i - c_t^i - a_t^i = 0,$$

and then roll the budget constraint forward one period and then substitute the result for a_t^i back into the period t budget constraint:

$$\begin{aligned} 0 &= y_{t+1}^i + R_t a_t^i - c_{t+1}^i - a_{t+1}^i \\ \implies a_t^i &= \frac{c_{t+1}^i + a_{t+1}^i - y_{t+1}^i}{R_t}, \end{aligned}$$

put back into (8):

$$y_t^i + R_{t-1} a_{t-1}^i - c_t^i - \left(\frac{c_{t-1}^i + a_{t-1}^i - y_{t+1}^i}{R_t} \right) = 0.$$

Do this again for a_{t+1}^i to get

$$\begin{aligned} y_t^i + \frac{y_{t+1}^i}{R_t} + R_{t-1} a_{t-1}^i - c_t^i - \frac{c_{t+1}^i}{R_t} - \frac{1}{R_t} \left(\frac{c_{t+2}^i + a_{t+2}^i - y_{t+2}^i}{R_{t+1}} \right) &= 0 \\ \Leftrightarrow y_t^i + \frac{y_{t+1}^i}{R_t} + \frac{y_{t+2}^i}{R_t R_{t+1}} + R_{t-1} a_{t-1}^i - c_t^i - \frac{c_{t+1}^i}{R_t} - \frac{c_{t+2}^i}{R_t R_{t+1}} - \frac{1}{R_t} \frac{1}{R_{t+1}} a_{t+2}^i &= 0, \end{aligned}$$

and eventually we have

$$\sum_{s=0}^{\infty} \frac{y_{t+s}^i R_t}{\prod_{j=0}^{s} R_{t+j}} + R_{t-1} a_{t-1}^i - \sum_{s=0}^{\infty} \frac{c_{t+s}^i R_t}{\prod_{j=0}^{s} R_{t+j}} - \frac{a_{t+\infty}^i}{R_t R_{t+1} ... R_{t+\infty-1}} = 0.$$

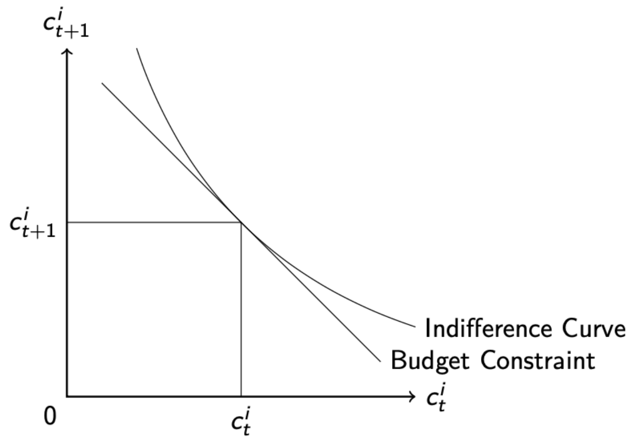
Rearrange and assume that $t-1$ is period 0 to get (10).

Now, how are we going to solve the household maximisation problem? We have four method available to us here.

1.4.1 Graphical approach

Consider two consecutive periods t and $t+1$ in the maximisation problem (7). From the utility function we can draw the indifference curves in $(c_t^i, c_{t+1}^i) \in \mathbb{R}^2$ space.

Figure 9: Graphical Solution to Household Problem



The utility function is

$$u(c_t^i) + \beta u(c_{t+1}^i) + \sum_{s=2}^{\infty} \beta^s u(c_{t+s}^i),$$

and the slope of an indifference curve can be calculated by total differentiation of the utility function and is given by

$$\begin{aligned} 0 &= u_{c,t} dc_t^i + \beta u_{c,t+1} dc_{t+1}^i \\ \implies \frac{dc_{t+1}^i}{dc_t^i} &= -\frac{1}{\beta} \frac{u_{c,t}}{u_{c,t+1}}, \end{aligned}$$

and this is what we call the marginal rate of substitution (MRS). We then add the budget constraint with a slope given by iterating the budget constraint forward:

$$a_t^i = R_{t-1}a_{t-1}^i + y_t^i - c_t^i,$$
$$a_{t+1}^i = R_t a_t^i + y_{t+1}^i - c_{t+1}^i,$$

to then get

$$\frac{a_{t+1}^i - y_{t+1}^i + c_{t+1}^i}{R_t} = R_{t-1}a_{t-1}^i + y_t^i - c_t^i,$$

where it's clear that

$$-\frac{1}{R_t}dc_{t+1}^i = dc_t^i$$
$$\implies \frac{dc_{t+1}^i}{dc_t^i} = -R_t,$$

which is the marginal rate of transformation (MRT). Use basic microeconomic theory to justify that the solution to the household's problem is where MRS = MRT:

$$\frac{1}{\beta}\frac{u_{c,t}}{u_{c,t+1}} = R_t$$
$$\Leftrightarrow u_{c,t} = \beta R_t u_{c,t+1}, \tag{11}$$

which is the consumption Euler equation.

1.4.2 Direct substitution/"sledgehammer" approach

Next is the most brute-force method of solving the household's problem. Simply rearrange the budget constraint (8) to get c_t^i in terms of the other variables, and then substitute into the objective function (7):

$$\max_{\{a_{t+s}\}} \sum_{s=0}^{\infty} \beta^s u(R_{t+s-1}a_{t+s-1}^i + y_{t+s}^i - a_{t+s}^i).$$

Differentiating the above summation term with respect to a_t, and setting the derivative equal to zero gives

$$-u_{c,t} + u_{c,t+1}\beta R_t = 0,$$

and after rearranging we get

$$u_{c,t} = \beta R_t u_{c,t+1},$$

which is simply (11).

1.4.3 Value function approach

This is the dynamic programming approach, which has a large range of uses in macroeconomics.[9] Write the value function as

$$V(a_{t-1}^i) = \max_{a_t^i} \left[u(R_{t-1}a_{t-1}^i + y_t^i - a_t^i) + \beta V(a_t^i) \right], \tag{12}$$

noting that a_t^i is the state variable and c_t^i is the control variable. The first order condition (FOC) with respect to assets a_t^i is

$$\begin{aligned} 0 &= -u_{c,t} + \beta V'(a_t^i) \\ \implies u_{c,t} &= \beta V'(a_t^i). \end{aligned} \tag{13}$$

As is usual in dynamic programming, we do not know the form of the value function $V(a_{t-1}^i)$, but we do know its first derivative $V'(a_{t-1}^i)$. Differentiating the value function (12) yields

$$V'(a_{t-1}^i) = u_{c,t}R_{t-1},$$

and if we roll one period ahead

$$V'(a_t^i) = u_{c,t+1}R_t,$$

[9] See *Recursive Macroeconomic Theory* by Ljungqvist and Sargent, or *Recursive Methods in Economic Dynamics* by Stokey et al.

then substitute into (13), we get

$$u_{c,t} = \beta R_t u_{c,t+1},$$

which is nothing but the consumption Euler equation.

1.4.4 The Lagrangian approach

This should be very familiar from microeconomics and macroeconomics in undergraduate studies. Begin by setting up the Lagrangian:

$$\mathcal{L}^i = \sum_{s=0}^{\infty} \beta^s u(c^i_{t+s}) + \sum_{s=0}^{\infty} \lambda^i_{t+s} \beta^s \left(R_{t+s-1} a_{t+s-1} + y^i_{t+s} - c^i_{t+s} - a^i_{t+s} \right).$$

This is the present value formulation of the Lagrangian as the Lagrangian multiplier, λ^i_{t+s}, is discounted by β^s back to its present value. It is equally valid to work with the current value Lagrangian and write the second term without discounting, i.e. $\tilde{\lambda}^i_{t+s} = \lambda^i_{t+s}\beta^s$. They are mathematically equivalent but sometimes it is more convenient to work with one than the other. The FOCs with respect to c^i_t, c^i_{t+1}, and a^i_t are

$$\begin{aligned} u_{c,t} &= \lambda^i_t, \\ u_{c,t+1} &= \lambda^i_{t+1}, \\ \lambda^i_{t+1}\beta R_t - \lambda^i_t &= 0. \end{aligned}$$

Do some substitution and rearranging and then we get

$$u_{c,t} = \beta R_t u_{c,t+1},$$

which is the consumption Euler equation.

1.4.5 Implications of the consumption Euler equation

We will soon assign particular functional forms to the utility function, but from (11) we can already notice some of the major implications of the neoclassical model for consumption. We can see that what determines the growth in the marginal utility of consumption (which as we shall see is closely linked to consumption) is the interest rate, R_t. In our model we have assumed that the consumer can only invest in one asset, a_t^i. However, equation (11) holds for any asset the consumer invests in so we should think of R_t more widely as the return on any asset.

To see this more clearly, assume that $R_t = \bar{R}$ and that $\beta\bar{R} = 1$. This then implies that $\mathbb{E}_t\left[u_{c,t+1}/u_{c,t}\right] = 1$ so that agents do not expect their marginal utility to change between time periods. As a consequence, they are not expecting their consumption to change either. Similar reasoning suggests that if $\beta R_t > 1$ then the expectation of the ratio of marginal utilities of consumption is less than one, which given that marginal utility is declining in consumption (i.e. $u'' < 0$) implies that agents must be expecting consumption to increase. Similarly, if $\beta R_t < 1$ then consumption is expected to fall.

In all cases, the only thing which determines consumption growth is the rate of return/interest rate and not income. The rationale for the interest rate effect is as follows. If consumers know that savings this period are going to earn a high rate of return, there is an incentive for them to save more by having lower consumption. For a given end of period consumption level, the lower the level of initial consumption the faster is the growth rate.

1.4.6 Econometric evidence on the consumption Euler equation

The first paper to examine the consumption Euler equation was Hall (1978). He focused on utility functions which were well approximated by quadratic functions and assumed a constant interest rate which satisfies $\beta R = 1$. The result of this model is that consumption changes should be unpredictable. This paper sparked one of the largest literatures in applied econometrics. Hall found that consumption growth was unpredicted by income growth, but could be forecast by stock market prices. He interpreted this as a mild victory for the model. Subsequent work has been less kind to the model and has found that consumption growth does display a small but significant dependence on past income growth. However, the overall prediction that agents try and smooth their consumption over the business cycle is partly correct (see Table 1).

1.4.7 Taking the model to general equilibrium

The consumption Euler equation explains the dynamics of consumption of an individual i. To make further progress we make the simplifying assumption that utility is logarithmic in consumption, i.e., $u(c_t^i) = \log c_t^i$. In this case, the marginal utility of consumption is given by $u'(c_t^i) = 1/c_t^i$ and the consumption Euler equation is therefore

$$c_{t+1}^i = \beta R_t c_t^i.$$

General equilibrium requires that all individuals satisfy their Euler equations for consumption and that markets clear (Walras' Law), which is achieved here by the interest rate adjusting to clear the market. Since there is no aggregate savings device, market clearing requires that individual net claims must sum to zero and $\sum_i a_t^i = 0,\ \forall t$. In this case, all the endowment is consumed each period and $\sum_i y_t^i = \sum_i c_t^i,\ \forall t$. When we aggregate the individual consumption Euler equations with logarithmic utility, we find that $\sum_i c_{t+1}^i = \beta R_t \sum_i c_t^i$, and hence:

$$\sum_i y_{t+1}^i = \beta R_t \sum_i y_t.$$

Defining $\bar{y}_t$ and $\bar{y}_{t+1}$ as the average endowments in periods t and $t+1$, we see that the rate of interest is determined by the ratio of endowments in the two periods

$$\beta R_t = \frac{\bar{y}_{t+1}}{\bar{y}_t}.$$

1.5 Comments and key readings

The simple example illustrates a lot of what will become familiar in macroeconomics. The dominant approach is to view outcomes as the result of purposeful interaction of many agents in many markets. The two main elements are optimisation and equilibrium: (i) taking some sets X and Z as given, agents optimise, and (ii) there is a consistency requirement on X and Z such as Walrasian market clearing consistency or Nash equilibrium. Modern macroeconomics stresses the importance of dynamic optimisation, motivated in part by the Lucas critique (absence of stable behavioural equations) and the notion of intertemporal substitution.

Key readings for this section are Barro and King (1984), Hall (1978), Hall (1988), Lucas (1978), Mankiw (1990), Mankiw et al. (1985), and Romer (2012) (chapters 4 and 5).

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2 The Diamond Overlapping Generations Model

2.1 Introduction

The endowment economy we examined in the previous section had no firms and only financial capital. This isn't a crazy idea if labour is supplied inelastically and is the only input in production. However, physical capital is clearly important in the economy. We now allow agents in the economy to accumulate physical capital. To keep things simple, however, we will continue to assume that labour supply is exogenous. There are two important classes of models with capital accumulation: the overlapping generation (OLG) model and the representative agent model. We will start with the OLG model in this section.

Peter Diamond produced a version of the OLG model introduced by Samuelson in which the savings rate is endogenous and can change with other parameters of the economy – addressing one of the biggest weaknesses of the Solow-Swan model.

Neoclassical growth models such as the OLG model typically have two entities (firms and households[10]) and three markets: goods, labour, and capital. We can generally disregard financial markets as they will be redundant. While our analysis of the endowment economy focussed mainly on the determination of interest rates, these models focus extensively on the determination of consumption and the capital stock. In the baseline case we will assume no population growth and no technological progress. We will later consider these as extensions. Agents in the OLG model live for two periods and must make decisions in the first period of their lives about their consumption in both periods of life. Sounds quite morbid. Individuals who have substantial income in the first periods of life may save some of this in the form of capital or lending and are able to consume more than they otherwise might in the second period of life. People live two periods in these economies because this is the smallest number of periods that permits a savings decision.

[10]We can add government into the model too.

2.2 The basic two-period OLG model

The basic OLG model is described as follows. Let there be an infinite sequence of time, $t = 0, 1, 2, ..., \infty$. The generation born in period t is referred to as generation t. There are $N(t)$ members of generation t, and people live for 2 periods, and generation t is young in t and old in $t+1$. Generation t does not exist in period $t+2$.

A member h of generation t has utility

$$u_t^h(c_t^h(t), c_t^h(t+1)), \tag{14}$$

where, to clarify the notation, $c_t^h(t+1)$ denotes the consumption of the aggregate consumption good by individual h of generation t in period $t+1$. Production takes place in competitive firms with homogenous of degree 1 (HOD1) production technology with constant returns to scale (CRS), implying that they do not produce economic profits.[11] Production in period t is given by

$$Y(t) = F(K(t), L(t)),$$

where $L(t)$ is the total labour used in production and $K(t)$ is the total capital.

Individuals are endowed with lifetime endowment of labour given by

$$l_t^h = [l_t^h(t), l_t^h(t+1)].$$

[11] It's worth expanding on how and why we can assume this. Suppose firms have the production technology,

$$f(x_t) = x,$$

and firms want to maximise profits, $\Pi = px - wx$. Clearly, if $p > w$, then no profitable production plan exists for this firm. So $p \leq w$ is a necessary condition, in which case max profits will be zero, and will occur for any CRS technology. Why? Suppose we have (p, w) where $\Pi > 0$, then

$$\Pi^* = pf(x^*) - wx^* > 0,$$

and suppose we scale up production by $\lambda > 1$, so our profits will be

$$pf(\lambda x^*) - w\lambda x^* = \lambda(pf(x^*) - wx^*) = \lambda\Pi^* > \Pi^*.$$

This means that if profits are ever positive, they can always be scaled up, and are unbounded and no maximal production plan will exist. So, the only nontrivial profit maximising position for a CRS firm is one involving zero profits.

Total labour is given as

$$L(t) = \sum_{h=1}^{N(t)} l_t^h(t) + \sum_{h=1}^{N(t-1)} l_{t-1}^h(t).$$

Aggregate labour of the young at time t is the first component of the RHS, and the aggregate labour of the old is the second component of the RHS. We also assume that $K(t)$ depreciates fully. This assumption removes the complication of a capital market between members of different generations.

The economy has the following resource constraint:

$$Y(t) = F(K(t), L(t)) \geq \sum_{h=1}^{N(t)} c_t^h(t) + \sum_{h=1}^{N(t-1)} c_{t-1}^h(t) + K(t+1).$$

The production of period t goes either to consumption of the young or the old or to capital for use in period $t+1$.

We assume that the economic organisation of the economy is one of perfectly competitive markets where individuals are owners of their own labour. Members of generation t earn income in period t by offering all their labour endowment to firms at market wage, w_t, and use income to fuel consumption in period t, to fund borrowing and lending to other members of generation t, and for accumulation of private capital. The budget constraint for individual h when they're young is

$$w_t l_t^h(t) = c_t^h(t) + a^h(t) + k^h(t+1), \tag{15}$$

where $a^h(t)$ are net asset holdings of individual h. $a^h(t) < 0$ implies net borrowing from other members of generation t. Because of the overlapping nature of the generations, borrowing and lending can only occur among members of the same generation.

Suppose that a young person of generation t lends some goods to an old member of generation $t-1$ in period t with the expectation of being paid back in periods $t+1$. In period $t+1$, this rather naive young person hunts for the member of generation $t-1$ so that he/she can be paid back. Unfortunately, members of generation $t-1$ are now all dead and the dead cannot be forced to pay back their debts. Individuals know this and will not make loans to members of other generations. Individuals cannot borrow or lend across generations so

$$\sum_{h=1}^{N(t)} a^h(t) = 0.$$

In period $t+1$, a member of generation t has income from the labour supplied in period $t+1$, from interest earned on any loans that were made in period t, and from the rent on capital that they accumulated in period t. Since this is the last period of life, all income will be consumed. Therefore, the budget constraint for generation t individual in period $t+1$ is:

$$c_t^h(t+1) = w_{t+1} l_t^h(t+1) + R_t a^h(t) + R_{t+1} k^h(t+1), \tag{16}$$

where R_t is the interest paid on loans between period t and $t+1$.

Individuals are assumed to have perfect foresight in the sense that they know, when young, what wages and rents will be when they are old. In addition, no fraud is permitted so that all loans are paid back with the agreed upon interest.

Factor prices are determined by their marginal products due to competitive equilibrium:

$$w_t = F_L(K(t), L(t)), \tag{17}$$

$$R_t = F_K(K(t), L(t)), \tag{18}$$

where $F_i(\cdot, \cdot)$ is the partial derivative of the production function with respect to its i-th component.

We can combine the budget constraints of the young and old. From (15):

$$a^h(t) = w_t l_t^h(t) - c_t^h(t) - k^h(t+1),$$

and substitute this expression into (16) to get:

$$c_t^h(t+1) = w_{t+1} l_t^h(t+1) + R_t w_t l_t^h(t) - R_t c_t^h(t) - R_t k^h(t+1) + R_{t+1} k^h(t+1),$$

collecting terms, we can yield an expression for $c_t^h(t)$,

$$c_t^h(t) = \frac{w_{t+1} l_t^h(t+1) - c_t^h(t+1)}{R_t} + w_t l_t^h(t) - k^h(t+1)\left[1 - \frac{R_{t+1}}{R_t}\right].$$

Since we assume that there are no arbitrage opportunities, the return on capital should equal the return on loans amongst members of a particular cohort, $R_t = R_{t+1}$. Thus the budget constraint becomes:

$$c_t^h(t) + \frac{c_t^h(t+1)}{R_t} = w_t l_t^h(t) + \frac{w_{t+1} l_t^h(t+1)}{R_t}. \tag{19}$$

In words: The present value of lifetime consumption must equal the present value of lifetime wage income.

A competitive equilibrium consists of a sequence of prices

$$\{w_t, R_t\}_{t=0}^{\infty},$$

and quantities

$$\left\{\{c_t^h(t)\}_{h=1}^{N(t)}, \{c_{t-1}^h(t)\}_{h=1}^{N(t-1)}, K(t+1)\right\}_{t=0}^{\infty},$$

such that each member h of each generation $t > 0$ maximises utility (14) subject to their lifetime budget constraint given by (19), and so that the equilibrium conditions

$$\begin{aligned}
R_{t+1} &= R_t, \\
w_t &= F_L(K(t), L(t)), \\
R_t &= F_K(K(t), L(t)), \\
L(t) &= \sum_{h=1}^{N(t)} l_t^h(t) + \sum_{h=1}^{N(t-1)} l_{t-1}^h(t),
\end{aligned}$$

hold each period.

Note that in the above definition we did not define the individual holdings of either lending or of capital. This is because they offer exactly the same return and there are an infinite number of distributions of lending and capital holdings among members of a generation that would meet the equilibrium conditions. Two example distributions for an economy where all members of a generation are identical are i) person $h = 1$ borrows from everyone else and holds all the capital; and ii) no one borrows and each person holds $K(t+1)/N(t)$ units of capital. These two distributions would result in the same total capital stock and the same equilibrium as the above definition.

Now, substitute the lifetime budget constraint (19) into the utility function, to set up household h of generation t's problem

$$\max_{c_t^h(t)} \; u\left(c_t^h(t), R_t w_t l_t^h(t) - w_{t+1} l_t^h(t+1) - R_t c_t^h(t)\right),$$

where, for individual h, the assumption of perfect foresight means that the values of all the other parameters are known. The FOC is:

$$u_1\left(c_t^h(t), R_t w_t l_t^h(t) + w_{t+1} l_t^h(t+1) - R_t c_t^h(t)\right) = R_t u_2\left(c_t^h(t), R_t w_t l_t^h(t) + w_{t+1} l_t^h(t+1) - R_t c_t^h(t)\right), \tag{20}$$

where $u_i(\cdot,\cdot)$ is the partial derivative of the utility function with respect to its i-th element. Using the budget constraint when young (15), we can find a savings function for individual h of generation t, $s_t^h(\cdot)$, where

$$s_t^h(w_t, w_{t+1}, R_t) = a^h(t) + k^h(t+1).$$

Summing the savings of all members of generation t, we define an aggregate savings function $S(\cdot)$, as equal to

$$S_t(\cdot) = \sum_{h=1}^{N(t)} s_t^h(\cdot) = \sum_{h=1}^{N(t)} a^h(t) + \sum_{h=1}^{N(t)} k^h(t+1).$$

Given that, in equilibrium,

$$\sum_{h=1}^{N(t)} a^h(t) = 0,$$

and

$$K(t+1) = \sum_{j=1}^{N(t)} k^h(t+1),$$

the aggregate savings equation can be written as

$$S_t(w_t, w_{t+1}, R_t) = K(t+1).$$

Substituting R_{t+1} for R_t, and using the equilibrium conditions for factor prices ((17) and (18)) in periods t and $t+1$, we can write aggregate savings as:

$$S_t\left(\underbrace{F_L(K(t), L(t))}_{w_t}, \underbrace{F_L(K(t+1), L(t+1))}_{w_{t+1}}, \underbrace{F_K(K(t+1), L(t+1))}_{R_t}\right) = K(t+1).$$

The above expression gives $K(t+1)$ as an implicit functions of the labour supplies in each periods, $L_t(t), L_{t-1}(t), L_t(t+1)$, the parameters of the utility functions and the production function, and $K(t)$. Since, as the model is constructed, all of these except $K(t)$ are constants through time, one can find the capital stock in $t+1$ as a function of the capital stock in time t:

$$K(t+1) = G(K(t)). \tag{21}$$

This is a first-order difference equation/law of motion that describes the growth path of the economy.

2.2.1 An example OLG economy

Suppose that the agents in our model possessed log-utility:

$$u(c_t) = \ln c_t,$$

and the production technology is of Cobb-Douglas form:

$$F(K(t), L(t)) = K(t)^\alpha L(t)^{1-\alpha}, \ \alpha \in (0, 1).$$

Our problem would be

$$\max_{c_t^h(t)} \ \ln c_t^h(t) + \beta \ln c_t^h(t+1),$$

and using our lifetime budget constraint (19) we can write this as

$$\max_{c_t^h(t)} \ln c_t^h(t) + \beta \ln \left(R_t w_t l_t^h(t) - w_{t+1} l_t^h(t+1) - R_t c_t^h(t) \right),$$

and with the following FOC:

$$\begin{aligned} 0 &= \frac{1}{c_t^h(t)} - \underbrace{\frac{\beta R_t}{R_t w_t l_t^h(t) - w_{t+1} l_t^h(t+1) - R_t c_t^h(t)}}_{c_t^h(t+1)} \\ \implies 1 &= \beta \frac{R_t c_t^h(t)}{c_t^h(t+1)}. \end{aligned}$$

The above equation is nothing but the consumption Euler equation. Now, substitute the optimal consumption given by the Euler equation back into the budget constraint:

$$\begin{aligned} c_t^h(t) + \frac{c_t^h(t+1)}{R_t} &= w_t l_t^h(t) + \frac{w_{t+1} l_t^h(t+1)}{R_t} \\ \implies c_t^h(t) + \frac{1}{R_t}\left[\beta R_t c_t^h(t)\right] &= w_t l_t^h(t) + \frac{w_{t+1} l_t^h(t+1)}{R_t} \\ c_t^h(t)(1+\beta) &= w_t l_t^h(t), \end{aligned}$$

where we also assume that the agent does not work when they're old, so we have

$$c_t^h(t) = \frac{w_t l_t^h(t)}{1+\beta}. \tag{22}$$

Now that have consumption per period for an individual h of generation t in period t, we want to pin down aggregate savings, which help us get the law of motion of capital in this model. But first, we need our factor prices:

$$\begin{aligned} \frac{\partial Y(t)}{\partial K(t)} &= R_t = \alpha \left[\frac{K(t)}{L(t)}\right]^{\alpha-1} = \alpha k(t)^{\alpha-1}, \\ \frac{\partial Y(t)}{\partial L(t)} &= w_t = (1-\alpha) \left[\frac{K(t)}{L(t)}\right]^{\alpha} = (1-\alpha) k(t)^{\alpha}, \end{aligned}$$

and from our household FOC, we have

$$c_t^h(t) = \frac{w_t l_t^h(t)}{1+\beta} = \frac{l_t^h(t)}{1+\beta}(1-\alpha)k(t)^\alpha,$$

and aggregating across the cohort yields

$$\begin{aligned} C_t(t) &= \frac{1}{1+\beta}(1-\alpha)K(t)^\alpha L(t)^{1-\alpha} \\ &= \left(\frac{1-\alpha}{1+\beta}\right) Y(t). \end{aligned}$$

So savings is given by

$$\begin{aligned} S(t) &= Y(t) - C_t(t) \\ &= Y(t) - \left(\frac{1-\alpha}{1+\beta}\right) Y(t) \\ &= \left(\frac{\alpha+\beta}{1+\beta}\right) Y(t), \end{aligned}$$

and since $S(t) = K(t+1)$,

$$K(t+1) = \left(\frac{\alpha+\beta}{1+\beta}\right) Y(t).$$

If we assume that labour is supplied inelastically by the young, then the law of motion of capital can be written as

$$K(t+1) = \left(\frac{\alpha+\beta}{1+\beta}\right) K(t)^\alpha. \tag{23}$$

The steady state capital stock, $\bar{K}$, is given by

$$\begin{aligned} \bar{K} &= \left(\frac{\alpha+\beta}{1+\beta}\right) \bar{K}^\alpha \\ \bar{K}^{1-\alpha} &= \left(\frac{\alpha+\beta}{1+\beta}\right) \\ \therefore \bar{K} &= \left(\frac{\alpha+\beta}{1+\beta}\right)^{\frac{1}{1-\alpha}}. \end{aligned} \tag{24}$$

In other words, $\bar{K}$ satisfies the condition $\Delta K(t+1) = 0$:

$$\Delta K(t+1) = 0 = K(t+1) - K(t),$$

and this satisfies these conditions for two values of $\bar{K}$: $\bar{K} = 0$ and the value for $\bar{K}$ in (24). Actually, also, we could log-linearise the law of motion of capital (23):

$$\begin{aligned} \ln K(t+1) &= \ln\left(\frac{\alpha+\beta}{1+\beta}\right) + \alpha \ln K(t) \\ \ln \bar{K} + \frac{1}{\bar{K}}(K(t+1) - \bar{K}) &\approx \ln\left(\frac{\alpha+\beta}{1+\beta}\right) + \alpha \ln \bar{K} + \frac{\alpha}{\bar{K}}(K(t) - \bar{K}), \end{aligned}$$

and we know from (24) that in the steady state $\ln \bar{K} = \ln\left(\frac{\alpha+\beta}{1+\beta}\right) + \alpha \ln \bar{K}$, so we have:

$$\begin{aligned} \frac{1}{\bar{K}}(K(t+1) - \bar{K}) &= \frac{\alpha}{\bar{K}}(K(t) - \bar{K}) \\ \therefore \hat{K}(t+1) &= \alpha \hat{K}(t). \end{aligned}$$

2.2.2 Convergent dynamics in the OLG model

The behaviour of this model out of a steady state is similar to that of the Solow-Swan model. If the initial capital stock is between the two steady states, $0 < K(0) < \left(\frac{\alpha+\beta}{1+\beta}\right)^{\frac{1}{1-\alpha}}$, the capital stock will grow, converging on the positive steady state. This can be seen by simply looking for the range of initial capital stocks for which $K(t+1) > K(t)$, or where

$$\Delta K(t+1) = \left(\frac{\alpha+\beta}{1+\beta}\right) K(t)^{\alpha} - K(t) > 0.$$

This condition holds for positive $K(t)$ when $K(t) < \left(\frac{\alpha+\beta}{1+\beta}\right)^{\frac{1}{1-\alpha}}$. In addition, the rate of growth of the capital stock declines as it grows. Define the gross rate of growth of capital as $\Delta K(t) = K(t+1)/K(t)$. This can be written as

$$\Delta K(t) = \left(\frac{\alpha+\beta}{1+\beta}\right) \frac{K(t)^{\alpha}}{K(t)} = \left(\frac{\alpha+\beta}{1+\beta}\right) K(t)^{\alpha-1}.$$

Taking the derivative of the growth rate with respect to the capital stock yields:

$$\frac{d\Delta K(t)}{dK(t)} = (\alpha - 1)\left(\frac{\alpha + \beta}{1 + \beta}\right) K(t)^{\alpha - 2} < 0.$$

As in the Solow-Swan model, the larger the initial capital stock, the slower the growth rate of capital. In addition, since output is defined by a Cobb-Douglas technology, the gross growth rate of output, $\Delta Y(t) = Y(t+1)/Y(t)$ is equal to

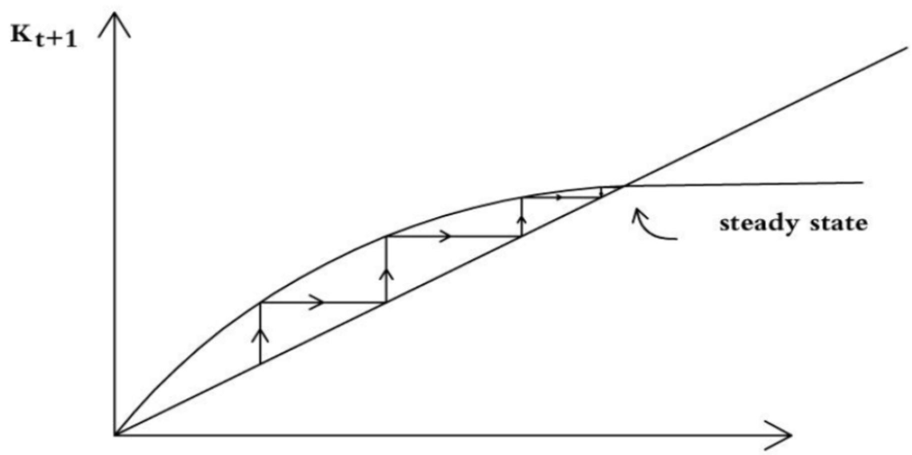
$$\Delta Y(t) = \frac{K(t+1)^{\alpha} L(t+1)^{1-\alpha}}{K(t)^{\alpha} L(t)^{1-\alpha}} = \frac{K(t+1)^{\alpha}}{K(t)^{\alpha}} = \Delta K(t)^{\alpha},$$

where the second equality is given because we have inelastic labour supply equal to unity. The derivative of the gross growth rate of output with respect to the capital stock is:

$$\frac{d\Delta Y(t)}{dK(t)} = \frac{d\left[\left(\frac{\alpha+\beta}{1+\beta}\right) K(t)^{\alpha-1}\right]^{\alpha}}{dK(t)} = \alpha \left[\left(\frac{\alpha + \beta}{1 + \beta}\right) K(t)^{\alpha - 1}\right]^{\alpha - 1} (\alpha - 1)\left(\frac{\alpha + \beta}{1 + \beta}\right) K(t)^{\alpha - 2} < 0,$$

so output growth slows as the capital stock increases.

Figure 10: Convergence in a Simple OLG Model



As Figure 10 shows, we have monotonic convergent dynamics, although there is a possibility of dynamic inefficiency as it may be possible to generate Pareto improvements by transferring resources from each young generation to the current old generation.[12]

2.3 Fiscal policy and non-Ricardian equivalence

With the possibility of efficiency gains by reallocating resources from the old to the young, we now amend the basic OLG model slightly to see the effects of government spending. To make things simple, I will use a change of notation here.[13] Let fiscal policy in the two period OLG model be given by $\{G_t, T_t^0, T_t^1, B_t\}$ where G_t is government spending which directly benefits the young (such as schooling), T_t^0 is taxes on the young, T_t^1 is taxes on the old, B_t is government debt. Interest on government debt is given by $R_t^b = R_t$, so the government budget constraint is:

$$B_{t+1} = G_t - T_t^0 - T_t^1 + R_t^b B_t.$$

The household problem is now:

$$\max_{C_t^0, C_{t+1}^1} \; u(C_t^0 + G_t) + \beta u(C_{t+1}^1),$$

subject to

$$C_t^0 + B_{t+1}^1 + K_{t+1}^1 = w_t - T_t^0, \tag{25}$$

$$C_{t+1}^1 = (B_{t+1}^1 + K_{t+1}^1) R_{t+1} - T_t^1, \tag{26}$$

where B_{t+1}^1 is saving by the household in the form of government debt. Assuming a well-behaved utility function and $T_t^1 = 0$ so there are no taxes on the old, the FOC with respect to K_{t+1}^1 yields the consumption Euler equation:

$$C_{t+1}^1 = \beta \underbrace{R_{t+1}}_{F_K(K_{t+1}^1)} (C_t^0 + G_t), \tag{27}$$

[12]See the discussion in Section 3.8.

[13]Here, I also use the start of period notation for government debt.

and with the household budget constraints and market clearing factor prices, we can find the law of motion of capital. From (26) and (27), we have

$$(B_{t+1}^1 + K_{t+1}^1)R_{t+1} = \beta R_{t+1}(C_t^0 + G_t),$$

and then substitute in the value for C_t^0 from (25),

$$\begin{aligned}
(B_{t+1}^1 + K_{t+1}^1)R_{t+1} &= \beta R_{t+1}(w_t - T_t^0 - B_{t+1}^1 - K_{t+1}^1 + G_t) \\
(B_{t+1}^1 + K_{t+1}^1)R_{t+1} + \beta R_{t+1} K_{t+1}^1 &= \beta R_{t+1}(w_t - T_t^0 - B_{t+1}^1 + G_t) \\
B_{t+1}^1 + K_{t+1}^1 + \beta K_{t+1}^1 &= \beta(w_t - T_t^0 - B_{t+1}^1 + G_t) \\
K_{t+1}^1 &= \frac{\beta}{1+\beta}(w_t - T_t^0 - B_{t+1}^1 + G_t) - \frac{B_{t+1}^1}{1+\beta},
\end{aligned}$$

which gives:

$$K_{t+1}^1 = \frac{\beta}{1+\beta}\left(F_L(K_t^1) - T_t^0 + G_t\right) - B_{t+1}^1.$$

Ricardian equivalence states that it does not matter whether a given sequence of government spending is funded through taxes or debt. To see where this fails in OLG models, we fix the sequence of government spending to $G_t = \bar{G}$ at time t, and $G_{t+i} = 0$, $\forall i > 0$. Now consider two financing schemes. The first funds the one-off government spending by a tax on the young so that $T_t^0 = G_t$. The second places no taxes in period t and instead borrows $B_{t+1} = G_t$ and taxes the young $R_{t+1}^b G_t$ in period $t+1$ to repay the debt. If Ricardian equivalence holds then these two financing schemes will have equivalent aggregate effects. In the first case, government policy does nothing:

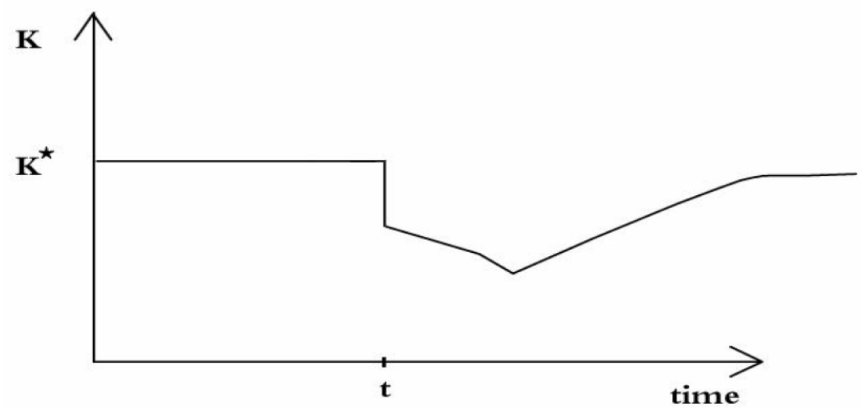
$$K_{t+1}^1 = \frac{\beta}{1+\beta}\left(F_L(K_t^1) - T_t^0 + G_t\right) = \frac{\beta}{1+\beta}F_L(K_t^1),$$

so if the economy is in steady state at time t it will stay there. In contrast, with the second policy we have that:

$$
\begin{aligned}
K_{t+1}^1 &= \frac{\beta}{1+\beta} F_L(K_t^1) - \frac{1}{1+\beta} G_t \\
K_{t+2}^1 &= \frac{\beta}{1+\beta} \left(F_L(K_t^1) - T_{t+1}^0 \right) \\
K_{t+i}^1 &= \frac{\beta}{1+\beta} F_L(K_{t+i-1}^1), \ \forall i > 3.
\end{aligned}
$$

So financing matters and Ricardian equivalence fails to hold in the OLG model. If the economy starts in steady state then it will deviate from steady state for several periods. Figure 11 illustrates this.

Figure 11: Path of Capital Stock After a Bond Financed Fiscal Expansion



2.4 Adding technological change

The model so far has abstracted from technological change. In general, we can think of technological change as entering the production function $Y_t = F(K_t, L_t, \theta_t)$, where θ_t is a technology term that is given exogenously. Fortunately, there are cases where this allows for a simple characterisation of a balanced growth path that satisfies the Kaldor facts. Example production functions which account for technological changed are the Hicks-neutral production function $Y_t = \theta_t F(K_t, L_t)$, the capital-augmenting production function $Y_t = F(\theta_t K_t, L_t)$, and the labour-augmenting production function

$Y_t(K_t, \theta_t L_t)$. The latter of these technologies allows a characterisation consistent with the Kaldor stylised facts!

Consider the labour augmenting production function. Also, let's assume that households in that model possess a constant-relative risk aversion (CRRA) utility function of the form

$$U(c) = \frac{c^{1-\sigma}}{1-\sigma},$$

and let's assume that the law of motion for technological change is:

$$\theta_{t+1} = (1+g)\theta_t,$$

where g is the growth rate of the economy. Now, in a model with exogenous labour, we have the following equilibrium condition (from the consumption Euler equation):

$$\begin{aligned}
u'(C_t) &= \beta R_{t+1} u'(C_{t+1}) \\
u'(w_t - K^1_{t+1}) &= \beta R_{t+1} u'(K^1_{t+1} R_{t+1}) \\
u'(\theta_t F_L(K^1_t, \theta_t L_t) - K^1_{t+1}) &= \beta F_K(K^1_{t+1}, \theta_{t+1} L_{t+1}) u'(K_{t+1} F_K(K^1_{t+1}, \theta_{t+1} L_{t+1})) \\
u'\left(\theta_t F_L\left(\frac{K^1_t}{\theta_t}, 1\right) - K^1_{t+1}\right) &= \beta F_K\left(\frac{K^1_{t+1}}{\theta_{t+1}}, 1\right) u'\left(K^1_{t+1} F_K\left(\frac{K^1_{t+1}}{\theta_{t+1}}, 1\right)\right), \qquad (28)
\end{aligned}$$

since labour is exogenous at $L_t = 1,\ \forall t$. To see why we can write this, consider the case where Y_t is a Cobb-Douglas technology:

$$\begin{aligned}
F(K_t, \theta_t L_t) &= Y_t = K_t^\alpha (\theta_t L_t)^{1-\alpha}, \\
\implies F_L(K_t, \theta_t L_t) &= (1-\alpha) K_t^\alpha \theta_t^{1-\alpha} L_t^{-\alpha} \\
&= (1-\alpha) \left(\frac{K_t}{L_t}\right)^\alpha \theta_t^{1-\alpha} \\
\implies F_L(K_t, \theta_t) &= (1-\alpha) K_t^\alpha \theta_t^{1-\alpha} \\
&= (1-\alpha) \frac{K_t^\alpha}{\theta_t^{\alpha-1}},
\end{aligned}$$

and divide through by θ_t to get:

$$\begin{aligned}(1-\alpha)\frac{K_t^\alpha}{\theta_t^{\alpha-1}}\theta_t^{-1} &= (1-\alpha)\left(\frac{K_t}{\theta_t}\right)^\alpha \\ &= F_L\left(\frac{K_t}{\theta_t},1\right).\end{aligned}$$

We could also easily show this for marginal product of capital. It's also worth noting that the model will have a balanced growth path property if it can be written as a dynamic equation in K_t/θ_t. When the production function has constant returns to scale then derivatives are HOD0, which means that a doubling of all inputs leads to a doubling of output, and doubling of factor prices and resources has no effect in input demand.[14]

Now, assuming we have CRRA utility, (28) yields:

$$\left(\theta_t F_L\left(\frac{K_t^1}{\theta_t},1\right) - K_{t+1}^1\right)^{-\sigma} = \beta F_K\left(\frac{K_{t+1}^1}{\theta_{t+1}},1\right)\left(K_{t+1}^1 F_K\left(\frac{K_{t+1}^1}{\theta_{t+1}},1\right)\right)^{-\sigma},$$

divide through by $\theta_{t+1}^{-\sigma}$ from our law of motion of technological growth:

$$\begin{aligned}\left(\frac{\theta_t}{\theta_{t+1}}F_L\left(\frac{K_t^1}{\theta_t},1\right) - \frac{K_{t+1}^1}{\theta_{t+1}}\right)^{-\sigma} &= \beta F_K\left(\frac{K_{t+1}^1}{\theta_{t+1}},1\right)\left(\frac{K_{t+1}^1}{\theta_{t+1}} F_K\left(\frac{K_{t+1}^1}{\theta_{t+1}},1\right)\right)^{-\sigma}, \\ \Leftrightarrow \left(\frac{1}{1+g}F_L\left(\frac{K_t^1}{\theta_t},1\right) - \frac{K_{t+1}^1}{\theta_{t+1}}\right)^{-\sigma} &= \beta F_K\left(\frac{K_{t+1}^1}{\theta_{t+1}},1\right)\left(\frac{K_{t+1}^1}{\theta_{t+1}} F_K\left(\frac{K_{t+1}^1}{\theta_{t+1}},1\right)\right)^{-\sigma},\end{aligned}$$

which is now a first order difference equation in K_t/θ_t with only a minor change compared to what we saw in the baseline OLG model without technological change. If we assumed log utility, where $\sigma = 1$ and a Cobb-Douglas production technology, then the law of motion for capital relative to technology becomes:

$$\frac{K_{t+1}^1}{\theta_{t+1}} = \frac{\beta(1-\alpha)}{(1+\beta)(1+g)}\left(\frac{K_t^1}{\theta_t}\right)^\alpha.$$

Not much has changed, however. $g > 0$ changes the speed of convergence but nothing else. The model with exogenous technological change has the same qualitative properties as the model without it.

[14]Recall your microeconomics about HOD1, HOD0, and homothetic functions.

2.5 Comments and key readings

An OLG model allows us to make the savings decision endogenous to the model in a relatively simple way. Since agents live only two periods, their optimisation problem involves only those two periods. The overlapping nature of the economy and the fact that the old are holding all the capital that they saved from the previous periods gives the model some persistence. In the version shown here, we did not make the labour supply decision endogenous, but this can be done relatively easily, since it adds only two more variables to the decision problem of each agent: the labour to supply when young and that when old.

Key readings for this section are far and wide. See Acemoglu (2009), Blanchard and Fischer (1989), Ljungqvist and Sargent (2018), McCandless (2008), and Romer (2012). For a more rigorous treatment of OLG models see McCandless and Wallace (1992) and Sargent (1987).

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3 The Ramsey-Cass-Koopmans Model

3.1 Introduction

The Ramsey-Cass-Koopmans model (henceforth referred to as just the Ramsey model) is the basic model of a competitive capitalist economy. Competitive firms rent capital and hire labour to produce and sell output, and a fixed number of infinitely lived households supply labour, hold capital, consume, and save. This model, which was developed by Ramsey (1928), Cass (1965), and Koopmans (1965), avoids all market imperfections and all issues raised by heterogeneous households and links among generations. It provides us with an excellent benchmark model to build on from. Thus, there will be no exogenous dynamics for now, so all the dynamics will be induced by the mechanism of capital accumulation, which will feedback on interest rates and savings decisions. Our ultimate aim is to understand the dynamic properties of the Walrasian equilibrium: Is it stable, and does it allow for growth? To what extent does growth generated in the model have properties similar to observed growth?

3.2 Households

To keep our analysis simple, for now, we assume a constant large number of households (i.e., the population of households does not grow[15]) all bundled in a single representative agent. We have already stated the behaviour of households, so we can go ahead and write the problem of the representative agent household as:

$$\max_{\{c_t\}} \int_0^\infty u(c(t)) \exp(-\rho t) dt, \tag{29}$$

subject to

$$c(t) + I(t) = w(t)l + r(t)k(t) + \Pi(t),$$
$$\dot{k}(t) = I(t) - \delta k(t),$$

[15] You could easily relax this assumption. Romer (2012) does a full setup of the Ramsey model with population growth.

where the objective is the present discounted value of utility in continuous time, $u(\cdot)$ is the instantaneous utility function, and ρ is the discount factor. The first constraint is the standard budget constraint requiring consumption and investment to be equal to labour income, capital income, plus any profits from firms owned by the household. The latter of which is treated as exogenous by the household, and in equilibrium will be zero because firms with constant returns to scale in competitive markets don't make profits. The second constraint defines the law of motion of capital with $\dot{k}(t) = \frac{dk(t)}{dt}$, and states the change in capital with respect to time is a function of investment less depreciation. The two constraints can be combined by substitution:

$$\dot{k}(t) = w(t)l + (r(t) - \delta)k(t) + \Pi(t) - c(t). \tag{30}$$

We will impose regularity conditions on the utility function: $u' > 0$ and $u'' < 0$. Our now-familiar forms of utility, CRRA and log utility, will abide by these conditions which will help us achieve nice analytical results. Finally, note that labour l is supplied inelastically by the household.

3.2.1 Technical aside: CRRA utility

It's worth pointing out some basic characteristics of the CRRA utility functional form:

$$u(c(t)) = \frac{c(t)^{1-\sigma}}{1-\sigma}, \ \sigma > 0.$$

In the context of the Ramsey model, this functional form is needed for the economy to converge to a balanced growth path. It is known as constant-relative-risk-aversion because the coefficient of relative risk aversion,

$$-c\frac{u''(c)}{u'(c)}, \tag{31}$$

is σ, and thus is independent of c.

Since there is no uncertainty in this model, the household's attitude toward risk is not directly relevant. But σ also determines the household's willingness to shift consumption between different periods. When σ is smaller, marginal utility falls more slowly as consumption rises, and so the household is more willing to allow its consumption to vary over time. If σ is close to zero, for example, utility is almost linear in c, and so the household is willing to accept large swings in consumption to take advantage of small differences between the discount rate and the rate of return on savings. It is for this reason that σ is also known as the inverse of the elasticity of substitution.

Three additional features of the CRRA utility function are worth mentioning. First, $c^{1-\sigma}$ is increasing in c if $\sigma < 1$ but decreasing if $\sigma > 1$; dividing $c^{1-\sigma}$ by $1-\sigma$ thus ensures that the marginal utility of consumption is positive regardless of the value of σ. Second, in the special case of $\sigma \to 1$, the CRRA utility function simplifies to log utility, $\ln c$; this is often a useful case to consider.

3.3 Firms

Firms' behaviour is relatively simple in the Ramsey model. At each point in time they employ the stocks of labour and capital, pay them their marginal products, and sell the resulting output. Because the production function has constant returns to scale (CRS) and the economy is competitive, firms earn zero profits – as previously stated. Thus, the problem of the firm is:

$$\max_{k(t),l(t),y(t)} \Pi(t) = y(t) - w(t)l(t) - r(t)k(t), \tag{32}$$

subject to

$$y(t) = F(k(t), l(t)), \tag{33}$$

$$\lim_{k(t)\to\infty} F_k(k(t), l(t)) = 0, \tag{34}$$

where (34) is basically to ensure that the firms' production technology satisfies the Inada conditions: assumptions about the shape of the production functions which guarantee the stability of a growth path.[16]

[16]Formally, we want $F(0,0) = 0$, $F(\cdot)$ is concave, $\lim_{k(t)\to 0} F_k(k(t), l(t)) = \infty$, and the condition in (34). Thankfully, Cobb-Douglas production technology satisfies these conditions.

3.4 Equilibrium

Equilibrium is a sequence of wages and rental rates $\{w(t), r(t)\}$ and allocations $\{k(t), c(t), I(t)\}$ which satisfy optimality and market clearing. The firm's first order conditions are as usual, and thus the real interest rate and what is paid to capital at time t is:

$$r(t) = F_k(k(t), l(t)),$$

and labour's marginal product is:

$$w(t) = F_l(k(t), l(t)).$$

To solve the household problem, we set up the Hamiltonian[17]:

$$\mathcal{H} = u(c(t)) \exp(-\rho t) + \lambda(t)(w(t)l + (r(t) - \delta)k(t) + \Pi(t) - c(t)), \tag{35}$$

and taking the derivative of the Hamiltonian with respect to the control variable $c(t)$ and setting it equal to zero gives the consumption FOC:

$$\mathcal{H}_{c(t)} = u'(c(t)) \exp(-\rho t) - \lambda(t) = 0, \tag{36}$$

and differentiating the Hamiltonian with respect to the state variable $k(t)$, using our rules for differentiating Hamiltonian functions, gives:

$$\mathcal{H}_{k(t)} = \lambda(t)(r(t) - \delta) = -\dot{\lambda}(t), \tag{37}$$

[17]We could have equivalently set up a current-value Hamiltonian:

$$\bar{\mathcal{H}} = u(c(t)) + \lambda(t)(w(t)l + (r(t) - \delta)k(t) + \Pi(t) - c(t)),$$

which would have given us the following FOCs:

$$\bar{\mathcal{H}}_{c(t)} = u'(c(t)) - \lambda(t) = 0,$$
$$\bar{\mathcal{H}}_{k(t)} = \lambda(t)(r(t) - \delta) = \rho\lambda(t) - \dot{\lambda}(t),$$

and the transversality/"no-Ponzi" condition:

$$\lim_{t\to\infty} \exp(-\rho t)\lambda(t)k(t) = 0$$

and where we have the "no-Ponzi"/transversality condition:

$$\lim_{t\to\infty} \exp\left(\int_0^t (\delta - r_\tau) d\tau\right) k(t) = 0. \tag{38}$$

There are now seven equations in seven unknowns $\{r(t), w(t), k^f(t), k^h(t), c(t), l(t), \lambda(t)\}$ where $k^f(t)$ is the capital stock that solves the firm problem and $k^h(t)$ is the capital stock that solves the household problem. We summarise the equations which constitute equilibrium below:

$$F_K(k^f(t), l(t)) = r(t), \tag{39}$$

$$F_L(k^f(t), l(t)) = w(t), \tag{40}$$

$$u'(c(t)) \exp(-\rho t) = \lambda(t), \tag{41}$$

$$\lambda(t)(r(t) - \delta) = -\dot{\lambda}(t), \tag{42}$$

$$w(t)l + (r(t) - \delta)k(t) + \Pi(t) - c(t) = \dot{k}^h(t), \tag{43}$$

$$l(t) = l, \tag{44}$$

$$k^f(t) = k^h(t). \tag{45}$$

Let's to try to solve for equilibrium by first starting with the household's problem. From (36) we know:

$$\lambda(t) = u'(c(t)) \exp(-\rho t),$$

and if we differentiate (remember to use product rule here) this with respect to t, we get

$$\begin{aligned} \frac{d\lambda(t)}{dt} &= \dot{\lambda}(t) = u''(c(t))\dot{c}(t) \exp(-\rho t) - \rho u'(c(t)) \exp(-\rho t). \\ \implies \dot{c}(t) &= \frac{\dot{\lambda}(t) + \rho u'(c(t)) \exp(-\rho t)}{u''(c(t)) \exp(-\rho t)} \end{aligned}$$

Now, substitute our value for $\dot{\lambda}(t)$ from (37) (be careful of the minus sign):

$$\begin{aligned}\dot{c}(t) &= \frac{\rho u'(c(t))\exp(-\rho t) - u'(c(t))\exp(-\rho t)(r(t)-\delta)}{u''(c(t))\exp(-\rho t)} \\ &= \frac{\rho u'(c(t)) - u'(c(t))(r(t)-\delta)}{u''(c(t))} \\ &= \frac{u'(c(t))\left[\rho - r(t) + \delta\right]}{u''(c(t))},\end{aligned}$$

and then divide both the LHS and RHS by $c(t)$ to get:

$$\frac{\dot{c}(t)}{c(t)} = \frac{u'(c(t))}{u''(c(t))c(t)}\left[\rho - r(t) + \delta\right] = \frac{1}{\sigma(c(t))}\left[r(t) - \rho - \delta\right], \tag{46}$$

where

$$\sigma(c(t)) = -c(t)\frac{u''(c(t)}{u'(c(t))}.$$

(46) is nothing but the familiar Euler equation, simply written in continuous time, and $\sigma(c(t))$ is the definition of the coefficient of relative risk aversion given in (31)! It states that the rate of consumption growth is higher if the rate of interest is high, if depreciation is low, or if the discount factor is low. The intuition for these effects is the same as we saw in the simple discrete time general equilibrium/Robinson Crusoe-like model in the first section. Note that $\sigma(c(t))$ depends only on the first and second derivatives of the utility function. The CRRA form of utility is particular neat here as $\sigma(c(t))$ is constant and independent of consumption.

Under CRRA preferences, the equilibrium conditions become:

$$\dot{c}(t) = \frac{F_K(k(t), l) - \rho - \delta}{\sigma}c(t), \tag{47}$$

$$\dot{k}(t) = F(k(t), l) - c(t) - \delta k(t), \tag{48}$$

$$\lim_{t\to\infty} \exp\left(\int_0^\infty (\delta - F_K(k(\tau), l)d\tau\right) k(t) = 0 \tag{49}$$

3.5 The dynamics of the economy

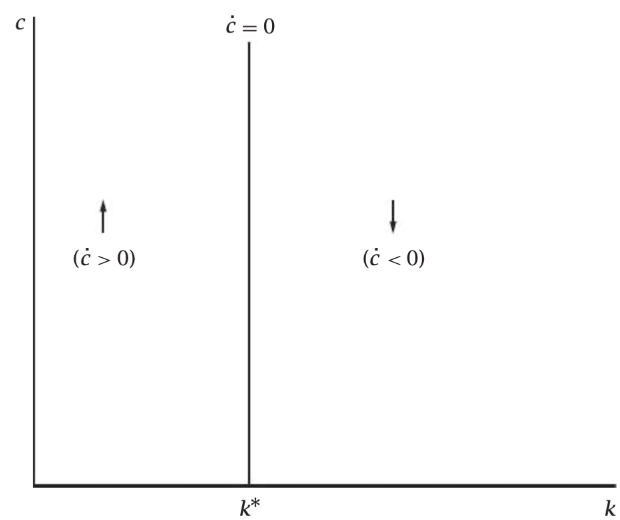
To make further progress with the Ramsey model, we will need to consider the steady state. The steady state is the point at which $\dot{c}(t) = \dot{k}(t) = 0$. Looking at our equilibrium conditions at the steady state we have:

$$0 = \frac{F_K(k,l) - \rho - \delta}{\sigma} c, \tag{50}$$

$$0 = F(k,l) - c - \delta k, \tag{51}$$

and note that we have gotten rid of all time-subscripts. This is common notation in macroeconomics, because the non-time denoted variables are referred to as steady state equilibrium values.[18] Now, we know that in order for (50) to hold,[19] it must be that $F_K(k,l) = r = \rho + \delta$. Let k^* denote this level of k. When k exceeds k^*, $F_K(k,l)$ is less than $\rho + \delta$, and so $\dot{c}$ is negative; when k is lower than k^*, $F_K(k,l)$ is more than $\rho + \delta$, and so $\dot{c}$ is positive. Figure 12 illustrates these dynamics.

Figure 12: Dynamics of c



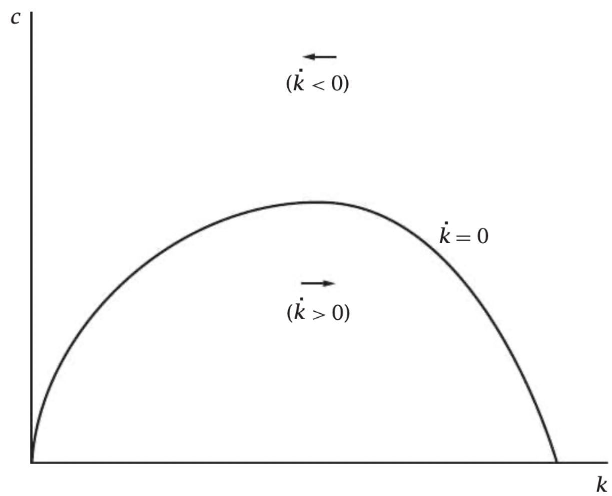
As in the Solow-Swan model, $\dot{k}$ equals actual investment minus break-even investment. Here,

[18]Or barred variables, such as $\bar{x}$ being the steady state value of x_t.

[19]Implicitly we are assuming that $c \neq 0$. Well, $c = 0$ is mathematically feasible, but economically of no interest to us as our economy would be in a situation with no consumption and no capital.

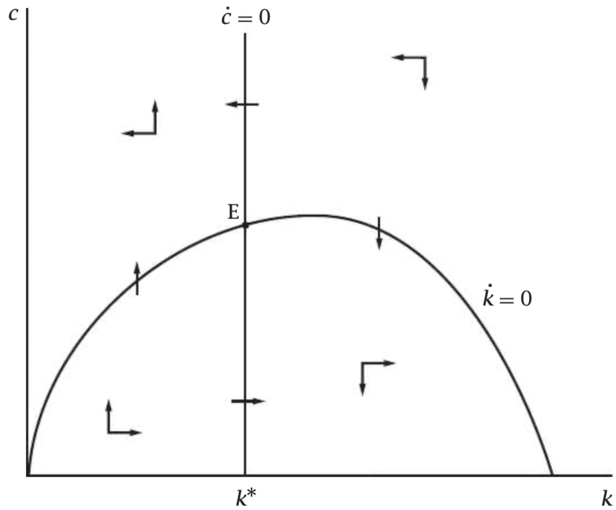
without population growth, actual investment is output minus consumption, and break-even investment is simply δk. From (51), $\dot{k}$ is zero when consumption equals the difference between the actual output and break-even investment lines when you plot a simple Solow-Swan diagram like as in Figure 10. This value of c is increasing in k until $F_K(k, l) = \delta$ (the golden-rule level of k) and is then decreasing. When c exceeds the level that yields $\dot{k} = 0$, k is falling; when c is lower than this level, k is rising. For k sufficiently large, break-even investment exceeds total output, and so $\dot{k}$ is negative for all positive values of c. Figure 13 plots these dynamics.

Figure 13: Dynamics of k



Now if we combine the information from Figures 12 and 13, we can illustrate a phase diagram which describe the convergence-to-equilibrium dynamics of the Ramsey model. This is done in Figure 14.

Figure 14: Dynamics of c and k



Notice that Figure 14 is drawn with k^* (the level of k that implies $\dot{c} = 0$) less than the golden-rule level of k. To see that this must be the case, recall that k^* is defined by $F_K(k^*, l) = \rho + \delta$, and that the golden-rule k is defined by $F_K(k^{GR}, l) = \delta$. Since $F_{KK}(k, l) < 0$, k^* is less than k^{GR} if $\rho + \delta$ is greater than δ, which it obviously is. This is equivalent to arguing that $\rho > 0$, which is a key assumption we make to ensure that lifetime utility does not diverge.[20]

3.5.1 The saddle path

Figure 15 shows how c and k evolve over time to satisfy households' intertemporal Euler equation and the law of motion of capital for various given initial values of c and k. We can trace out the paths for points such as A, B, C, and D using the phase diagram, where we see the economy trail off to the upper-left and down-right quadrants. But we notice there is some critical point between C and D – point F in the diagram – such that at that level of initial c, the economy converges to the stable point, point E. We can suppose that for any positive initial level of k, there is a unique initial level of c that is consistent with the Euler equation, the law of motion of capital, households' budget constraint, and the requirement that k not be negative. The function giving this initial c as a function of k is known

[20] In a model with population growth and no depreciation, this condition is $\rho - n - (1 - \sigma)g > 0$, where n is the population growth rate.

as the saddle path; and is shown in Figure 16.

Figure 15: The behaviour of c and k for various initial values of c

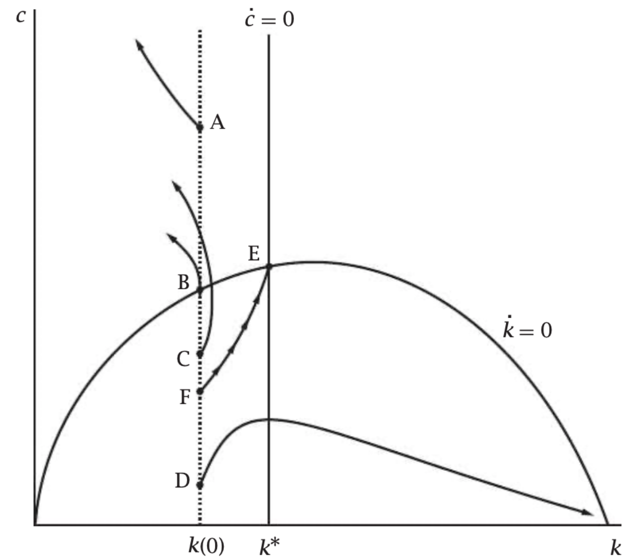
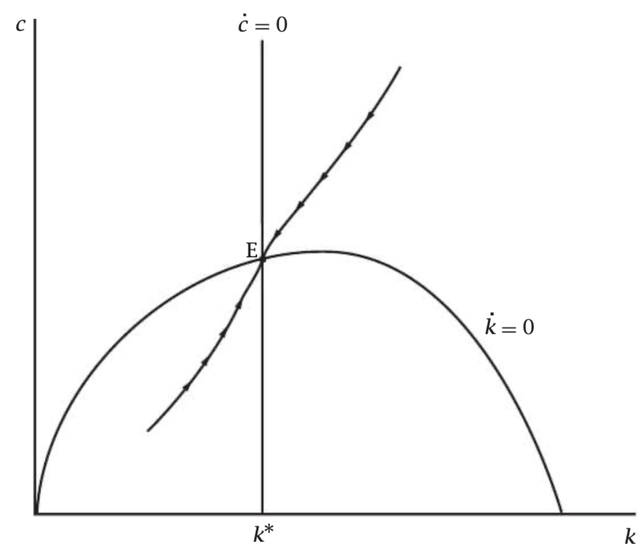


Figure 16: The saddle path



3.6 Adding government to the model

Thus far, we have left government out of our model. Yet, modern economies devote their resource not just to investment and private consumption but also to public uses. It is thus natural to extend our model to include a government sector.

Assume that the government buys output at rate $G(t)$, and that government purchases are assumed not to affect utility from private consumption; this can occur if the government devotes the goods to some activity that does not affect utility at all, or if utility equals the sum of utility from private consumption and utility from government provided goods. The purchases are financed by lump-sum taxes $T(t)$ which directly equal $G(t)$; thus the government always runs a balanced budget. Assume that the representative that the representative agent has preferences given by

$$\int_0^\infty \exp(-\rho t)\left[\ln C(t) + \ln G(t)\right] dt,$$

where ρ is the discount factor, $C(t)$ is private consumption, and $G(t)$ is government spending. Production in this economy is given by a constant returns to scale production technology satisfying the Inada conditions. Households supply an exogenous amount of labour in every period. There is no population growth, no technological change, and capital depreciates at the rate $\delta > 0$. Suppose the economy is initially at the steady state and subsequently the government increases spending to a permanently higher level.

How will this permanent increase in government spending affect the new steady-state level of output? Begin by assuming competitive markets, which allows to write the representative household's problem as

$$\max_{\{C_t\}} \int_0^\infty \exp(-\rho t)[\ln C(t) + \ln G(t)]dt,$$

subject to

$$\dot{K}(t) = w(t)L(t) + r(t)K(t) + \Pi(t) - C(t) - \delta K(t) - T(t),$$

and so our present-value Hamiltonian is

$$\mathcal{H} = \exp(-\rho t)\left\{\ln C(t) + \ln G(t)\right\} + \lambda(t)\left[w(t)L(t) + r(t)K(t) + \Pi(t) - C(t) - \delta K(t) - T(t)\right],$$

where labour is standardised to unity so $L(t) = 1, \forall t$, $r(t)$ is the return on capital, $\Pi(t)$ is firms' profits (and are zero due to perfect competition). As stated, government expenditure does not affect utility from private consumption; utility is additively separable between private consumption and government expenditure. Therefore, we can take the derivative of the Hamiltonian with respect to the control variable, $C(t)$, to yield our first FOC:

$$\mathcal{H}_{C(t)} = \frac{\exp(-\rho t)}{C(t)} = \lambda(t), \tag{52}$$

and taking the derivative with respect to the state variable $K(t)$ yields our second FOC:

$$\mathcal{H}_{K(t)} = \lambda(t)(r(t) - \delta) = -\dot{\lambda}_t \tag{53}$$

Differentiating (52) with respect to time yields

$$\dot{\lambda}(t) = \frac{-\rho\exp(-\rho t)C(t) - \dot{C}(t)\exp(-\rho t)}{C(t)^2},$$

and we can combine this with (53) to yield

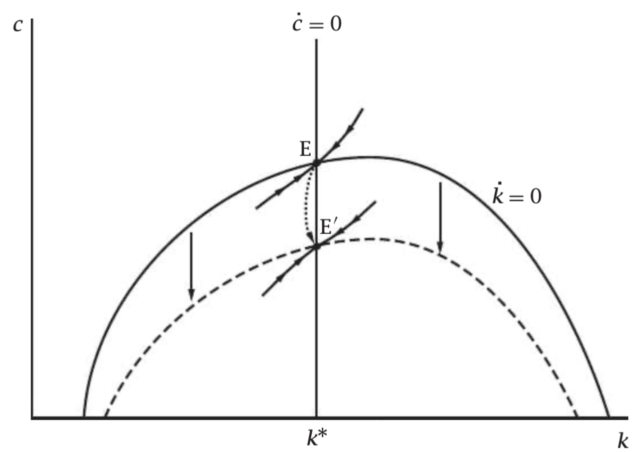
$$\begin{aligned}
\lambda(t)(\delta - r(t)) &= \frac{-\rho\exp(-\rho t)C(t) - \dot{C}(t)\exp(-\rho t)}{C(t)^2} \\
\Leftrightarrow \frac{\exp(-\rho t)}{C(t)}(\delta - r(t)) &= \frac{-\rho\exp(-\rho t)C(t) - \dot{C}(t)\exp(-\rho t)}{C(t)^2} \\
(\delta - r(t)) &= \frac{-\rho C(t)^2}{C(t)^2} - \frac{C(t)\dot{C}(t)}{C(t)^2} \\
\delta - r(t) + \rho &= -\frac{\dot{C}(t)}{C(t)} \\
\implies \dot{C}(t) &= C(t)(r(t) - \rho - \delta).
\end{aligned}$$

Set $\dot{C}(t) = 0$ and our law of motion equation $\dot{K}(t) = 0$, and we have two equations describing equilibrium dynamics[21]:

$$\dot{C}(t) = C(t)(F_K(K(t), L) - \delta - \rho) = 0,$$
$$\dot{K}(t) = F(K(t), L) - C(t) - \delta K(t) - T(t) = 0.$$

These dynamics are identical to an economy where government expenditures do not enter the household's utility at all. Thus, any increases in $G(t)$ will simply be funded by an increase in $T(t)$ and a downward shift in the locus of equilibria dictated by $\dot{K}(t) = 0$. That is, the economy will undergo a crowding out affect and household private consumption will fall instantly by the same amount as $G(t)$ increases. The dynamics are illustrated in Figure 17.

Figure 17: The Effects of a Permanent Increase in Government Purchases



In words: we know that in response to a permanent and surprise increase in government spending, C must jump so that the economy is on its new saddle path. If not, then as before, either capital would become negative at some point or households would accumulate infinite wealth. In this case, the adjust takes a simple form: C falls by the amount of the increase in G, and the economy is immediately on its new balanced growth path. Households do not have an opportunity to smooth their consumption

[21] Here I simply rewrite the law of motion of capital as being total output minus consumption (which is actual investment), and minus break-even investment $\delta K(t)$ minus taxes $T(t)$.

moving from the initial equilibrium E to E' due to the unanticipated increase in G. Intuitively, the permanent increases in government purchases and taxes reduce households' lifetime wealth (there's a crowding out effect). And because the increases in purchases and taxes are permanent, there is no scope for households to raise their utility by adjusting the time pattern of their consumption. Thus the size of the immediate fall in consumption is equal to the full amount of the increase in government purchases, and the capital stock and the real interest rate are unaffected.

3.7 Considering technological progress in the Ramsey model

The baseline Ramsey model we developed had a time invariant steady state level. In order to match the Kaldor stylised facts, we need the steady state to constantly shift and grow. How would we do this? We could assume that the production technology features a labour augmenting technology term such as:

$$Y(t) = F(K(t), A(t)L),$$

where A grows exogenously by the rate γ:

$$\frac{\dot{A}(t)}{A(t)} = \gamma.$$

How would our new steady state convergence equations look like?

Under CRRA preferences, our equilibrium conditions are:

$$\begin{aligned}
\frac{\dot{C}(t)}{C(t)} &= \frac{F_K(K(t), A(t)L) - \delta - \rho}{\sigma}, \\
\dot{K}(t) &= F(K(t), A(t)L) - C(t) - \delta K(t) - T(t), \\
\lim_{t\to\infty} &= \exp\left(\int_0^\infty (\delta - F_K(K(\tau), A(t)L))d\tau\right) K(t) = 0,
\end{aligned}$$

and consider the following transformation of our variables:

$$\begin{aligned}
\tilde{K}(t) &= \frac{K(t)}{A(t)}, \\
\implies \dot{K}(t) &= \dot{A}(t)\tilde{K}(t) + A(t)\dot{\tilde{K}}(t), \\
\implies F_L(K(t), A(t)L) &= A(t)F_L(\tilde{K}(t), L),
\end{aligned}$$

and

$$\begin{aligned}
\tilde{C}(t) &= \frac{C(t)}{A(t)}, \\
\implies \dot{C}(t) &= \dot{A}(t)\tilde{C}(t) + A(t)\dot{\tilde{C}}(t), \\
\implies F_K(K(t), A(t)L) &= F_K(\tilde{K}(t), L).
\end{aligned}$$

Using our transformed variables, equilibrium is now

$$\frac{\dot{\tilde{C}}(t)}{\tilde{C}(t)} = \frac{F_K(\tilde{K}(t), L) - \sigma\gamma - \delta\rho}{\sigma}, \tag{54}$$

$$\dot{\tilde{K}}(t) = F(\tilde{K}(t), L) - \tilde{C}(t) - (\delta + \gamma)\tilde{K}(t). \tag{55}$$

With this tweak, our Ramsey model now manages to reproduce the Kaldor stylised facts of persistent growth. Our model exhibits transitional dynamics by converging to a balanced growth path which is constantly growing at rate γ. Success!

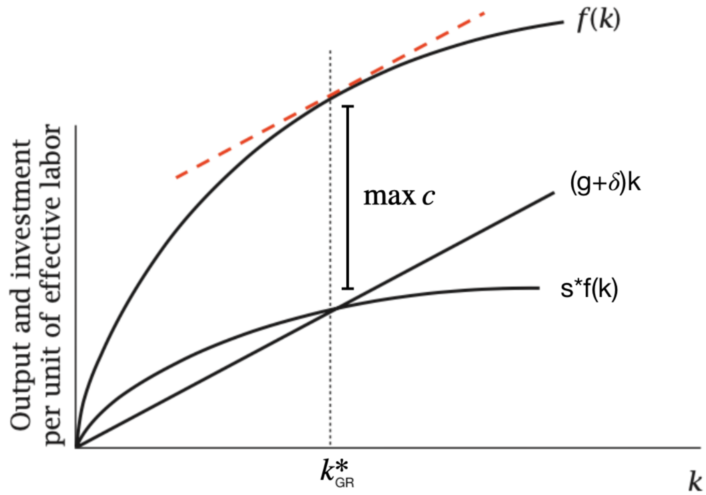
But this was done in a pretty ad-hoc way... So maybe it's too early to celebrate.

3.8 The social optimum and the golden-rule level of capital

One thing worth mentioning about the Ramsey model is a note on dynamic efficiency. Throughout this section, we've been comparing the Ramsey model to the Solow-Swan model – and for obvious reasons; the two models are very similar. The primary difference between the two models is that in the Solow-Swan model saving in each period is some fixed proportion of total output, s, whereas in the

Ramsey model savings/investment fluctuates depending on the capital stock. Recall your undergraduate macroeconomics and look at Figure 18, which illustrates a savings rate, s^*, which coincides with the "golden rule" of investment: the level of investment which permits the highest level of consumption and golden rule level of capital, k^*_{GR} in the Solow-Swan model. Savings levels above or below s^* are of course possible in the Solow-Swan model, and the model does provide for a balanced growth path which will converge to a steady state k that is either above or below k^*_{GR}. That is to say that the Solow-Swan model may be dynamically inefficient.[22]

Figure 18: Steady State (Golden Rule) in the Solow-Swan Model



Now consider the plot of the convergent dynamics of the Ramsey model in Figure 16. In the Ramsey model, it is not possible to be on the balanced growth path with a capital stock above k^*. This is because, as we now know, savings in the Ramsey model is derived from the behaviour of households whose utility depends on their consumption, and there are no externalities. As a result, it cannot be an equilibrium for the economy to follow a path where higher consumption can be attained in every period; if the economy were on such a path, households would reduce their saving and take advantage of this opportunity. Also, remember that k^* in the Ramsey model is lower than the golden rule level of capital corresponding to k^*_{GR} in the Solow-Swan model.

[22] An economy is said to be dynamically inefficient if it is saving too much.

If the initial capital stock exceeds the golden rule of level of capital (if $k(0)$ is greater than the k associated with the peak of the $\dot{k} = 0$ locus), initial consumption is above the level needed to keep k constant; thus $\dot{k}$ is negative, and k gradually approaches k^*, which is below the golden-rule level. Finally the fact that k^* is less than the golden-rule capital stock implies that the economy does not converge to the balanced growth path that yields the maximum sustainable level of c.

We saw the intuition for this result earlier, but just to repeat: Consider the baseline Ramsey model with no technological growth. In this case, k^* is defined by $F_K(k^*, l) = \rho + \delta$ and k^{GR} is defined by $F_K(k^{GR}, l) = \delta$. A condition for convergence is that $\rho + \delta < \delta \Leftrightarrow \rho < 0$. Since $k^* < k^{GR}$, an increase in saving starting at $k = k^*$ would cause consumption per worker to eventually rise above its previous level and remain there. But because households value present consumption more than future consumption, the benefit of the eventual permanent increase in consumption is bounded. At some point – specifically, when k exceeds k^* – the tradeoff between the temporary short-term sacrifice and the permanent long-term gain is sufficiently unfavourable that accepting it reduces rather than raises lifetime utility. Thus k converges to a value below the golden-rule level. Because k^* is the optimal level of k for the economy to converge to, it is known as the "modified golden-rule capital stock."

3.9 Comments and key readings

The Ramsey model obviously isn't very realistic. For one, the assumption that agents have perfect foresight is pretty hard to reconcile. But it offers a very important foundation for when we move onto models in a stochastic setting, where agents have to form rational expectations about the future based on information known today (the real business cycle model).

Key readings for the Ramsey model are Blanchard and Fischer (1989), McCandless (2008), and Romer (2012).

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4 Endogenous Growth

4.1 Introduction

The neoclassical models presented in the previous sections takes the rate of technological change as being determined exogenously. There is good reason, however, to believe that technological change depends on economic decisions, because it comes from industrial innovations made by profit-seeking firms, and depends on the funding of science, the accumulation of human capital, and other such activities. What we want to do is endogenise growth in our macroeconomic model (a concept we will frequently revisit), so that it is determined within the model and not just exogenously assumed.

Incorporating endogenous technology into growth theory forces us to deal with the difficult phenomenon of increasing returns to scale. More specifically, people must be given an incentive to improve technology. But because the aggregate production function F exhibits constant returns to scale (CRS) in K and L alone, Euler's Theorem tells us that it will take all of the economy's output to pay capital and labour their marginal products in producing final output, leaving nothing over to pay for the resources used in improving technology.[23] Thus, a theory of endogenous technology cannot be based on the usual theory of competitive equilibrium, which requires that all factors be paid their marginal products.

We will develop two approaches to tackle this challenge: the AK model and the varieties model.

4.2 The AK model based on Arrow (1962) and Frankel (1962)

Arrow's (1962) approach to endogenous growth was to propose that technological progress is an unintended consequence of producing new capital goods, a phenomenon dubbed "learning by doing".

[23]Euler's Theorem states that if F is homogeneous of degree 1 (HOD1) in K and L (the definition of CRS) then:

$$F_K(K,L)K + F_L(K,L)L = F(K,L). \tag{56}$$

The marginal products of K and L are F_K and F_L, respectively. So if K and L are paid their marginal products then the LHS is the total payment to capital plus the total payment to labour, and the equation states that these payments add up to total output. To verify Euler's Theorem take the equation

$$F(\lambda K, \lambda L) = \lambda F(K,L), \tag{57}$$

which defines HOD1, and differentiate both sides with respect to λ at the point $\lambda = 1$. Since the (57) must hold for all $\lambda > 0$, the two derivatives must be equal, which implies (56).

Learning by doing was assumed to be purely external to the firms responsible for it. That is, if technological progress depends on the aggregate production of capital, and firms are all very small, then they can be assumed all to take the rate of technological progress as being given independently of their own production of capital goods. So each firm maximises profit by paying K and L their marginal products, without offering any additional payment for their contribution to technological progress.

Learning by doing formed the basis of the first model of endogenous growth theory, which is known as the AK model. The AK model assumes that when people accumulate capital, learning by doing generates technological progress that tends to raise the marginal product of capital, thus offsetting the tendency for the marginal product to diminish when technology is unchanged. The model results in a production function of the form $Y = AK$, in which the marginal product of capital is equal to the constant A.

The AK model predicts that a country's long-run growth rate will depend on economic factors such as thrift and the efficiency of resource allocation. In subsequent sections we will develop alternative models of endogenous growth that emphasise not thrift and efficiency, but creativity and innovation, which we see as the main driving forces behind economic growth. But given its historical place as the first endogenous growth model, the AK paradigm is an important part of any economist's toolkit. Accordingly, we devote this section to developing the AK model.

We begin by assuming that firm output is given by

$$F(K_t, A) = AK_t. \tag{58}$$

Fundamental to the AK model is that K embodies both physical and human capital, and thus firms' production technology is a special case of the Cobb-Douglas production function $Y_t = AK_t^{\alpha} L_t^{1-\alpha}$ where $\alpha = 1$, and hence the model's name. Since $\alpha = 1$, the model relies on constant returns to scale production. The fact that the return on capital is now a constant, A, eliminates any potential for there being transition dynamics.

Households in the AK model behave as they do in the Ramsey model or Solow-Swan model, choosing consumption to maximise the present discounted valued of utility subject to a budget constraint a transversality condition. Households maximise their present discounted value of utility

$$\max_{\{C_t\}} \int_0^\infty \exp(-\rho t) u(C_t) dt,$$

subject to a budget constraint and a transversality condition. The optimality condition is from the now familiar consumption Euler equation. With CRRA preferences this is:

$$\frac{\dot{C}_t}{C_t} = \frac{r_t - \rho - \delta}{\sigma}, \tag{59}$$

Since the firms' production technology is (58), their profit maximisation problem can be written as:

$$\arg\max_{K_t} \; AK_t - rK_t,$$

where the rental rate of capital is given by:

$$r_t = F_K(A_t, A) = A.$$

Thus, the law of motion for capital is

$$\dot{K}_t = AK_t - C_t - \delta K_t. \tag{60}$$

The model does not have a steady state but it has a balanced growth path:

$$\frac{\dot{C}_t}{C_t} = \frac{A - \rho - \delta}{\sigma} = g, \tag{61}$$

$$\frac{\dot{K}_t}{K_t} = \underbrace{A - \frac{C_t}{K_t}}_{sA} - \delta = g \tag{62}$$

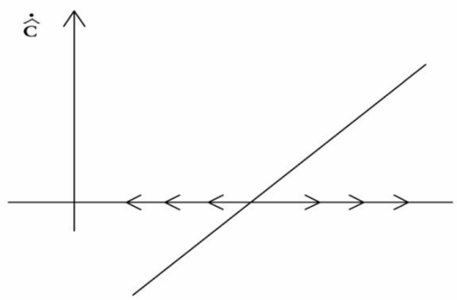
$$\implies \frac{C_t}{K_t} = A - \delta - g. \tag{63}$$

What about transition dynamics? Assuming log utility for simplicity and defining $\tilde{C}_t = \frac{C_t}{K_t}$, we know that:

$$\frac{\dot{\tilde{C}}_t}{\tilde{C}_t} = \frac{K_t\dot{C}_t - C_t\dot{K}_t}{K_t^2}\frac{K_t}{C_t} = \frac{\dot{C}_t}{C_t} - \frac{\dot{K}_t}{K_t} = A - \rho - \delta\left(A - \frac{C_t}{K_t} - \delta\right) = \tilde{C}_t - \rho. \tag{64}$$

When we plot $\dot{\tilde{C}}_t/\tilde{C}_t$ as a function of $\tilde{C}_t$, we see that the relationship is upward sloping. Hence there are no transitional dynamics and economy jumps directly to the balanced growth path, as shown in Figure 19.

Figure 19: Dynamics in the AK Model



4.2.1 Microfoundations of the AK model

The most straightforward microfoundation for the AK model is the idea is that knowledge accumulates through accumulating capital:

$$\begin{aligned} Y_t &= F(K_t, A_tL_t), \\ A_t &= \psi\frac{K_{t-1}}{L_{t-1}}. \end{aligned}$$

Implicit in this is an assumption there is an externality in capital accumulation. Knowledge advances at the aggregate level as the capital stock increases, but the effect for an individual firm is so small that they do no internalise it in their own (firm-specific) investment decisions. With a CRS production function:

$$
\begin{aligned}
\frac{Y_t}{L_t} &= \frac{1}{L_t} F(K_t, A_t L_t) \\
&= \frac{K_t}{L_t} F\left(1, \frac{A_t L_t}{K_t}\right) \\
&= \frac{K_t}{L_t} F\left(1, \psi \frac{K_{t-1} L_t}{L_{t-1} K_t}\right) \\
&\approx \frac{K_t}{L_t} F\left(1, \psi \frac{1}{1+g}\right),
\end{aligned} \tag{65}
$$

which is of the desired AK form.

A different argument is based on human capital, assuming that it accumulates over time and plays a part in production. Production depends on physical and human capital, the latter of which can be increased through investment. The structural equations of this one-sector model are therefore:

$$Y_t = F(K_t, H_t), \tag{66}$$

$$Y_t = C_t + I_t^K + I_t^H, \tag{67}$$

$$\dot{K}_t = I_t^K - \delta K_t, \tag{68}$$

$$\dot{H}_t = I_t^H - \delta H_t. \tag{69}$$

None of that should be too controversial, so let's go ahead and set up the present value Hamiltonian of the household problem:

$$\mathcal{H} = u(C_t)\exp(-\rho t) + \lambda_{1,t}(F(K_t, H_t) - C_t - I_t^K - I_t^H) + \lambda_{2,t}(I_t^K - \delta K_t) + \lambda_{3,t}(I_t^H - \delta H_t),$$

and this problem has the following FOCs:

$$u'(C_t)\exp(-\rho t) = \lambda_{1,t}, \tag{70}$$

$$\lambda_{1,t} = \lambda_{2,t}, \tag{71}$$

$$\lambda_{1,t} = \lambda_{3,t}, \tag{72}$$

$$\lambda_{1,t}F_K(K_t, H_t) - \lambda_{2,t}\delta = -\dot{\lambda}_{2,t}, \tag{73}$$

$$\lambda_{1,t}F_H(K_t, H_t) - \lambda_{3,t}\delta = -\dot{\lambda}_{3,t}. \tag{74}$$

These conditions imply that $\lambda_{2,t} = \lambda_{3,t}$ so the marginal products of physical and human capital are equalised and $F_k(K_t, H_t) = F_H(K_t, H_t)$. It follows that K_t/H_t is constant because of the HOD1 of $F(\cdot)$. With a constant K_t/H_t the production function works like in the AK model. More interesting would be a two-sector model in which the first sector products output (for consumption o investment) and the second sector produces human capital (e.g. schools). Then K_t/H_t becomes a state variable, and there can be transition dynamics.

4.2.2 Neoclassical models vs the AK model

In this section, we briefly reflect on a now closed debate between advocates of the neoclassical approach and those of the AK model.

The AK model can first account for persistent and positive growth rates of output (we could also show positive growth rates for output per capita) – something that we observe empirically, and a feature which neoclassical models cannot address. Where the AK model struggles is with its inability to explain convergence, which is another important empirical observation. Neoclassical models such as the Solow-Swan and Ramsey model predict that economies with lower GDP and capital per-capita levels undergo rapid growth initially, before plateauing out as the economy reaches its steady state. The main mechanism for this convergence dynamic is the fact that production in neoclassical model had constant returns to scale in capital and labour together. In other words, the further an economy is away from its steady state, the faster it grows as it has a relatively high marginal product of capital.

Now consider the case of two geographically proximate economies with similar characteristics (e.g.

states in the US), such as similar population growth, savings, and depreciation rates. Suppose that one economy had lower GDP and capital per-capita than the other. Under the AK model and constant returns to scale, the two economies would never converge to one another as they grow at exogenous rates. Whereas under a neoclassical model, the two economies would tend toward their steady states – which is something that we observe in the data.

So, to conclude: AK models can explain long-run growth, but can't say much for convergence. However, neoclassical models can explain convergence but do not have a convincing story for long-run technological growth. As I previously mentioned, the fact that the AK model lumps physical and human capital together under a catch-all capital term is the model's primary weakness.

4.3 The fixed varieties model

The objective here is to build a tractable model where changes in the production possibility set are endogenously determined as a response to economic incentives. The difficulty is that we need some sort of imperfection in the goods market if technological change is to be remunerated. Otherwise, if factors are paid their marginal product and there are constant returns to scale, this exhausts the output. We therefore use a monopolistic-competitive setup where: (i) final goods production is competitive but needs intermediate goods, and (ii) intermediate goods are produced by a monopolist.

4.3.1 Firms

We first setup the varieties model where the number of goods is fixed. Some of the setup we establish here will make a comeback later on when we look at the New Keynesian model – where we have final good products and intermediate goods. For now, let's assume that the production of final goods are:

$$Y_t = L_t^{1-\alpha} \sum_{i=1}^{N_i} X_{i,t}^{\alpha}, \tag{75}$$

where N_i is the number of intermediate goods. The production functions shows increasing returns to specialisation, as seen for the first time by Adam Smith in the pin factory. Firms that produce final goods maximise their profits:

$$\max_{L_t,\{X_{i,t}\}_{i=1}^{N_i}} Y_t - w_t L_t - \sum_{i=1}^{N_i} P_{i,t} X_{i,t},$$

subject to the production function (75). The Lagrangian for this problem is

$$\mathcal{L} = Y_t - w_t L_t - \sum_{i=1}^{N_i} P_{i,t} X_{i,t} + \lambda_t \left(Y_t - L_t^{1-\alpha} \sum_{i=1}^{N_i} X_{i,t}^{\alpha} \right),$$

giving the FOCs:

$$\begin{aligned}
\mathcal{L}_{L_t} &= -w_t - (1-\alpha) L_t^{-\alpha} \lambda_t \sum_{i=1}^{N_i} X_{i,t}^{\alpha} = 0, \\
\mathcal{L}_{X_{i,t}} &= -P_{i,t} - \alpha \lambda_t L_t^{1-\alpha} X_{i,t}^{\alpha-1} = 0, \\
\mathcal{L}_{Y_t} &= 1 + \lambda_t = 0, \\
\implies w_t &= (1-\alpha) L_t^{-\alpha} \sum_{i=1}^{N_i} X_{i,t}^{\alpha}, \qquad (76) \\
\implies P_{i,t} &= \alpha L_t^{1-\alpha} X_{i,t}^{\alpha-1}. \qquad (77)
\end{aligned}$$

Intermediate goods, which are slightly differentiated from one another, are produced using a simply technology that transforms one unit of the final good into one unit of $X_{i,t}$. The profit maximisation problem of the intermediate good monopolistically competitive firm is therefore:

$$\max_{P_{i,t},X_{i,t}} \Pi_t = P_{i,t} X_{i,t} - y_{i,t},$$

subject to

$$\begin{aligned}
y_{i,t} &= X_{i,t}, \\
P_{i,t} &= \alpha L_t^{1-\alpha} X_{i,t}^{\alpha-1}.
\end{aligned}$$

The Lagrangian for intermediate firm i is:

$$\mathcal{L} = P_{i,t}X_{i,t} - X_{i,t} + \lambda_t \left(P_{i,t} - \alpha L_t^{1-\alpha} X_{i,t}^{\alpha-1}\right),$$

and the FOCs are:

$$\mathcal{L}_{P_{i,t}} = X_{i,t} + \lambda_t = 0, \tag{78}$$

$$\mathcal{L}_{X_{i,t}} = P_{i,t} - 1 - \lambda_t \alpha(\alpha-1) L_t^{1-\alpha} X_{i,t}^{\alpha-2} = 0. \tag{79}$$

Combining the FOCs we have

$$\begin{aligned} 1 &= P_{i,t} + X_{i,t}\alpha(\alpha-1)L_t^{1-\alpha}X_{i,t}^{\alpha-2} \\ &= P_{i,t} + \alpha(\alpha-1)L_t^{1-\alpha}X_{i,t}^{\alpha-1} \\ &= P_{i,t} + \alpha^2 L_t^{1-\alpha}X_{i,t}^{\alpha-1} - \alpha L_t^{1-\alpha}X_{i,t}^{\alpha-1}, \end{aligned}$$

and since $P_{i,t} = \alpha L_t^{1-\alpha} X_{i,t}^{\alpha-1}$, we have:

$$\begin{aligned} 1 &= \alpha P_{i,t} \\ \implies P_{i,t} &= \frac{1}{\alpha}. \end{aligned} \tag{80}$$

Plugging this value for $P_{i,t}$ into one second constraint gives:

$$\begin{aligned} \frac{1}{\alpha} &= \alpha L_t^{1-\alpha} X_{i,t}^{\alpha-1} \\ \frac{1}{\alpha^2} &= L_t^{1-\alpha} X_{i,t}^{\alpha-1} \\ \frac{L_t^{\alpha-1}}{\alpha^2} &= X_{i,t}^{\alpha-1} \\ \implies X_{i,t} &= L_t \alpha^{\frac{2}{1-\alpha}}. \end{aligned} \tag{81}$$

4.3.2 Households

The household maximises the present discounted value of its utility from consumption in the standard way:

$$\max_{\{C_t, L_t, A_t\}} \int_0^\infty u(C_t) \exp(-\rho t) dt,$$

subject to

$$\begin{aligned} C_t + \dot{A}_t &= w_t L_t + r_t A_t + \Pi_t, \\ L_t &= \bar{L}, \end{aligned}$$

noting that profits $\Pi_t > 0$ in equilibrium because the intermediate producers are monopolistically competitive. The consumer Euler equation is standard:

$$\frac{\dot{C}_t}{C_t} = r_t - \rho.$$

4.3.3 Equilibrium

Equilibrium is a sequence of prices $\{P_{i,t}, w_t, r_t\}$ and a set of quantities $\{Y_t, \{X_{i,t}\}_{i=1}^{N_i}, C_t, A_t, L_t\}$ such that: Given prices, $Y_t, \{X_{i,t}\}_{i=1}^{N_i}, L_t$ solves the problem of final good firms; given perceived demand, $\{X_{i,t}\}_{i=1}^{N_i}$ solves the intermediate goods firms' problem; given prices, C_t, A_t solves the households problem; and, markets clear.

Putting everything together defines the equilibrium:

$$X_{i,t} = \bar{L}\alpha^{\frac{2}{1-\alpha}}, \tag{82}$$

$$P_{i,t} = \frac{1}{\alpha}, \tag{83}$$

$$Y_t = \bar{L}N_i\alpha^{\frac{2\alpha}{1-\alpha}}, \tag{84}$$

$$C_t = Y_t - N_i X_{i,t} = \bar{L}N_i\alpha^{\frac{2\alpha}{1-\alpha}} - N_i\bar{L}\alpha^{\frac{2}{1-\alpha}}, \tag{85}$$

$$A_t = 0, \tag{86}$$

$$r_t = \rho, \tag{87}$$

$$w_t = (1-\alpha)N_i\alpha^{\frac{2\alpha}{1-\alpha}}, \tag{88}$$

which is static and exhibits no growth.

4.4 Endogenous number of varieties

To get growth up and running we endogenise N_i, the number of varieties or intermediate goods. We allow monopolistically competitive firms to create a new variety a cost of η, where η is sufficiently small to ensure existence of equilibrium. The per-period profit of a new variety to the intermediate firm is:

$$\begin{aligned}
\Pi_{i,t} &= P_{i,t}X_{i,t} - X_{i,t} \\
&= \frac{1}{\alpha}\bar{L}\alpha^{\frac{2}{1-\alpha}} - \bar{L}\alpha^{\frac{2}{1-\alpha}} \\
&= \bar{L}\alpha^{\frac{2}{1-\alpha}-1} - \bar{L}\alpha^{\frac{2}{1-\alpha}} \\
\therefore \Pi_i &= \bar{L}\alpha^{\frac{2}{1-\alpha}}\left(\frac{1-\alpha}{\alpha}\right),
\end{aligned} \tag{89}$$

which in present value terms is

$$\int_0^\infty \Pi_i \exp(-rt)dt = \frac{\Pi_i}{r}.$$

The intermediate firm will bring new varieties of intermediate goods to the market as long as the present value of profits exceeds the entry cost, i.e. equilibrium requires $\Pi_i/r = \eta$ in which case consumption has to grow according to:

$$\frac{\dot{C}_t}{C_t} = \frac{\Pi_i}{\eta} - \rho \equiv g. \tag{90}$$

Can we build an equilibrium in which consumption, output, and the number of varieties grows at rate g? The resource constraint in the growing economy is:

$$\begin{aligned} C_t &= Y_t - N_t X_{i,t} \\ &= \bar{L} N_t \left(\alpha^{\frac{2\alpha}{1-\alpha}} - \alpha^{\frac{2}{1-\alpha}} \right) - \underbrace{\dot{N}_t \eta}_{\text{Cost of new varieties}} \\ &= N_t \left\{ \bar{L} \left(\alpha^{\frac{2\alpha}{1-\alpha}} - \alpha^{\frac{2}{1-\alpha}} \right) - g\eta \right\}. \end{aligned} \tag{91}$$

It's clear that N_t will grow at the same rate as C_t. Since $Y_t = \bar{L} N_t \alpha^{\frac{2\alpha}{1-\alpha}}$ we also have Y_t growing at the same rate and we have succeeded in finding an equilibrium where everything grows. In this equilibrium we have profits being given by (89) so the economy grows at the rate:

$$g \equiv \frac{\Pi_i}{\eta} - \rho \Leftrightarrow \bar{L} \alpha^{\frac{2}{1-\alpha}} \left(\frac{1-\alpha}{\alpha\eta} \right) - \rho. \tag{92}$$

4.4.1 Efficiency concerns in the varieties model

We now examine whether these equilibrium are efficient (they will not be). The first calculation is a static optimum for the model with a fixed number of varieties. For a given N_i is the household getting the maximal amount of consumption for its labour input? Maximising consumption involves solving the following problem:

$$\arg \max_{Y_t, L_t, \{X_{i,t}\}_{i=1}^{N_i}} Y_t - \sum_{i=1}^{N_i} X_{i,t},$$

subject to

$$
\begin{aligned}
Y_t &= L_t^{1-\alpha} \sum_{i=1}^{N_i} X_{i,t}^{\alpha}, \\
L_t &= \bar{L}.
\end{aligned}
$$

We know from our previous problem that from the FOCs we have:

$$
\begin{aligned}
X_{i,t}^* &= \bar{L} \alpha^{\frac{1}{1-\alpha}}, \\
Y_t^* &= \bar{L} N_i \alpha^{\frac{\alpha}{1-\alpha}}, \\
C_t^* &= \bar{L} N_i \left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}} \right),
\end{aligned}
$$

and from the expanding varieties model with monopolistic competition we had:

$$
X_{i,t} = \bar{L} \alpha^{\frac{2}{1-\alpha}},
$$

and since $\alpha^{\frac{2}{1-\alpha}} < \alpha^{\frac{1}{1-\alpha}}$ we have that $X_{i,t} < X_{i,t}^*$. In other words, the economy with imperfect supply of intermediate goods produces too few intermediate goods relative to the efficient level. This should not be too surprising – the intermediate firms artificially restricts supply of the intermediate goods to maximise profits. Consumption is similarly restricted because $C_t < C_t^*$, which is easiest to check numerically for all $\alpha \in (0, 1)$.

The second calculation asks whether the growing economy is dynamically efficient, i.e. does it grow at the optimal rate? To answer this we solve for the growth rate that maximises the present discounted value of utility. Under logarithmic utility for simplicity we have:

$$
\max_{\{C_t, N_t\}} \int_0^{\infty} (\log C_t) \exp(-\rho t) dt,
$$

subject to

$$\begin{aligned} Y_t &= C_t + \eta \dot{N}_t \\ \implies \dot{N}_t &= \frac{Y_t - C_t}{\eta}, \end{aligned}$$

where Y_t is the statically efficient output level. The Hamiltonian is:

$$\mathcal{H} = (\log C_t) \exp(-\rho t) + \lambda_t \left(\frac{\bar{L} N_t \alpha^{\frac{\alpha}{1-\alpha}} - C_t}{\eta} \right),$$

and its FOCs are:

$$\begin{aligned} \mathcal{H}_{C_t} &= \frac{\exp(-\rho t)}{C_t} - \frac{\lambda_t}{\eta} = 0 \\ \implies \frac{\dot{C}_t}{C_t} &= -\frac{\dot{\lambda}_t}{\lambda_t} - \rho, \qquad (93) \\ \mathcal{H}_{N_t} &= \lambda_t \frac{\bar{L} \alpha^{\frac{\alpha}{1-\alpha}}}{\eta} = -\dot{\lambda}_t \\ \implies -\frac{\dot{\lambda}_t}{\lambda_t} &= \frac{\bar{L} \alpha^{\frac{\alpha}{1-\alpha}}}{\eta}, \qquad (94) \end{aligned}$$

and therefore, combining the two FOCs, the optimal growth rate is:

$$\frac{\dot{C}_t^*}{C_t^*} = \frac{\bar{L} \alpha^{\frac{\alpha}{1-\alpha}}}{\eta} - \rho. \qquad (95)$$

Compared to what happens in the monopoly model, dynamic efficiency requires the economy to grow faster. This is because

$$\frac{\dot{C}_t^*}{C_t^*} = \frac{\bar{L} \alpha^{\frac{\alpha}{1-\alpha}}}{\eta} - \rho > \frac{\dot{C}_t}{C_t} = \frac{\bar{L} \alpha^{\frac{2}{1-\alpha}}}{\eta} - \rho.$$

We see that monopoly power in the intermediate goods market creates two distinct distortions. Not only is an efficiently low quantity of intermediate goods produced, the growth in variety of those goods is also inefficiently restricted.

4.5 Comments and key readings

There was quite a lot of digest in this section. We looked at the strengths and weaknesses of the AK model against the neoclassical Solow-Swan and Ramsey models. There is a large literature for further study and research including: *Introduction to Modern Economic Growth* by Acemoglu (2009), *Endogenous Growth Theory* by Aghion and Howitt (1997), *Economic Growth* by Barro and Sala-i-Martin (2003), Rebelo (1991), and *Advanced Macroeconomics* by D. H. Romer (2012).

For varieties model, the above textbooks are also very good. In addition, see Aghion and Howitt (1992) and P. M. Romer (1990). The key message to takeaway from the varieties model (with endogenous growth) is that we had a loss of efficiency as soon as we introduced monopolistic competition. Keep this in mind when we introduce monopolistic competition to the Real Business Cycle model.

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5 Primer to DSGE Models

Before we tackle our first dynamic stochastic general equilibrium (DSGE) model, the Real Business Cycle model, it's worth going over a few important mathematical concepts. Some of the concepts are essential to understand now, but some of the other concepts, such as solution methods for DSGE models can be revisited later. But it's good to be aware of them now, and keep them in mind as we move on in the course.

5.1 Vector autoregressions

As we saw in the first section, AR models are useful tools for understanding the dynamics of individual variables such as output or consumption, but they ignore the interrelationships between variables. A vector autoregression (VAR)[24] model captures the dynamics of n different variables allowing each variable to depend on lagged values of all variables. More specifically, with VAR models we can examine the impulse responses of all n variables to all n shocks. Consider the following simple VAR(1) model with two variables and one lag:

$$\begin{aligned} y_{1,t} &= a_{11}y_{1,t-1} + a_{12}y_{2,t-1} + e_{1,t} \\ y_{2,t} &= a_{21}y_{1,t-1} + a_{22}y_{2,t-1} + e_{2,t}, \end{aligned}$$

where $e_{1,t}$ and $e_{2,t}$ are shocks to the system. What are these shocks? They could be shocks which macroeconomists are interested in such as: policy changes not due to the systematic component of policy captured by the VAR system; changes in preferences such as attitudes to consumption, saving, work, or leisure; technology shocks – random increases or decreases in the efficiency with which firms produce goods and services; or, shocks to various frictions, such as increases or decreases in the efficiency with which various markets operate.

The time series perspective – that business cycle are being determined by various random shocks which are propagated throughout the economy over time[25] – is central to how modern macroeconomists

[24] Invented by Sims (1980) in his seminal piece "Macroeconomics and Reality."

[25] i.e. The Frisch-Slutsky paradigm: shocks to the economy causes impulses which lead to a propagation mechanism

now view economic fluctuations. VARs are a very common framework for modelling macroeconomic dynamics and the effects of shocks. But while VARs can describe how things work, they cannot explain why things work that way – hence why we need models based on economic theory (DSGE models!).

These VAR models were introduced to the economics discipline by Christopher Sims in 1980. Sims was telling macroeconomists to "get real." He criticised the widespread use of highly specialised macro-models that made very strong identifying restrictions (in the sense that each equation in the model usually excluded most of the model's other variables from the right-hand side) as well as very strong assumptions about the dynamic nature of these relationships. VARs were an alternative that allowed one to model macroeconomic data accurately, without having to impose lots of incredible restrictions. In the phrase used in an earlier paper by Sargent and Sims (who shared the Nobel prize) it was "macro modelling without pretending to have too much a priori theory". We will see that VARs are not theory free. But they do make the role of theoretical identifying assumptions far clearer than was the case for the types of models Sims was criticising.

5.1.1 Matrix representation of VARs and the vector moving average representation

Let's consider our simple VAR(1) model:

$$
\begin{aligned}
y_{1,t} &= a_{11}y_{1,t-1} + a_{12}y_{2,t-1} + e_{1,t},\ \ e_{1,t} \sim N(0,\sigma_1^2) \\
y_{2,t} &= a_{21}y_{1,t-1} + a_{22}y_{2,t-1} + e_{2,t},\ \ e_{2,t} \sim N(0,\sigma_2^2)
\end{aligned}
$$

based on the structure of the economy, which in turn results in fluctuations.

which we can express more compactly using matrices. Let

$$\mathbf{Y}_t = \begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix},$$
$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$
$$\mathbf{e}_t = \begin{bmatrix} e_{1,t} \\ e_{2,t} \end{bmatrix},$$

and so we can write the simple VAR(1) model as:

$$\mathbf{Y}_t = \mathbf{A}\mathbf{Y}_{t-1} + \mathbf{e}_t. \tag{96}$$

VARs express variables as a function of what happened yesterday and today's shocks. But what happened yesterday depended on yesterday's shocks and on what happened the day before, and so on. So with a bit of recursion, and like we do with AR(1) models, we can express the VAR(1) model as a vector moving average (VMA) model:

$$\begin{aligned}
\mathbf{Y}_t &= \mathbf{e}_t + \mathbf{A}\mathbf{Y}_{t-1} \\
&= \mathbf{e}_t + \mathbf{A}\left[\mathbf{e}_{t-1} + \mathbf{A}\mathbf{Y}_{t-2}\right] \\
&= \mathbf{e}_t + \mathbf{A}\mathbf{e}_{t-1} + \mathbf{A}^2\left[\mathbf{e}_{t-2} + \mathbf{A}\mathbf{Y}_{t-3}\right] \\
&\vdots \\
\mathbf{Y}_t &= \mathbf{e}_t + \mathbf{A}\mathbf{e}_{t-1} + \mathbf{A}^2\mathbf{e}_{t-2} + \mathbf{A}^3\mathbf{e}_{t-3} + ... + \mathbf{A}^t\mathbf{e}_0.
\end{aligned}$$

This makes it clear how today's values for the series are the cumulation of all the shocks from the past. It is also useful for deriving predictions about the properties of VARs.

5.1.2 Impulse response functions

Suppose there is an initial shock identified as:

$$\mathbf{e}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and then all shocks terms are zero afterwards, i.e. $\mathbf{e}_t = \mathbf{0}$, $\forall t > 0$. Using our VMA representation we see that the response in $\mathbf{Y}_t$ after n periods is

$$\mathbf{A}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

So the impulse response function (IRF) for VARs are directly analogous to the IRFs for AR(1) models that we looked at before.

VARs are often used for forecasting. Suppose we observe our vector of variables $\mathbf{Y}_t$. What is our forecast for $\mathbf{Y}_{t+1}$? Using forward iteration, the model for next period is:

$$\mathbf{Y}_{t+1} = \mathbf{A}\mathbf{Y}_t + \mathbf{e}_{t+1}.$$

But because $\mathbb{E}_t \mathbf{e}_{t+1} = 0$, an unbiased forecast at time t is $\mathbf{A}\mathbf{Y}_t$. In other words, $\mathbb{E}_t \mathbf{Y}_{t+1} = \mathbf{A}\mathbf{Y}_t$. The same reasoning tells us that $\mathbf{A}^2\mathbf{Y}_t$ is an unbiased forecast of $\mathbf{Y}_{t+2}$, and $\mathbf{A}^3\mathbf{Y}_t$ is in unbiased forecast of $\mathbf{Y}_{t+3}$, and so on. So once a VAR is estimated and organised in this form, it is very easy to construct forecasts.

The model (96) we've been looking at may seem like a small subset of all possible VARs because it doesn't have a constant term and only has lagged values from one period ago. However, we can easily add a third variable here which takes the constant value 1 each period. The equation for the constant term will just state that it equals its own lagged values. So this formulation actually incorporates models with constant terms. What about more than one lagged term? It turns out the first-order matrix representation can represent VARs with longer lags. Consider the two-lag system:

$$y_{1,t} = a_{11}y_{1,t-1} + a_{12}y_{1,t-2} + a_{13}y_{2,t-1} + a_{14}y_{2,t-2} + e_{1,t}$$
$$y_{2,t} = a_{21}y_{1,t-1} + a_{22}y_{1,t-2} + a_{23}y_{2,t-1} + a_{24}y_{2,t-2} + e_{2,t},$$

and define the vector

$$\mathbf{Z}_t = \begin{bmatrix} y_{1,t} \\ y_{1,t-1} \\ y_{2,t} \\ y_{2,t-1} \end{bmatrix}.$$

This system can be represented in matrix form as

$$\mathbf{Z}_t = \mathbf{A}\mathbf{Z}_{t-1} + \mathbf{e}_t, \tag{97}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; \mathbf{e}_t = \begin{bmatrix} e_{1,t} \\ 0 \\ e_{2,t} \\ 0 \end{bmatrix}.$$

The representation (97) is called the “companion form” matrix representation.

5.1.3 Interpreting shocks

The system we’ve been looking at is usually called a reduced-form VAR model. It is a purely econometric model, without any theoretical element. How should we interpret it? One interpretation is that $\epsilon_{1,t}$ is a shock that affects only $y_{1,t}$ on impact and $\epsilon_{2,t}$ is a shock that affects only $y_{2,t}$ on impact. For instance, one can use the IRFs generated from an inflation-output VAR to calculate the dynamic effects of “a shock to inflation” and “a shock to output”.

But other interpretations are available. For instance, one might imagine that the true shocks generating inflation and output are an "aggregate supply" shock and an "aggregate demand" shock and that both of these shocks have a direct effect on both inflation and output. How would we identify these "structural" shocks and their impulse responses?

Suppose reduced-form and structural shocks are related by

$$
\begin{aligned}
e_{1,t} &= c_{11}\epsilon_{1,t} + c_{12}\epsilon_{2,t} \\
e_{2,t} &= c_{21}\epsilon_{1,t} + c_{22}\epsilon_{2,t},
\end{aligned}
$$

and in matrix form we can write this as

$$\mathbf{e}_t = \mathbf{C}\boldsymbol{\epsilon}_t.$$

These two VMA representations describe the data equally well:

$$
\begin{aligned}
\mathbf{Y}_t &= \mathbf{e}_t + \mathbf{A}\mathbf{e}_{t-1} + \mathbf{A}^2\mathbf{e}_{t-2} + \mathbf{A}^3\mathbf{e}_{t-3} + ... + \mathbf{A}^t\mathbf{e}_0, \\
\Leftrightarrow \mathbf{Y}_t &= \mathbf{C}\boldsymbol{\epsilon}_t + \mathbf{A}\mathbf{C}\boldsymbol{\epsilon}_{t-1} + \mathbf{A}^2\mathbf{C}\boldsymbol{\epsilon}_{t-2} + \mathbf{A}^3\mathbf{C}\boldsymbol{\epsilon}_{t-3} + ... + \mathbf{A}^t\mathbf{C}\boldsymbol{\epsilon}_0.
\end{aligned}
$$

We can interpret the model as one with reduced form shocks, $\mathbf{e}_t$, and IRFs given by $\mathbf{A}^n$; or as a model with structural shocks, $\boldsymbol{\epsilon}_t$, and IRFs are given by $\mathbf{A}^n\mathbf{C}$. We could do this for any $\mathbf{C}$ if we knew the structural shocks.

Another way to see how reduced-form shocks can be different from structural shocks is if there are contemporaneous interactions between variables, which is likely in macroeconomics. Consider the following model:

$$
\begin{aligned}
y_{1,t} &= a_{12}y_{2,t} + b_{11}y_{1,t-1} + b_{12}y_{2,t-1} + \epsilon_{1,t} \\
y_{2,t} &= a_{21}y_{1,t} + b_{21}y_{1,t-1} + b_{22}y_{2,t-1} + \epsilon_{2,t},
\end{aligned}
$$

which can be written in matrix form as:

$$\mathbf{A}\mathbf{Y}_t = \mathbf{B}\mathbf{Y}_{t-1} + \boldsymbol{\epsilon}_t,$$

where

$$\mathbf{A} = \begin{bmatrix} 1 & -a_{12} \\ -a_{21} & 1 \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Now, if we estimate the "reduced-form" VAR model,

$$\mathbf{Y}_t = \mathbf{D}\mathbf{Y}_{t-1} + \mathbf{e}_t,$$

then the reduced-form shocks and coefficients are:

$$\mathbf{D} = \mathbf{A}^{-1}\mathbf{B},$$
$$\mathbf{e}_t = \mathbf{A}^{-1}\boldsymbol{\epsilon}_t.$$

Again, the following two decompositions both describe the data equally well:

$$\mathbf{Y}_t = \mathbf{e}_t + \mathbf{D}\mathbf{e}_{t-1} + \mathbf{D}^2\mathbf{e}_{t-2} + \mathbf{D}^3\mathbf{e}_{t-3} + ...,$$
$$\Leftrightarrow \mathbf{Y}_t = \mathbf{A}^{-1}\boldsymbol{\epsilon}_t + \mathbf{D}\mathbf{A}^{-1}\boldsymbol{\epsilon}_{t-1} + \mathbf{D}^2\mathbf{A}^{-1}\boldsymbol{\epsilon}_{t-2} + ... + \mathbf{D}^t\mathbf{A}^{-1}\boldsymbol{\epsilon}_0.$$

For the structural model, the impulse responses to the structural shocks from n periods are given by $\mathbf{D}^n\mathbf{A}^{-1}$. This is true for any matrix $\mathbf{A}$.

So why should we care about this? There seems to be no problem with forecasting with reduced-form VARs: Once you know the reduced-form shocks and how they affected today's value of the variables, you can use the reduced-form coefficients to forecast, right? The problem comes when you start asking "what if" questions/counterfactuals. For example, "what happens if there is a shock to the first variable in the VAR?" In practice, the error series in a reduced-form VAR are usually correlated with each other. So are you asking "What happens when there is a shock to the first variable only?"

or, are you asking “What usually happens when there is a shock to the first variable given that this is usually associated with a corresponding shock to the second variable?”

Most interesting questions about the structure of the economy relate to the impact of different types of shocks that are uncorrelated with each other. A structural identification that explains how the reduced-form shocks are actually combinations of uncorrelated structural shocks is far more likely to give clear and interesting answers.

5.1.4 Structural VARs: A general formulation

In its general formulation, the structural VAR (SVAR) is:

$$\underset{n\times n}{\mathbf{A}}\underset{n\times 1}{\mathbf{Y}_t} = \underset{n\times n}{\mathbf{B}}\underset{n\times 1}{\mathbf{Y}_{t-1}} + \underset{n\times n}{\mathbf{C}}\underset{n\times 1}{\boldsymbol{\epsilon}_t}. \tag{98}$$

The mode is fully described by the following parameters: n^2 parameters in $\mathbf{A}$, n^2 parameters in $\mathbf{B}$, n^2 parameters in $\mathbf{C}$, and $\frac{n^2+n}{2}$ parameters in $\boldsymbol{\Sigma}$, which describes the pattern of variances in covariances underlying the shock terms. Adding all these together, we see that the most general form of the structural VAR is a model with $3n^2 + \frac{n^2+n}{n}$ parameters. Estimating the reduced-form VAR,

$$\mathbf{Y}_t = \mathbf{D}\mathbf{Y}_{t-1} + \mathbf{e}_t,$$

gives us information on $n^2 + \frac{n^2+n}{2}$ parameters: the coefficients in $\mathbf{D}$ and the estimated variance-covariance matrix of the reduced form errors. To obtain information about structural shocks, we thus need to impose $2n^2$ a priori theoretical restrictions on our SVAR. This will leave us with $n^2 + \frac{n^2+n}{2}$ known reduced-form parameters and $n^2 + \frac{n^2+n}{2}$ structural parameters that we want to know. This can be expressed as $n^2 + \frac{n^2+n}{2}$ equations in $n^2 + \frac{n^2+n}{2}$ unknowns, so we can get a unique solution. e.g., Asserting that the reduced-form VAR is the structural model is the same as imposing the $2n^2$ a priori restrictions that $\mathbf{A} = \mathbf{C} = \mathbf{I}_n$.

SVARs generally identify their shocks as coming from distinct independent sources, and thus assume that they are uncorrelated. The error series in reduced-form VARs are usually correlated with each other. One way to view these correlations is that the reduced-form errors are combinations of a set of statistically independent structural errors. The most popular SVAR method is the recursive identification method. This method (used in the Sims 1980 paper) uses simple regression techniques to construct a set of uncorrelated structural shocks directly from the reduced-form shocks. This method sets $\mathbf{A} = \mathbf{I}_n$ and constructs the matrix $\mathbf{C}$ so that the structural shocks will be uncorrelated.

5.1.5 The Cholesky decomposition

It's probably best to go through an example. Start with a reduced-form VAR with three variables and the errors, $e_{1,t}, e_{2,t}$, and $e_{3,t}$:

$$\begin{aligned} &\mathbf{Y}_t = \mathbf{D}\mathbf{Y}_{t-1} + \mathbf{e}_t, \\ \Leftrightarrow &\begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} e_{1,t} \\ e_{2,t} \\ e_{3,t} \end{bmatrix}, \end{aligned} \tag{99}$$

where the joint distribution of $\mathbf{e}_t$ is:

$$\begin{bmatrix} e_{1,t} \\ e_{2,t} \\ e_{3,t} \end{bmatrix} \sim N\left(\mathbf{0}, \mathbf{\Sigma}\right), \ \mathbf{\Sigma} = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_y^2 & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_z^2 \end{bmatrix}.$$

Now, we want to express the shocks $\mathbf{e}_t$ as a function of structural shocks. Apply the following restriction:

$$\begin{aligned} e_{1,t} &= c_{11}\epsilon_{1,t} \\ e_{2,t} &= c_{21}\epsilon_{1,t} + c_{22}\epsilon_{2,t} \\ e_{3,t} &= c_{31}\epsilon_{1,t} + c_{32}\epsilon_{2t} + c_{33}\epsilon_{3,t}, \end{aligned}$$

or, in matrix form:

$$\begin{aligned} \begin{bmatrix} e_{1,t} \\ e_{2,t} \\ e_{3,t} \end{bmatrix} &= \begin{bmatrix} c_{11} & 0 & 0 \\ c_{21} & c_{22} & 0 \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \\ \epsilon_{3,t} \end{bmatrix}, \\ \Leftrightarrow \mathbf{e}_t &= \mathbf{C}\boldsymbol{\epsilon}_t, \end{aligned} \tag{100}$$

where the joint distribution of $\boldsymbol{\epsilon}_t$ is:

$$\begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \\ \epsilon_{3,t} \end{bmatrix} \sim N \left(\mathbf{0}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right).$$

How do we get $\mathbf{C}$? Estimation is one option, but the easier way is to use the Choleski decomposition for the variance-covariance matrix $\boldsymbol{\Sigma}$:

$$\begin{aligned} \boldsymbol{\Sigma} &= \mathbf{C}\mathbf{C}^\top, \\ \implies \mathbf{C}^{-1}\boldsymbol{\Sigma}(\mathbf{C}^\top)^{-1} &= \mathbf{I}_n. \end{aligned}$$

To see how this works, first note that the transpose Equation (100) is:

$$\begin{aligned} \mathbf{e}_t^\top &= (\mathbf{C}\boldsymbol{\epsilon}_t)^\top \\ &= \boldsymbol{\epsilon}_t^\top \mathbf{C}^\top, \end{aligned}$$

so post multiply (100) with $\mathbf{e}_t^\top$ to get:

$$\mathbf{e}_t\mathbf{e}_t^\top = \mathbf{C}\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_t^\top\mathbf{C}^\top, \tag{101}$$

or equivalently:

$$\begin{bmatrix} e_{1,t} \\ e_{2,t} \\ e_{3,t} \end{bmatrix} \begin{bmatrix} e_{1,t} & e_{2,t} & e_{3,t} \end{bmatrix} = \begin{bmatrix} c_{11} & 0 & 0 \\ c_{21} & c_{22} & 0 \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \\ \epsilon_{3,t} \end{bmatrix} \begin{bmatrix} \epsilon_{1,t} & \epsilon_{2,t} & \epsilon_{3,t} \end{bmatrix} \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ 0 & c_{22} & c_{32} \\ 0 & 0 & c_{33} \end{bmatrix},$$

and if we take expectations of (101), we get:

$$\begin{aligned} \mathbb{E}_t \left[\mathbf{e}_t \mathbf{e}_t^\top\right] &= \mathbb{E}_t \left[\mathbf{C}\boldsymbol{\epsilon}_t \boldsymbol{\epsilon}_t^\top \mathbf{C}^\top\right] \\ \Leftrightarrow \boldsymbol{\Sigma} &= \mathbf{C}\mathbf{I}_n \mathbf{C}^\top = \mathbf{C}\mathbf{C}^\top. \end{aligned} \tag{102}$$

Identification done! We have shown that we can get $\mathbf{C}$ by a Choleski decomposition for the variance-covariance matrix.

So, from (99), if we substitute in equation (100), we have

$$\begin{aligned} \begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} c_{11} & 0 & 0 \\ c_{21} & c_{22} & 0 \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \\ \epsilon_{3,t} \end{bmatrix} \\ \Leftrightarrow \mathbf{Y}_t &= \mathbf{D}\mathbf{Y}_{t-1} + \mathbf{C}\boldsymbol{\epsilon}_t, \end{aligned}$$

which can be transformed into:

$$\mathbf{C}^{-1}\mathbf{Y}_t = \mathbf{C}^{-1}\mathbf{D}\mathbf{Y}_{t-1} + \boldsymbol{\epsilon}_t, \tag{103}$$

which is nothing but a SVAR with what macro-econometricians like to call a "short-run restriction."

Note now that, by construction, the $\boldsymbol{\epsilon}_t$ shocks constructed in this way are uncorrelated with each other. This method posits a sort of "causal chain" of shocks. The first shock affects all of the variables at time t. The second only affects two of them at time t, and the last shock only affects the last variable at time t.

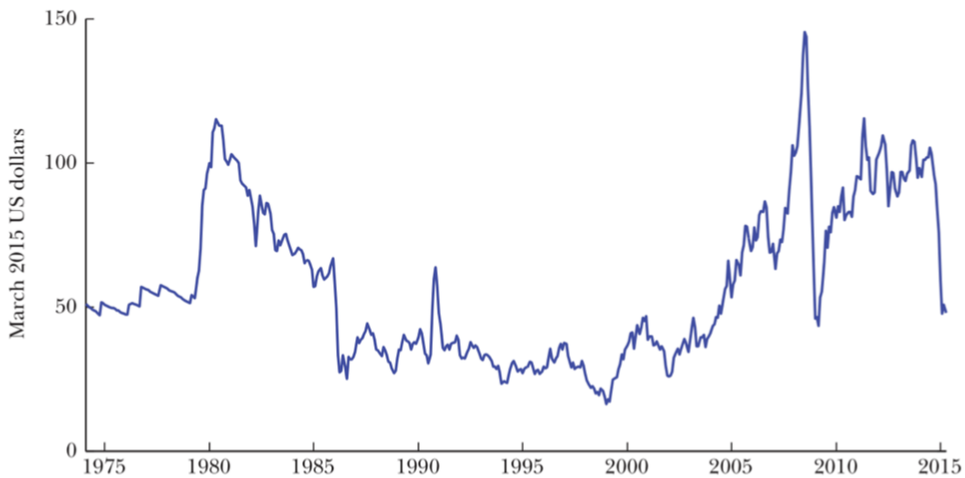
There is a serious drawback to this however: The causal ordering is not unique. Any one of the VAR variables can be listed first, and any one can be listed lasted. This means there are $n! = 1 \times 2 \times 3 \times ... \times n$

possible recursive orderings. We need to think very carefully about our own prior thinking about causation!

5.1.6 Example: Kilian (2009) and Baumeister and Kilian (2016)

Oil shocks – large run-ups and subsequent declines in the price of crude oil – regularly receive attention. Many recessions have been preceded by an increase in the price of oil. Why exactly this has occurred is not obvious: oil usage is actually a relatively small input compared to GDP.

Figure 20: Inflation-Adjusted WTI Price of Crude Oil (1974.1-2015.3)



Source: US Energy Information Administration. Note: The West Texas Intermediate (WTI) oil price series has been deflated with the seasonally adjusted US consumer price index for all urban consumers.

Empirical studies, prior to Kilian's, generally asked the question "what are the effects of an oil price shock?" Kilian (2009) and Baumeister and Kilian (2016) asked "what is an oil price shock and are there different kinds of oil price shocks?" He uses VAR analysis to distinguish between shocks to oil prices due to global demand, shocks due to oil supply, and shocks due to speculation in the oil market.

The three variable, monthly VAR model of Killian is based on the following:

$$\mathbf{z}_t = \begin{bmatrix} \Delta prod_t \\ rea_t \\ rpo_t \end{bmatrix},$$

where $\Delta prod_t$ is the growth rate of oil production, rea_t is real global economic activity, and rpo_t is the real price of oil.

The VAR structure is

$$\mathbf{A}_0 \mathbf{z}_t = \boldsymbol{\alpha} + \sum_{i=1}^{24} \mathbf{A}_i \mathbf{z}_{t-i} + \boldsymbol{\epsilon}_t,$$

where $\boldsymbol{\epsilon}_t$ are the structural shocks, and $\mathbf{A}_0$ is a lower-triangular matrix,

$$\mathbf{A}_0 = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}.$$

Kilian's identifying assumptions are:

- Oil production does not respond within the month to world demand and oil prices.
- World demand is affected within the month by oil production, but not by oil prices.
- Oil prices responded immediately to oil production and world demand.

It follows that if $\mathbf{A}_0$ is lower-triangular, then so is its inverse, $\mathbf{A}_0^{-1}$. Thus, the reduced form model is

$$\mathbf{z}_t = \mathbf{A}_0^{-1} \boldsymbol{\alpha} + \sum_{i=1}^{24} \mathbf{A}_0^{-1} \mathbf{A}_i \mathbf{z}_{t-i} + \mathbf{A}_0^{-1} \epsilon_t.$$

Reduced-form shocks, $\mathbf{e}_t$, are related to the structural shocks, $\boldsymbol{\epsilon}_t$, by

$$\begin{aligned}\mathbf{e} &= \mathbf{A}_0^{-1}\boldsymbol{\epsilon}_t \\ \Leftrightarrow \begin{bmatrix} e_t^{\Delta prod} \\ e_t^{rea} \\ e_t^{rpo} \end{bmatrix} &= \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \epsilon_t^{\Delta prod} \\ \epsilon_t^{rea} \\ \epsilon_t^{rpo} \end{bmatrix}.\end{aligned}$$

As you can see, the oil production reduced-form shock, $e_t^{\Delta prod}$, is considered a structural shock ($e_t^{\Delta prod} = \epsilon_t^{\Delta prod}$); the reduced form economic activity shock, e_t^{rea}, combines the structural oil shock and the structural activity shock, ϵ_t^{rea}; and the reduced form oil price shock, e_t^{rpo}, is a combination of all three structural shocks.

So, relative to the general model of an SVAR,

$$\mathbf{AY}_t = \mathbf{BY}_{t-1} + \mathbf{C}\boldsymbol{\epsilon}_t,$$

where are our $2n^2 = 18$ identifying restrictions? Well, we set $\mathbf{C} = \mathbf{I}$, assuming contemporaneous interactions between the variables (9 restrictions); we assumed that $\mathbf{A}_0$ is a lower triangle matrix (3 restrictions); we assume that the diagonal of $\mathbf{A}_0$ are unit coefficients (3 restrictions); and we assume that the structural shocks are orthogonal, i.e., 3 off-diagonal elements of $\boldsymbol{\Sigma}$ are zero (3 restrictions. Thus we get our 18 restrictions.

In addition to the standard IRFs, Kilian shows how the real price oil can be decomposed into components related to these three shocks. How did he do this? Recall the VMA representation:

$$\mathbf{Y}_t = \boldsymbol{\epsilon}_t + \mathbf{A}\boldsymbol{\epsilon}_{t-1} + \mathbf{A}^2\boldsymbol{\epsilon}_{t-2} + ... + \mathbf{A}^t\boldsymbol{\epsilon}_0.$$

One can do this calculation three times, each time with only one type of shock “turned on” and the others set to zero. Adding these up, one will get the realised values of $\mathbf{Y}_t$. Alternatively, one can do a dynamic simulation of the model

$$\mathbf{Y}_t = \mathbf{AY}_{t-1} + \boldsymbol{\epsilon}_t,$$

in each case letting $\boldsymbol{\epsilon}_t$ represent one of the realised historical shocks with the others set to zero.

The main results from Kilian's finds were:

- Shocks to oil supply have limited effects on oil prices and have been of negligible importance in driving oil prices over time.
- Both global demand and speculative oil price shocks can have significant effects on oil prices, but speculative oil price shocks have limited effects on global economic activity.
- Speculative oil-market shocks have accounted for most of the month-to-month movements in oil prices.
- Steady increase in oil prices from 2000 onwards was mostly due to strong global demand.
- How the economy reacts to an "oil price shock" will depend on the origins of that shock.

The last point helps to explain why the world economy survived a long period of increasing oil prices in the 2000s without going into recession (due to oil shocks – the 2008 GFC had little to do with oil prices).

5.1.7 Another VAR example: Stock and Watson (2001)

Stock and Watson in their *Journal of Economic Perspectives* 2001 piece, "Vector Autoregressions", examine the effect of monetary policy shocks. The paper is a useful introduction to VAR methods. You can think of these VARs as useful in two ways. First, an exercise in positive analysis: monetary policy co-moves with lots of other macro variables, by only identifying the structural or exogenous shocks to policy can we discover its true effects. Second, there's the normative analysis perspective: It may help a policy maker to answer the question "If I choose to raise interest rates by 25 basis points today, what is likely to happen over the next year to inflation and output relative to the case where I keep rates unchanged? Should I do this or not?" Essentially, this is a question about impulse responses.

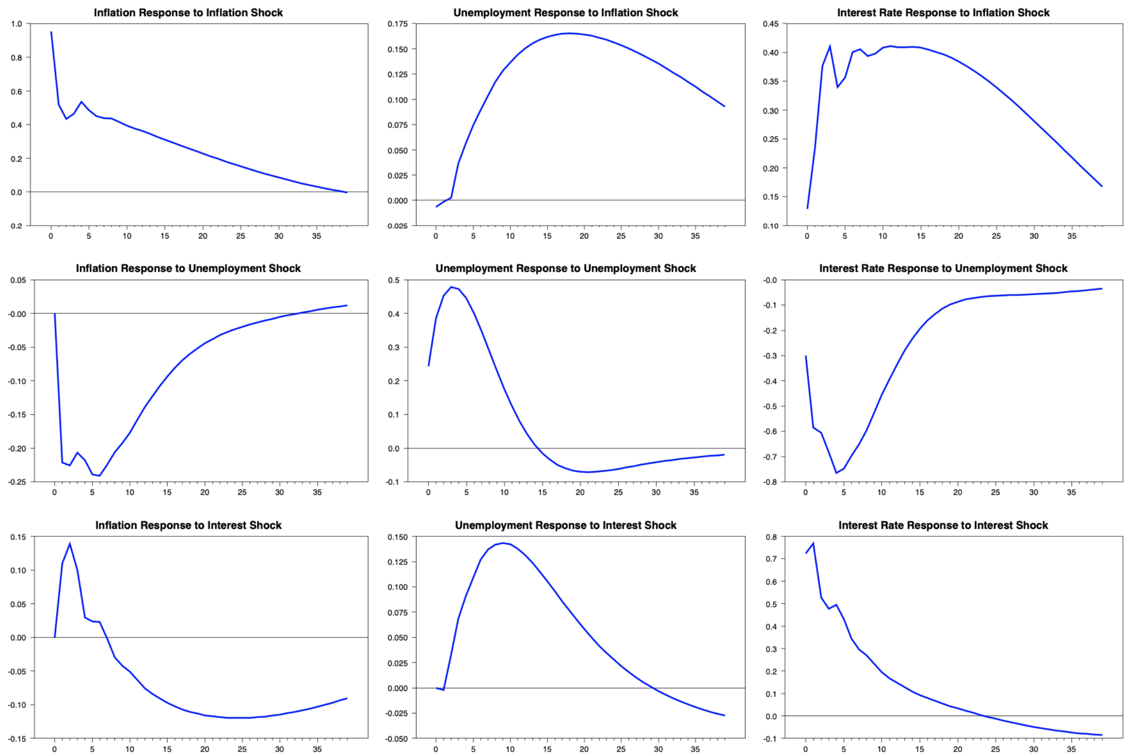
Stock and Watson's VAR features monthly data on inflation, π_t, the unemployment rate, u_t, and the federal funds rate, i_t. They posit a lower-triangle causal chain of the form:

$$\begin{aligned}\mathbf{A}\mathbf{Z}_t &= \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \pi_t \\ u_t \\ i_t \end{bmatrix} \\ &= \mathbf{B}\mathbf{Z}_{t-1} + \boldsymbol{\epsilon}_t.\end{aligned}$$

Their identification assumptions are: (i) inflation depends only on lagged values of the other variables (perhaps motivated by the idea of sticky prices); (ii) unemployment depends on contemporaneous inflation but not the funds rate; and (iii) the funds rate depends on both contemporaneous inflation and unemployment (Fed using its knowledge about the current state of the economy when it is setting rates). The IRFs from the Stock and Watson paper are reproduced in Figure 21.

Figure 21: IRFs from Recursive VAR, First Identification

Order is Inflation, Unemployment, Interest Rate



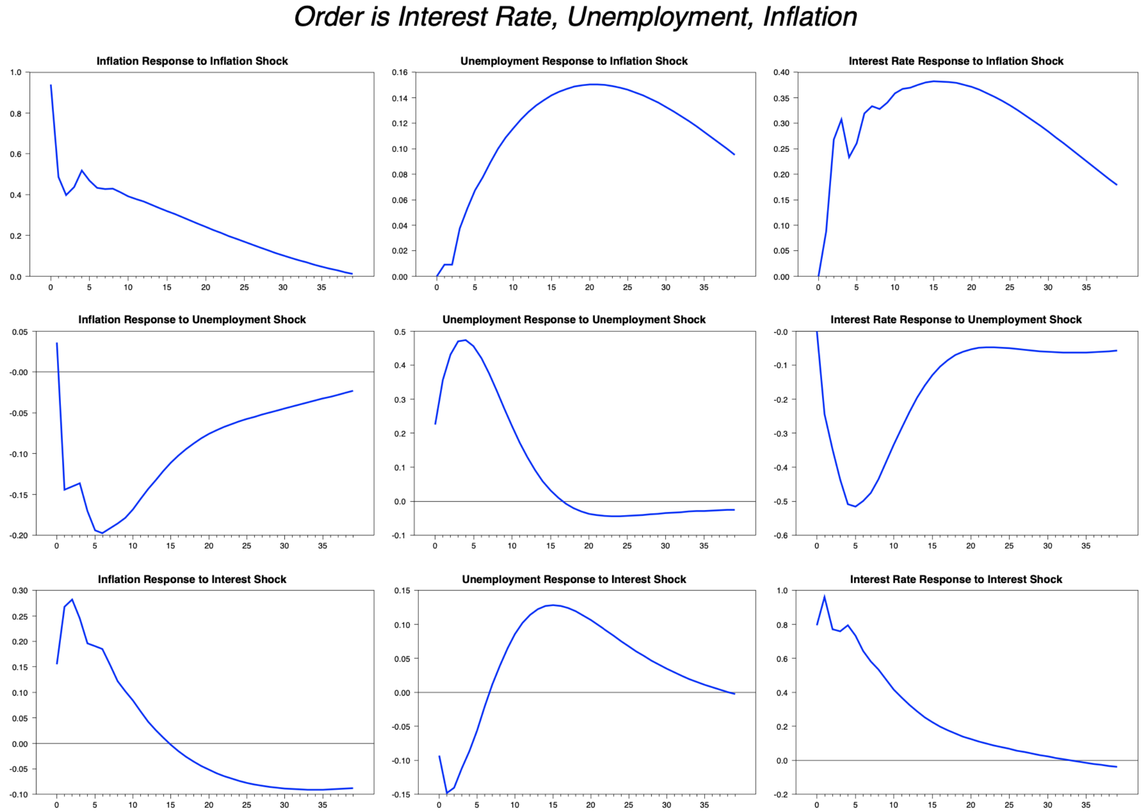
Results reproduced by Whelan (2016)

Looking at the IRFs, we see that most of the results make sense: An increase in the interest rate leads to a rise in unemployment and a delayed decline in inflation. But, the short-run response of the inflation rate, however, is a bit puzzling: the interest rate increase seems to raise the inflation rate for a few periods before it falls.

This "price puzzle" result has been obtained in a number of other VAR studies. It provides a good illustration of the potential limitations of VAR analysis. Some think the explanation is that the Fed is acting on information not captured in the VAR (e.g., information about commodity prices) and that this information may provide signals of future inflationary pressures. Thus, interest rate increases could occur just before an increase in inflation. The VAR may be capturing this pattern and confusing causation and correlation. Indeed, subsequent research has managed to solve the puzzle by factoring in commodity prices into inflation measures.

Secondly, as eluded to earlier, the ordering of a VAR is very important. Consider the IRFs in Figure 22.

Figure 22: IRFs from Recursive VAR, Second Identification



Results reproduced by Whelan (2016)

Here the ordering has been changed to: 1) interest rates, 2) unemployment, and 3) inflation. A researcher could rationalise this ordering on the grounds that the Fed can only respond to the economy with a lag because it takes time to collate data on the economy, but that inflation should be able to respond immediately to economic events. This sounds reasonable enough, but the results from this identification don't make much sense: the interest rate shock raises inflation now for almost four years and unemployment drops for a while after the increase in interest rates!

5.1.8 Long-run restrictions and the Blanchard-Quah method

The identifying assumptions in the recursive VAR approach require knowledge of how certain variables react in an instantaneous way to certain shocks. Sometimes, because certain variables are sluggish or because information about some variables is only available with a lag, we can be pretty confident about these restrictions. But often they are pure guesswork. Economic theory gives us little guidance – in fact, economic theory usually tells us about how variables react in the long-run rather than what will happen contemporaneously. For example, shocks in the IS-LM model or aggregate demand shocks have no effect on output and a positive effect on prices in the long run. This suggests an alternative approach: use these theoretically-inspired long-run restrictions to identify shocks and impulse responses.

Consider the VAR model:

$$\mathbf{Z}_t = \mathbf{B}\mathbf{Z}_{t-1} + \mathbf{C}\boldsymbol{\epsilon}_t, \tag{104}$$

where the variance-covariance matrix of the structural shocks is:

$$\mathbb{E}_t\left[\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_t^\top\right] = \begin{bmatrix} \mathbb{E}_t[\epsilon_1^2] & \mathbb{E}_t[\epsilon_1\epsilon_2] \\ \mathbb{E}_t[\epsilon_2\epsilon_1] & \mathbb{E}_t[\epsilon_2^2] \end{bmatrix} = \mathbf{I}_2,$$

so the structural shocks are uncorrelated and have unit variance. Note that the variance-covariance matrix of the observed reduced-form errors is:

$$\boldsymbol{\Sigma} = \mathbb{E}_t[\mathbf{e}_t\mathbf{e}_t^\top] = \mathbb{E}_t\left[(\mathbf{C}\boldsymbol{\epsilon}_t)(\mathbf{C}\boldsymbol{\epsilon}_t)^\top\right] = \mathbf{C}\mathbb{E}_t[\boldsymbol{\epsilon}_t\boldsymbol{\epsilon}_t^\top]\mathbf{C}^\top = \mathbf{C}\mathbf{C}^\top,$$

and as we saw before, the observed variance-covariance structure of the reduced-form shocks tells us something about how they are related to the uncorrelated, unit variance, structural shocks.

Now, suppose

$$\mathbf{Z}_t = \begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix},$$

then the long-effect of the shock on y_t is the sum of its effects of Δy_t, $\Delta y_{t+1}, \Delta y_{t+2}$, and so on. The long-run effect is the sum of the impulse responses, and the impulse responses for the model (104) are:

$\mathbf{C}$ in the impact period, $\mathbf{BC}$ after one period, $\mathbf{B}^2\mathbf{C}$ after two periods, and so on. This implies the IRFs are given by $\mathbf{B}^n\mathbf{C}$ after n periods. Thus, the long-run level effects are:

$$\mathbf{D} = (\mathbf{I} + \mathbf{B} + \mathbf{B}^2 + \mathbf{B}^3 + ... + \mathbf{B}^n)\mathbf{C}.$$

If the eigenvalues of $\mathbf{B}$ are inside unit circle, then

$$\mathbf{I} + \mathbf{B} + \mathbf{B}^2 + \mathbf{B}^3 + ... + \mathbf{B}^n = (\mathbf{I} - \mathbf{B})^{-1},$$

which is the matrix equivalent to the scalar case,

$$1 + a + a^2 + a^3 + ... + a^n = \frac{1}{1-a}.$$

Thus, the long-run responses are

$$\mathbf{D} = (\mathbf{I} - \mathbf{B})^{-1}\mathbf{C}.$$

Now, note that

$$\mathbf{D}\mathbf{D}^\top = (\mathbf{I} - \mathbf{B})^{-1}\mathbf{C}\mathbf{C}^\top \left((\mathbf{I} - \mathbf{B})^{-1}\right)^\top.$$

But, we know that $\mathbf{C}\mathbf{C}^\top = \mathbf{\Sigma}$, the variance-covariance matrix of the reduced-form shocks, which can be estimated, so we have:

$$\mathbf{D}\mathbf{D}^\top = (\mathbf{I} - \mathbf{B})^{-1}\mathbf{\Sigma} \left((\mathbf{I} - \mathbf{B})^{-1}\right)^\top. \tag{105}$$

Now, make a restriction about the long-run effects described in $\mathbf{D}$: Assume that $\mathbf{D}$ is lower triangular so only the first shock has a long-run effect on the first variable, and only the first and second shocks have long-run effects on the second variable, and so on. In the two variable case, this is just:

$$\mathbf{D} = \begin{bmatrix} d_{11} & 0 \\ d_{21} & d_{22} \end{bmatrix}.$$

Since we assume that $\mathbf{D}$ is lower triangular, a unique lower-triangle matrix $\mathbf{D}$, when post-multiplied by its transpose, will equal symmetric matrix, $\mathbf{D}\mathbf{D}^\top$. This is known as the Cholesky factor of the symmetric matrix. Typically, in most software packages, $\mathbf{D}$ can be calculated as the Cholesky factor of the known matrix $(\mathbf{I}-\mathbf{B})^{-1}\mathbf{\Sigma}\left((\mathbf{I}-\mathbf{B})^{-1}\right)^\top$.

Now, recall that $\mathbf{D}=(\mathbf{I}-\mathbf{B})^{-1}\mathbf{C}$, so the crucial matrix, $\mathbf{C}$, defining the structural shocks can be calculated as

$$\mathbf{C}=(\mathbf{I}-\mathbf{B})\mathbf{D}.$$

5.1.9 Examples of long-run restrictions and the Blanchard-Quah method

Blanchard and Quah (1989) (BQ) used a two-variable VAR in the log-difference of GDP, Δy_t, and the unemployment rate, U_t (and was entered in levels form). Because the VAR is estimated to be stationary (eigenvalues inside unit circle) both structural shocks have zero long-run effect on the unemployment rate. The lower diagonal assumption thus implies that of the two structural shocks, only one of them could have a long-run effect on the level of output. BQ labelled this the "supply shock" while the shock that has no effect on long-run output was labelled the "demand shock":

$$\begin{bmatrix}\Delta y_t\\ U_t\end{bmatrix}=\begin{bmatrix}d_{11} & 0\\ d_{21} & d_{22}\end{bmatrix}\begin{bmatrix}\epsilon_{s,t}\\ \epsilon_{d,t}\end{bmatrix}, \tag{106}$$

where $\epsilon_{s,t}$ is the supply shock, and $\epsilon_{d,t}$ is the demand shock. The relative importance of supply verses demand shocks in determining output is a long-running theme in macroeconomics. Keynesians emphasise the importance of demand shocks while more classically-orientated economists, such as advocates for the RBC approach, see supply shocks as being more important. BQ's results implied that demand shocks were responsible for the vast majority of short-run fluctuations.

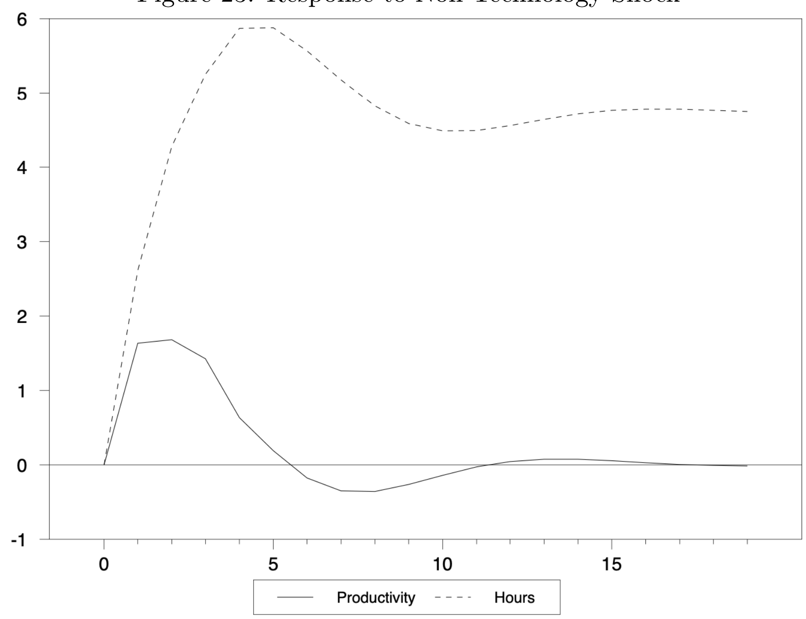
Now we look at Galí (1999), who suggested that BQ's formulation was a little bit restrictive. The assumption that neither supply nor demand shocks can change unemployment rates in the long-run may not be correct. Galí's paper applied a similar analysis to BQ, but for a formulation that moved a bit closer to the debate about RBC models and their predictions for the labour market. RBC models assume technology shocks drive the business cycle, and explain why hours worked are higher in booms than in recessions (i.e., make hay while the sun shines).

Galí's long-run restriction was as follows:

$$\begin{bmatrix} \Delta \ln\left(\frac{y}{h}\right) \\ \Delta \ln(h) \end{bmatrix} = \begin{bmatrix} d_{11} & 0 \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} \epsilon_s \\ \epsilon_d \end{bmatrix}. \tag{107}$$

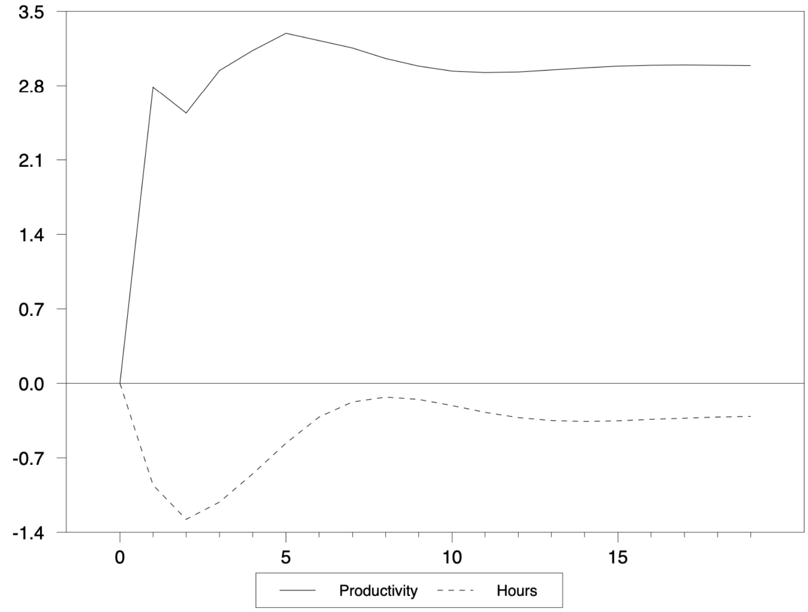
The lower-diagonal assumption about long-run responses now means that the supply shock (now called the "technology shock") can affect productivity in the long-run, while the non-technology shock cannot. The model lets the data dictate the long-run effects of technology and non-technology shocks on hours worked. If the technology shock increases hours worked then that's essentially a score for the neoclassical camp. If hours worked falls, then that's a win for the Keynesians.

Figure 23: Response to Non-Technology Shock



Source: Galí (1999)

Figure 24: Response to Technology Shock



Source: Galí (1999)

Looks like that's a win for the Keynesians. A positive technology shock actually decreases hours worked. Short-run output seems to be demand-driven not supply-driven. More efficiency means that demanded output can be supplied with less labour. Furthermore, non-technology shocks seem to cause both output and productivity to rise in the short-run.

5.2 Solving models with Rational Expectations

5.2.1 Moving beyond VARs

Having described econometric methods for measuring the shocks that hit the macroeconomy and their dynamic effects, we now turn to developing theoretical models that explain these patterns. This requires models with explicit dynamics and with stochastic shocks. Obviously, VARs are dynamic stochastic models, however they are econometric models, not theoretical models, and they have their limitations (as we previously saw). They do not explicitly characterise the underlying decisions rules adopted by firms and households – i.e., they don't tell us how or why things happen. This "why" element is crucial if the stories underlying our forecasts or analysis of policy effects are to be believed.

The goal of the modern DSGE approach is to develop models that can explain macroeconomic dynamics as well as the VAR approach, but that are based upon the fundamental idea of optimising firms and households.

5.2.2 Introducing expectations

A key sense in which DSGE models differ from VARs is that while VARs just have backward-looking dynamics, DSGE models have both backward-looking and forward-looking dynamics. The backward-looking dynamics stem, for instance, from identities linking today's capital stock with last period's capital stock and this period's investment. For example:

$$K_t = (1 - \delta)K_{t-1} + I_t.$$

The forward-looking dynamics stem from optimising behaviour: What agents expect to happen tomorrow is very important for what they decide to do today – think about our consumption Euler equation. Modelling this idea requires an assumption about how people formulate their expectations.

Almost all economic transactions rely crucially on the fact that the economy is not a "one-period game". Economic decisions have an intertemporal element to them. A key issue in macroeconomics is how people formulate expectations about them in the presence of uncertainty. Prior to the 1970s, this aspect of macroeconomic theory was largely ad hoc. Generally, it was assumed that agents used some simple extrapolative rule whereby the expected future value of a variable was close to some weighted average of its recent past values – e.g., recall how agents formed inflationary expectations in the AD-AS model.

This approach was criticised in the 1970s by economists such as Robert Lucas[26] and Thomas Sargent. Lucas and Sargent instead promoted the use of an alternative approach which they called

[26]See also the "Lucas Critique". In a nutshell, the Lucas' Critique states that it is fraught with hazard to try to predict the effects of a policy change based on correlations (or regression coefficients) based on historical data. We say that a parameter is "structural" if it is invariant to the rest of the economic environment, and, in particular, the policy environment. A parameter is "reduced form" if it is not invariant to the environment, or, more generally, if that parameter cannot be mapped back into some economic primitive.

"Rational Expectations." In economics, rational expectations usually means two things: (i) Agents use publicly available information in an efficient manner. Thus, they do not make systematic mistakes when formulating expectations; and, (ii) That agents understand the structure of the model economy and base their expectations of variables on this knowledge.

Rational Expectations clearly is a strong assumption. No one truly understand the structure of an economy – not even macroeconomists. But one reason for using Rational Expectations as a baseline assumption is that once one has specified a particular model of the economy, any other assumption about expectations means that people are making systematic errors, which seems inconsistent with rationality. In other words, we think it's entirely reasonable to presume that agents are optimising to get what's best for them. We can easily disagree on what "the best" is for them, but I think we can agree that they will [try to] act optimally.

5.2.3 First-order stochastic difference equations

A lot of models in economics take the form:

$$y_t = x_t + a\mathbb{E}_t y_{t+1}, \tag{108}$$

which just says that y today is determined by x and by tomorrow's expected value of y given the information we have today. But what determines this expected value? Rational Expectations implies a very specific answer. Under Rational Expectations, the agents in the economy understand the equation and formulate their expectation in a way that is consistent with it:

$$\mathbb{E}_t y_{t+1} = \mathbb{E}_t x_{t+1} + a\mathbb{E}_t \mathbb{E}_{t+1} y_{t+2},$$

where we can simplify the second expression on the RHS by the law of iterated expectations (LIE):[27]

$$\mathbb{E}_t y_{t+1} = \mathbb{E}_t x_{t+1} + a\mathbb{E}_t y_{t+2}.$$

[27] LIE in a nutshell: It is not rational for me expect to have a different expectation next period for y_{t+2} than the one that I have today.

Substituting our expression for $\mathbb{E}_t y_{t+1}$ into our expression for y_t yields:

$$y_t = x_t + a\mathbb{E}_t x_{t+1} + a^2 \mathbb{E}_t y_{t+2},$$

and if we kept repeating this by substituting for $\mathbb{E}_t y_{t+2}$, then $\mathbb{E}_t y_{t+3}$, and so on, we would get:

$$\begin{aligned} y_t &= x_t + a\mathbb{E}_t x_{t+1} + a^2 \mathbb{E}_t x_{t+2} + ... + a^{N-1}\mathbb{E}_t x_{t+N-1} + a^N \mathbb{E}_t y_{t+N}, \\ \Leftrightarrow y_t &= \sum_{j=0}^{N-1} a^j \mathbb{E}_t x_{t+j} + a^N \mathbb{E}_t y_{t+N}, \end{aligned}$$

where usually we assume that

$$\lim_{N\to\infty} a^N \mathbb{E}_t y_{t+N} = 0.$$

So, the solution is:

$$y_t = \sum_{k=0}^{\infty} a^k \mathbb{E}_t x_{t+k}. \tag{109}$$

This solution underlies the logic of a very large amount of modern macroeconomics.

5.2.4 Example of a first-order difference equation

Consider an asset[28] that can be purchased today for price P_t and which yields a dividend D_t. Suppose there is a close alternative to this asset that will yield a guaranteed rate of return of r. Then, a risk neutral investor will only invest in the asset if it yields the same rate of return, i.e., if

$$\frac{D_t + \mathbb{E}_t P_{t+1}}{P_t} = 1 + r. \tag{110}$$

We can rearrange this to get:

$$P_t = \frac{D_t}{1+r} + \frac{\mathbb{E}_t P_{t+1}}{1+r},$$

[28] This is a simple “Lucas tree” type of asset.

and then iterating forward we get:

$$P_t = \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^{j+1} \mathbb{E}_t D_{t+j}. \tag{111}$$

This equation, which states that asset prices should equal a discounted present-value sum of expected future dividends is usually known as the dividend-discount model.

5.2.5 Forward and backward solutions

The model

$$y_t = x_t + a\mathbb{E}_t y_{t+1} \tag{112}$$

can also be written as:

$$y_t = x_t + ay_{t+1} + a\epsilon_{t+1},$$

where ϵ_{t+1} is a forecast error that cannot be predicted at date t. Moving the time subscripts back one period and rearranging this yields:

$$y_t = a^{-1} y_{t-1} - a^{-1} x_{t-1} - \epsilon_t.$$

This backward-looking equation which can also be solved via recursive substitution to give:

$$y_t = -\sum_{j=0}^{\infty} a^{-j} \epsilon_{t-j} - \sum_{j=0}^{\infty} a^{-j} x_{t-j}. \tag{113}$$

The forward and backward solutions are both correct solutions to the first-order stochastic difference equation (as are all linear combinations of them). Which solution we choose to work with depends on the value of the parameter a. If $|a| > 1$, then the weights on future values of x_t in the forward solution will explode. In this case, it is most likely that the forward solution will not converge to a finite sum. Even if it does, the idea that today's value of y_t depends more on values of x_t far in the distant future than it does on today's values i not one that we would be comfortable with. In this case, practical applications should focus on the backwards solutions.

However, the equation holds for any set of shocks ϵ_t such that $\mathbb{E}_{t-1}\epsilon_t = 0$. So the solution is indeterminate: We can't actually predict what will happen with y_t even if we knew the full path for x_t.

But if $|a| < 1$, then the weights in the backwards solution are explosive and the forward solution is the one to focus on. Also, this solution is determinate. Knowing the path of x_t will tell you the path of y_t. In most cases, it is assumed that $|a| < 1$, and we can assume that

$$\lim_{n\to\infty} a^n \mathbb{E}_t y_{t+n} = 0,$$

amounts to a statement that y_t can't grow too fast.

What if it doesn't hold? Then the solution can have other elements. Let

$$y_t^* = \sum_{j=0}^{\infty} a^j \mathbb{E}_t x_{t+j},$$

and let $y_t = y_t^* + b_t$ be any other solution. The solution must satisfy

$$y_t^* + b_t = x_t + a\mathbb{E}_t y_{t+1}^* + a\mathbb{E}_t b_{t+1}.$$

By construction, one can show that $y_t^* = x_t + a\mathbb{E}_t y_{t+1}^*$. Now, the above equation means that the additional component satisfies

$$b_t = a\mathbb{E}_t b_{t+1},$$

and because $|a| < 1$, this means that b is always expected to get bigger in absolute value, going to infinity in expectation. This is a bubble. Note that the term bubble is usually associated with irrational behaviour by investors. But in this simple model, the agents have rational expectations. This is a rational bubble.

There may be restrictions in the real economy that stop b from growing forever. But constant growth is not the only way to satisfy $b_t = a\mathbb{E}_t b_{t+1}$. The following process also works:

$$b_{t+1} = \begin{cases} (aq)^{-1} b_t + e_{t+1}, & \text{w.p. } q, \\ e_{t+1}, & \text{w.p. } 1-q, \end{cases}$$

where $\mathbb{E}_t e_{t+1} = 0$. This is a bubble that everyone knows is going to crash eventually. And even then, a new bubble can get going. Imposing $\lim_{n\to\infty} a^n \mathbb{E}_t y_{t+n} = 0$ rules out bubbles of this (or any other) form.

5.2.6 The DSGE recipe

The forward solution to (112),

$$y_t = \sum_{j=0}^{\infty} a^j \mathbb{E}_t x_{t+j},$$

provides useful insights into how the variable y_t is determined. However, without some assumptions about how x_t evolves over time, it cannot be used to give precise predictions about the dynamics of y_t (and ideally, we want to be able to simulate the behaviour of y_t).

One reason why there is a strong linkage between DSGE modelling and VARs is because we assume that the exogenous "driving variables" such as x_t are generated by backward-looking time series models like in VARs. Consider for instance the case where the process driving x_t is AR(1),

$$x_t = \rho x_{t-1} + \epsilon_t, \quad |\rho| < 1.$$

In this case, we have

$$\mathbb{E}_t x_{t+j} = \rho^j x_t.$$

Now the model's solution can be written as

$$y_t = \left[\sum_{j=0}^{\infty} (a\rho)^j \right] x_t,$$

and because $|a\rho| < 1$, the infinite sum converges to

$$\sum_{j=0}^{\infty}(a\rho)^j = \frac{1}{1-a\rho}.$$

Which should look familiar if you did undergrad macro – it's how we derived the Keynesian multiplier formula. So, in this case, the model solution is

$$y_t = \frac{1}{1-a\rho}x_t.$$

Macroeconomists call this a reduced-form solution for the model. Together with the equation describing the evolution for x_t, it can be easily simulated on a computer (Dynare will do this for you automatically).

While this example is obviously very simple, it illustrates the general principle for getting predictions from DSGE models:

1. Obtain structural equations involving expectations of future driving variables (in this case, the $\mathbb{E}_t x_{t+j}$ terms).

2. Make assumptions about the time series process for the driving variables (in this case, x_t).

3. Solve for a reduced-form solution that can be simulated on the computer along with the driving variables.

Finally, note that the reduced-form of this model also has a VAR-like representation, which can be shown as follows

$$\begin{aligned} y_t &= \frac{1}{1-a\rho}(px_{t-1} + \epsilon_t) \\ &= \rho y_{t-1} + \frac{1}{1-a\rho}\epsilon_t. \end{aligned}$$

So both the x_t and y_t series have purely backward-looking representations. Even this simple model helps to explain how theoretical models tend to predict that the data can be described well using a

VAR.

5.2.7 Second order stochastic difference equations

First, let's define the term "jump variable," as it's a concept that will pop up a lot when solving DSGE models. Variables that are characterised by

$$y_t = \sum_{j=0}^{\infty} a^j \mathbb{E}_t x_{t+j}, \tag{114}$$

are jump variables. They only depend on what's happening today what's expected to happen tomorrow. If expectations about the future change, they will jump. Nothing that happened in the past will restrict their movement. This may be an okay characterisation of financial variables like stock prices but it's harder to argue with it as a description of variables in the real economy like employment, consumption, or investment.

Many models in macroeconomics feature variables which depend on both the expected future value and their past values. They are characterised by second-order difference equations of the form

$$y_t = ay_{t-1} + b\mathbb{E}_t y_{t+1} + x_t \tag{115}$$

Here's one way of solving second order stochastic difference equations. Suppose there was a value, λ, such that the expression,

$$v_t = y_t - \lambda y_{t-1},$$

followed a first order stochastic difference equation of the form:

$$v_t = \alpha \mathbb{E}_t v_{t+1} + \beta x_t.$$

If such a value of λ existed, we would know how to solve for v_t, and then back out the values for y_t.

From the fact that $y_t = v_t + \lambda y_{t-1}$, we can rewrite the original equation as:

$$\begin{aligned} v_t + \lambda y_{t-1} &= a y_{t-1} + b(\mathbb{E}_t v_{t+1} + \lambda y_t) + x_t \\ &= a y_{t-1} + b\mathbb{E}_t v_{t+1} + b\lambda(v_t + \lambda y_{t-1}) + x_t, \end{aligned}$$

which after rearranging yields:

$$(1 - b\lambda)v_t = b\mathbb{E}_t v_{t+1} + x_t + (b\lambda^2 - \lambda + a)y_{t-1}, \tag{116}$$

which is now a first order stochastic difference equation in v_t! So just to recap, we postulated that there existed a λ such that the variable, v_t, it defined followed a first order stochastic difference equation, and whereby it satisfies the condition:

$$b\lambda^2 - \lambda + a = 0.$$

This is a quadratic equation, so there are two values of λ that satisfy it. For either of these values, we can characterise v_t by

$$\begin{aligned} v_t &= \frac{b}{1 - b\lambda}\mathbb{E}_t v_{t+1} + \frac{1}{1 - b\lambda}x_t \\ &= \frac{b}{1 - b\lambda}\left[\frac{b}{1 - b\lambda}\mathbb{E}_t v_{t+2} + \frac{1}{1 - b\lambda}x_{t+1}\right] + \frac{1}{1 - b\lambda}x_t \\ &= \frac{b}{1 - b\lambda}\left[\frac{b}{1 - b\lambda}\left[\frac{b}{1 - b\lambda}\mathbb{E}_t v_{t+3} + \frac{1}{1 - b\lambda}x_{t+2}\right] + \frac{1}{1 - b\lambda}x_{t+1}\right] + \frac{1}{1 - b\lambda}x_t \\ &\vdots \\ \implies v_t &= \frac{1}{1 - b\lambda}\sum_{j=0}^{\infty}\left(\frac{b}{1 - b\lambda}\right)^j \mathbb{E}_t x_{t+j}, \end{aligned}$$

which as you can see, is a jump variable, and y_t obeys:

$$y_t = \lambda y_{t-1} + \frac{1}{1 - b\lambda}\sum_{j=0}^{\infty}\left(\frac{b}{1 - b\lambda}\right)^j \mathbb{E}_t x_{t+j}.$$

Usually, only one of the potential values of λ is less than one in absolute value, so this delivers the unique stable solution.[29]

5.3 Systems of stochastic difference equations

5.3.1 Introduction

Thus far, we have only looked at a single equation linking two variables. However, it turns out that the logic of the first-order stochastic difference equation underlies the solution methodology for just about all rational expectations models. Suppose one has a vector of variables:

$$\mathbf{Z}_t = \begin{bmatrix} z_{1,t} \\ z_{2,t} \\ \vdots \\ z_{n,t} \end{bmatrix}.$$

It turns out that a lot of macroeconomic models can be represented by an equation of the form

$$\mathbf{Z}_t = \mathbf{B}\mathbb{E}_t\mathbf{Z}_{t+1} + \mathbf{X}_t, \tag{117}$$

where $\mathbf{B}$ is an $n \times n$ matrix. The logic of recursive or iterated substitution can also be applied to this model to give a solution of the form:

$$\mathbf{Z}_t = \sum_{j=0}^{\infty} \mathbf{B}^j \mathbb{E}_t \mathbf{X}_{t+j}. \tag{118}$$

5.3.2 Eigenvalues and eigenvectors

As with the single-equation model, this will only give you a stable non-explosive solution under certain conditions. A value, λ_i, is an eigenvalue of the matrix $\mathbf{B}$ if there exists an "eigenvector" $\mathbf{e}_i$ such that:

$$\mathbf{B}\mathbf{e}_i = \lambda_i \mathbf{e}_i.$$

[29] If this all seems a bit abstract now, don't worry. We will go more in-depth in the next section.

Many $n \times n$ matrices have n distinct eigenvalues. Denote by $\mathbf{P}$ the matrix that has as its columns n eigenvectors corresponding to these eigenvalues. In this case:

$$\mathbf{BP} = \mathbf{P\Lambda},$$

where

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & & & \mathbf{O} \\ & \lambda_2 & & \\ & & \ddots & \\ \mathbf{O} & & & \lambda_n \end{bmatrix}$$

is a diagonal matrix of eigenvalues, and$\mathbf{O}$ denotes triangular matrices.[30] Now assume that we can write:

$$\mathbf{B} = \mathbf{P\Lambda P}^{-1}. \tag{119}$$

This tells us something about the relationship between eigenvalues and higher powers of $\mathbf{B}$, because:

$$\mathbf{B}^n = \mathbf{P\Lambda}^n\mathbf{P}^{-1} = \mathbf{P} \begin{bmatrix} \lambda_1^n & & & \mathbf{O} \\ & \lambda_2^n & & \\ & & \ddots & \\ \mathbf{O} & & & \lambda_n^n \end{bmatrix} \mathbf{P}^{-1}.$$

So, the difference between lower and higher powers of $\mathbf{B}$ is that the higher powers depend on the eigenvalues taken to the power of n. If all of the eigenvalues are inside the unit circle (i.e., less than one in absolute value) then all of the entries in $\mathbf{B}^n$ will tend towards zero as $n \to \infty$. So, a condition that ensures that model of the form (117) has unique stable forward-looking solution is that the eigenvalues of $\mathbf{B}$ are all inside the unity circle.

[30] Just be aware that sometimes I will use the matrix $\mathbf{O}$ to signify a square null matrix; other times, such as this, I use it in substitution for a void of null values. These two cases should be clear from context. For more information on this notation see Turkington (2013).

How are eigenvalues calculated? Consider a simple 2×2 matrix,

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

and suppose that $\mathbf{A}$ has two eigenvalues, λ_1 and λ_2, and define $\boldsymbol{\lambda}$ as the vector:

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}.$$

The fact that there are eigenvectors which when multiplied by $\mathbf{A} - \boldsymbol{\lambda}\mathbf{I}$ equal a vector of zeros (i.e., we have $(\mathbf{A} - \boldsymbol{\lambda}\mathbf{I})\mathbf{e}_i = \mathbf{0}$) means that the determinant of the matrix,

$$\mathbf{A} - \boldsymbol{\lambda}\mathbf{I} = \begin{bmatrix} a_{11} - \lambda_1 & a_{12} \\ a_{21} & a_{22} - \lambda_2 \end{bmatrix}$$

equals zero, i.e.,

$$\det(\mathbf{A} - \boldsymbol{\lambda}\mathbf{I}) = 0.$$

So, solving the quadratic formula:

$$(a_{11} - \lambda_1)(a_{22} - \lambda_2) - a_{12}a_{21} = 0,$$

gives the two eigenvalues of $\mathbf{A}$.

5.3.3 The Binder-Pesaran method

Consider a vector $\mathbf{Z}_t$ characterised by

$$\mathbf{Z}_t = \mathbf{A}\mathbf{Z}_{t-1} + \mathbf{B}\mathbb{E}_t\mathbf{Z}_{t+1} + \mathbf{H}\mathbf{X}_t. \tag{120}$$

The restriction to one-lag and one-lead form is apparent, and the companion matrix trick can be used to allow this model to represent models with n leads and lags. In this sense, this equation summarises all possible linear rational expectations models.

Binder and Pesaran (1996) solved this model in a manner exactly analogous to the second-order difference equation discussed earlier: find a matrix $\mathbf{C}$ such that $\mathbf{W}_t = \mathbf{Z}_t - \mathbf{C}\mathbf{Z}_{t-1}$ obeys a first-order matrix equation of the form

$$\mathbf{W}_t = \mathbf{F}\mathbb{E}_t\mathbf{W}_{t+1} + \mathbf{G}\mathbf{X}_t.$$

In other words, we transform the problem of solving the "second-order" system in equation into a simpler first-order system.

What must matrix $\mathbf{C}$ be? Using the fact that $\mathbf{Z}_t = \mathbf{W}_t + \mathbf{C}\mathbf{Z}_{t-1}$, the model can be rewritten as:

$$\begin{aligned}\mathbf{W}_t + \mathbf{C}\mathbf{Z}_{t-1} &= \mathbf{A}\mathbf{Z}_{t-1} + \mathbf{B}(\mathbb{E}_t\mathbf{W}_{t+1} + \mathbf{C}\mathbf{Z}_t) + \mathbf{H}\mathbf{X}_t \\ &= \mathbf{A}\mathbf{Z}_{t-1} + \mathbf{B}(\mathbb{E}_t\mathbf{W}_{t+1} + \mathbf{C}(\mathbf{W}_t + \mathbf{C}\mathbf{Z}_{t-1})) + \mathbf{H}\mathbf{X}_t.\end{aligned}$$

This rearranges to:

$$(\mathbf{I} - \mathbf{B}\mathbf{C})\mathbf{W}_t = \mathbf{B}\mathbb{E}_t\mathbf{W}_{t+1} + (\mathbf{B}\mathbf{C}^2 - \mathbf{C} + \mathbf{A})\mathbf{Z}_{t-1} + \mathbf{H}\mathbf{X}_t.$$

Because $\mathbf{C}$ is the matrix such that $\mathbf{W}_t$ follows a first-order forward-looking matrix equation, it follows that

$$\mathbf{B}\mathbf{C}^2 - \mathbf{C} + \mathbf{A} = \mathbf{O}.$$

This "matrix quadratic equation" can be solved to give $\mathbf{C}$. Solving these equations is non-trivial, however. One method uses the fact that $\mathbf{C} = \mathbf{B}\mathbf{C}^2 + \mathbf{A}$ to solve iteratively is as follows. Provide an initial guess, say $\mathbf{C}_0 = \mathbf{I}$, and then iterate on $\mathbf{C}_n = \mathbf{B}\mathbf{C}_{n-1}^2 + \mathbf{A}$ until all the entries in $\mathbf{C}_n$ converge. Once we know $\mathbf{C}$, we have:

$$\mathbf{W}_t = \mathbf{F}\mathbb{E}_t\mathbf{W}_{t+1} + \mathbf{G}\mathbf{X}_t,$$

where

$$\mathbf{F} = (\mathbf{I} - \mathbf{BC})^{-1}\mathbf{B},$$

$$\mathbf{G} = (\mathbf{I} - \mathbf{BC})^{-1}\mathbf{H}.$$

Assuming that all the eigenvalues of $\mathbf{F}$ are inside the unit circle, this has a stable forward-looking solution:

$$\mathbf{W}_t = \sum_{j=0}^{\infty} \mathbf{F}^j \mathbb{E}_t[\mathbf{G}\mathbf{X}_{t+j}],$$

which can be written in terms of the original equation as:

$$\mathbf{Z}_t = \mathbf{C}\mathbf{Z}_{t-1} + \sum_{j=0}^{\infty} \mathbf{F}^j \mathbb{E}_t[\mathbf{G}\mathbf{X}_{t+j}].$$

Finally, consider the case in which the driving variables $\mathbf{X}_t$ follow a VAR representation of the form:

$$\mathbf{X}_t = \mathbf{D}\mathbf{X}_{t-1} + \boldsymbol{\epsilon}_t,$$

where $\mathbf{D}$ has eigenvalues inside the unit circle. This implies $\mathbb{E}_t\mathbf{X}_{t+j} = \mathbf{D}^j\mathbf{X}_t$, so the model solution is:

$$\mathbf{Z}_t = \mathbf{C}\mathbf{Z}_{t-1} + \left[\sum_{j=0}^{\infty} \mathbf{F}^j \mathbf{G}\mathbf{D}^j\right] \mathbf{X}_t.$$

The infinite sum in this equation will converge to a matrix $\mathbf{P}$, so the model has a reduced-form representation:

$$\mathbf{Z}_t = \mathbf{C}\mathbf{Z}_{t-1} + \mathbf{P}\mathbf{X}_t,$$

which can be simulated along with the VAR process for the driving variables. This provides a relatively simple recipe for simulating DSGE models: Specify the $\mathbf{A}, \mathbf{B}$, and $\mathbf{H}$ matrices; solve for $\mathbf{C}, \mathbf{F}$, and $\mathbf{G}$; specify a VAR process for the driving variables; and then obtain the reduced form representations.

5.4 Comments and key readings

All these mathematical "prerequisites" are certainly a lot to digest. If some of it has gone over your head then not to worry – we'll revisit some of these concepts later on with some more examples. Without further ado, let's move onto the RBC model...

In addition to the examples shown in this section, other really good papers to read to get a better understanding of how VAR models work are Eichenbaum and Evans (1995) and C. D. Romer and D. H. Romer (2004). Christiano et al. (1999, 2005) are really good too (and highly important papers in modern macroeconomic research), but are technically quite challenging. Also, they talk a lot about the effects of monetary policy, which I've wanted to avoid because our m

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6 The Real Business Cycle Model

6.1 Introduction

As we saw in the first section, modern economies undergo significant short-run fluctuations in aggregate output and employment. What's more is that these fluctuations don't really follow a pattern that we can heuristically predict or forecast easily. We do, however, know that these fluctuations have some intriguing characteristics. Understanding the causes and characteristics of these aggregate fluctuations is a central goal of macroeconomics. Critically, by understanding these factors, we can build models which can replicate business cycle moments and to hopefully consider optimal policy responses to these fluctuations.

In this section (and the ones that follow), we develop the leading theories concerning the causes and nature of macroeconomic fluctuations. We have so far worked with rudimentary general equilibrium/Walrasian models, and have slowly been increasing our proficiency by building evermore sophisticated models. Now, we are ready to take on the challenge of building a Walrasian model in order to explain business cycles.

First, it's important to state the assumptions we'll be making. Our Walrasian model will feature perfectly competitive markets without externalities, asymmetric information, missing markets, or other imperfections. All the neoclassical models we've looked at so far have featured these assumptions, so can we pick a familiar to model to build upon? The Ramsey model seems like a very good candidate to start with. We know that absent of any shocks, the Ramsey model with converge to a balanced growth path, and then grows smoothly. It then seems sensible to incorporate business cycle fluctuations and shocks into the Ramsey model. From there, we can also look at things like worker productivity and government purchases. Because the Ramsey model features no money or prices,[31] the shocks we will introduce to the Ramsey model will all be in real terms. These shocks will thus change the actual productive capacity of the economy. Hence, the modified Ramsey model is known as the Real Business Cycle (RBC) model.

The RBC model also features one other significant departure from the Ramsey model: labour will

[31] More accurately, the Ramsey model does not feature any nominal variables. Output is treated as a numeraire good. Households consume and save it, and get paid in units of it. Think of it like something like rice or corn.

be endogenous and will be allowed to vary. Instead of just optimising over consumption in each period, households in the RBC model will be able to choose how much time they would like to allocate between working and leisure. The motivation for this is, again, wanting to build a model which can explain the business cycle facts we looked at in the opening section. We saw that despite investment and output fluctuating quite wildly throughout the business cycle, working hours were quite invariant. We hope that our RBC model can capture this phenomenon.

As we will soon find out, however, the RBC model does a pretty poor job of explaining actual fluctuations. Thus, we will have to move beyond the baseline RBC model to far more sophisticated models – as is standard in the discipline of macroeconomics. At the same time, however, what these models are trying to accomplish remains the ultimate goal of business cycle research: building a general equilibrium model from microeconomic foundations and a specification of the underlying shocks that explains, both qualitatively and quantitatively, the main features of macroeconomic fluctuations. Despite its empirical failings, the RBC model established a research agenda which remains as the central orthodoxy in macroeconomics to this day: the research of dynamic stochastic general equilibrium (DSGE) models. The RBC model represented such a significant departure from the models that came before it, that many economists see it as the progenitor of the current DSGE paradigm.

6.2 The social planner's (centralised) problem

There are a few ways to set up an RBC model: either from the perspective of a benevolent social planner[32] which is able to allocate all resources in the economy, or by setting up competitive markets and finding market equilibria. The Ramsey social planner seeks to maximise social welfare subject to the economy's resource constraints; in competitive markets, agents optimise their utility or profit given their endowments. In the RBC model, both approaches yield the same outcome – an important point that we will later come back to.[33] For now, to keep things simple, we will set up the RBC model from the perspective of the Ramsey social planner.

[32] Often referred to as the "Ramsey social planner."

[33] In macroeconomic models, solving the Ramsey planner's problem yields the social welfare maximising, Pareto-efficient solution. This is because in other models we will look at, markets are not fully competitive or efficient, so the competitive equilibrium will be unable to achieve a first-best outcome. Here, in the RBC model where all markets are complete and efficient, the Ramsey policy coincides with the competitive market solution.

The Ramsey planner faces the following problem:

$$\arg\max_{\{C_t, N_t\}} \mathbb{E}_t \left[\sum_{i=0}^{\infty} \beta^i (U(C_{t+i}) - V(N_{t+i})) \right], \tag{121}$$

where C_t is consumption, N_t is hours worked, and β is the representative household's rate of time preference (their discount factor). In words, the Ramsey planner wishes to maximise households' welfare by assigning the optimal amounts of consumption and labour supply. Furthermore, the Ramsey planner wishes to maximise (121) subject to the following economy-wide resource constraints:

$$Y_t = C_t + I_t, \tag{122}$$

$$Y_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha}, \tag{123}$$

$$K_t = I_t + (1-\delta) K_{t-1}, \tag{124}$$

and a process for the technology shock term A_t:

$$\log A_t = (1-\rho) \log \bar{A} + \rho \log A_{t-1} + \epsilon_t, \ \epsilon_t \sim IID(0, \sigma_a^2), \tag{125}$$

where Y_t is output, I_t is investment into new capital, A_t is a technology term, K_t is productive capital, and δ is the depreciation rate.

The question that arises now is: how do we go about maximising (121)? The main issue is that we have a stream of future consumption and labour decisions to make, constrained to the fact that we don't know what A_t will be in the future. Technically, the best way to solve this problem is using stochastic dynamic programming[34], but we don't have time for that. Instead, we will use a trick and simplification: we treat the Ramsey problem as a deterministic problem and then substitute $\mathbb{E}_t X_{t+i}$ for X_{t+i}. Can we do this? Sure.

Suppose

$$G(x) = \sum_{j=1}^{N} p_j F(a_j, x),$$

[34] Try reading Stokey, Lucas, and Prescott – but good luck. It's a tough read.

which is maximised by setting

$$G'(x) = \sum_{j=1}^{N} p_j F'(a_j, x) = \mathbb{E}_t F'(x) = 0,$$

so, the FOCs for maximising $\mathbb{E}_t F(x)$ are just $\mathbb{E}_t F'(x) = 0$.

Now, we can combine our constraints to simply get:

$$A_t K_{t-1}^{\alpha} N_t^{1-\alpha} = C_t + K_t - (1-\delta)K_{t-1}. \tag{126}$$

Then, we can set up the Ramsey planner's problem as a Lagrangian:

$$\begin{aligned}\mathcal{L} = \mathbb{E}_t &\left[\sum_{i=0}^{\infty} \beta^i (U(C_{t+i}) - V(N_{t+i}))\right] \\ &+ \mathbb{E}_t \left[\sum_{i=0}^{\infty} \beta^i \lambda_{t+i} \left\{A_{t+i} K_{t+i-1}^{\alpha} N_{t+i}^{1-\alpha} + (1-\delta)K_{t+i-1} - C_{t+i} - K_{t+i}\right\}\right].\end{aligned} \tag{127}$$

But this is still a hideous equation to work with. It involves two infinite sums, so technically there is an infinite number of FOCs for current and future expected values of consumption, capital, and labour. So, what can we do? This is macroeconomics, so we will use another trick/simplification.

We want to take a snapshot of how the variables behave in the period in which we are optimising in, t. Most of our variables are denoted in period t with the subscript t, so they're fine. But we have K_{t-1} and A_{t-1} in the law of motion equations for capital and technology, respectively. So, what we can do is set up the Lagrange with the objective function based in period t, a single constraint dated in period t, and then we can add in a second constraint from period $t+1$. Then, the period t variables appear as:

$$\begin{aligned}\mathcal{L} = U(C_t) - V(N_t) &+ \lambda_t(A_t K_{t-1}^{\alpha} N_t^{1-\alpha} + (1-\delta)K_{t-1} - C_t - K_t) \\ &+ \beta \mathbb{E}_t \left[\lambda_{t+1}\left(A_{t+1} K_t^{\alpha} N_{t+1}^{1-\alpha} + (1-\delta)K_t - C_{t+1} - K_{t+1}\right)\right].\end{aligned}$$

After that, the period t variables don't ever appear again. So, the FOCs for the period t variables consist of differentiating this equation with respect to these variables and setting the derivatives equal to zero. Then, the period $t+i$ variables appear exactly as the period t variables do, except that they are in expectation form and they are multiplied by the discount rate β^i. But this means that the FOCs for the period $t+i$ variables will be identical to those for period t variables. So differentiating this equation gives us the equations for the optimal dynamics at all times.

Thus, we yield the following FOCs:

$$
\begin{aligned}
\mathcal{L}_{C_t} &= U'(C_t) - \lambda_t = 0, && (128)\\
\mathcal{L}_{K_t} &= -\lambda_t + \beta \mathbb{E}_t \left[\lambda_{t+1} \left(\alpha \frac{Y_{t+1}}{K_t} + 1 - \delta \right) \right] = 0, && (129)\\
\mathcal{L}_{N_t} &= -V'(N_t) + \lambda_t (1-\alpha) \frac{Y_t}{N_t} = 0, && (130)\\
\mathcal{L}_{\lambda_t} &= A_t K_{t-1}^{\alpha} N_t^{1-\alpha} + (1-\delta) K_{t-1} - C_t - K_t = 0. && (131)
\end{aligned}
$$

Easy!

6.3 The Keynes-Ramsey condition (consumption Euler equation)

Define the marginal value of an additional unit of capital next year as

$$R_{t+1} = \alpha \frac{Y_{t+1}}{K_t} + 1 - \delta,$$

then the FOC for capital (129) can be written as:

$$\lambda_t = \beta \mathbb{E}_t \left[\lambda_{t+1} R_{t+1} \right],$$

and this can be combined with the FOC for consumption (128) to yield:

$$U'(C_t) = \beta \mathbb{E}_t \left[U'(C_{t+1}) R_{t+1} \right], \tag{132}$$

which is nothing but the consumption Euler equation – sometimes referred to as the Keynes-Ramsey condition. As a quick refresher, we can interpret the Keynes-Ramsey condition as: decreasing consumption by Δ today, at a loss of $U'(C_t)\Delta$ in utility; invest to get $R_{t+1}\Delta$ tomorrow; that investment is worth $\beta\mathbb{E}_t\left[U'(C_{t+1})R_{t+1}\Delta\right]$ in terms of utility today; and, along the optimal path, an agent must be indifferent between these options.

If we assume CRRA utility and a simple linear technology for the disutility from labour, we can write the utility function as:

$$U(C_t) - V(N_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \eta N_t,$$

then the Keynes-Ramsey condition (132) becomes:

$$C_t^{-\sigma} = \beta\mathbb{E}_t\left[C_{t+1}^{-\sigma}R_{t+1}\right],$$

and the intratemporal Euler equation for labour and leisure, derived from (130), becomes:

$$-\eta + C_t^{-\sigma}(1-\alpha)\frac{Y_t}{N_t} = 0.$$

6.4 Equilibrium and log-linearisation

The RBC model can be defined by the following seven equations:

$$Y_t = C_t + I_t, \tag{133}$$

$$Y_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha}, \tag{134}$$

$$K_t = I_t + (1-\delta)K_{t-1}, \tag{135}$$

$$R_t = \alpha\frac{Y_t}{K_{t-1}} + 1 - \delta, \tag{136}$$

$$C_t^{-\sigma} = \beta\mathbb{E}_t\left[C_{t+1}^{-\sigma}R_{t+1}\right], \tag{137}$$

$$\frac{Y_t}{N_t} = \frac{\eta}{1-\alpha}C_t^{\sigma}, \tag{138}$$

$$\log A_t = (1-\rho)\log\bar{A} + \rho\log A_{t-1} + \epsilon_t, \tag{139}$$

so we have seven equations in seven unknown variables. Notice that a lot of the RBC model equations are non-linear – and we haven't discussed any strategies of solving systems of stochastic non-linear equations. So what can we do? Again, this is macroeconomics, so there's a trick: we linearise the model equations via log-linearisation, from which we can then solve the model.

The idea is to use Taylor series approximations. In general, any non-linear function $F(x_t, y_t)$ can be approximated around any point $F(x_t^*, y_t^*)$ using the formula:

$$\begin{aligned}F(x_t, y_t) = &F(x_t^*, y_t^*) + F_x(x_t^*, y_t^*)(x_t - x_t^*) + F_y(x_t^*, y_t^*)(y_t - y_t^*) \\ &+ F_{xx}(x_t^*, y_t^*)(x_t - x_t^*)^2 + F_{yy}(x_t^*, y_t^*)(y_t - y_t^*)^2 + F_{xy}(x_t^*, y_t^*)(x_t - x_t^*)(y_t - y_t^*) + ...\end{aligned}$$

If the gap between (x_t, y_t) and (x_t^*, y_t^*) is small, then terms in second and higher powers and cross-terms will all be very small and can be ignored (i.e. a first-order Taylor series approximation will suffice), leaving something like:

$$F(x_t, y_t) \approx \alpha + \beta_1 x_t + \beta y_t.$$

But if we linearise about a point that (x_t, y_t) is far away from (if F is very non-linear), then this approximation will not be accurate.

DSGE models use a particular version of this technique. They take logs and then linearise the logs of variables about a simple "steady-state" path in which all real variables are growing at the same rate. The steady-state path is relevant because the stochastic economy will, on average, tend to fluctuate around the values given by this path, making the approximation an accurate one. This will give us a set of linear equations in terms of deviations of the logs of these variables from their steady-state values.

Remember that log-differences are approximately percentage deviations:

$$\log X - \log Y \approx \frac{X - Y}{Y},$$

so this approach gives us a system that expresses variables in terms of their percentage deviations from the steady-state paths. It can be thought of as giving a system of variables that represents the business-cycle component of the model! Coefficients are elasticities and IRFs are easy to interpret. Also, believe it or not, log-linearisation is easy – we won't have to take a lot of derivatives.

From here, it's important to note down some notation. Let "hatted" variables (e.g. $\hat{X}_t$) denote log-deviations of variables from their steady state values, denoted by a "bar" (e.g. $\bar{X}$):

$$\hat{X}_t = \log X_t - \log \bar{X}.$$

The key to the log-linearisation method is that every variable can be written as:

$$X_t = \bar{X}\frac{X_t}{\bar{X}} = \bar{X}e^{\hat{X}_t},$$

and the big trick is that a first-order Taylor approximation of $e^{\hat{X}_t}$ is given by:

$$e^{\hat{X}_t} \approx 1 + \hat{X}_t.$$

So, we can write variables as:

$$X_t \approx \bar{X}(1 + \hat{X}_t).$$

The next trick is for variables multiplying each other such as:

$$X_t Y_t \approx \bar{X}\bar{Y}(1 + \hat{X}_t)(1 + \hat{Y}_t) \approx \bar{X}\bar{Y}(1 + \hat{X}_t + \hat{Y}_t),$$

because you set terms like $\hat{X}_t\hat{Y}_t = 0$ since we're looking at small deviations from steady-state and multiplying these small deviations together gives a term close to zero.

Anything else? Nope, that's it. It's also worth noting, however, that there are a few ways to do log-linearisation. The above gives a short-cut, broad picture approach to log-linearisation.[35] It's probably best that we go through a few examples in order to nail down how log-linearisation works.

[35] "A Toolkit for Analyzing Nonlinear Dynamic Stochastic Models Easily" by Harold Uhlig (1995) gives a very rigorous treatment of log-linearisation.

6.4.1 The Uhlig method

Start with

$$Y_t = C_t + I_t,$$

now, we could take logs and then do some total derivatives to log-linearise, but it's far easier to use the methodology explained above (often referred to as the Uhlig method). Rewrite our equation as:

$$\begin{aligned}\bar{Y}e^{\hat{Y}_t} &= \bar{C}e^{\hat{C}_t} + \bar{I}e^{\hat{I}_t} \\ \Leftrightarrow \bar{Y}(1+\hat{Y}_t) &= \bar{C}(1+\hat{C}_t) + \bar{I}(1+\hat{I}_t),\end{aligned}$$

and we know that in the steady state $\bar{Y} \equiv \bar{C} + \bar{I}$, so terms cancel out, so

$$\begin{aligned}\bar{Y}\hat{Y}_t &= \bar{C}\hat{C}_t + \bar{I}\hat{I}_t \\ \therefore \hat{Y}_t &= \frac{\bar{C}}{\bar{Y}}\hat{C}_t + \frac{\bar{I}}{\bar{Y}}\hat{I}_t. \qquad (140)\end{aligned}$$

6.4.2 The Taylor expansion (standard) method

This method works particularly well when you have multiplicative terms. So let's start with our production technology:

$$Y_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha},$$

and then take logs:

$$\log Y_t = \log A_t + \alpha \log K_{t-1} + (1-\alpha)\log N_t,$$

where we know that

$$\ln X_t = \ln \bar{X} + \frac{X_t - \bar{X}}{\bar{X}},$$

and so we have:

$$\ln \bar{Y} + \frac{Y_t - \bar{Y}}{\bar{Y}} = \ln \bar{A} + \frac{A_t - \bar{A}}{\bar{A}} + \alpha\left[\ln \bar{K} + \frac{K_{t-1} - \bar{K}}{\bar{K}}\right] + (1-\alpha)\left[\ln \bar{N} + \frac{N_t - \bar{N}}{\bar{N}}\right],$$

and we know that in the steady state we have $\ln \bar{Y} = \ln \bar{A} + \alpha \ln \bar{K} + (1-\alpha) \ln \bar{N}$, so

$$\begin{aligned} \frac{Y_t - \bar{Y}}{\bar{Y}} &= \frac{A_t - \bar{A}}{\bar{A}} + \alpha \frac{K_{t-1} - \bar{K}}{\bar{K}} + (1-\alpha) \frac{N_t - \bar{N}}{\bar{N}} \\ \Leftrightarrow \hat{Y}_t &= \hat{A}_t + \alpha \hat{K}_{t-1} + (1-\alpha) \hat{N}_t. \end{aligned} \tag{141}$$

6.4.3 The total derivative method

This method is a bit of a headache, but it does come in handy when we have to deal with messy expressions. It essentially uses the fact that the differential of a variable, say X_t, about its steady state can be written as $\frac{1}{\bar{X}} dX_t$, where $dX_t = X_t - \bar{X}$. Again, it's better to demonstrate this, so let's take:

$$R_t = \alpha \frac{Y_t}{K_{t-1}} + 1 - \delta,$$

and don't bother taking logs (since we don't have to deal with any power terms); just take the total derivative:

$$\begin{aligned} dR_t &= \alpha \frac{1}{\bar{K}} dY_t - \alpha \frac{\bar{Y}}{\bar{K}^2} dK_{t-1} \\ \Leftrightarrow R_t - \bar{R} &= \alpha \frac{1}{\bar{K}} (Y_t - \bar{Y}) - \alpha \frac{\bar{Y}}{\bar{K}^2} (K_{t-1} - \bar{K}), \end{aligned}$$

and then divide the LHS and RHS by $\bar{R}$, and then do some manipulation to the terms on the RHS:

$$\begin{aligned} \frac{R_t - \bar{R}}{\bar{R}} &= \frac{1}{\bar{R}} \left[\alpha \frac{1}{\bar{K}} (Y_t - \bar{Y}) \frac{\bar{Y}}{\bar{Y}} - \alpha \frac{\bar{Y}}{\bar{K}^2} (K_{t-1} - \bar{K}) \right] \\ \Leftrightarrow \hat{R}_t &= \frac{1}{\bar{R}} \left[\alpha \frac{\bar{Y}}{\bar{K}} \hat{Y}_t - \alpha \frac{\bar{Y}}{\bar{K}} \hat{K}_{t-1} \right], \end{aligned}$$

and then clean up a bit to get:

$$\hat{R}_t = \frac{\alpha}{\bar{R}} \frac{\bar{Y}}{\bar{K}} \left[\hat{Y}_t - \hat{K}_{t-1} \right]. \tag{142}$$

Now let's look at the Keynes-Ramsey condition since it does have an exponent term:

$$C_t^{-\sigma} = \beta \mathbb{E}_t \left[C_{t+1}^{-\sigma} R_{t+1} \right],$$

and then take logs:[36]

$$-\sigma \ln C_t = \ln \beta - \sigma \mathbb{E}_t \ln C_{t+1} + \mathbb{E}_t \ln R_{t+1},$$

then take total derivatives:

$$\begin{aligned}
\frac{-\sigma}{\bar{C}} dC_t &= \frac{-\sigma}{\bar{C}} \mathbb{E}_t dC_{t+1} + \frac{1}{\bar{R}} \mathbb{E}_t dR_{t+1} \\
\Leftrightarrow -\sigma \frac{C_t - \bar{C}}{\bar{C}} &= -\sigma \frac{\mathbb{E}_t C_{t+1} - \bar{C}}{\bar{C}} + \frac{\mathbb{E}_t R_{t+1} - \bar{R}}{\bar{R}} \\
-\sigma \hat{C}_t &= -\sigma \mathbb{E}_t \hat{C}_{t+1} + \mathbb{E}_t \hat{R}_{t+1} \\
\hat{C}_t &= \mathbb{E}_t \hat{C}_{t+1} - \frac{1}{\sigma} \mathbb{E}_t \hat{R}_{t+1}. \qquad (143)
\end{aligned}$$

Again, not too difficult since the terms were multiplicative.

Those with eagle eyes are probably outraged that I've seemingly ignored Jensen's Inequality[37] by taking logs of the expected future value of consumption and the interest rate. In general, you would be correct: the log of an expectation is not equal to the expectation of a log term. So let's log-linearise the Keynes-Ramsey condition without applying logs. To start, replace the variables C_t, C_{t+1}, and R_{t+1} using the Uhlig/substitution method and write

$$\begin{aligned}
\left(\bar{C} \exp\left\{\hat{C}_t\right\}\right)^{-\sigma} &= \beta \mathbb{E}_t \left[\left(\bar{C} \exp\left\{\hat{C}_{t+1}\right\}\right)^{-\sigma} \bar{R} \exp\left\{\hat{R}_{t+1}\right\}\right] \\
\left(\exp\left\{\hat{C}_t\right\}\right)^{-\sigma} &= \mathbb{E}_t \left[\left(\exp\left\{\hat{C}_{t+1}\right\}\right)^{-\sigma} \exp\left\{\hat{R}_{t+1}\right\}\right] \\
\mathbb{E}_t \left[\exp\left\{\hat{R}_{t+1}\right\}\right] &= \left(\frac{\mathbb{E}_t \left[\exp\left\{\hat{C}_{t+1}\right\}\right]}{\exp\left\{\hat{C}_t\right\}}\right)^{\sigma} \\
\mathbb{E}_t \left[\exp\left\{\hat{R}_{t+1}\right\}\right] &= \frac{\mathbb{E}_t \left[\exp\left\{\sigma \hat{C}_{t+1}\right\}\right]}{\exp\left\{\sigma \hat{C}_t\right\}}.
\end{aligned}$$

[36] I've taken natural logs here to show that it doesn't matter whether you take logs with base 10 or e.

[37] Recall that:

$$\ln \mathbb{E}[x] \geq \mathbb{E}[\ln x].$$

Now apply the approximation by replacing the exponent terms:

$$\begin{aligned}
1 + \mathbb{E}_t \hat{R}_{t+1} &= \frac{1 + \sigma \mathbb{E}_t \hat{C}_{t+1}}{1 + \sigma \hat{C}_t} \\
\left(1 + \sigma \hat{C}_t\right)\left(1 + \mathbb{E}_t \hat{R}_{t+1}\right) &= 1 + \sigma \mathbb{E}_t \hat{C}_{t+1} \\
1 + \mathbb{E}_t \hat{R}_{t+1} + \sigma \hat{C}_t + \underbrace{\sigma \hat{C}_t \mathbb{E}_t \hat{R}_{t+1}}_{=0} &= 1 + \sigma \mathbb{E}_t \hat{C}_{t+1} \\
\hat{C}_t &= \mathbb{E}_t \hat{C}_{t+1} - \frac{1}{\sigma} \mathbb{E}_t \hat{R}_{t+1},
\end{aligned}$$

which is what we had above.

6.4.4 Taylor approximation method: Single variable case

We won't use this method for the RBC model, but it can come in handy in the future. Consider the following non-linear first-order difference equation:

$$X_t = f(X_{t-1}),$$

where f is any non-linear functional form you can think of (something not too crazy, though). First-order Taylor expansion of the RHS about the steady-state gives:

$$X_t \approx f(\bar{X}) + f'(\bar{X})(X_{t-1} - \bar{X}),$$

and in the steady state if we assume $\bar{X} = f(\bar{X})$,then our Taylor expansion becomes:

$$\begin{aligned}
X_t &\approx \bar{X} + f'(\bar{X})(X_{t-1} - \bar{X}) \\
\Leftrightarrow X_t - \bar{X} &\approx f'(\bar{X})(X_{t-1} - \bar{X})
\end{aligned}$$

then divide this by $\bar{X}$:

$$\frac{X_t - \bar{X}}{\bar{X}} \approx f'(\bar{X}) \frac{\bar{X}_{t-1} - \bar{X}}{\bar{X}},$$

and with a bit cleaning up we have:

$$\hat{X}_t = f'(\bar{X})\hat{X}_{t-1}. \tag{144}$$

Consider the following example:

$$K_t = (1-\delta)K_{t-1} + AK_{t-1}^{\alpha},$$

and then apply the formula in (144) to get:

$$\hat{K}_t = \left[1 - \delta + \alpha A\bar{K}^{\alpha-1}\right]\hat{K}_{t-1}$$

6.4.5 Taylor approximation method: Multivariate case

The Taylor approximation has a vector version as well as a scalar version. Suppose have:

$$X_t = f(X_{t-1}, Y_t),$$

where f is a non-linear function. The vector (bivariate) version of a first-order Taylor expansion about the steady-state is:

$$X_t = f(\bar{X}, \bar{Y}) + f_X(\bar{X}, \bar{Y})(X_{t-1} - \bar{X}) + f_Y(\bar{X}, \bar{Y})(Y_t - \bar{Y}),$$

and again, set the steady state condition $\bar{X} = f(\bar{X}, \bar{Y})$, and with a bit of rearranging we get:

$$X_t - \bar{X} = f_X(\bar{X}, \bar{Y})(X_{t-1} - \bar{X}) + f_Y(\bar{X}, \bar{Y})(Y_t - \bar{Y}),$$

and then divide through by $\bar{X}$:

$$\frac{X_t - \bar{X}}{\bar{X}} = f_X(\bar{X}, \bar{Y})\frac{(X_{t-1} - \bar{X})}{\bar{X}} + f_Y(\bar{X}, \bar{Y})\frac{(Y_t - \bar{Y})}{\bar{X}},$$

use the steady-state trick ("create something out of nothing") on the second term on the RHS:

$$\frac{X_t - \bar{X}}{\bar{X}} = f_X(\bar{X}, \bar{Y})\frac{(X_{t-1} - \bar{X})}{\bar{X}} + f_Y(\bar{X}, \bar{Y})\frac{(Y_t - \bar{Y})}{\bar{X}}\frac{\bar{Y}}{\bar{Y}},$$

and then clean up

$$\hat{X}_t \approx f_X(\bar{X}, \bar{Y})\hat{X}_{t-1} + f_Y(\bar{X}, \bar{Y})\frac{\bar{Y}}{\bar{X}}\hat{Y}_t \tag{145}$$

Consider the following example

$$K_t = (1-\delta)K_{t-1} + sZ_t K_{t-1}^{\alpha},$$

and so taking partial derivatives and following formula in (145) gives:

$$\hat{K}_t = \left[(1-\delta) + \alpha s \bar{Z}\bar{K}^{\alpha-1}\right]\hat{K}_{t-1} + \left[s\bar{K}^{\alpha}\right]\frac{\bar{Z}}{\bar{Y}}\hat{Z}_t.$$

6.5 Log-linearised system and the steady state

The full log-linearised system is given by following seven equations:

$$\hat{Y}_t = \frac{\bar{C}}{\bar{Y}}\hat{C}_t + \frac{\bar{I}}{\bar{Y}}\hat{I}_t, \tag{146}$$

$$\hat{Y}_t = \hat{A}_t + \alpha\hat{K}_{t-1} + (1-\alpha)\hat{N}_t, \tag{147}$$

$$\hat{K}_t = \frac{\bar{I}}{\bar{K}}\hat{I}_t + (1-\delta)\hat{K}_{t-1}, \tag{148}$$

$$\hat{R}_t = \frac{\alpha}{\bar{R}}\frac{\bar{Y}}{\bar{K}}\left[\hat{Y}_t - \hat{K}_{t-1}\right], \tag{149}$$

$$\hat{C}_t = \mathbb{E}_t\hat{C}_{t+1} - \frac{1}{\sigma}\mathbb{E}_t\hat{R}_{t+1}, \tag{150}$$

$$\hat{N}_t = \hat{Y}_t - \sigma\hat{C}_t, \tag{151}$$

$$\hat{A}_t = \rho\hat{A}_{t-1} + \epsilon_t. \tag{152}$$

We are almost ready to take this basic RBC model to a computer (e.g., Dynare). We simply need to calibrate the model (macroeconomist speak for assigning values to our parameters), and to solve for steady state values. We need to calculate $\frac{\bar{C}}{\bar{Y}}, \frac{\bar{I}}{\bar{K}}, \frac{\alpha}{\bar{R}}\frac{\bar{Y}}{\bar{K}}$. We can do this by taking the original non-linearised RBC model and figuring out what things look like along a zero-growth path.

Start with the steady-state interest rate. This is linked to consumption behaviour via the Keynes-Ramsey condition:

$$1 = \beta \mathbb{E}_t \left[\left(\frac{C_t}{C_{t+1}} \right)^{\sigma} R_{t+1} \right].$$

Because we have no trend growth in technology in our model, the steady-state features consumption, investment, and output all taking on constant values with no uncertainty. Thus, in steady state, we have $\bar{C}_t = \bar{C}_{t+1} = \bar{C}$, so

$$\bar{R} = \frac{1}{\beta}, \tag{153}$$

i.e. in a no-growth economy, the rate of return on capital is determined by the rate of time preference.

Next, take the equation for the rate of return on capital (in period t):

$$R_t = \alpha \frac{Y_t}{K_{t-1}} + 1 - \delta.$$

In the steady state we have:

$$\bar{R} = \frac{1}{\beta} = \alpha \frac{\bar{Y}}{\bar{K}} + 1 - \delta,$$

thus, with a bit of rearranging we get:

$$\frac{\bar{Y}}{\bar{K}} = \frac{\beta^{-1} + \delta - 1}{\alpha}. \tag{154}$$

So we have

$$\begin{aligned} \frac{\alpha}{\bar{R}} \frac{\bar{Y}}{\bar{K}} &= [\alpha\beta] \left[\frac{\beta^{-1} + \delta - 1}{\alpha} \right] \\ &= 1 - \beta(1 - \delta), \end{aligned} \tag{155}$$

which is one of the steady-state values we needed.

Now, look at the law of motion of capital:

$$K_t = I_t + (1-\delta)K_{t-1},$$

and use the fact that in the steady state we have $\bar{K}_t = \bar{K}_{t-1} = \bar{K}$, so:

$$\frac{\bar{I}}{\bar{K}} = \delta, \tag{156}$$

which is also what we were looking for.

Putting things together, we have:

$$\frac{\bar{I}}{\bar{Y}} = \frac{\frac{\bar{I}}{\bar{K}}}{\frac{\bar{Y}}{\bar{K}}} = \frac{\delta}{\frac{\beta^{-1}+\delta-1}{\alpha}} = \frac{\alpha\delta}{\beta^{-1}+\delta-1}, \tag{157}$$

and

$$\frac{\bar{C}}{\bar{Y}} = 1 - \frac{\bar{I}}{\bar{Y}} = 1 - \frac{\alpha\delta}{\beta^{-1}+\delta-1}. \tag{158}$$

So the final, log-linearised RBC model is:

$$\hat{Y}_t = \left[1 - \frac{\alpha\delta}{\beta^{-1}+\delta-1}\right]\hat{C}_t + \left[\frac{\alpha\delta}{\beta^{-1}+\delta-1}\right]\hat{I}_t, \tag{159}$$

$$\hat{Y}_t = \hat{A}_t + \alpha\hat{K}_{t-1} + (1-\alpha)\hat{N}_t, \tag{160}$$

$$\hat{K}_t = \left[\frac{\alpha\delta}{\beta^{-1}+\delta-1}\right]\hat{I}_t + (1-\delta)\hat{K}_{t-1}, \tag{161}$$

$$\hat{R}_t = [1-\beta(1-\delta)]\left[\hat{Y}_t - \hat{K}_{t-1}\right], \tag{162}$$

$$\hat{C}_t = \mathbb{E}_t\hat{C}_{t+1} - \frac{1}{\sigma}\mathbb{E}_t\hat{R}_{t+1}, \tag{163}$$

$$\hat{N}_t = \hat{Y}_t - \sigma\hat{C}_t, \tag{164}$$

$$\hat{A}_t = \rho\hat{A}_{t-1} + \epsilon_t. \tag{165}$$

We will explore the performance of this model via numerical simulation. But first, let's compare the decentralised RBC model to this setup with the Ramsey social planner.

6.6 The decentralised model

As I mentioned previously, there are alternatives in how to set up the RBC model, but these will give us the same outcome. In this section, I will now setup the RBC model without the Ramsey social planner. General equilibrium will be achieved via competitive markets as households and firms optimise over their endowments. Furthermore, I will make the assumption that firms own the capital stock in the economy, while the households own the firms – again, whether the firms own the capital stock or households own the capital stock, both will lead to the same outcome.

6.6.1 The household problem

The representative household allocates its time between work and leisure, and it picks a stream of consumption $\{C_t\}_{t=0}^{\infty}$ to maximise its present discounted value of lifetime utility. In exchange for supplying labour, the household earns a wage, w_t, which it takes as given. The household purchases one-period bonds, B_t, which pays out a gross interest rate of R_{t-1}. R_{t-1} is the interest rate known at $t-1$ which pays out in t when the bond matures. The household also takes the interest rate as given. Additionally, since the representative agent owns firms, it earns firms' profit in the form of dividend imputations, Π_t. So, the household problem can be written as:

$$\arg \max_{\{C_t, N_t B_t\}} \mathbb{E}_t \left[\sum_{i=0}^{\infty} \beta^i \left(U(C_{t+i}) - V(N_{t+i}) \right) \right],$$

subject to

$$C_t + B_t \leq w_t N_t + \Pi_t + R_{t-1} B_{t-1}. \tag{166}$$

Note the timing I have used in the budget constraint. I assume that the bonds purchased in period $t-1$ pay out R_{t-1} in period t, and that the household will purchase in a new bundle of bonds, B_t, which will pay R_t in period $t+1$. Forming a Lagrangian (and using the trick in (127)) gives us:

$$\begin{aligned}\mathcal{L} = U(C_t) - V(N_t) + \lambda_t \left(w_t N_t + \Pi_t + R_{t-1} B_{t-1} - C_t - B_t \right) \\ + \beta \mathbb{E}_t \left[\lambda_{t+1} \left(w_{t+1} N_{t+1} + \Pi_{t+1} + R_t B_t - C_{t+1} - B_{t+1} \right) \right],\end{aligned}$$

and the following FOCs:

$$\mathcal{L}_{C_t} = U'(C_t) - \lambda_t = 0, \tag{167}$$

$$\mathcal{L}_{N_t} = -V'(N_t) + \lambda_t w_t = 0, \tag{168}$$

$$\mathcal{L}_{B_t} = -\lambda_t + \beta \mathbb{E}_t \lambda_{t+1} R_t = 0, \tag{169}$$

$$\mathcal{L}_{\lambda_t} = w_t N_t + \Pi_t + R_{t-1} B_{t-1} - C_t - B_t = 0. \tag{170}$$

These seem very familiar. They're essentially identical to the problem which the Ramsey planner solved.

6.6.2 The firm problem

There is a representative firm. The firm wants to maximise the present discounted value of real net profits. It discounts future cash flows by the stochastic discount factor. The way we'll define the stochastic discount factor puts cash flows (measured in goods) in terms of current consumption. The stochastic discount factor is:

$$M_{t,t+i} = \beta^i \frac{\mathbb{E}_t U'(C_{t+i})}{U'(C_t)},\ i > t,$$

where t is the current period. Why do the firms use this formulation for the stochastic discount factor? Because this is how consumers value future dividend flows. One unit of dividends returned to the household at time $t+i$ generates $U'(C_{t+i})$ additional units of utility, which must be discounted back to the present period (which we assume to be 0), by β^i. Dividing by $U'(C_t)$ gives the current consumption equivalent value of the future utils.[38]

The firm produced output, Y_t, with a CRS production function,

$$Y_t = A_t F(K_{t-1}, N_t),$$

with the usual assumptions that we make. It hires labour, purchases new capital goods, and issues one-period debt promises, D_t. The firm also pays gross interest, R_{t-1}, on debt issued in the previous

[38] We will explore this further when look at macro-finance.

period, and the interest paid on debt is equal to the interest paid on bonds due to a no-arbitrage condition. The firm's problem can be written as:

$$\max_{\{N_t, I_t, D_t, K_t\}} V_t = \mathbb{E}_t \sum_{i=0}^{\infty} M_{t+i} \left(A_{t+i} F(K_{t-1+i}, N_{t+i}) - w_{t+i} N_{t+i} - I_{t+i} + D_{t+i} - R_{t-1+i} D_{t-1+i} \right),$$

subject to

$$K_t = I_t + (1-\delta) K_{t-1}.$$

Rearranging the law of motion for capital, and substituting for I_t in the objective function gives us:

$$\begin{aligned}
&\max_{\{N_t, D_t, K_t\}} V_t = \mathbb{E}_t \sum_{i=0}^{\infty} M_{t+i} \begin{pmatrix} A_{t+i} F(K_{t-1+i}, N_{t+i}) - K_{t+i} + (1-\delta) K_{t-1+i} \\ -w_{t+i} N_{t+i} + D_{t+i} - R_{t-1+i} D_{t-1+i} \end{pmatrix} \\
\Leftrightarrow &\max_{\{N_t, D_t, K_t\}} V_t = A_t F(K_{t-1}, N_t) - K_t + (1-\delta) K_{t-1} - w_t N_t + D_t - R_{t-1} D_{t-1} \\
&\qquad + \mathbb{E}_t \left[M_{t+1} \left(A_{t+1} F(K_t, N_{t+1}) - K_{t+1} + (1-\delta) K_t - w_{t+1} N_{t+1} + D_{t+1} - R_t D_t \right) \right],
\end{aligned}$$

which basically says that the firm's revenue each period is equal to output. Its costs each period are the wage bill, investment in new physical capital, and servicing costs on its debt. It can raise its cash flow by issuing new debt.

The FOCs from the firm problem are:

$$\frac{\partial V_t}{\partial N_t} = A_t F_N(K_{t-1}, N_t) - w_t = 0,$$

$$\implies w_t = A_t F_N(K_{t-1}, N_t), \tag{171}$$

$$\frac{\partial V_t}{\partial D_t} = 1 - \mathbb{E}_t M_{t+1} R_t = 0$$

$$\Leftrightarrow 1 = \mathbb{E}_t \beta R_t \frac{U'(C_{t+1})}{U'(C_t)}$$

$$\implies U'(C_t) = \beta \mathbb{E}_t R_t U'(C_{t+1}), \tag{172}$$

$$\frac{\partial V_t}{\partial K_t} = -1 + \mathbb{E}_t M_{t+1} A_{t+1} F_K(K_t, N_{t+1}) + (1-\delta) = 0$$

$$\Leftrightarrow 1 = \mathbb{E}_t \beta \frac{U'(C_{t+1})}{U'(C_t)} A_{t+1} F_K(K_t, N_{t+1}) + (1-\delta)$$

$$\implies U'(C_t) = \beta \mathbb{E}_t U'(C_{t+1}) A_{t+1} F_K(K_t, N_{t+1}) + (1-\delta). \tag{173}$$

Let's interpret these FOCs a bit. (171) is pretty intuitive: The wage rate w_t is equal to the marginal productivity of labour. However, look at (172) and (173) – they're essentially the same, and must therefore hold in equilibrium as long as the household is optimising. This means that the amount of debt the firm issues is indeterminate, since the condition will hold for any choice of D_t. This is essentially the Modigliani-Miller theorem:[39] it doesn't matter how the firm finances its purchases of new capital – debt or equity – and hence the debt/equity mix is indeterminate.

6.6.3 Technology process

In order to close the model, we need to specify a stochastic process for the exogenous variable(s). The only exogenous variable in this model is A_t, so let's assume:

$$\ln A_t = \rho \ln A_{t-1} + \epsilon_t. \tag{174}$$

[39] "The Cost of Capital, Corporation Finance and the Theory of Investment" (1958, *AER*)

6.6.4 Equilibrium

A competitive equilibrium is a set of prices $\{R_t, w_t\}$ and allocations $\{C_t, N_t, K_t, D_t, B_t\}$ taking K_{t-1}, D_{t-1}, B_{t-1}, A_{t-1}, and the stochastic process for A_t as given; the optimality conditions (167)-(173); the labour and bond market clearing conditions ($N_t^d = N_t^s$ and $B_t = D_t,\ \forall t$); and both budget constraints holding with equality.

Consolidating the household and firm budget constraints gives:

$$C_t + B_t = w_t N_t + R_{t-1} B_{t-1} + A_t F(K_{t-1}, N_t) - w_t N_t - I_t + D_t - R_{t-1} D_{t-1}$$

$$\implies A_t F(K_{t-1}, N_t) = C_t + I_t, \tag{175}$$

in other words, bond market-clearing plus both budget constraints holding just gives the standard accounting identity that output must be consumed or invested.

If you combine the household's FOC for labour supply (168) with the firm's FOC, you get:

$$V'(N_t) = U'(C_t) A_t F_N(K_{t-1}, N_t). \tag{176}$$

The FOC for bonds/debt (173) along with the FOC for the firm's choice of its capital stock (172) imply that:

$$\beta \mathbb{E}_t U'(C_{t+1}) A_{t+1} F_K(K_t, N_{t+1}) + (1-\delta) = \beta \mathbb{E}_t R_t U'(C_{t+1}),$$

which can be rewritten as:

$$R_t = \frac{\mathbb{E}_t U'(C_{t+1}) A_{t+1} F_K(K_t, N_{t+1}) + (1-\delta)}{\mathbb{E}_t U'(C_{t+1})}. \tag{177}$$

Putting all the equations together, the equilibrium conditions for the decentralised RBC model are:

$$U'(C_t) = \beta \mathbb{E}_t U'(C_{t+1}) A_{t+1} F_K(K_t, N_{t+1}) + (1-\delta), \tag{178}$$

$$V'(N_t) = U'(C_t) A_t F_N(K_{t-1}, N_t), \tag{179}$$

$$K_t = A_t F(K_{t-1}, N_t) - C_t + (1-\delta) K_{t-1}, \tag{180}$$

$$\ln A_t = \rho \ln A_{t-1} + \epsilon_t, \tag{181}$$

$$Y_t = A_t F(K_{t-1}, N_t), \tag{182}$$

$$Y_t = C_t + I_t, \tag{183}$$

$$U'(C_t) = \beta \mathbb{E}_t U'(C_{t+1}) R_t, \tag{184}$$

$$w_t = A_t F_N(K_{t-1}, N_t), \tag{185}$$

$$R_t = A_t F_K(K_{t-1}, N_t). \tag{186}$$

But these are nothing the same as the equilibrium conditions[40] for when we solved for the Ramsey social planner. Why is this the case?

6.7 First and Second Welfare Theorems of Economics

Recall the fundamental welfare theorems of economics (from Mas-Colell et al. (1995)):

The First Fundamental Welfare Theorem: If the economy is described by complete markets, no externalities or non-convexities then every equilibrium of the competitive market is socially optimal.

The Second Fundamental Welfare Theorem: If household preferences and firm production sets are convex, there is a complete set of markets with publicly known prices, and every agent acts as a price taker, then any Pareto optimal outcome can be achieved as a competitive equilibrium if appropriate lump-sum transfers of wealth are arranged.

The result that the competitive equilibrium of a representative agent economy and that of a perfectly competitive one, that is otherwise identical, is not surprising. The statement of the second

[40] You might be looking at the equations for R_t, (177) and (186), and asking "how are they the same?". Once you account for depreciation, roll back the marginal product of capital back to period t, the two expressions equalise.

fundamental welfare theorem holds for finite dimensional economies. Our economies have an infinite number of periods and, therefore, an infinite number of goods. The conditions for existence of a competitive equilibrium in infinite horizon economies are somewhat more complex than those for finite dimensional ones and some extra assumptions are required.[41] Here, we simply assume that a competitive equilibrium exists and are interested in its relationship to a Ramsey planner economy.

The first fundamental welfare theorem tells us that that any competitive equilibrium is necessarily Pareto optimal, so that the equilibrium found using a decentralised economy with factor and goods markets is also Pareto optimal. The second welfare theorem tells us that, since the production technologies and preferences are the same between the centralised and decentralised economies, then with the right initial wealth conditions, the competitive economy can achieve an equilibrium that is identical to the Ramsey planner economy.

The second fundamental welfare theorem permits us to use a representative agent economy to mimic a competitive economy. Since the second fundamental theorem is carefully worded, it should be clear that using a representative agent economy will not always give the appropriate results. If the economy is not perfectly competitive, if part of the economy has some monopoly power, or if there are some external or internal restrictions that prevent some agents from being perfectly competitive, then the equilibrium found by the decentralised economy will not necessarily be achievable with a representative agent economy.[42]

However, when the conditions are right (like in the RBC model), solving a representative agent economy is often technically much simpler than solving a decentralised economy. In this case, the second fundamental welfare theorem states that, with appropriate initial conditions, the solution of the representative agent economy is one for the decentralised economy.

[41] See Stokey et al. (1989) chapter 16 for details.

[42] It is partly for this reason that neoclassical models (which invariably satisfy the second welfare theorem) are better understood and articulated than Keynesian models. The latter involve many departures from the welfare theorems and as a result it is far harder to characterise the equilibrium properties of such models.

6.8 Assessing the RBC model

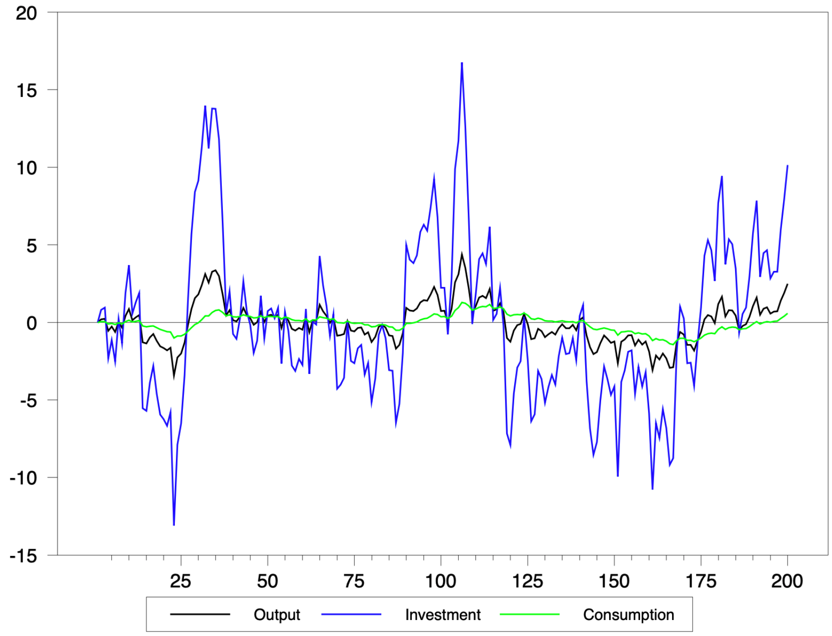
Let's return to our log-linearised RBC model given by (159)-(165):

$$
\begin{aligned}
\hat{Y}_t &= \left[1 - \frac{\alpha\delta}{\beta^{-1}+\delta-1}\right]\hat{C}_t + \left[\frac{\alpha\delta}{\beta^{-1}+\delta-1}\right]\hat{I}_t, \\
\hat{Y}_t &= \hat{A}_t + \alpha\hat{K}_{t-1} + (1-\alpha)\hat{N}_t, \\
\hat{K}_t &= \left[\frac{\alpha\delta}{\beta^{-1}+\delta-1}\right]\hat{I}_t + (1-\delta)\hat{K}_{t-1}, \\
\hat{R}_t &= [1-\beta(1-\delta)]\left[\hat{Y}_t - \hat{K}_{t-1}\right], \\
\hat{C}_t &= \mathbb{E}_t\hat{C}_{t+1} - \frac{1}{\sigma}\mathbb{E}_t\hat{R}_{t+1}, \\
\hat{N}_t &= \hat{Y}_t - \sigma\hat{C}_t, \\
\hat{A}_t &= \rho\hat{A}_{t-1} + \epsilon_t,
\end{aligned}
$$

and let's see how the model performs. If we parameterise the model with the following values: $\alpha = \frac{1}{3}$, $\beta = 0.99$, $\delta = 0.015$, $\rho = 0.95$, and $\sigma = 1$, and simulate, what do we get?

Figure 25 shows results from a 200-period simulation of the RBC model. It demonstrates the main successful feature of the RBC model: It generates actual business cycles they look very realistic! Reasonable parameterisations of the model can roughly match the magnitude of observed fluctuations in output, and the model can match the fact that investment is far more volatile than consumption. You can see why RBC models were met with high praise when Kydland and Prescott (1982) and Hansen (1985) brought forward the RBC research agenda.

Figure 25: RBC Models Generate Cycles with Volatile Investment



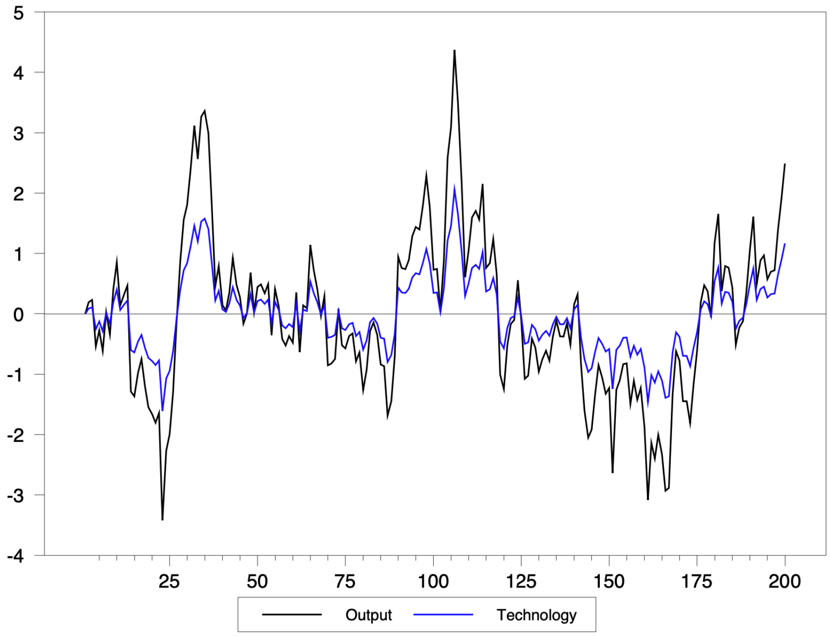
Source: Whelan (2016)

But, despite the successes of the RBC model, they also had some major weaknesses which were heavily criticised by the Keynesians. One reason is that they have not quite lived up to the hype of their early advocates. Part of that hype stemmed from the idea that RBC models contained important propagation mechanisms[43] for turning technology shocks into business cycles. The idea was that increases in technology induced extra output through higher capital accumulation and by incentivising people to work more. In other words, some of the early research suggested that even in a world of IID technology shocks, one would expect RBC models to still generate business cycles.

But, as you can see on Figure 26, the RBC model's output fluctuations follow technology fluctuations quite closely. This implies that any propagation mechanisms of the RBC model, besides the technology shock, is extremely weak.

[43] Recall the Frisch-Slutsky paradigm.

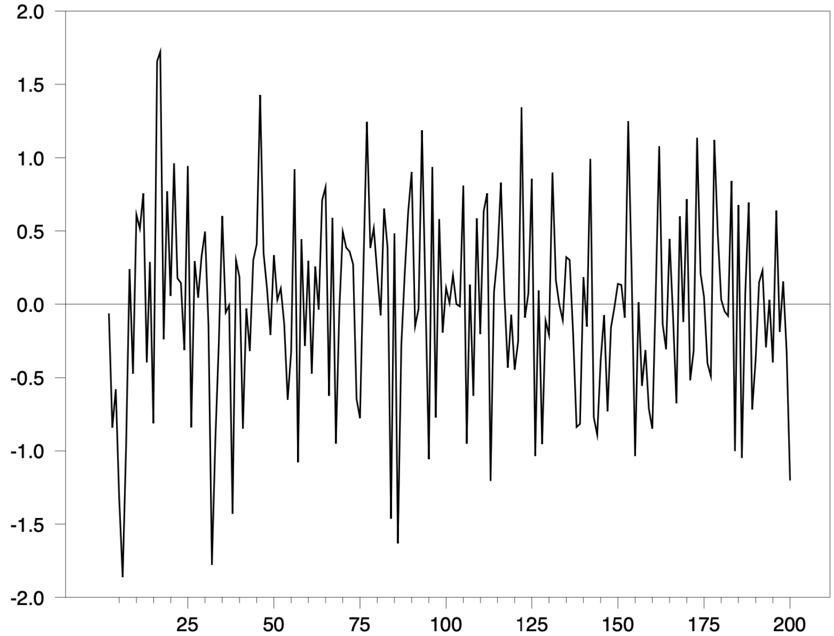
Figure 26: RBC Cycles Rely Heavily on Technology Fluctuations



Source: Whelan (2016)

Cogley and Nason (1995) noted another fact about business cycles that the RBC model does not match: output growth is positively autocorrelated (albeit not very autocorrelated – an autocorrelation coefficient of 0.34 – but still statistically significant). But RBC models do not generate this pattern (see Figure 27). They can only do so if one simulates a technology process that has a positively autocorrelated growth rate.

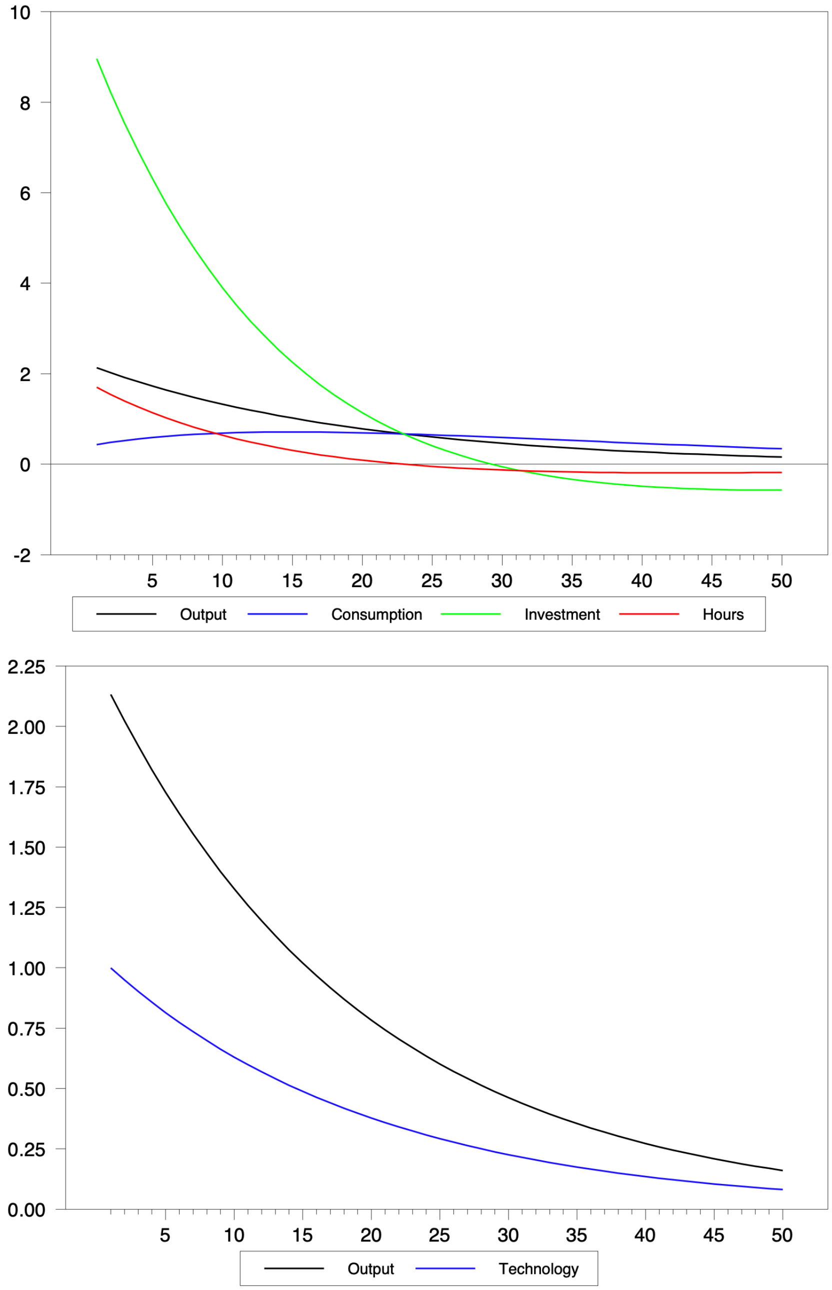
Figure 27: RBCs Do Not Generate Positively Autocorrelated Growth



Source: Whelan (2016)

Cogley and Nason relate this back to the IRFs generated by RBC models. As you can see from 28, the response of output to the technology shock pretty much matches the response of technology itself. Cogley and Nason argue that one needs to instead have "hump-shaped" responses to shocks – a growth rate increase needs to be followed by another growth rate increase – if a model is to match the facts about autocorrelated output growth. The responses to technology shocks do not deliver this. Also, while we don't have other shocks in the model (e.g. government spending shocks), Cogley and Nason show that RBC models don't generate hump-shaped responses for these either.

Figure 28: IRFs to Technology Shock



Source: Whelan (2016)

The key takeaway from the Cogley and Nason critique is that RBC models follow the notion of "you are what you eat" or WYGIWYPI – what you get is what you put in. In other words, if you put into the neoclassical propagation mechanism a volatile and very persistent productivity shock then output fluctuations will also be very volatile and persistent. But if you put in random productivity shocks then the model will provide basically random output fluctuations. In other words, the neoclassical models reliance on capital accumulation as a propagation mechanism adds very little persistence. Is this a problem?

If we could be sure that productivity shocks really were very volatile and persistent, then the fact that the Kydland and Prescott or Hansen model does not not provide a propagation would not be a problem. The problem is there is very little evidence in favour of aggregate technology shocks: i) if business cycles are caused by productivity shocks how do we explain recessions – technical regressions? Is that plausible? (ii) different industries use different technologies. Why should all industries simultaneously experience a positive productivity shock? Are aggregate technology shocks realistic? (iii) are these shocks just oil price shifts? (iv) according to RBC theories the Solow residual should be exogenous. That is, it should not be influenced by any other variable such as monetary policy, government expenditure, and so on. For the US there is strong evidence that this is not the case, that the Solow residual is predictable and further that it is predictable by demand variables. If this is the case then it cannot be interpreted as a pure productivity shock.

What about other moments and correlations? Remember Table 2? Let's compare the US business cycle moments with the moments generated by the RBC model. For consistency I will refer to the data and RBC model from Sims (2017).

Table 3: RBC Model Moments

Series	Std. Dev.	Rel. Std. Dev.	Corr w/ y_t	Autocorr	Corr w/ Y_{t-4}	Corr w/ Y_{t+4}
Output	0.017	1.00	1.00	0.72	0.13	0.13
Consumption	0.006	0.35	0.95	0.78	0.34	-0.03
Investment	0.056	3.29	0.99	0.71	0.04	0.20
Hours	0.007	0.41	0.98	0.71	0.00	-0.23
Productivity	0.010	0.59	0.99	0.74	-0.50	0.06
Wage	0.010	0.59	0.99	0.74	0.74	0.06
1+Interest Rate	0.001	0.03	0.96	0.71	-0.05	0.26
TFP	0.012	0.69	0.99	0.72	0.11	0.15

US Business Cycle Moments

Series	Std. Dev.	Rel. Std. Dev.	Corr w/ y_t	Autocorr	Corr w/ Y_{t-4}	Corr w/ Y_{t+4}
Output	0.017	1	1.00	0.85	0.07	0.11
Consumption	0.009	0.53	0.76	0.79	0.07	0.22
Investment	0.047	2.76	0.79	0.87	-0.10	0.26
Hours	0.019	1.12	0.88	0.90	0.29	-0.03
Productivity	0.011	0.65	0.42	0.72	-0.50	0.35
Wage	0.009	0.53	0.10	0.73	-0.10	0.10
1 + Interest Rate	0.004	0.24	0.00	0.42	0.27	-0.25
Price Level	0.009	0.53	-0.13	0.91	0.09	-0.41
TFP	0.012	0.71	0.76	0.75	-0.34	0.34

Moments generated by model calibrated and simulated by Sims (2017). All series are HP filtered, with the data from 1948q1 to 2010q3.

As we saw from the simulation plots, the model does a good job at matching the volatilities of output, consumption, and investment, and we can see from the above table that it also does well with labour productivity and TFP volatility. The RBC model also does well with own autocorrelations – the series are all persistent with first order autocorrelation coefficients typically in the neighbourhood of 0.75. Lastly, the model captures the fact that most quantity series are quite procyclical, though these correlations are too high in the model relative to the data.

Where the model really struggles with are with factor prices. Look at wages and interest rates in the data – there is almost no correlation with output – yet, in the RBC model, wages and interest rates are highly correlated with output: Almost one-to-one. There is some evidence which suggests that aggregate wage data understates the procyclicality of wages due to a composition bias,[44] so this could be forgiven. But that doesn't explain the discrepancy for interest rates. The model also gets the relative standard deviation of interest rates to output standard deviation wrong too (the second columns): the model doesn't generate enough volatility in interest rates. Finally, the RBC model does not generate enough volatility in work hours. In the data, hours are actually slightly more volatile than output, but in the RBC model hours are about half as volatile as output.

6.9 Comments and key readings

RBC analysis has been very controversial but also extremely influential. As is often the case with the neoclassical program, it is important to discriminate between methodological innovations and economic theories. The RBC program instigated by Prescott has been controversial for three reasons: (i) reliance on productivity shocks to explain the business cycle; (ii) use of competitive equilibrium models which satisfy the conditions of the Fundamental Welfare Theorems implying business cycles are optimal; and (iii) the eschewing of econometrics in favour of calibration. Another key feature is the use of computer simulations to assess theoretical models. It is now more than 30 years since the seminal RBC paper of Kydland and Prescott. This paper seems to have had three long-run impacts: i) a reassessment of the relative roles of supply and demand shocks in causing business cycles ii) widespread use of computer simulations to assess macroeconomic models iii) widespread use of non-econometric tools to assess the success of a theory. The RBC program is still a very active research area but current models are far more sophisticated in their market structure and while they still have an important role for productivity shocks, additional sources of uncertainty are allowed.

In addition to the Cogley and Nason critique, RBC models have also been criticised by Jordi Galí – and other Keynesian economists – for failing to explain the labour market response to technology shocks. Galí used VARs to show that hours worked tends to decline after a positive technology shock in strong contrast to the RBC model's predictions.

There are currently a number of branches of research aimed at fixing the deficiencies of the RBC approach. Some of them involve putting extra bells and whistles on the basic market-clearing RBC

[44] See "Measuring the Cyclicality of Real Wages: How Important is Composition Bias" by Solon et al. (1994).

approach: Examples include variable capital utilisation, lags in investment projects, habit persistence in consumer utility, indivisible labour, and so on. Adding these elements tends to strengthen the propagation mechanism element of the model. But they ultimately fall short in reconciling the RBC's performance with the data.

The second approach is to depart more systematically from the basic RBC approach by adding rigidities and frictions into the model, such as sticky prices and wages. We will explore this in more depth when we move onto the New Keynesian DSGE model.

The literature is abundant with papers on RBC research. The key readings are listed below.

McCandless (2008) *ABCs of RBCs*: As the title of the book suggests, it's entirely dedicated to the RBC model, and also extends the basic model to improve its performance. McCandless even has a couple chapters introducing Keynesian-like assumptions, and a simple open-economy model.

Romer (2012) *Advanced Macroeconomics*: Chapter 5 gives a complete walkthrough of the baseline RBC model, and even provides an analytical solution by assuming that $\delta = 1$. Romer also provides some background on the RBC model, as well as explaining the merits and weaknesses of the RBC model.

Kydland and Prescott (1982) "Time to Build and Aggregate Fluctuations": The progenitor of the RBC revolution. The paper is extremely dated, and quite difficult to read. It presents the baseline RBC model with technology shocks. They found that simulated data from their model show the same patterns of volatility, persistence, and co-movement as are present in US data. The paper surprised macroeconomists as the paper presented a model with no money, nominal frictions, or a policy institution (no monetary or fiscal policy).

Hansen (1985) "Indivisible Labor and the Business Cycle": The standard model by Kydland and Prescott (1982) featured "divisible labour": Households voluntarily choose the amount of hours they work. Divisible labour households willingly substitute leisure time between periods in response to changes in factor prices. However, in the data, sufficient intertemporal substitution of leisure was not found (Ashenfelter 1984; Hall 1988). As such, the model cannot explain large fluctuations of hours worked, existence of unemployed workers, nor fluctuations in unemployment. Also, the model could not explain small fluctuations of productivity and wages relative to hours worked. Hansen introduced indivisible labor: In a certain period, indivisible labour households either work full time or not work at all – they are not able to work an intermediate amount of hours. Which households work full time or not is determined by a lottery. The results are displayed in Table 4. The Hansen model became the de-facto RBC model, but still featured a lot of the weaknesses we discussed.

Table 4: Indivisible Labour Improves RBC Model Performance

Standard deviations in percent (a) and correlations with output (b) for U.S. and artificial economies.

	Quarterly U.S. time series[a] (55,3–84,1)		Economy with divisible labor[b]		Economy with indivisible labor[b]	
Series	(a)	(b)	(a)	(b)	(a)	(b)
Output	1.76	1.00	1.35 (0.16)	1.00 (0.00)	1.76 (0.21)	1.00 (0.00)
Consumption	1.29	0.85	0.42 (0.06)	0.89 (0.03)	0.51 (0.08)	0.87 (0.04)
Investment	8.60	0.92	4.24 (0.51)	0.99 (0.00)	5.71 (0.70)	0.99 (0.00)
Capital stock	0.63	0.04	0.36 (0.07)	0.06 (0.07)	0.47 (0.10)	0.05 (0.07)
Hours	1.66	0.76	0.70 (0.08)	0.98 (0.01)	1.35 (0.16)	0.98 (0.01)
Productivity	1.18	0.42	0.68 (0.08)	0.98 (0.01)	0.50 (0.07)	0.87 (0.03)

Source: Hansen (1985)

Mehra and Prescott (1985) "The Equity Premium: A Puzzle": RBC models are successful at mimicking the cyclical behaviour of level quantities. But this paper shows that utility specifications in RBC models have counterfactual implications for asset prices. These utility specifications are not consistent with the difference between the average return to stocks and bonds. Mehra and Prescott follow this research up in their 2003 piece.

Greenwood et al. (1988) "Investment, Capacity Utilization, and the Real Business Cycle": Attempts to explain or correct the baseline RBC model. Exogenous technology shocks proposed by Kydland and Prescott (1982) are far too large in reality. In response, GHH propose that the Solow residual contains an endogenous component such as capacity utilisation. By doing so, a small technology shock which matches the data can still have a large impact on the macroeconomy.

King, Plosser, et al. (1988) "Production, Growth and Business Cycles": Uses a simplified version of the Kydland and Prescott model and drops non-central ideas such as "time-to-build" in investment, non-separable utility in leisure, and technology shocks that include both a permanent and transitory component. The model reproduces the first-order features of US business cycles. Consumption, investment, and hours worked are all procyclical. Consumption is less volatile than output, investment is much more volatile than output, and hours worked are only slightly less volatile than output. Furthermore, recessions in the model last for about one year, just as in US data.

Benhabib et al. (1991) "Homework in Macroeconomics: Household Production and Aggregate Fluctuations": According to the baseline RBC model, the correlation between consumption and labour is high. Yet, in reality, this does not seem to be the case. Household labour is not included in the official statistics, such as housekeeping or child-rearing, but this may be important for the determination of macroeconomic variables. BRW's approach was to include home production in the RBC model, and make consumption an aggregate of both consumption of market goods and consumption of home production goods. The model with home production manages to break the strong positive correlation between consumption and labour, as well as labour and wages.

Cogley and Nason (1995) "Output Dynamics in Real-Business-Cycle Models": We've covered this paper when we assessed the RBC model. It's one of the more scathing criticisms of the RBC framework.

Merz (1995) "Search in the Labor Market and the Real Business Cycle": The simple RBC model cannot explain unemployment. In the real world, workers don't use their entire time endowment to work – they optimally choose between work and leisure. Merz used search and match models to explain unemployment. We will look at this further when we look at labour markets.

Galí (1999) "Technology, Employment, and the Business Cycle: Do Technology Shocks Explain Aggregate Fluctuations?": We've covered Galí's criticisms too. This was essentially the nail in the coffin for the RBC model, and fuelled the New Keynesian framework.

King and Rebelo (1999) "Resuscitating Real Business Cycles": The title is quite self explanatory. As taken from the abstract of the paper: This chapter exposits the basic RBC model and shows that it requires large technology shocks to produce realistic business cycles. While Solow residuals are sufficiently volatile, these imply frequent technological regress. Productivity studies permitting unobserved factor variation find much smaller technology shocks, suggesting the imminent demise of real business cycles. However, we show that greater factor variation also dramatically amplifies shocks: a RBC model with varying capital utilisation yields realistic business cycles from small, nonnegative changes in technology.

Rebelo (2005) "Real Business Cycle Models: Past, Present and Future": This paper gives a comprehensive literature review of RBC research.

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7 Solving DSGE Models

7.1 Introduction

The solution of many discrete time DSGE macroeconomic models is a system of non-linear difference equations. One method for approximating the solution to these models is by log-linearising the system of equations about a point (typically the steady state), thereby translating the system of non-linear difference equations into a system of (approximately) linear difference equations. In this chapter, we describe how to arrive at the approximate policy functions/decisions rules once the system of equations has been transformed into a log-linearised system.

The method that we are primarily going to use for solving linear DSGE models is known as the method of undetermined coefficients. It was originally presented by McCallum (1983) and developed by Christiano (2002). What is probably the clearest exposition of the method is by Uhlig (1998), although he used a solution technique different from that of Christiano. Basically, a linear form for the solution is assumed and the method finds the coefficients for the solution of this form. The assumption of a linear form for the solution is not a very great jump, since linear models generally provide linear solutions.

The method of undetermined coefficients was not the first method in the literature for solving rational expectations models. The first method in economics for solving these linear rational expectations models comes from Blanchard and Kahn (1980), who used techniques that were in the engineering literature and applied them to macroeconomic models. Christiano uses solution techniques similar to those of Blanchard and Kahn for solving his undetermined coefficients problems. A good expanded explanation of these methods can be found in Blake and Fernandez-Corugedo (2010).

7.2 The general form and Blanchard-Kahn condition

Let $\mathbf{X}_t$ be an $(n+m) \times 1$ vector of variables expressed as expressed as percentage deviations from steady state. Let n be the number of "jump" or "forward-looking" variables, while m is the number of states or predetermined variables. In a deterministic growth model, for example, $n = 1$ (consumption) and $m = 1$ (the capital stock), while in the stochastic growth model $n = 1$ (consumption) and $m = 2$

(capital stock and TFP). We partition the vector of variables into two parts: $\mathbf{X}_{1,t}$ is an $n \times 1$ vector containing the jump variables, while $\mathbf{X}_{2,t}$ is an $m \times 1$ vector containing the state variables. The linearised solution takes the form:

$$\mathbb{E}_t \begin{bmatrix} \mathbf{X}_{1,t+1} \\ \mathbf{X}_{2,t+1} \end{bmatrix}_{(n+m)\times 1} = \underset{(n+m)\times(n\times m)}{\mathbf{B}} \begin{bmatrix} \mathbf{X}_{1,t} \\ \mathbf{X}_{2,t} \end{bmatrix}_{(n\times m)\times 1}$$

$$\Leftrightarrow \mathbb{E}_t \mathbf{X}_{t+1} = \mathbf{B}\mathbf{X}_t. \tag{187}$$

We can typically derive a closed form expression for $\mathbf{B}$ in terms of the underlying parameters of the model once it has been log-linearised (which we have previously done). But this does not mean that we have the solution to the model. $\mathbf{B}$ tells us how the variables in the system will evolve given an initial starting point. But we only have the initial starting point for the state variables – we do not know where to "start" the jump variables. We have to work harder to figure that out, essentially imposing a terminal condition of non-explosion. The rest of what we do in this chapter is working out how to find that starting position for the non-predetermined variables.

Recall the definition of eigenvalues and eigenvectors. An eigenvalue is a scalar, λ, and an eigenvector is a vector, $\mathbf{e}$, which jointly satisfy:

$$\mathbf{B}\mathbf{e}_i = \lambda_i \mathbf{e}_i,$$

$$(\mathbf{B} - \boldsymbol{\lambda}\mathbf{I})\mathbf{e}_i = \mathbf{0}, \quad i = 1, ..., n+m.$$

Unless you've made a mistake, there will be the same number of distinct eigenvalues as there are rows/columns in the square matrix $\mathbf{B}$ (in this case there will be $n+m$ eigenvalues), some of which may be complex. There will also be the same number of distinct eigenvectors as there are rows/columns of $\mathbf{B}$. Index these eigenvalues/eigenvectors by $k = 1, ..., n+m$. The above definition will hold for each

$k = 1, ..., n+m$. In other words:

$$\mathbf{B}\mathbf{e}_1 = \lambda_1 \mathbf{e}_1,$$

$$\mathbf{B}\mathbf{e}_2 = \lambda_2 \mathbf{e}_2,$$

$$\vdots$$

$$\mathbf{B}\mathbf{e}_{n+m} = \lambda_{n+m} \mathbf{e}_{n+m},$$

which means that we can write these up as follows:

$$\mathbf{B}\begin{bmatrix} e_{1,1} & e_{2,1} & \cdots & e_{n+m,1} \\ e_{1,2} & e_{2,2} & \cdots & e_{n+m,2} \\ \vdots & \vdots & \ddots & \vdots \\ e_{1,n+m} & e_{2,n+m} & \cdots & e_{n+m,n+m} \end{bmatrix} = \begin{bmatrix} e_{1,1} & e_{2,1} & \cdots & e_{n+m,1} \\ e_{1,2} & e_{2,2} & \cdots & e_{n+m,2} \\ \vdots & \vdots & \ddots & \vdots \\ e_{1,n+m} & e_{2,n+m} & \cdots & e_{n+m,n+m} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_{n+m} \end{bmatrix}.$$

Use the following notation to clean things up a bit:

$$\mathbf{P} = \begin{bmatrix} e_{1,1} & e_{2,1} & \cdots & e_{n+m,1} \\ e_{1,2} & e_{2,2} & \cdots & e_{n+m,2} \\ \vdots & \vdots & \ddots & \vdots \\ e_{1,n+m} & e_{2,n+m} & \cdots & e_{n+m,n+m} \end{bmatrix},$$

$$\mathbf{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_{n+m} \end{bmatrix},$$

and using this notation, we have:

$$\mathbf{BP} = \mathbf{P\Lambda},$$
$$\mathbf{B} = \mathbf{P\Lambda P}^{-1}. \tag{188}$$

Note that we can arrange the eigenvalues and eigenvectors in whatever order you want, so long as the k-th column of $\mathbf{P}$ corresponds with the k-th eigenvalue which occupies the (k,k) position of $\mathbf{\Lambda}$. As such, it is helpful to "order" the eigenvalues from smallest to largest.[45] More generally, let

$$\mathbf{\Lambda} = \begin{bmatrix} \underset{S\times S}{\mathbf{\Lambda}_1} & \mathbf{O} \\ \mathbf{O} & \underset{U\times U}{\mathbf{\Lambda}_2} \end{bmatrix},$$

where $\mathbf{\Lambda}_1$ is a $S \times S$ diagonal matrix containing the S stable eigenvalues, while $\mathbf{\Lambda}_2$ is a $U \times U$ diagonal matrix containing the U unstable eigenvalues (obviously, $S + U = n + m$, but neither n nor m are necessarily guaranteed to equal S or U, respectively).

Using the eigenvalue/eigenvector decomposition of $\mathbf{B}$, we can rewrite the system as follows:

$$\mathbb{E}_t\mathbf{X}_{t+1} = \mathbf{P\Lambda P}^{-1}\mathbf{X}_t. \tag{189}$$

Premultiply each side by $\mathbf{P}^{-1}$ to get:

$$\mathbb{E}_t\mathbf{P}^{-1}\mathbf{X}_{t+1} = \mathbf{\Lambda P}^{-1}\mathbf{X}_t,$$

and then define the auxiliary vector $\mathbf{Z}_t$ as follows:

$$\mathbf{Z}_t = \mathbf{P}^{-1}\mathbf{X}_t,$$

[45] In terms of absolute value, that is. If there are complex parts of the eigenvalues, order them by modulus, where the modulus is the square root of the sum of squared non-complex and complex components. e.g. If $y = x + zi$, then the modulus is $\sqrt{x^2 + z^2}$. If $z = 0$, then the modulus is just the absolute value.

and so we have

$$
\begin{aligned}
\mathbb{E}_t \mathbf{Z}_{t+1} &= \mathbf{\Lambda} \mathbf{Z}_t \qquad (190) \\
\Leftrightarrow \mathbb{E}_t \begin{bmatrix} \mathbf{Z}_{1,t+1} \\ \mathbf{Z}_{2,t+1} \end{bmatrix} &= \begin{bmatrix} \mathbf{\Lambda}_1 & \mathbf{O} \\ \mathbf{O} & \mathbf{\Lambda}_2 \end{bmatrix} \begin{bmatrix} \mathbf{Z}_{1,t} \\ \mathbf{Z}_{2,t} \end{bmatrix}.
\end{aligned}
$$

We've partitioned $\mathbf{Z}_t$ into two parts: $\mathbf{Z}_{1,t}$ is partitioned as the first S variables in $\mathbf{Z}_t$, while $\mathbf{Z}_{2,t}$ are the second U elements of $\mathbf{Z}_t$. Because we've effectively rewritten this as a VAR(1) process with a diagonal coefficient matrix, $\mathbf{Z}_{1,t}$ and $\mathbf{Z}_{2,t}$ evolve independently of one another. We can write the expected values updating forward in time as:

$$
\mathbb{E}_t \mathbf{Z}_{1,t+T} = \mathbf{\Lambda}_1^T \mathbf{Z}_{1,t}, \qquad (191)
$$

$$
\mathbb{E}_t \mathbf{Z}_{2,t+T} = \mathbf{\Lambda}_2^T \mathbf{Z}_{2,t}. \qquad (192)
$$

Because the eigenvalues in $\mathbf{\Lambda}_1$ are all stable (absolute value less than 1), $\mathbf{\Lambda}_1^T \to 0$ as $T \to \infty$. The same does not hold true for the second expression, which contains the explosive eigenvalues. Because the eigenvalues in $\mathbf{\Lambda}_2$ are all unstable, $\mathbb{E}_t \mathbf{Z}_{2,t+T} \to \infty$ as T grows, unless $\mathbf{Z}_{2,t} = 0$. But we cannot let $\mathbf{Z}_{2,t+T} \to \infty$ while simultaneously being consistent with the transversality conditions and/or feasibility constraints.

For further clarity, let's write out what $\mathbf{P}^{-1}$ is:

$$
\underset{(S+U)\times(n+m)}{\mathbf{P}^{-1}} = \begin{bmatrix} \underset{S\times n}{\mathbf{G}_{11}} & \underset{S\times m}{\mathbf{G}_{12}} \\ \underset{U\times n}{\mathbf{G}_{21}} & \underset{U\times m}{\mathbf{G}_{22}} \end{bmatrix}.
$$

Since $S + U = n + m$, this is obviously still a square matrix, but the individual partitions need not necessarily be square matrices. Recall that there are S stable eigenvalues and U unstable ones, while there are n jump variables and m state variables. Let's write out in long hand what the $\mathbf{Z}$'s are:

$$\underset{S\times 1}{\mathbf{Z}_{1,t}} = \underset{S\times n}{\mathbf{G}_{11}}\underset{n\times 1}{\mathbf{X}_{1,t}} + \underset{S\times m}{\mathbf{G}_{12}}\underset{m\times 1}{\mathbf{X}_{2,t}},$$

$$\underset{U\times 1}{\mathbf{Z}_{2,t}} = \underset{U\times n}{\mathbf{G}_{21}}\underset{n\times 1}{\mathbf{X}_{1,t}} + \underset{U\times m}{\mathbf{G}_{22}}\underset{m\times 1}{\mathbf{X}_{2,t}}.$$

As noted above, the transversality/feasibility conditions require that $\mathbf{Z}_{2,t} = 0$. We can use this to then solve for the initial position of the jump variables $\mathbf{X}_{1,t}$ in terms of the given initial conditions of the states $\mathbf{X}_{2,t}$:

$$\mathbf{0} = \mathbf{G}_{21}\mathbf{X}_{1,t} + \mathbf{G}_{22}\mathbf{X}_{2,t},$$

and solving this yields

$$\mathbf{G}_{21}\mathbf{X}_{1,t} = -\mathbf{G}_{22}\mathbf{X}_{2,t}.$$

Provided that $\mathbf{G}_{21}$ is a square matrix, we can invert $\mathbf{G}_{21}$ and then we can solve this as:

$$\mathbf{X}_{1,t} = -\mathbf{G}_{21}^{-1}\mathbf{G}_{22}\mathbf{X}_{2,t}. \tag{193}$$

In other words, this is our linearised policy function. For a given state vector (i.e., given values of $\mathbf{X}_{2,t}$) this will tell us what the value of the jump variables need to be.

Now, what does it mean for $\mathbf{G}_{21}$ to be square/invertible? Recall that the dimension of $\mathbf{G}_{21}$ is $U \times n$, where U is the number of unstable eigenvalues and n is the number of jump variables. Put differently, we must have an equal number of unstable eigenvalues as we do jump variables – this is known as the Blanchard-Kahn condition, and it is required for saddle path stability. If we don't have enough unstable eigenvalues, there will be an infinite number of solutions. If we have too many unstable eigenvalues, there will be no solution.

7.2.1 Example: Deterministic growth model

Consider the Robinson Crusoe model (non-stochastic neoclassical growth model) with CRRA preferences, Cobb-Douglas production, and the level of technology normalised to unity.[46] It can be reduced to a system of non-linear difference equations:

$$
\begin{aligned}
C_t^{-\sigma} &= \beta C_{t+1}^{-\sigma}\Big(\underbrace{\alpha K_{t+1}^{\alpha-1} + (1-\delta)}_{R_{t+1}}\Big), \\
K_{t+1} &= K_t^{\alpha} - C_t + (1-\delta)K_t.
\end{aligned}
$$

Log-linearisation of these equations about the steady state yields:

$$
\begin{aligned}
-\sigma \ln C_t &= \ln\beta - \sigma \ln C_{t+1} + \ln R_{t+1} \\
-\frac{\sigma}{\bar{C}} dC_t &= -\frac{\sigma}{\bar{C}} dC_{t+1} + \frac{1}{\bar{R}} dR_{t+1} \\
-\sigma \frac{C_t - \bar{C}}{\bar{C}} &= -\sigma \frac{C_{t+1} - \bar{C}}{\bar{C}} + \frac{R_{t+1} - \bar{R}}{\bar{R}} \\
-\sigma \hat{C}_t &= -\sigma \hat{C}_{t+1} + \hat{R}_{t+1} \\
\Leftrightarrow \hat{C}_t &= \hat{C}_{t+1} - \frac{\beta}{\sigma}(\alpha-1)\bar{R}\hat{K}_{t+1}, \qquad (194)
\end{aligned}
$$

and[47]

$$
\begin{aligned}
\hat{K}_{t+1} &= \left[\alpha \bar{K}^{\alpha-1} + 1 - \delta\right]\hat{K}_t + [-1]\frac{\bar{C}}{\bar{K}}\hat{C}_t \\
\hat{K}_{t+1} &= \left[\bar{R} + 1 - \delta\right]\hat{K}_t - \frac{\bar{C}}{\bar{K}}\hat{C}_t \\
\hat{K}_{t+1} &= \frac{1}{\beta}\hat{K}_t - \frac{\bar{C}}{\bar{K}}\hat{C}_t, \qquad (195)
\end{aligned}
$$

[46] Unlike previous sections, I assume a start of period notation for capital. This makes the exposition slightly easier. Functionally, it's the same as using the end of period notation.

[47] Recall the Taylor approximation method for the multivariate case:

$$\hat{X}_t = f_X(\bar{X},\bar{Y})\hat{X}_{t-1} + f_Y(\bar{X},\bar{Y})\frac{\bar{Y}}{\bar{X}}\hat{Y}_t.$$

where $\bar{R} = \alpha \bar{K}^{\alpha-1}$ (the steady state marginal product of capital) and $\frac{\bar{C}}{\bar{K}}$ is the steady state ratio of consumption to capital, both of which are functions of underlying parameters of the model. This can be re-arranged into the VAR(1) form, $\mathbb{E}\mathbf{X}_{t+1} = \mathbf{B}\mathbf{X}_t$, as:

$$\mathbb{E}_t \begin{bmatrix} \hat{C}_{t+1} \\ \hat{K}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 - \frac{\bar{C}}{\bar{K}} \frac{\beta(\alpha-1)\bar{R}}{\sigma} & \frac{(\alpha-1)\bar{R}}{\sigma} \\ -\frac{\bar{C}}{\bar{K}} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} \hat{C}_t \\ \hat{K}_t \end{bmatrix}.$$

We assume the following parameterisation: $\sigma = 1$, $\beta = 0.95$, $\delta = 0.1$, and $\alpha = 0.33$. These values then imply that $\bar{K} = 3.16$ and $\bar{C} = 1.146$. The numerical values of this matrix are easily seen to be:

$$\mathbf{B} = \begin{bmatrix} 1.0352 & -0.1023 \\ -0.3625 & 1.0526 \end{bmatrix}.$$

The MATLAB function "`[lam,V,j]=eig_order(M);`" will produce a diagonal matrix of eigenvalues ordered from smallest to largest (this is the output matrix "`lam`") and the matrix of eigenvectors corresponding with these eigenvalues (the output matrix "`V`" will be the matrix of eigenvectors). The output "`j`" is the index of the first unstable eigenvalue. Remember that we can write $\mathbf{B}$ as

$$\mathbf{B} = \mathbf{P}\mathbf{\Lambda}\mathbf{P}^{-1},$$

and using MATLAB, the eigenvalues of $\mathbf{B}$ come out to be 0.85 and 1.24, so the Blanchard-Kahn condition for saddle path stability are satisfied (i.e., one explosive root, one stable root). We find that $\mathbf{P}^{-1}$ is:

$$\mathbf{P}^{-1} = \begin{bmatrix} -1.0759 & -0.5462 \\ 1.0547 & -0.5861 \end{bmatrix},$$

and our matrix $\mathbf{Z}_t = \mathbf{P}^{-1}\mathbf{X}_t$ takes the following form:

$$Z_{1,t} = -1.0759\hat{C}_t - 0.5462\hat{K}_t$$

$$Z_{2,t} = 1.0547\hat{C}_t - 0.5861\hat{K}_t.$$

Using eigenvalue decomposition, we know that (with only two variables the diagonal matrices of eigenvalues are just scalars):

$$Z_{1,t+T} = \lambda_1^T Z_{1,t},$$
$$Z_{2,t+T} = \lambda_2^T Z_{2,t}.$$

Satisfaction of the transversality and feasibility conditions requires that $Z_{2,t} = 0$. This means our linearised policy function is:

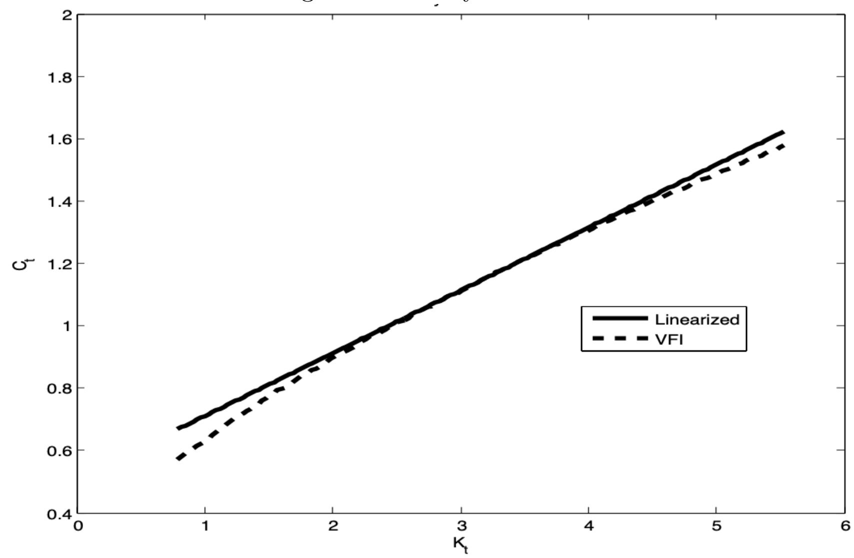
$$\hat{C}_t = \frac{0.5861}{1.0547}\hat{K}_t = 0.5557\hat{K}_t$$

As the above expression is log-linear, we need to adjust to get the policy function in terms of levels:

$$\begin{aligned}
\frac{C_t - \bar{C}}{\bar{C}} &= 0.5557\frac{K_t - \bar{K}}{\bar{K}} \\
C_t - \bar{C} &= 0.5557\frac{\bar{C}}{\bar{K}}K_t - 0.5557\bar{C} \\
C_t &= 0.4443\bar{C} + 0.5557\frac{\bar{C}}{\bar{K}}K_t.
\end{aligned}$$

Below is a plot of this linearised policy function and the policy function retrieved for the same parameterisation of the model using value function iteration. As you can see, the linearised policy function performs pretty well, especially near the steady state. The linear approximation grows worse as σ increases (the policy function becomes more concave).

Figure 29: Policy Functions



Source: Sims (2017)

7.2.2 Example: Stochastic growth model

Now consider the model with stochastic TFP shocks. The non-linear system of difference equations can be written as:

$$C_t^{-\sigma} = \beta \mathbb{E}_t C_{t+1}^{-\sigma} \left(\alpha A_{t+1} K_{t+1}^{\alpha-1} + (1-\delta) \right),$$
$$K_{t+1} = A_t K_t^{\alpha} - C_t + (1-\delta) K_t,$$
$$\ln A_t = \rho \ln A_{t-1} + e_t,$$

where e_t is a white noise process, and we assume that $\bar{A} = 1$, which means that the mean of the log of technology is zero. One can show that the log-linearised equations are:

$$\hat{C}_t = \hat{C}_{t+1} - \frac{\beta \bar{R}}{\sigma} \tilde{A}_{t+1} - \frac{\beta(\alpha-1)\bar{R}}{\sigma} \hat{K}_{t+1},$$
$$\hat{K}_{t+1} = \bar{K}^{\alpha-1} \hat{A}_t - \frac{\bar{C}}{\bar{K}} \hat{C}_t + \frac{1}{\beta} \hat{K}_t,$$
$$\hat{A}_t = \rho \hat{A}_{t-1} + e_t.$$

Isolating the $t+1$ variables, to get the VAR(1) form, we get:

$$\mathbb{E}_t \begin{bmatrix} \hat{C}_{t+1} \\ \hat{K}_{t+1} \\ \hat{A}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 - \frac{\beta(\alpha-1)\bar{R}}{\sigma}\frac{\bar{C}}{\bar{K}} & \frac{(\alpha-1)\bar{R}}{\sigma} & \frac{\beta\bar{R}(\rho+(\alpha-1)\bar{K}^{\alpha-1}}{\sigma} \\ -\frac{\bar{C}}{\bar{K}} & \frac{1}{\beta} & \bar{K}^{\alpha-1} \\ 0 & 0 & \rho \end{bmatrix} \begin{bmatrix} \hat{C}_t \\ \hat{K}_t \\ \hat{A}_t \end{bmatrix}.$$

Using the same parameterisation as in the deterministic example with $\rho = 0.95$, we find that the eigenvalues of this matrix are 0.8512, 0.95, and 1.2367, so the Blanchard-Kahn conditions are satisfied. The inverse of the matrix of eigenvectors are seen to be:

$$\mathbf{P}^{-1} = \begin{bmatrix} -1.0759 & -0.5462 & 3.5671 \\ 0 & 0 & 3.4172 \\ 1.0547 & -0.5861 & -0.6041 \end{bmatrix},$$

and the components of the $\mathbf{Z}$ matrix are then:

$$\underset{2\times 1}{\mathbf{Z}_{1,t}} = \begin{bmatrix} -1.0759 \\ 0 \end{bmatrix} \hat{C}_t + \begin{bmatrix} -0.5462 & 3.5671 \\ 0 & 3.4172 \end{bmatrix} \begin{bmatrix} \hat{K}_t \\ \hat{A}_t \end{bmatrix},$$

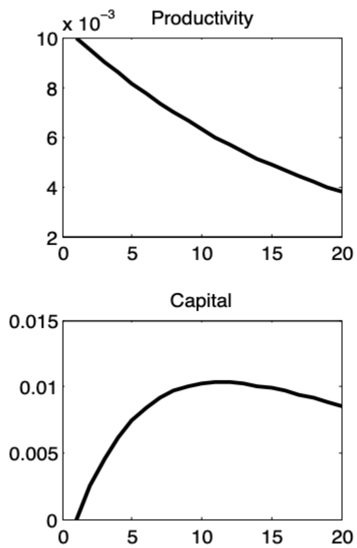
$$\underset{1\times 1}{Z_{2,t}} = 1.0547\hat{C}_t + \begin{bmatrix} -0.5861 & -0.6041 \end{bmatrix} \begin{bmatrix} \hat{K}_t \\ \hat{A}_t \end{bmatrix}.$$

Stability requires that $Z_{2,t} = 0$ since that is associated with the explosive eigenvalue. We can then solve for the policy function as:

$$\begin{aligned} \hat{C}_t &= -\frac{1}{1.0547} \begin{bmatrix} -0.5861 & -0.6041 \end{bmatrix} \begin{bmatrix} \hat{K}_t \\ \hat{A}_t \end{bmatrix}, \\ &= 0.5557\hat{K}_t + 0.5728\hat{A}_t. \end{aligned}$$

Given this policy function and an initial condition for $\hat{K}_t$, we can shock $\hat{A}_t$ and then let the system play out. Below are the impulse responses to a one unit shock to technology:

Figure 30: IRFs to a Technology Shock



Source: Sims (2017)

7.2.3 Dealing with static variables

Static variables are defined as variables in the model which only show up at time t. They are not explicitly forward-looking (jump variables) or explicitly backward-looking (state variables), though these variables are often implicitly forward-looking through their dependence on jump variables (like consumption). The basic strategy is the simply solve for the static variables in terms of the jump and state variables. You proceed by finding the policy functions for the jump variables just like we did above.

Some kinds of static variables are easier to deal with than others. The easy ones are variables which are essentially just log-linear combinations of the jump and state variables. An example is output:

$$\begin{aligned} Y_t &= A_t K_t^{\alpha} \\ \implies \hat{Y}_t &= \hat{A}_t + \alpha \hat{K}_t. \end{aligned}$$

Another easy one is investment. From the log-linearised aggregate resource constraint, we know that:

$$\hat{Y}_t = \frac{\bar{C}}{\bar{Y}}\hat{C}_t + \frac{\bar{I}}{\bar{Y}}\hat{I}_t.$$

Here $\bar{I} = \delta\bar{K}$. We can then back out investment as:

$$\hat{I}_t = \frac{\bar{Y}}{\bar{I}}\hat{Y}_t - \frac{\bar{C}}{\bar{I}}\hat{C}_t.$$

In other words, once we know $\hat{C}_t$ and $\hat{I}_t$, we have $\hat{Y}_t$. It is this reason that sometimes these variables are called "redundant" variables, because they are simply linear combinations of jump variables and state variables.

Some static variables present more headaches. An example of one is from a model in which there is variable labour. As an example, suppose that the within period utility function takes the form:

$$U(C_t, N_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \psi\frac{N_t^{1+\theta}}{1+\theta},$$

and the production function for this economy takes the form:

$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha}.$$

One can show that the consumption Euler equation and capital accumulation equation takes the following form:

$$\begin{aligned} C_t^{-\sigma} &= \beta\mathbb{E}_t C_{t+1}^{-\sigma}\left(\alpha A_{t+1}K_{t+1}^{\alpha-1}N_{t+1}^{1-\alpha} + (1-\delta)\right), \\ K_{t+1} &= A_t K_t^{\alpha} N_t^{1-\alpha} - C_t + (1-\delta)K_t. \end{aligned}$$

The question is: how do we deal with the N_t? The answer is is that there is also a first order condition for optimal labour supply. We can show that it takes the following form:

$$\psi N_t^{\theta} = C_t^{-\sigma}(1-\alpha)A_t K_t^{\alpha} N_t^{-\alpha}.$$

In words, this says that the marginal disutility from work equals the marginal utility of consumption times the real wage (the marginal product of capital). What you can do is to solve for N_t in terms of the jump and state variables:

$$\begin{aligned} N_t^{\theta+\alpha} &= \frac{1}{\psi} C_t^{-\sigma}(1-\alpha) A_t K_t^{\alpha} \\ N_t &= \left(\frac{1}{\psi} C_t^{-\sigma}(1-\alpha) A_t K_t^{\alpha} \right)^{\frac{1}{\theta+\alpha}} . \end{aligned}$$

Give this, we can substitute this whenever N_t shows up in the first order conditions and you're back to the kind of system we previously had, though it is more complicated. In practice, the easier thing to do is often to log-linearise all the equations first, and then eliminate the log-linearised $\hat{N}_t$.

7.2.4 Getting the dynamics right

Suppose we want to construct impulse responses or simulate data from the linearised model. As an example, suppose that we take the deterministic growth model and want to compute what happens in expectation if the capital stock starts out below the steady state. The simple thing to do would be to start at some $\hat{K}_0$, set $\hat{C}_0 = -\mathbf{G}_{21}^{-1}\mathbf{G}_{22}\hat{K}_0$, and then trace out expected future dynamics as:

$$\begin{bmatrix} \hat{C}_t \\ \hat{K}_t \end{bmatrix} = \mathbf{B}^t \begin{bmatrix} \hat{C}_0 \\ \hat{K}_0 \end{bmatrix} = \mathbf{B}^t \begin{bmatrix} -\mathbf{G}_{21}^{-1}\mathbf{G}_{22}\hat{K}_0 \\ \hat{K}_0 \end{bmatrix} .$$

This is analytically correct, but is prone to numerical problems. Why? Recall the whole idea of saddle path stability. If you are at all off the policy function, even by a very small amount, the system eventually explodes (due to the presence of unstable eigenvalues/roots in $\mathbf{B}$). In practice, there will be small numerical errors in the policy function $-\mathbf{G}_{21}^{-1}\mathbf{G}_{22}$. Like, numerical errors to several decimal places, but the system still can't tolerate these, particularly at longer horizons. If we do the exercise, everything will look great for about 100 periods, but out at longer horizons the system starts to explode.

There is a straightforward way of dealing with this and avoiding the potential for explosion that results from small numerical errors. Consider the general case. Decompose $\mathbf{B}$ into blocks:

$$\mathbf{B} = \begin{bmatrix} \underset{n\times n}{\mathbf{B}_{11}} & \underset{n\times m}{\mathbf{B}_{12}} \\ \underset{m\times n}{\mathbf{B}_{21}} & \underset{m\times m}{\mathbf{B}_{22}} \end{bmatrix},$$

write out the original system out in long hand using this notation:

$$\mathbb{E}_t \mathbf{X}_{1,t+1} = \mathbf{B}_{11}\mathbf{X}_{1,t} + \mathbf{B}_{12}\mathbf{X}_{2,t},$$
$$\mathbb{E}_t \mathbf{X}_{2,t+1} = \mathbf{B}_{21}\mathbf{X}_{1,t} + \mathbf{B}_{22}\mathbf{X}_{2,t}.$$

Now, plug in the policy function to eliminate $\mathbf{X}_{1,t}$ in both expressions:

$$\mathbb{E}_t \mathbf{X}_{1,t+1} = (-\mathbf{B}_{11}\mathbf{G}_{21}^{-1}\mathbf{G}_{22} + \mathbf{B}_{12})\mathbf{X}_{2,t},$$
$$\mathbb{E}_t \mathbf{X}_{2,t+1} = (-\mathbf{B}_{21}\mathbf{G}_{21}^{-1}\mathbf{G}_{22} + \mathbf{B}_{22})\mathbf{X}_{2,t}.$$

Define a new matrix, $\mathbf{A}$, as follows:

$$\mathbf{A} = \begin{bmatrix} \underset{n\times n}{\mathbf{O}} & -\mathbf{B}_{11}\mathbf{G}_{21}^{-1}\mathbf{G}_{22} + \mathbf{B}_{12} \\ \underset{m\times n}{\mathbf{O}} & -\mathbf{B}_{21}\mathbf{G}_{21}^{-1}\mathbf{G}_{22} + \mathbf{B}_{22} \end{bmatrix}.$$

Then, write the system as:

$$\mathbb{E}_t \begin{bmatrix} \mathbf{X}_{1,t+1} \\ \mathbf{X}_{2,t+1} \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathbf{X}_{1,t} \\ \mathbf{X}_{2,t} \end{bmatrix}.$$

Effectively what this does is imposes the policy function so that you can write the AR coefficient matrix with only coefficients on $\mathbf{X}_{2,t}$, the vector of states. This turns out to eliminate the problem. You can then proceed as follows – you can start the system as some arbitrary value of the state, start the controls at the appropriate place given the policy function, and then iterate forward using $\mathbf{A}$ instead of $\mathbf{B}$.

7.3 Solving the RBC model

We just discussed how static and redundant variables need to be "eliminated" to solve for the linearised policy functions. This is correct and can be done by hand, but it is algebraically intense and annoying. Below we discuss a way in which to do this that just involves manipulation of a few matrices. Recall that our RBC model is comprised of the following log-linearised set of equations:[48]

$$
\begin{aligned}
\hat{C}_t &= \mathbb{E}_t\hat{C}_{t+1} - \frac{1}{\sigma}\mathbb{E}_t\hat{R}_{t+1}, \\
\hat{K}_{t+1} &= \left[\frac{\alpha\delta}{\beta^{-1}+\delta-1}\right]\hat{I}_t + (1-\delta)\hat{K}_t, \\
\hat{A}_t &= \rho\hat{A}_{t-1} + \epsilon_t, \\
\hat{N}_t &= \hat{Y}_t - \sigma\hat{C}_t, \\
\hat{Y}_t &= \hat{A}_t + \alpha\hat{K}_t + (1-\alpha)\hat{N}_t, \\
\hat{Y}_t &= \left[1 - \frac{\alpha\delta}{\beta^{-1}+\delta-1}\right]\hat{C}_t + \left[\frac{\alpha\delta}{\beta^{-1}+\delta-1}\right]\hat{I}_t, \\
\hat{w}_t &= \hat{A}_t + \alpha\hat{K}_t - \alpha\hat{N}_t, \\
\hat{R}_t &= [1-\beta(1-\delta)]\left[\hat{Y}_t - \hat{K}_t\right].
\end{aligned}
$$

Let's stack all of these up into vectors. Let:

$$
\mathbf{X}_t = \begin{bmatrix} \hat{C}_t \\ \hat{K}_t \\ \hat{A}_t \\ \hat{N}_t \\ \hat{Y}_t \\ \hat{I}_t \\ \hat{w}_t \\ \hat{R}_t \end{bmatrix},
$$

[48] I've added an equation for wages. It's not necessary, but we may as well include both factor prices. Also, I've adjusted the timing of capital here.

and note that we haven't ordered these randomly. We've started with the forward-looking jump variable, then the two state variables, and then the redundant/static variables. We can write out the log-linearised conditions in matrix form as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sigma} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbb{E}_t \begin{bmatrix} \hat{C}_{t+1} \\ \hat{K}_{t+1} \\ \hat{A}_{t+1} \\ \hat{N}_{t+1} \\ \hat{Y}_{t+1} \\ \hat{I}_{t+1} \\ \hat{w}_{t+1} \\ \hat{R}_{t+1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-\delta & 0 & 0 & 0 & \frac{\alpha\delta}{\beta^{-1}+\delta-1} & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 & 0 & 0 & 0 \\ -\sigma & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & \alpha & 1 & (1-\alpha) & -1 & 0 & 0 & 0 \\ \left[1-\frac{\alpha\delta}{\beta^{-1}+\delta-1}\right] & 0 & 0 & 0 & -1 & \left[\frac{\alpha\delta}{\beta^{-1}+\delta-1}\right] & 0 & 0 \\ 0 & \alpha & 1 & -\alpha & 0 & 0 & -1 & 0 \\ 0 & -\left[1-\beta(1-\delta)\right] & 0 & 0 & \left[1-\beta(1-\delta)\right] & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \hat{C}_t \\ \hat{K}_t \\ \hat{A}_t \\ \hat{N}_t \\ \hat{Y}_t \\ \hat{I}_t \\ \hat{w}_t \\ \hat{R}_t \end{bmatrix},$$

or, more compactly as:

$$\mathbf{A}_0 \mathbb{E}_t \mathbf{X}_{t+1} = \mathbf{D}_0 \mathbf{X}_t.$$

There are a lot of rows of zeros in the $\mathbf{A}_0$ coefficient matrix – these rows correspond to the redundant variables. Note, however, that we can decompose these matrices as follows: Let n be the number of jump variables (here it's one, $\hat{C}_t$), m be the number of states (here it's two, $\hat{K}_t$ and $\hat{A}_t$), and q be the number of redundant/static variables (here five). We can then write:

$$\mathbf{A}_0 = \begin{bmatrix} \underset{(n+m)\times(n+m)}{\mathbf{A}_{11}} & \underset{(n+m)\times q}{\mathbf{A}_{12}} \\ \underset{q\times(n+m)}{\mathbf{O}} & \underset{q\times q}{\mathbf{O}} \end{bmatrix},$$

$$\mathbf{D}_0 = \begin{bmatrix} \underset{(n+m)\times(n+m)}{\mathbf{D}_{11}} & \underset{(n+m)\times q}{\mathbf{D}_{12}} \\ \underset{q\times(n+m)}{\mathbf{D}_{21}} & \underset{q\times q}{\mathbf{D}_{22}} \end{bmatrix}.$$

Now, let $\mathbf{Y}_t$ be the $(n+m)\times 1$ vector of jump and state variables, and $\mathbf{X}_t$ be the $q \times 1$ vector of redundant variables. We can write this out as:

$$\begin{bmatrix} \underset{(n+m)\times(n+m)}{\mathbf{A}_{11}} & \underset{(n+m)\times q}{\mathbf{A}_{12}} \\ \underset{q\times(n+m)}{\mathbf{O}} & \underset{q\times q}{\mathbf{O}} \end{bmatrix} \mathbb{E}_t \begin{bmatrix} \mathbf{Y}_{t+1} \\ \mathbf{X}_{t+1} \end{bmatrix} = \begin{bmatrix} \underset{(n+m)\times(n+m)}{\mathbf{D}_{11}} & \underset{(n+m)\times q}{\mathbf{D}_{12}} \\ \underset{q\times(n+m)}{\mathbf{D}_{21}} & \underset{q\times q}{\mathbf{D}_{22}} \end{bmatrix} \begin{bmatrix} \mathbf{Y}_t \\ \mathbf{X}_t \end{bmatrix}.$$

From this, we can see that:

$$\mathbf{O} = \mathbf{D}_{21}\mathbf{Y}_t + \mathbf{D}_{22}\mathbf{X}_t,$$

and since $\mathbf{D}_{22}$ is a square matrix, we can (in principle) invert it, so we have:

$$\mathbf{X}_t = -\mathbf{D}_{22}^{-1}\mathbf{D}_{21}\mathbf{Y}_t.$$

In other words, we can write the vector redundant variables as a linear combination of the jump and state variables! Note that the dimension of $-\mathbf{D}_{22}^{-1}\mathbf{D}_{21}\mathbf{Y}_t$ is $q \times 1$. Hence, we can write:

$$\begin{bmatrix} \underset{(n+m)\times(n+m)}{\mathbf{A}_{11}} & \underset{(n+m)\times q}{\mathbf{A}_{12}} \\ \underset{q\times(n+m)}{\mathbf{O}} & \underset{q\times q}{\mathbf{O}} \end{bmatrix} \mathbb{E}_t \begin{bmatrix} \mathbf{Y}_{t+1} \\ -\mathbf{D}_{22}^{-1}\mathbf{D}_{21}\mathbf{Y}_{t+1} \end{bmatrix} = \begin{bmatrix} \underset{(n+m)\times(n+m)}{\mathbf{D}_{11}} & \underset{(n+m)\times q}{\mathbf{D}_{12}} \\ \underset{q\times(n+m)}{\mathbf{D}_{21}} & \underset{q\times q}{\mathbf{D}_{22}} \end{bmatrix} \begin{bmatrix} \mathbf{Y}_t \\ -\mathbf{D}_{22}^{-1}\mathbf{D}_{21}\mathbf{Y}_t \end{bmatrix},$$

or:

$$(\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{D}_{22}^{-1}\mathbf{D}_{21})\mathbb{E}_t\mathbf{Y}_{t+1} = (\mathbf{D}_{11} - \mathbf{D}_{12}\mathbf{D}_{22}^{-1}\mathbf{D}_{21})\mathbf{Y}_t.$$

The dimensions work out:

- $\mathbf{A}_{11}$ is $(n+m) \times (n+m)$;
- $\mathbf{A}_{12}$ is $(n+m) \times q$;
- $\mathbf{D}_{22}^{-1}\mathbf{D}_{21}$ is $q \times (n+m)$;
- Hence, $\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{D}_{22}^{-1}\mathbf{D}_{21}$ is $(n+m) \times (n+m)$;
- $\mathbf{D}_{11}$ is $(n+m) \times (n+m)$;
- $\mathbf{D}_{12}$ is $(n+m) \times q$;
- $\mathbf{D}_{22}^{-1}\mathbf{D}_{21}$ is $q \times (n+m)$;
- Hence, $\mathbf{D}_{11} - \mathbf{D}_{12}\mathbf{D}_{22}^{-1}\mathbf{D}_{21}$ is $(n+m) \times (n+m)$.

Since these are both square, we can invert to form:

$$\mathbb{E}_t \mathbf{Y}_{t+1} = \mathbf{B}\mathbf{Y}_t,$$

where $\mathbf{B} = (\mathbf{A}_{11} - \mathbf{A}_{12}\mathbf{D}_{22}^{-1}\mathbf{D}_{21})^{-1}(\mathbf{D}_{11} - \mathbf{D}_{12}\mathbf{D}_{22}^{-1}\mathbf{D}_{21})$. In other words, what we've done here is system reduction – we've reduced the system back to the VAR(1) in only the jumps and states, and given $\mathbf{B}$ can solve for the policy function mapping the states into the jump variables exactly as before. Given this new matrix $\mathbf{B}$ only in the system of jump and state variables, we can find the policy function just as before.

Recall, that in state space form in terms of jump and state variables, we can write the state as:

$$\mathbb{E}_t \mathbf{X}_{2,t+1} = \mathbf{B}_{21}\mathbf{X}_{1,t} + \mathbf{B}_{22}\mathbf{X}_{2,t},$$

where $\mathbf{X}_{1,t}$ was the $n \times 1$ vector of jump variables and $\mathbf{X}_{2,t}$ were the $m \times 1$ vector of states. Using the policy function mapping the states into the jumps, we can write this as:

$$\mathbb{E}_t \mathbf{X}_{2,t+1} = (\mathbf{B}_{21}\mathbf{\Phi} + \mathbf{B}_{22})\mathbf{X}_{2,t},$$

where $\mathbf{\Phi} = -\mathbf{G}_{21}^{-1}\mathbf{G}_{22}$, $\mathbf{X}_{1,t} = \mathbf{\Phi}\mathbf{X}_{2,t}$, and the $\mathbf{G}$ matrices correspond to different blocks of the inverse matrix of eigenvectors of $\mathbf{B}$ appropriately sorted. Now, we we may want to write this expression without expectation operators and instead with shocks. We know that:

$$\mathbf{X}_{2,t} = (\mathbf{B}_{21}\mathbf{\Phi} + \mathbf{B}_{22})\mathbf{X}_{2,t-1} + \mathbf{H}_0\boldsymbol{\epsilon}_t,$$

where $\boldsymbol{\epsilon}_t$ is a $k \times 1$ vector shocks (in the baseline RBC model $k = 1$, the TFP shock), and $\mathbf{H}_0$ is $m \times k$. In the baseline RBC model if the elements of the states are capital and productivity, we know that $\mathbf{H}_0 = \begin{bmatrix} 0 & 1 \end{bmatrix}$. Since we know that $\mathbf{X}_{1,t} = \mathbf{\Phi}\mathbf{X}_{2,t}$, we can write:

$$\mathbf{X}_{1,t} = \mathbf{\Phi}(\mathbf{B}_{21}\mathbf{\Phi} + \mathbf{B}_{22})\mathbf{X}_{2,t-1} + \mathbf{\Phi}\mathbf{H}_0\boldsymbol{\epsilon}_t,$$

and we can stack to write:

$$\begin{bmatrix} \mathbf{X}_{1,t} \\ \mathbf{X}_{2,t} \end{bmatrix} = \begin{bmatrix} \mathbf{\Phi}(\mathbf{B}_{21}\mathbf{\Phi} + \mathbf{B}_{22}) \\ \mathbf{B}_{21}\mathbf{\Phi} + \mathbf{B}_{22} \end{bmatrix} \mathbf{X}_{2,t-1} + \begin{bmatrix} \mathbf{\Phi}\mathbf{H}_0 \\ \mathbf{H}_0 \end{bmatrix} \boldsymbol{\epsilon}_t.$$

Now, we need to get the redundant/static variables back in. Recall that we can write:

$$\mathbf{X}_t = -\mathbf{D}_{22}^{-1}\mathbf{D}_{21}\mathbf{Y}_t,$$

and let's define $\mathbf{\Psi} = -\mathbf{D}_{22}^{-1}\mathbf{D}_{21}$. This matrix is $q \times (n + m)$. Let's decompose it as follows:

$$\mathbf{\Psi} = \begin{bmatrix} \underset{q\times n}{\mathbf{\Psi}_{11}} & \underset{q\times m}{\mathbf{\Psi}_{12}} \end{bmatrix}.$$

In other words, we can write the redundant/state variables as:

$$\mathbf{X}_t = \mathbf{\Psi}_{11}\mathbf{X}_{1,t} + \mathbf{\Psi}_{12}\mathbf{X}_{2,t},$$

But using the policy function, we have:

$$\mathbf{X}_t = (\mathbf{\Psi}_{11}\mathbf{\Phi} + \mathbf{\Psi}_{12})\mathbf{X}_{2,t},$$

and lagging $\mathbf{X}_{2,t}$ we have:

$$\mathbf{X}_t = (\mathbf{\Psi}_{11}\mathbf{\Phi} + \mathbf{\Psi}_{12})(\mathbf{B}_{21}\mathbf{\Phi} + \mathbf{B}_{22})\mathbf{X}_{2,t-1} + (\mathbf{\Psi}_{11}\mathbf{\Phi} + \mathbf{\Psi}_{12})\mathbf{H}_0\boldsymbol{\epsilon}_t.$$

Hence, we can characterise the solution as:

$$\mathbf{W}_t = \mathbf{F}\mathbf{W}_{t-1} + \mathbf{J}\boldsymbol{\epsilon}_t, \tag{196}$$

where

$$\mathbf{W}_t = \begin{bmatrix} \mathbf{X}_{1,t} \\ \mathbf{X}_{2,t} \\ \mathbf{X}_t \end{bmatrix}, \; \mathbf{F} = \begin{bmatrix} \mathbf{\Phi}(\mathbf{B}_{21}\mathbf{\Phi} + \mathbf{B}_{22}) \\ \mathbf{B}_{21}\mathbf{\Phi} + \mathbf{B}_{22} \\ (\mathbf{\Psi}_{11}\mathbf{\Phi} + \mathbf{\Psi}_{12})(\mathbf{B}_{21}\mathbf{\Phi} + \mathbf{B}_{22}) \end{bmatrix}, \; \mathbf{J} = \begin{bmatrix} \mathbf{\Phi}\mathbf{H}_0 \\ \mathbf{H}_0 \\ (\mathbf{\Psi}_{11}\mathbf{\Phi} + \mathbf{\Psi}_{12})\mathbf{H}_0 \end{bmatrix}.$$

We can then use this formulation to produce impulse responses and model simulations.

7.4 Comments and key readings

As observed, solving DSGE models can be algebraically intense, and keeping track of all the matrix manipulations can be tricky. In reality, we will rarely do this by hand – software is getting increasingly better at solving models using the methods we just discussed. Something like Dynare will solve your model, simulate shocks, plot impulse responses, and so on. Dynare will even be able to estimate models using Bayesian methods for calibration.

Key readings for this chapter are, once again, far and wide. But a few to focus on would be McCandless (2008), Uhlig (1998), and the fantastic set of notes by Eric Sims (2017). Just be aware that everyone uses different notation for their matrices and decompositions, and there are many different approaches to getting the same result.

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8 Modelling the Labour Market

8.1 Introduction

In the previous sections we saw that while the baseline RBC model did a decent job of explaining some business cycle facts, its main failures were due to the labour market. In particular, the model fails to convincingly account for the large fluctuations in employment over the business cycle in the absence of any movements in wages. Exactly where to progress from this point is not clear and a plethora of different approaches exist. In this section we introduce three extensions to the baseline RBC model, and they primarily focus on labour market dynamics and introducing unemployment. We choose these approaches for the following reasons: (i) they are all choice theoretic, (ii) they all assume Rational Expectations, and (iii) they each differ in their welfare implications. We focus on three models: Hansen's indivisible labour with lotteries RBC model; the Shapiro and Stiglitz (1984) efficiency wages model; and, the Diamond-Mortensen-Pissarides search and match model.

8.2 Hansen's RBC model with indivisible labour

8.2.1 Motivation

Quick recap: the standard RBC models proposed by Kydland and Prescott, and Lucas and Prescott motivated that under the assumption of competitive markets, shocks to productivity lead to changes in economic growth. Standard RBC models feature "divisible labour" – households voluntarily choose the amount of hours they work. This has long been a point of criticism of these models (which we covered in the previous chapters).

Divisible labour households willingly substitute leisure time between periods in response to changes in factor prices (wages and interest rates). However, sufficient intertemporal substitution of leisure was not found to support this claim (Ashenfelter 1984; Hall 1988). Thus, the standard model cannot explain large fluctuations of hours worked, existence of unemployed workers, or fluctuations in unemployment. The model also cannot explain small fluctuations in productivity and wages relative to hours worked. Kydland and Prescott attempted to explain this puzzle, but were unable to account for these observations.

Hansen (1985) introduced "indivisible labour" into the standard RBC model, and the model quickly become a standard model used by RBC researchers. The idea indivisible labour was that in a certain period, households either work full time or they do not work at all – they are unable to work an intermediate amount of hours. Fluctuations in aggregate labour hours arise from households entering and leaving unemployment, which was a consistent feature of US post-war labour market data. The model could account for large aggregate fluctuations in hours worked, relative to productivity, while also having a small intertemporal elasticity of substitution for leisure on the part of individuals. This follows because the utility function of the representative agent implies an elasticity of substitution between leisure in different periods that is infinite.[49]

To rephrase, standard RBC models focused on the "intensive margin" – households that adjust their labour supply with respect to factor prices – however not much focus was on the "extensive margin" – households entering in or out of employment. This was a big oversight (e.g., female labour supply). Consider the following decomposition from Hansen's paper:

$$\operatorname{Var}(\log H_t) = \underbrace{\operatorname{Var}(\log h_t)}_{\approx 20\%} + \underbrace{\operatorname{Var}(\log N_t)}_{\approx 55\%} + \underbrace{2\operatorname{Cov}(\log h_t, \log N_t)}_{\approx 25\%},$$

where H_t is total hours worked, h_t is average hours worked, and N_t is number of individuals at work. Most households either work full time or not at all. This may be due to the presence of non-convexities either in individual preferences for leisure or in the production technology. For example, marginal productivity of labour could be high early in the week and it could be low late in the week, and this would imply a convex production function at first and then concave after. Hansen's model assumes non-convexity based on the property of preferences, and that individual households have preferences defined at two levels: full time work or no work at all. Thus, individuals can only adjust labour supply along the extensive margin.

[49] Note that in the this model utility functions of the representative agent and the utility function of the individual household are different.

8.2.2 The baseline model

Hansen's paper follows an inductive reasoning argument: the real economy has fluctuations along both the extensive and intensive margins, so compare two models – one with an intensive margin and one with an extensive margin. Thus, we can determine the importance of non-convexities for explaining labour variance. If both economies exhibit similar cyclical behaviour, then a model (or the real world) that incorporates both margins would also exhibit similar behaviour.

The following equations characterise the two economies:

$$f(\lambda_t, k_t, h_t) = \lambda_t k_t^{\theta} h_t^{1-\theta}, \tag{197}$$

$$c_t + i_t \leq f(\lambda_t, k_t, h_t), \tag{198}$$

$$k_{t+1} = (1-\delta)k_t + i_t, \;\; \delta \in (0,1), \tag{199}$$

$$\lambda_{t+1} = \gamma \lambda_t + \epsilon_{t+1}, \tag{200}$$

and where for simplicity, a single firm is assumed to exist. The technology shock follows a first-order Markov process, where the ϵ_t's are IID with distribution F. F has mean $1-\gamma$, and the unconditional mean of λ_t is equal to 1.

The base (divisible labour) model has a continuum of infinitely lived household along the closed set $[0,1]$ which populate the economy, where they maximise the following:

$$\mathbb{E}_t \sum_{t=0}^{\infty} \beta^t u(c_t, l_t),$$

where $\beta \in (0,1)$ is the discount factor and $l_t = 1 - h_t$. Utility in period t is given by:

$$u(c_t, l_t) = \log c_t + A \log l_t, \;\; A > 0, \tag{201}$$

and is subject to the following budget constraint:

$$c_t + i_t \leq w_t h_t + r_t k_t \tag{202}$$

and the law of motion of capital (199). The FOCs for the firm's maximisation problem imply that the wage and interest rate each period are equal to the marginal productivity of labour and capital, respectively. No externalities or distortions characterise the economy, so the competitive equilibrium can be considered as a Pareto optimum. Pareto optimum is the solution of maximising expected welfare of the representative agent subject to technology constraints. This completes the specification of the base model.

8.2.3 Economy with indivisible labour

For the indivisible labour model we apply a restriction that individuals can work full time, h_0, or not at all. There are some slight mathematical challenges: feasible equilibrium requires consumption possibilities set to be convex; and, trading work hours implies consumption possibilities set to be non-convex. Hansen resolves this by introducing a lottery to convexify the set – households choose lotteries to work or not rather than hours. Thus, households choose a probability of working, α_t.

The introduction of the lottery also implies that firms offer complete income insurance to households. Firms and households make and trade a contract that the households works h_0 with probably α_t, and that the household gets paid whether it works or not. All households are ex-ante identical (face the same α_t) but differ ex-post depending on outcome of lottery.

The utility function is now:

$$u(c_t, a_t) = \log c_t + A\alpha_t \log(1 - h_0), \tag{203}$$

and per capita hours worked in period t is given as:

$$h_t = \alpha h_0. \tag{204}$$

Other features of this economy are identical to the base model, described by (197)-(200).

Firms employ labour until:

$$f_h(\lambda_t, k_t, h_t) = w_t.$$

Due to the lottery system, the household budget constrain is slightly different compared to the baseline model:

$$c_t + i_t \leq w_t \alpha_t h_0 + r k_t. \tag{205}$$

Thus welfare is maximised by the following:

$$\mathbb{E}_t \sum_{t=0}^{\infty} \beta^t u(c_t, \alpha_t),$$

subject to (199) and (205).

8.2.4 Solving the model

As discussed earlier, a key property of this model is that the elasticity of substitution between leisure in different periods for the representative agent is infinite. To derive, begin by substituting $h_t = 1 - l_t$ into (204) to get:

$$\alpha_t = \frac{1 - l_t}{h_0},$$

and substituting α_t into (203) gets:

$$\begin{aligned} u(c_t, a_t) &= \log c_t + A \frac{(1 - l_t)}{h_0} \log(1 - h_0) \\ &= \log c_t + \underbrace{\frac{A}{h_0} \log(1 - h_0)}_{=\text{constant}} - \frac{A l_t}{h_0} \log(1 - h_0) \\ &= \log c_t + B l_t, \end{aligned}$$

where we ignore the constant term, and $B = -\frac{A}{h_0} \log(1 - h_0)$. This utility function is linear in leisure which implies an infinite elasticity of substitution between leisure in different periods. This follows no matter how small this elasticity is for the individuals populating the economy. Therefore, the elasticity of substitution between leisure in different periods for the aggregate economy is infinite and independent of the willingness of individuals to substitute leisure across time.[50]

[50] This was originally shown by Rogerson (1988). This result depends on the utility function being additively separable across time.

Hansen then goes onto solve the model by using the method of linear quadratic dynamic programming following Kydland and Prescott's procedure.[51] Key calibration parameters are $\delta = 0.025$, $\beta = 0.99$, and $A = 2$, which implies hours worked in the steady state (divisible labour case) is close to 1/3. h_0 was found to be 0.53, by equating hours of work in the steady state for both models equal to one another.

8.2.5 Key results

The table below presents the key findings from the Hansen paper. We've seen it before in Figure 4, but it's worth repeating the key points point here.

Figure 31: Comparison of Baseline RBC Model and Indivisible Labour Model

Standard deviations in percent (a) and correlations with output (b) for U.S. and artificial economies.

	Quarterly U.S. time series[a] (55,3–84,1)		Economy with divisible labor[b]		Economy with indivisible labor[b]	
Series	(a)	(b)	(a)	(b)	(a)	(b)
Output	1.76	1.00	1.35 (0.16)	1.00 (0.00)	1.76 (0.21)	1.00 (0.00)
Consumption	1.29	0.85	0.42 (0.06)	0.89 (0.03)	0.51 (0.08)	0.87 (0.04)
Investment	8.60	0.92	4.24 (0.51)	0.99 (0.00)	5.71 (0.70)	0.99 (0.00)
Capital stock	0.63	0.04	0.36 (0.07)	0.06 (0.07)	0.47 (0.10)	0.05 (0.07)
Hours	1.66	0.76	0.70 (0.08)	0.98 (0.01)	1.35 (0.16)	0.98 (0.01)
Productivity	1.18	0.42	0.68 (0.08)	0.98 (0.01)	0.50 (0.07)	0.87 (0.03)

Source: Hansen (1985)

The indivisible labour economy displays much more fluctuations than the baseline RBC model. The standard RBC model undershoots fluctuations and volatility drastically, while the indivisible labour model overshoots slightly – so things look promising. Observe the ratio of the standard deviation in hours worked to the standard deviation of productivity: the standard RBC model has a ratio of 1, while the indivisible model has a ratio of 2.7. For the record, the US economy has a ratio of 1.4. Hansen's paper seemingly addressed a lot of the weaknesses in the original Kydland and Prescott (1982) model, who despite easing intertemporal substitution of leisure, struggled to get the baseline RBC model's

[51] Linear quadratic dynamic programming is beyond the scope of this course. It's an alternative to the method of Blanchard and Kahn and the method of undetermined coefficients. McCandless (chapter 7) provides a good treatment.

ratio above 1.17. Conversely, the indivisible labour model delivers – as expected considering the assumptions – a high ratio.

However, the Hansen model still suffers from a lot of weaknesses that have plagued RBC models (see the chapter on the RBC model for an overview). Primarily, the propagation mechanism is still weak – productivity shocks are still too highly correlated with output fluctuations, and there is still too high a correlation between hours worked and wages.

Nevertheless, the Hansen model was a step in the right direction, and the model became a mainstream model in classrooms since its introduction.

8.3 Efficiency wages and the Shapiro-Stiglitz model

8.3.1 Motivation

Now we delve a bit deeper into mechanisms that can explain unemployment. If there is unemployment in a Walrasian labour market, unemployed workers immediately bid the wage down until supply and demand are in balance. Theories of unemployment can therefore be classified according to their view of why this mechanism fails to operate. Concretely, consider an unemployed worker who offers to work for a firm for slightly less (or more) than the firm is currently paying, and who is otherwise identical to the firm's current workers. The firm may not want to offer a different wage, however, due to there being costs and benefits to paying lower (or higher) wages. Theories in which there is a cost as well as a benefit to the firm of paying different wages are known as efficiency wage theories.

First, and most simply, a higher wage, for example, can increase workers' food consumption, and thereby cause them to better nourished and more productive. Obviously this possibility is not important in developed economies. Nonetheless, it provides a concrete example of an advantage of paying a higher wage. For that reason, it is often a useful reference point.

Second, a higher wage can increase workers' effort in situations where the firm cannot monitor them perfectly. In a Walrasian labour market, workers are indifferent about losing their jobs, since identical jobs are immediately available. thus, if the only way that firms can punish workers who exert low effort is bu firing them, workers in such a labour market have no incentive to exert effort. But if a firm pays more than the market-clearing wage, its jobs are valuable. Thus, its workers may choose to exert effort instead of shirking.

Third, paying a higher wage can improve workers' ability along dimensions the firm cannot observe. Specifically, if higher ability workers have higher reservation wages, offering a higher wage raises the average quality of the applicant pool, and thus raises the average ability of the workers the firm hires (Weiss 1980).

Finally, a high wage can build loyalty among workers and hence induce higher effort; conversely, a low wage can cause anger and desire for revenge, and thereby lead to shirking or sabotage. Akerlof and Yellen (1990) present extensive evidence that workers' effort is affected by such forces as anger, jealousy, and gratitude. For example, they describe studies showing that workers who believe they are underpaid sometimes perform their work in ways that are harder for them in order to reduce their employers' profits.

8.3.2 A simple efficiency wage model

We now turn to a model of efficiency wages. There is a large number, N, of identical competitive firms. The representative firm seeks to maximise its profits, which are given by:

$$\pi = Y - wL, \tag{206}$$

where Y is the firm's output, w is the wage that it pays, and L is the amount of labour it hires. A firm's output depends on the number of workers it employs and on their effort. For simplicity, we assume the firm's production technology is:

$$Y = F(eL),\ F'(\cdot) > 0,\ F''(\cdot) < 0,$$

where e denotes workers' effort. The crucial assumption of efficiency wage models is that effort depends positively on the wage the firm pays:

$$e = e(w),\ e'(\cdot) > 0.$$

We also assume that there are $\bar{L}$ identical workers, each of whom supplies 1 unit of labour inelastically.

The problem facing the representative firm is:

$$\arg\max_{L,w} \; F(e(w)L) - wL.$$

If there are unemployed workers, the firm can choose the wage freely. If unemployment is zero, on the other hand, the firm must pay at least the wage paid by other firms. When the firm is unconstrained, the FOCs for L and w are:

$$F'(e(w)L)e(w) - w = 0, \tag{207}$$

$$F'(e(w)L)e'(w)L - L = 0. \tag{208}$$

We can rewrite the first FOC (207) as:

$$F'(e(w)L) = \frac{w}{e(w)},$$

and by substituting this into (208) we get:

$$\frac{w}{e(w)}e'(w)L = L,$$

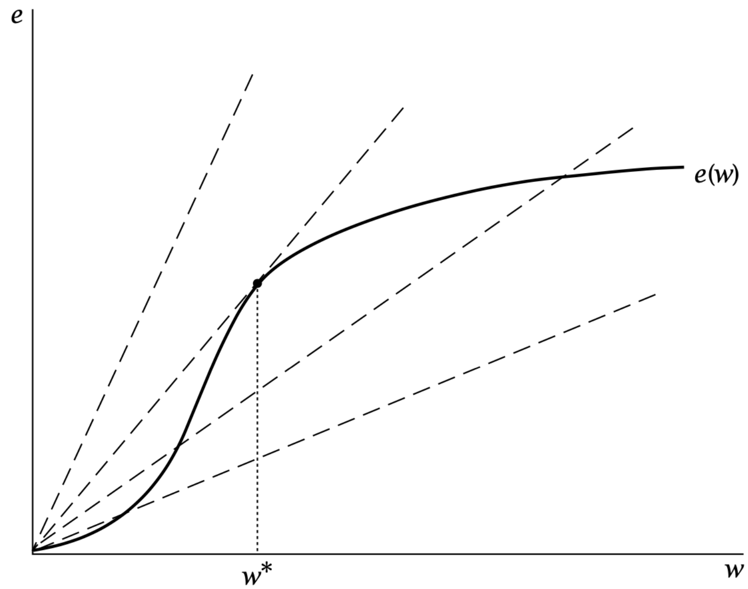
and then divide by L to get:

$$\frac{w}{e(w)}e'(w) = 1. \tag{209}$$

What this equation states is that at the optimum, the elasticity of effort with respect to the wage is 1. To understand this condition, note that output is a function of the quantity of effective labour, eL. The firm therefore wants to hire effective labour as cheaply as possible. When the firm hires a worker, it obtains $e(w)$ units of effective labour at a cost of w. Thus, the cost per unit of effect labour is $w/e(w)$. When the elasticity of $e$with respect to w is 1, a marginal change in w has no effect on this ratio; thus this is the FOC fo the problem of choosing w to minimise the cost of effective labour. The wage satisfying (209) is known as the efficiency wage.

Figure 32 shows the choice of w graphically in (w, e) space. The rays coming out of the origin are lines where the ratio of e to w is constant; the ratio is larger on the higher rays. Thus, the firm wants to choose w to attain a high a ray as possible. This occurs where the $e(w)$ function is just tangent to one of the rays – that is, where the elasticity of e with respect to w is 1.

Figure 32: Determination of the Efficiency Wage



Source: Romer (2012)

Note also that the FOC (207) states that the firm hires workers until the marginal product of effective labour equals its cost. This is analogous to the condition in a standard labour demand problem that the firm hires labour up to the point where the marginal product equals the wage.

Equations (207) and (209) describe the behaviour of a single firm. Scaling up to the economy wide equilibrium is straightforward. Let w^* and L^* denote the values of w and L that satisfy (207) and (209). Since firms are identical, each firm chooses these same values of w and L. Total labour demand is therefore NL^*. If labour supply, $\bar{L}$, exceeds this amount, firms are unconstrained in their choice of w. In this case, the wage is w^*, employment is NL^*, and there is unemployment of amount $\bar{L} - NL^*$. If NL^* exceeds $\bar{L}$, firms are constrained, and the wage is bid up to the point where demand and supply are in balance, and there is no unemployment.

This simple model shows how efficiency wages can give rise to unemployment. In addition, the model implies that the real wage is unresponsive to demand shifts. Suppose the demand for labour increases. Since the efficiency wage, w^*, is determined entirely by the properties of the effort function, $e(\cdot)$, there is no reason for firms to adjust their wages. Thus the model provides a candidate explanation of why shifts in labour demand lead to large movements in employment and small changes in the real wage. In addition, the fact that the real wage and effort do not change implies that the cost of a unit of effective labour does not change. As a result, in a model with price setting firms, the incentive to adjust prices is small.

Unfortunately, these results are less promising than they appear. The key difficulty is that they apply not just to the short-run but to the long-run too: the model implies that as economic growth shifts the demand for labour outward, the real wage remains unchanged and unemployment trends downward. Eventually, unemployment reaches zero, at which further increases in demand lead to increases in the real wage. In practice, however, we observe no clear trend in unemployment over extended periods. In other words, shifts in labour demand in the short-run only affect employment and not the real wage; and in the long-run it only seems to affect only real wages. This simple model does not explain this pattern.

8.3.3 Assumptions of the Shapiro-Stiglitz model

We now look at the Shapiro-Stiglitz model, which focuses on firms' monitoring ability (or lack thereof). Presenting a formal model of imperfect monitoring serves three purposes. First, it allows us to investigate whether this idea holds up under scrutiny. Second, it permits us to analyse additional questions. For example, only with a formal model can we ask whether government policies can improve welfare. Third, the mathematical tools the model employs are useful in other settings.

We first assume that the economy consists of a large number of workers, $\bar{L}$, and a large number of firms, N. Workers maximise their expected discounted utilities, and firms maximise their expected discounted profits. The model is set in continuous time. For simplicity, this analysis focuses on steady states.

Consider workers first. The representative worker's lifetime utility is:

$$U = \int_{t=0}^{\infty} \exp(-\rho t)u(t)dt, \ \rho > 0,$$

where $u(t)$ is instantaneous utility at time t, and ρ is the discount rate. Instantaneous utility is:

$$u(t) = \begin{cases} w(t) - e(t) & \text{if employed,} \\ 0 & \text{if unemployed,} \end{cases}$$

where w is the wage and e is the worker's effort. There are only two possible effort levels, $e = 0$ and $e = \bar{e}$. Thus, at any moment a worker must be in one of three states: employed and exerting effort (denoted E), employed and shirking (denoted S), or unemployed (denoted U).

A key ingredient of the model is its assumptions concerning workers' transitions between the three states. First, there is an exogenous rate at which jobs end. Specifically, if a worker begins working a job at some time, t_0 (and if the worker exerts effort), the probability that the worker is still employed in the job at some time later, t, is:

$$P(t) = \exp(-b(t - t_0)), \ b > 0. \tag{210}$$

This equation implies that $P(t + \tau)/P(t)$ equals $\exp(-b\tau)$, and thus that it is independent of t: if a worker is employed at some time, the probability that she is still employed at time τ later is $\exp(-b\tau)$ regardless of how long the worker has already been employed. This assumption that job breakups follow a Poisson process simplifies the analysis greatly, because it implies that there is no need to keep track of how long workers have been in their jobs.

An equivalent way to describe the process of job breakup is to say that it occurs with probability b per unit time, or to say that the hazard rate for job breakup is b. That is, the probability that an employed worker's job ends in the next dt units of time approaches $b \cdot dt$ as $dt \to 0$. To see that our assumptions imply this, note that (210) implies $P'(t) = -bP(t)$.

The second assumption concerning workers' transitions between states is that firms' detection of workers who are shirking is also a Poisson process. Specifically, detection occurs with probability q per unit time. q is exogenous, and detection is independent of job breakups. workers who are caught shirking are fired. Thus, if a worker is employed but shirking, the probability that she is still employed time τ later is $\exp(-q\tau)\exp(-b\tau)$, the probability that the worker has not been caught and fired times the probability that the job has not ended exogenously.

Third, unemployed workers find employment at rate a per unit time. Each worker takes a as given. In the economy as a whole, however, a is determined endogenously. When firms want to hire workers, they choose workers at random out of the pool of unemployed workers. Thus a is determined by the rate at which firms are hiring (which is determined by the number of employed workers and the rate at which jobs end) and the number of unemployed workers. Because workers are identical, the probability of finding a job does not depend on how workers become unemployed or on how long they are unemployed.

Firms' behaviour is simple. A firm's profits at time t are:

$$\pi(t) = F(\bar{e}L(t)) - w(t)\left[L(t) + S(t)\right], \; F'(\cdot) > 0, \; F''(\cdot) < 0, \tag{211}$$

where L is the number of employees who are exerting effort and S is the number who are shirking. The problem facing the firm is to set w sufficiently high that its workers do not shirk, and to choose L. Because the firm's decisions at any date affect profits only at that date, there is no need to analyse the present value of profits: the firm chooses w and L at each moment to maximise the instantaneous flow of profits.

The final assumption of the model is:

$$\begin{aligned} \bar{e}F'\left(\frac{\bar{e}\bar{L}}{N}\right) &> \bar{e}, \\ \implies F'\left(\frac{\bar{e}\bar{L}}{N}\right) &> 1. \end{aligned}$$

This condition states that if each firm hires $1/N$ of the labour force, the marginal product of labour exceeds the cost of exerting effort. Thus, in the absence of imperfect monitoring, there is full employment.

8.3.4 The values of E, U, and S

Let V_i denote the value of being in state i (for $i = (E, S, U)$). That is, V_i is the expected value of discounted lifetime utility from the present moment forward of a worker who is in state i. Because transitions among states follow Poisson processes, the V_i's do not depend on how long the worker has been in the current state or on the worker's prior history. And because we are focusing on steady states, the V_i's are constant over time.

To find V_E, V_S, and V_U, it is not necessary to analyse the various paths the worker may follow over the infinite future. Instead we can use dynamic programming. The central idea of dynamic programming is to look at only a brief interval of time and use the V_i's themselves to summarise what occurs after the end of the interval.[52] Consider first a worker that is employed and exerting effort at time 0. Suppose temporarily that time is divided into intervals of length Δt, and that a worker who loses her job during one interval cannot begin to look for a new job until the beginning of the next interval. Let $V_E(\Delta t)$ and $V_U(\Delta t)$ denote the values of employment and unemployment as of the beginning of an interval under this assumption. In a moment we will let $\Delta t \to 0$. When we do this, the constraint that a worker who loses her job during an interval cannot find a new job during the remainder of that interval becomes irrelevant. Thus $V_E(\Delta t) \to V_E$,

If a worker is employed in a job paying a wage of w, $V_E(\Delta t)$ is given by:

$$\begin{aligned} V_E(\Delta t) = &\int_{t=0}^{\Delta t} \exp(-bt)\exp(-\rho t)(w - \bar{e})dt \\ &+ \exp(-\rho\Delta t)\left[\exp(-b\Delta t)V_E(\Delta t) + (1 - \exp(-b\Delta t))V_U(\Delta t)\right]. \end{aligned} \tag{212}$$

The first term of this equation reflects utility during the interval $(0, \Delta t)$. The probability that the worker is still employed at time t is $\exp(-bt)$. if the worker is employed, flow utility is $w - \bar{e}$. Discounting this back to time 0 yields an expected contribution to lifetime utility of $\exp(-(\rho + b)t)(w - \bar{e})$. The second term reflects utility after Δt. At time Δt, the worker is employed with probability $\exp(-b\Delta t)$

[52] Here we look dynamic programming in a continuous context. We previously looked at the discrete time case, where we only looked at one period ahead. See Ljungqvist and Sargent (2018) for a proper treatment of dynamic programming.

and unemployed with probability $1 - \exp(-b\Delta t)$. Combining these probabilities with the V's and discounting yields the second term.

If we compute the integral in (212), we can rewrite the equation as:

$$V_E(\Delta t) = \frac{1}{\rho + b}\left(1 - \exp(-(\rho + b)\Delta t)\right)(w - \bar{e})$$
$$+ \exp(-\rho\Delta t)\left[\exp(-b\Delta t)V_E(\Delta t) + (1 - \exp(-b\Delta t))V_U(\Delta t)\right],$$

and solving this expression for $V_E(\Delta t)$ gives:

$$V_E(\Delta t) = \frac{1}{\rho + b}(w - \bar{e}) + \frac{1}{1 - \exp(-(\rho + b)\Delta t)}\exp(-\rho\Delta t)(1 - \exp(-b\Delta t))V_U(\Delta t). \tag{213}$$

As described above, V_E equals the limit of $V_E(\Delta t)$ as $\Delta t \to 0$. Similarly, V_U equals the limit of $V_U(\Delta t)$ as $t \to 0$. To find this limit, apply L'Hopital's rule to (213). This yields:

$$V_E = \frac{1}{\rho + b}\left[(w - \bar{e}) + bV_U\right]. \tag{214}$$

Intuitively: Think of an asset that pays dividends at rate $w - \bar{e}$ per unit time when the worker is employed and no dividends when the worker is unemployed. In addition, assume that the asset is being priced by risk-neutral investors with required rate of return ρ. Since the expected present value of lifetime dividends of this asset is the same as the worker's expected present value of lifetime utility, the asset's price must be V_E when the worker is employed and V_U when the worker is unemployed. For the asset to be held, it must provide an expected rate of return of ρ. That is, its dividends per unit time, plus any expected capital gains or losses per unit time, must equal ρV_E. When the worker is employed, dividends per unit time are $w - \bar{e}$, and there is a probability b per unit time of a capital loss of $V_E - V_U$. Thus,

$$\rho V_E = (w - \bar{e}) - b(V_E - V_U), \tag{215}$$

and rearranging this expression yields (214).

If the worker is shirking, the "dividend" is w per unit time, and the expected capital loss is $(b +$

$q)(V_S - V_U)$ per unit time. Thus, reasoning parallel to that used to derive (215) implies:

$$\rho V_S = w - (b + q)(V_S - V_U). \tag{216}$$

Finally, if the worker is unemployed, the dividend is zero and the expected capital gain (assuming that firms pay sufficiently high wages that employed workers exert effort) is $a(V_E - V_U)$ per unit time. Thus:

$$\rho V_U = a(V_E - V_U). \tag{217}$$

8.3.5 The no-shirking condition

The firm must pay enough that $V_E \geq V_S$; otherwise, its workers exert no effort and produce nothing. At the same time, since effort cannot exceed $\bar{e}$, there is no need to pay any excess over the minimum needed to induce effort. Thus the firm chooses w so that V_E equals V_S:

$$V_E = V_S.$$

Substitute in our expressions for V_E and V_S from (215) and (216) to yield:

$$\begin{aligned}(w - \bar{e}) - b(V_E - V_U) &= w - (b + q)(V_S - V_U) \\ \Leftrightarrow V_E - V_U &= \frac{\bar{e}}{q}. \end{aligned} \tag{218}$$

This equation implies that firms set wages high enough that workers strictly prefer employment to unemployment. Thus workers obtain rents. The size of the premium is increasing in the cost of exerting effort, $\bar{e}$, and decreasing in firms' efficacy in detecting shirkers, q.

The next step is to find what the wage must be for the rent to employment to equal $\bar{e}/q$. Equations (215) and (217) imply:

$$\rho(V_E - V_U) = (w - \bar{e}) - (a + b)(V_E - V_U). \tag{219}$$

It follows that for $V_E - V_U$ to equal $\bar{e}/q$, the wage must satisfy

$$w = \bar{e} + (a + b + \rho)\frac{\bar{e}}{q}. \tag{220}$$

Thus, the wage needed to induce effort is increasing in the cost of effort $\bar{e}$, the ease of finding jobs a, the rate of job breakup b, and the discount rate ρ, and decreasing in the probability that shirkers are detected q.

Next, write the rate at which the unemployed find jobs a in terms of employment per firm L. Use the fact that, since the economy is in steady state, movements along the extensive margin balance. The number of workers becoming unemployed per unit time is N (the number of firms) times L (the number of workers per firm) times b (the rate of job breakup). The number of unemployed workers finding jobs is $\bar{L} - NL$ times a. Equating these two quantities yields:

$$a = \frac{NLb}{\bar{L} - NL}, \tag{221}$$

which implies:

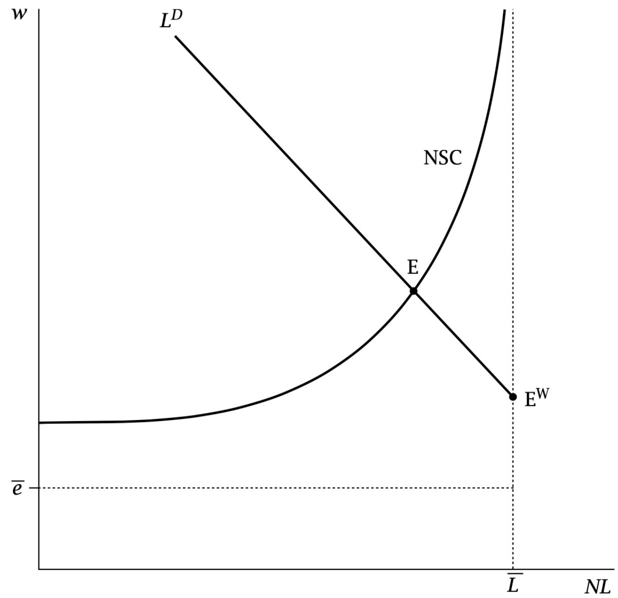
$$a + b = \frac{\bar{L}}{\bar{L} - NL}b,$$

and substituting this into (220) yields:

$$w = \bar{e} + \left(\rho + \frac{\bar{L}}{\bar{L} - NL}b\right)\frac{\bar{e}}{q}, \tag{222}$$

also known as the no-shirking condition (NSC). It shows, as a function of the levels of employment, the wage that firms must pay to induce workers to exert effort. When more workers are employed, there are fewer unemployed workers and more workers leaving their jobs; thus it is easier for unemployed workers to find employment. The wage needed to deter shirking is therefore an increasing function of employment. At full employment, unemployed workers find work instantly, and so there is no cost to being fired and no wage that can deter shirking. The set of points in (NL, w) space satisfying the NSC are shown in Figure 33.

Figure 33: The Shapiro-Stiglitz Model



Source: Romer (2012)

8.3.6 Closing the model

Firms hire workers up to the point where the marginal product of labour equals the wage. Equation (211) implies that when its workers are exerting effort, a firm's flow profits are:

$$\pi(t) = F(\bar{e}L) - wL.$$

Thus, the condition for the marginal product of labour equaling the wage is:

$$\begin{aligned} \frac{\partial \pi(t)}{\partial L} &= F'(\bar{e}L)\bar{e} - w = 0 \\ \implies F'(\bar{e}L)\bar{e} &= w. \end{aligned} \tag{223}$$

The set of points satisfying (223) (which is a simple labour demand curve) is also shown in Figure 33.

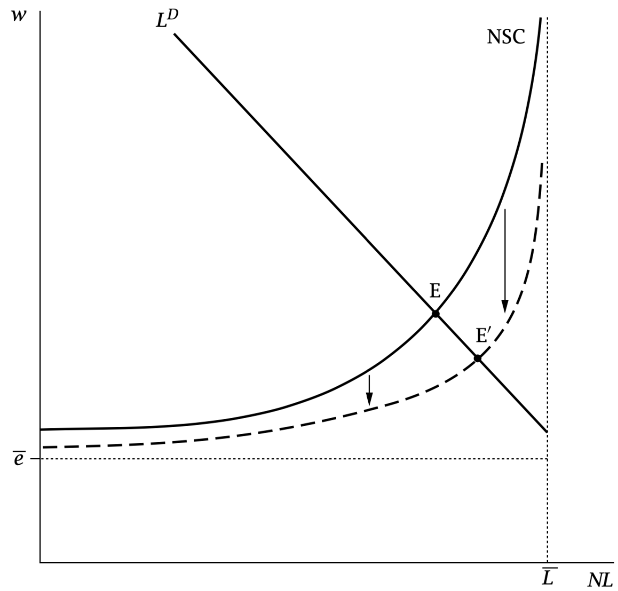
Labour supply is horizontal at $\bar{e}$ up to the number of workers $\bar{L}$, and then vertical. In the absence of imperfect monitoring, equilibrium occurs at the intersection of labour demand and supply. Our assumption that the marginal product of labour at full employment exceeds the disutility of effort ($F'(\bar{e}\bar{L}/N) > 1$) implies that this intersection occurs in the vertical part of the labour supply curve. The Walrasian equilibrium is shown as point E^W in the diagram.

With imperfect monitoring, equilibrium occurs at the intersection of the labour demand demand curve (Equation (223)) and the NSC (Equation (222)). This is shown as Point E in the diagram. At the equilibrium, there is unemployment. Unemployed workers strictly prefer to be employed at the prevailing wage and exert effort than to remain unemployed. Nonetheless, they cannot bid the wage down: firms know that if they hire additional workers at slightly less than the prevailing wage, the workers will prefer shirking to exerting effort. Thus the wage does not fall, and the unemployment remains.

Two examples may help to clarify the workings of the model. First, a rise in q – an increase in the probability per unit time that shirker is detected – shifts the no-shirking locus down and does not affect the labour demand curve. This is shown in Figure 34. The real wage falls and employment rises. As q approaches infinity, the probability that a shirker is detected in any finite length of time approaches 1. As a result, the no-shirking wage approaches $\bar{e}$ for any level of employment less than full employment. Thus, the economy approaches the Walrasian equilibrium.

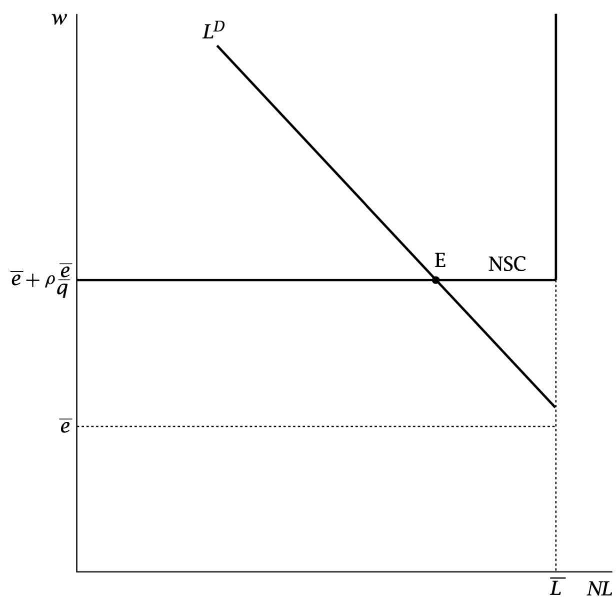
Second, if there is no turnover ($b = 0$), unemployed workers are never hired. As a result, the no-shirking wage in this case is $\bar{e} + \rho\bar{e}/q$. Intuitively, the gain from shirking relative to exerting effort is $\bar{e}$ per unit time. The cost is that there is probability q per unit time of becoming permanently unemployed and thereby losing the discounted surplus from the job, which is $(w - \bar{e})/\rho$. Equating the cost and benefit gives $w = \bar{e} + \rho\bar{e}/q$. This is shown in Figure 35.

Figure 34: The Effects of $q \uparrow$ in the Shapiro-Stiglitz Model



Source: Romer (2012)

Figure 35: The Shapiro-Stiglitz Model Without Turnover ($b = 0$)



Source: Romer (2012)

8.3.7 Implications of the Shapiro-Stiglitz model

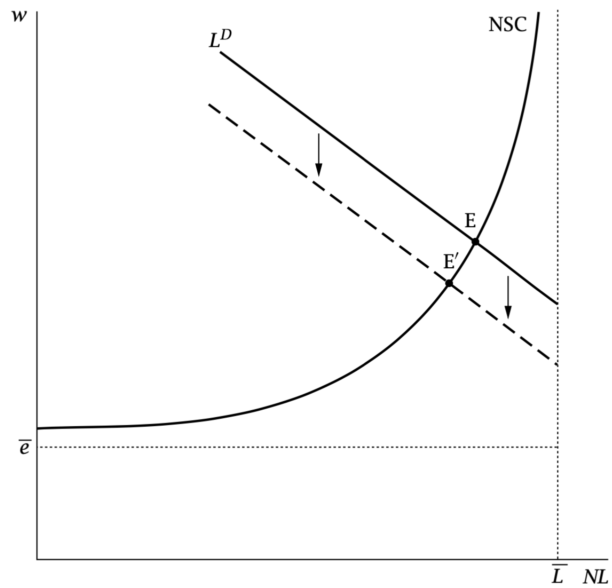
The model implies that there is equilibrium unemployment and suggests various factors that are likely to influence it. Thus the model has some promise as a candidate explanation of unemployment. Unfortunately, the model is so stylised that it is difficult to determine what level of unemployment it predicts.

With regard to short-run fluctuations, consider the impact of a fall in labour demand, shown in Figure 36. w and L move down along the NSC locus. Since labour supply is perfectly inelastic, employment necessarily responds more than it would without imperfect monitoring. Thus, the model suggests one possible reason that wages may respond less to demand driven output fluctuations than they would if workers were always on their labour supply curves.

Unfortunately, however, this effect appears to be quantitatively small. When unemployment is lower, a worker who is fired can find a new job more easily, and so the wage needed to prevent shirking is higher; this is the reason the NSC locus slopes up. Attempts to calibrate the model suggest that the locus is quite steep at the levels of unemployment we observe. That is, the model implies that the impact of a shift in labour demand falls mainly on wages and relatively little on employment.

Finally, the model implies that the decentralised equilibrium is inefficient. To see this, note that the marginal product of labour at full employment, $F'(\bar{e}\bar{L}/N)\bar{e}$, exceeds the cost to workers of supply effort, $\bar{e}$. Thus the first-best allocation is for everyone to be employed and exert effort. Of course, the government cannot bring this about by simply dictating that firms move down the labour demand curve until full employment is reached: tis policy causes workers to shirk, and thus results in zero output. But Shapiro and Stiglitz note that wage subsidies financed by lump-sum taxes or profits taxes improve welfare. This policy shifts the labour demand curve up, and thus increases the wage and employment along the NSC. Since the value of the additional output exceeds the opportunity cost of producing it, overall welfare rises. How the gain is divided between workers and firms depends on how the wage subsidies are financed.

Figure 36: The Effects of a Fall in Labour Demand in the Shapiro-Stiglitz Model



Source: Romer (2012)

8.4 Models of search and match

8.4.1 Motivation

The final departure of the labour market from Walrasian assumptions that we consider is the simple fact that workers and jobs are heterogeneous. In a frictionless labour market, firms are indifferent about losing their workers, since identical workers are costlessly available at the same wage; likewise, workers are indifferent about losing their jobs. These implications are obviously not accurate descriptions of actual labour markets.

When workers and jobs are highly heterogeneous, the labour market has little resemblance to a Walrasian market. Rather than meeting in centralised markets where employment and wages are determined by the intersections of supply and demand curves, workers and firms meet in a decentralised, one-on-one fashion, and engage in a costly process of trying to match up idiosyncratic preferences, skills, and needs. Since this process is not instantaneous, it results in some unemployment. In addition, it may have implications for how wages and employment respond to shocks.

In this section, we present a model of firm and worker heterogeneity and the matching process. Because modelling heterogeneity requires abandoning many of our usual tools, even a basic model is relatively complicated. As a result, the model here only introduces some of the issues involved. This class of models is known collectively as the Diamond-Mortensen-Pissarides search and match model.

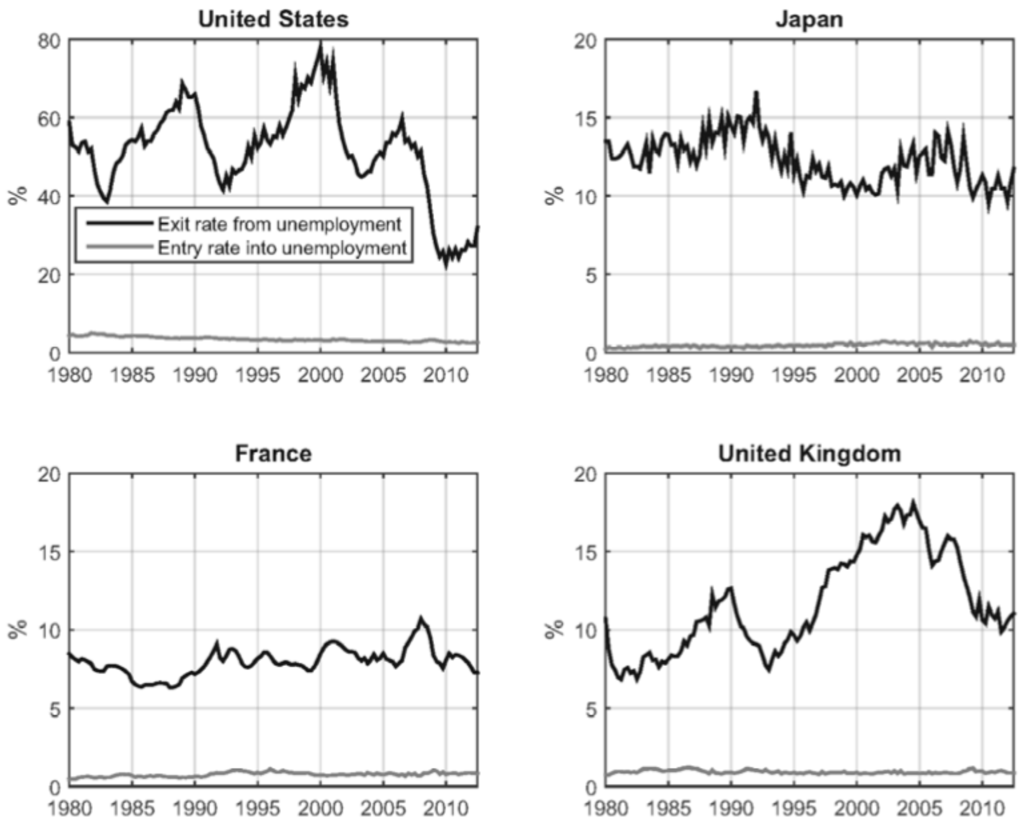
Search and match models have become common as a means of understanding the macroeconomics of the labour market. One reason for this is a number of empirical studies which have examined the behaviour of labour markets over the business cycle. These studies reveal a number of different phenomena such as:

- Even in recessions, large numbers of firms have unfilled vacancies and in booms some firms are laying off workers.

- In every period there are large gross labour market flows; movements in job creation and job destruction. The change in unemployment reflects net flows only (i.e., job destruction less job creation) and so is only a part of the overall labour market story.

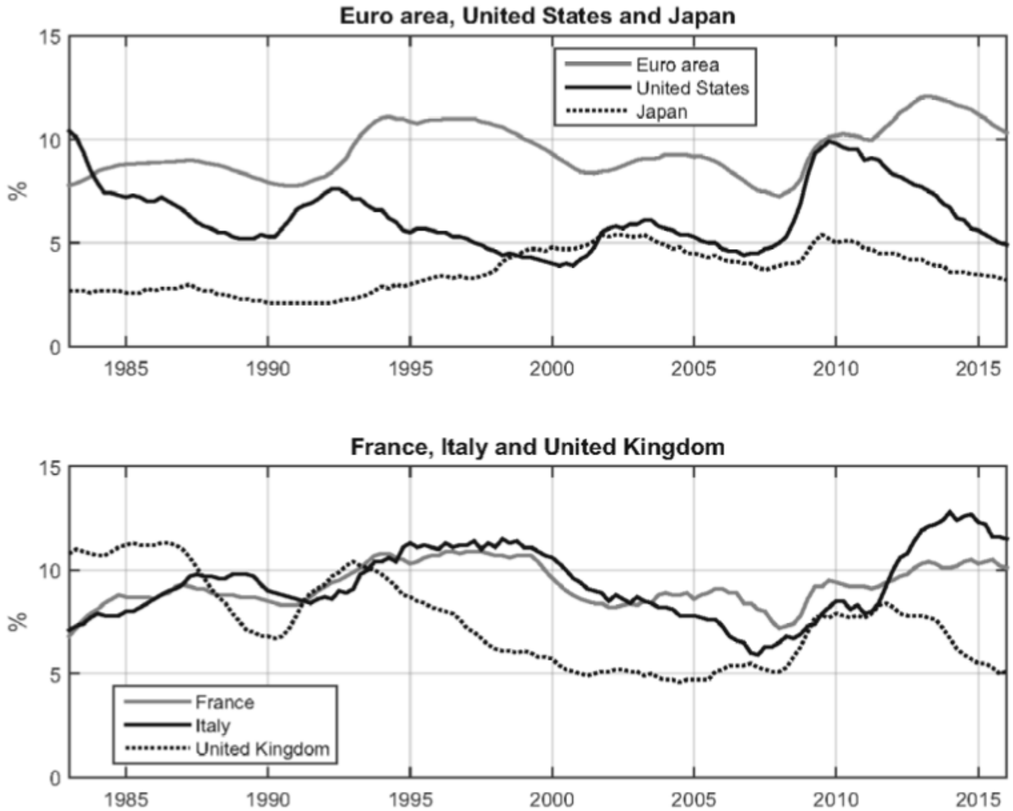
- Job creation is slightly procyclical but job destruction is strongly counter-cyclical with big spikes in recessions. In other words, big increases in unemployment are caused by occasional large periods of job destruction. The fact that job creation and job destruction have different cyclical properties suggests that labour market allocations are not well coordinated – it takes several periods for the unemployed to find vacancies.

Figure 37: Exit and Entry Rates to/from Unemployment



These facts imply that it is probably flows in and out of the labour market that are particularly important for understanding the cyclical behaviour of unemployment. Notice from Figure 37 that the exit rate from unemployment (we will define this shortly) is more cyclical than the entry rate into unemployment, and that to a first order approximation, the entry rate seems to be constant across time.

Figure 38: Historical Unemployment Rates



Finally, one other motivation for studying search and match models is that it allows us to consider externalities that may arise in the labour market. Search and match models inherently present frictions into the labour market, which may produce some inefficiencies that we see in the real world. For example, the wage bargaining process itself may be inefficient: the process may not internalise the effect of wages on the unemployed because they are not represented in the bargained contract between a hired worker and a firm (commonly known as the insider-outsider dilemma). A typical Walrasian labour market does not take into account these real frictions, and is unable to give much guidance in terms of policy recommendations to alleviate unemployment. By introducing labour market frictions, our model is able to capture one aspect of real life phenomena, and we can begin to consider optimal policy.

8.4.2 Definitions

Recall the definition of an unemployed individual: an individual who is without work, currently able to work, and seeking work. Figure 38 plots unemployment rates for a selection of developed economies. As previously stated, to understand the changes in the unemployment rate, we need to consider worker flows.

Let's define the entry rate into unemployment as δ_t, where

$$\delta_t = \frac{\text{flow of employed workers becoming unemployed during period } t}{\text{employed at the beginning of period } t}.$$

In other words, δ_t is the average probability of a worker becoming unemployed in period t. Next, let's define the exit rate from unemployment, p_t, as:

$$p_t = \frac{\text{flow of unemployed workers becoming employed during period } t}{\text{unemployed at the beginning of period } t},$$

and so p_t is the average probability of an unemployed person becoming employed in period t.

Linking the rate of unemployment with worker flows we get:

$$\begin{aligned} u_t - u_{t-1} &= \delta_t(1 - u_{t-1}) - p_t u_{t-1} \qquad (224) \\ \Leftrightarrow \Delta u_t &= \text{gross entry flow} - \text{gross exit flow}, \end{aligned}$$

where u_t is the unemployment rate. Using $\bar{\delta}$ and $\bar{p}$ to denote the mean transition rates to/out of unemployment, from (224), the mean unemployment rate is:

$$\begin{aligned} 0 &= \bar{\delta} - \bar{\delta}\bar{u} - \bar{p}\bar{u} \\ \bar{u} &= \frac{\bar{\delta}}{\bar{\delta} + \bar{p}}. \qquad (225) \end{aligned}$$

Figure 39: Average Entry and Exit Rates of Unemployment

So, we can conclude that entry and exit rates are quite informative on the conditions of the labour market. As an example, consider Figure 39 which shows the different average entry and exit rates of unemployment for a selection of countries. We can infer that the US tends to have a very flexible labour market, where it's quite easy to find and lose a job. Conversely, an economy like Italy seems to have a rigid labour market, where it's difficult to find jobs, but job duration seems to be quite long.

8.4.3 The matching function

Let the number of matches in each period be given by the matching function:

$$m(v_t, u_t) = v_t^{1-\xi} u_t^{\xi}, \quad 0 < \xi < 1, \tag{226}$$

where v_t is the number of job vacancies, and n_t and u_t is the number of employed and unemployed workers, respectively:

$$u_t = 1 - (1 - \delta) n_{t-1}. \tag{227}$$

The properties of the matching function are crucial to the model. In principle, it need not have constant returns to scale. When it exhibits increasing returns, there are thick-market effects: increases in the resources devoted to search make the matching process operate more effectively, in the sense that it yields more output (matches) per unit of input (unemployment and vacancies). When the matching function has decreasing returns, there are crowding effects.

The prevailing view, however, is that in practice constant returns is a reasonable approximation. For a large economy, over a relevant range, the thick-market and crowding effects may be relatively unimportant or may roughly balance. Empirical efforts to estimate the matching function have found no strong evidence of departures from constant returns.[53]

The assumption of constant returns implies that a single number, the ratio of vacancies to unemployment, summarises the tightness of the labour market. Define $\theta_t = v_t/u_t$ and note that constant returns imply:

$$\frac{m(v_t, u_t)}{v_t} = m\left(1, \frac{1}{v_t/u_t}\right) = q\left(\frac{v_t}{u_t}\right) = q(\theta_t), \tag{228}$$

which is the matching rate for vacancies (probability of filling a vacancy), and:

$$\frac{m(v_t, u_t)}{u_t} = m\left(\frac{v_t}{u_t}, 1\right) = p\left(\frac{v_t}{u_t}\right) = p(\theta_t), \tag{229}$$

which is the matching rate for the unemployed (probability of finding a job).

Our assumption that $m(v_t, u_t)$ exhibits constant returns and that it is increasing both arguments imply that $m(\theta_t)$ is increasing in θ_t, but that the increase is less than proportional. Thus, when the labour market is tighter (when θ_t is greater), the job-finding rate is higher and the vacancy filling rate is lower.

When macroeconomists want to assume a functional form for the matching function, they almost universally assume that it is Cobb-Douglas. We will take that approach here too.

[53] See for example "The Beveridge Curve" by Blanchard et al. (1989).

8.4.4 Employment accumulation

Each period a fraction, δ, of workers lose (exogenously) their job, and each period a fraction, $q(\theta_t)$, of vacancies are filled. Let us denote the law of motion of employment as:

$$n_t = (1 - \delta)n_{t-1} + q(\theta_t)v_t, \tag{230}$$

where the first term on the RHS – jobs that survive separation – is a stock, and the second term – new jobs created – is a flow object. Note that we use $q(\theta_t)$ and not $p(\theta_t)$ for the law of motion here.

8.4.5 Unemployment dynamics

Let's turn to look at what determines the unemployment rate. In this model, members of the households are either employed or unemployed, and so we have:

$$\begin{aligned} \text{Labour Force} &= \text{Employed} + \text{Unemployed} \\ \Leftrightarrow \bar{L} &= n_t + u_t, \end{aligned}$$

where we assume that the labour force is assumed to be constant, and no households are outside the labour force. Thus, in the model, the rates of employment and unemployment are related as:

$$1 = n_t^r + u_t^r,$$

and the unemployment rate is defined as:

$$u_t^r = 1 - n_t^r. \tag{231}$$

What about unemployment in the long-run? Begin by considering the stock of employed in the model given by the law of motion of employment (230):

$$n_t = (1 - \delta)n_{t-1} + q(\theta_t)v_t,$$

and note that job creation can alternatively be expressed as:

$$q(\theta_t)v_t = p(\theta_t)u_t.$$

In words, this just says that the entry matching rate for vacancies multiplied by the number of vacancies is equal to the matching rate of employment multiplied by the number of unemployed. This is intuitive. So, we can rewrite the law of motion for employment as:

$$n_t = (1-\delta)n_{t-1} + p(\theta_t)u_t. \tag{232}$$

Using (232), and normalising for the labour force, and evaluating at the steady state we have:

$$\delta n^r = p(\theta)u^r, \tag{233}$$

and since $n^r = 1 - u^r$ (from (231)), Equation (233) yields:

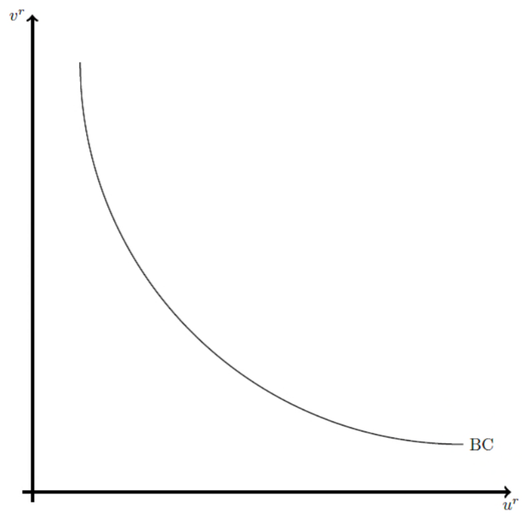
$$u^r = \frac{\delta}{\delta + p(\theta)}, \tag{234}$$

which is the long-run rate of unemployment, and it is determined by the job separation rate, δ, and the job creation rate, $p(\theta)$. Note that fiscal policy and labour market institutions are critical as they both affect $p(\theta)$ in this model. If the model had endogenous job separation, then δ would also be affected by policy and institutions.

Plotting (234) in (u^r, v^r) space gives us what is known as the Beveridge curve[54]:

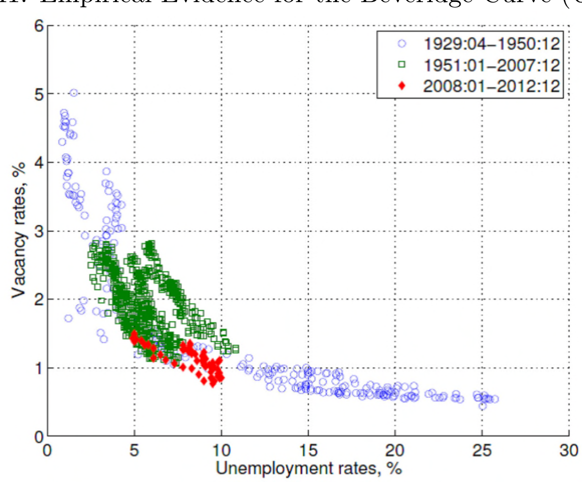
[54]Fun fact for the Oxonians: Baron William Henry Beveridge was Master of University College.

Figure 40: The Beveridge Curve



The Beveridge curve is fairly supported by the data as shown in Figure 41. In recent years, there has been some discussion about the accuracy of the Beveridge curve (about whether or not it's a "curve"). Such discussion is beyond the scope of this course, however.

Figure 41: Empirical Evidence for the Beveridge Curve (US data)



8.4.6 Households and firms

Our assumptions here are fairly standard. Households provide labour to the firm, and maximise their utility subject to a budget constraint. Firms maximise profits, and using labour inputs n_t to produce goods with a simple production technology:

$$y_t = a_t n_t^{\alpha}, \quad 0 < \alpha < 1. \tag{235}$$

Firms recruit workers by posting vacancies v_t at a flat cost κ, and face matching frictions $m(v_t, u_t)$.

Because a firm and a worker that meet are collectively better off if the firm hires the worker, they would be forgoing a mutually advantageous trade if the firm did not hire the worker. Thus the assumption that all meetings lead to hires is reasonable. But this does not uniquely determine the wage. The wage must be high enough such that the worker wants to work for the job, and low enough such that the firm wants to hire the worker. Because there is strictly positive surplus from the match, there is a range of wages that satisfy these requirements. Workers and firms bargain and negotiate wages in each period according to Nash Bargaining.

To explain Nash Bargaining, let $\mathcal{S}_t$ denote the sum of firm's surplus, $\mathcal{J}_t$, and the worker's surplus, $\mathcal{W}_t$:

$$\mathcal{S}_t = \mathcal{J}_t + \mathcal{W}_t.$$

The worker takes a fraction η of the total surplus:

$$\mathcal{W}_t = \eta \mathcal{S}_t,$$

and so by defining surpluses $\mathcal{J}_t$ and $\mathcal{W}_t$, we can derive the established wage, w_t (we will show this soon).

8.4.7 The social planner's problem

There are two ways to go about solving for equilibrium – a centralised solution via the Ramsey social planner, and through a competitive, decentralised process. Let's first solve the Ramsey planner's problem, and then compare it to the decentralised equilibrium. The results are quite illuminating.

The Ramsey planner chooses $\{c_t, n_t, v_t\}_{t=0}^{\infty}$ to maximise utility:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \ln c_{t+s},$$

subject to the economy wide resource constraint:

$$y_t = c_t + \kappa v_t,$$

and the law of motion of employment (230):

$$n_t = (1-\delta)n_{t-1} + q(\theta_t)v_t.$$

The Ramsey planner knows:

- the production technology (235): $y_t = a_t n_t^{\alpha}$;
- the matching function (226): $m(v_t, u_t) = v_t^{1-\xi} u_t^{\xi}$;
- the transition probabilities (228) and (229): $q(\theta_t) = \frac{m(v_t,u_t)}{v_t} = \theta_t^{-\xi}$ and $p(\theta_t) = \frac{m(v_t u_t)}{u_t} = \theta_t^{1-\xi}$, where $\theta_t = v_t/u_t$; and
- the amount of unemployed before hiring (227): $u_t = 1-(1-\delta)n_{t-1}$,

So, we can rewrite the constraints for the Ramsey planner as:

$$a_t n_t^{\alpha} = c_t + \kappa v_t, \tag{236}$$

and

$$
\begin{aligned}
n_t &= (1-\delta)n_{t-1} + \theta_t^{-\xi} v_t \\
&= (1-\delta)n_{t-1} + \left(\frac{v_t}{u_t}\right)^{-\xi} v_t \\
&= (1-\delta)n_{t-1} + \left(\frac{v_t}{1-(1-\delta)n_{t-1}}\right)^{-\xi} v_t.
\end{aligned} \tag{237}
$$

Note that the transition probabilities $q(\theta_t)$ and $p(\theta_t)$ are endogenous to the problem of the social planner. In other words, the Ramsey planner internalises/considers the effect of changes in vacancies and unemployment on the labour market transition probabilities and their effect on the equilibrium of the economy – the Ramsey planner internalises the effect of any potential search externalities on the equilibrium.

The Lagrangian for the Ramsey planner is:

$$
\mathcal{L} = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left\{ \ln c_{t+s} + \lambda_{1,t+s}\left(a_{t+s} n_{t+s}^{\alpha} - c_{t+s} - \kappa v_{t+s}\right) + \lambda_{2,t+s} \left[\begin{array}{c} (1-\delta)n_{t-1+s} \\ + \left(\frac{v_{t+s}}{1-(1-\delta)n_{t-1+s}}\right)^{-\xi} v_{t+s} - n_{t+s} \end{array} \right] \right\},
$$

where $\lambda_{1,t}$ and $\lambda_{2,t}$ are the Lagrangian multipliers, and the social planner chooses $\{c_t, n_t, v_t\}_{t=0}^{\infty}$. We know how to solve this problem – rewrite the problem as:

$$
\begin{aligned}
\mathcal{L} &= \ln c_t + \lambda_{1,t}\left(a_t n_t^{\alpha} - c_t - \kappa v_t\right) + \lambda_{2,t}\left[(1-\delta)n_{t-1} + \left(\frac{v_t}{1-(1-\delta)n_{t-1}}\right)^{-\xi} v_t - n_t\right] \\
&\quad + \beta \mathbb{E}_t \left[\lambda_{2,t+1}\left[(1-\delta)n_t + \left(\frac{v_{t+1}}{1-(1-\delta)n_t}\right)^{-\xi} v_{t+1} - n_{t+1}\right]\right],
\end{aligned} \tag{238}
$$

and our FOCs are:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t} &= \frac{1}{c_t} - \lambda_{1,t} = 0, && (239)\\
\frac{\partial \mathcal{L}}{\partial n_t} &= \alpha \lambda_{1,t} a_t n_t^{\alpha-1} - \lambda_{2,t} \\
&\quad + \beta \mathbb{E}_t \lambda_{2,t+1} \left[(1-\delta) - \xi \left(\frac{v_{t+1}}{1-(1-\delta)n_t} \right)^{-\xi-1} v_{t+1} \left(\frac{(1-\delta)v_{t+1}}{(1-(1-\delta)n_t)^2} \right) \right] = 0, && (240)\\
\frac{\partial \mathcal{L}}{\partial v_t} &= -\kappa \lambda_{1,t} + \lambda_{2,t} \left[\left(\frac{v_t}{1-(1-\delta)n_{t-1}} \right)^{-\xi} - \xi \left(\frac{v_t}{1-(1-\delta)n_{t-1}} \right)^{-\xi-1} \frac{v_t}{1-(1-\delta)n_{t-1}} \right] = 0. && (241)
\end{aligned}
$$

We can do some re-arranging and use our definitions of θ_t, $p(\theta_t)$, and $q(\theta_t)$ to get:

$$
\begin{aligned}
\lambda_{1,t} &= \frac{1}{c_t}, && (242)\\
\frac{\lambda_{2,t}}{\lambda_{1,t}} &= \alpha \frac{y_t}{n_t} + (1-\delta)\beta \mathbb{E}_t \frac{\lambda_{2,t+1}}{\lambda_{1,t}} \left[1 - \xi p(\theta_{t+1}) \right], && (243)\\
\frac{\kappa}{q(\theta_t)} &= \frac{\lambda_{2,t}}{\lambda_{1,t}} \left[1 - \xi \right], && (244)
\end{aligned}
$$

which describe the marginal utility of consumption, the marginal benefit of an additional worker, and the marginal benefit of posting a vacancy.

Combining our FOCs, we can derive an equilibrium condition. Begin by multiplying equation (243) with $\lambda_{1,t}/\lambda_{2,t}$ to obtain:

$$
1 = \frac{\lambda_{1,t}}{\lambda_{2,t}} \alpha \frac{y_t}{n_t} + (1-\delta)\beta \mathbb{E}_t \frac{\lambda_{2,t+1}}{\lambda_{2,t}} \left[1 - \xi p(\theta_{t+1}) \right], \tag{245}
$$

and note that from (244) we have:

$$
\frac{\lambda_{1,t}}{\lambda_{2,t}} = \frac{1-\xi}{\kappa / q(\theta_t)}, \tag{246}
$$

and so we combine equations (245) and (246) to get:

$$
\frac{\kappa}{q(\theta_t)} = (1-\xi)\alpha \frac{y_t}{n_t} + (1-\delta)\beta \frac{\kappa}{q(\theta_t)} \mathbb{E}_t \frac{\lambda_{2,t+1}}{\lambda_{2,t}} \left[1 - \xi p(\theta_{t+1}) \right]. \tag{247}
$$

Note that from (246) we also have:

$$\lambda_{2,t} = \frac{[\kappa/q(\theta_t)]}{1-\xi}\lambda_{1,t},$$

and so we can write (247) as:

$$\begin{aligned} \frac{\kappa}{q(\theta_t)} &= (1-\xi)\alpha\frac{y_t}{n_t} + (1-\delta)\beta\frac{\kappa}{q(\theta_t)}\mathbb{E}_t\frac{\lambda_{1,t+1}\left[\kappa/q(\theta_{t+1})\right]}{\lambda_{1,t}\left[\kappa/q(\theta_t)\right]}\left[1-\xi p(\theta_{t+1})\right] \\ \Leftrightarrow \frac{\kappa}{q(\theta_t)} &= (1-\xi)\alpha\frac{y_t}{n_t} + (1-\delta)\beta\mathbb{E}_t\frac{\lambda_{1,t+1}}{\lambda_{1,t}}\left[1-\xi p(\theta_{t+1})\right]\frac{\kappa}{q(\theta_{t+1})}. \end{aligned} \tag{248}$$

This equation is known as the job creation (JC) condition. There are a few things to note with the JC condition that we just derived. First, note that $\beta\mathbb{E}_t\frac{\lambda_{1,t+1}}{\lambda_{1,t}}$ is the stochastic discount factor or pricing kernel, $M_{t,t+1}$, that we used when we solved the firm problem in the decentralised RBC model — remember that $\lambda_{1,t}$ is nothing but the marginal utility from consumption. Secondly, the term $\left[1-\xi p(\theta_{t+1})\right]\kappa/q(\theta_{t+1})$ represents the net future benefit of a match and vacancy as the term $\xi p(\theta_{t+1})/q(\theta_{t+1}) = \xi\theta_{t+1}$ corrects for foregone search costs.

Now, the question we wish to ask is: Under what condition does a decentralised market economy replicate the Ramsey social planner allocation?

8.4.8 The decentralised equilibrium

Recall that the decentralised economy is such that each agent maximises its own objective function subject to its own constraint. We anticipate that externalities may arise since each agent fails to internalise the effect of its own choice on the economy.

Let's begin with the firms' problem: Each firm is small and takes transition probabilities as given, so the representative firm's problem is:

$$\arg\max_{\{n_t,v_t\}_{t=0}^{\infty}} \mathbb{E}_t\sum_{s=0}^{\infty}\beta^s\lambda_{1,t+s}\left(y_{t+s} - w_{t+s}n_{t+s} - \kappa v_{t+s}\right), \tag{249}$$

subject to:

$$n_t = (1-\delta)n_{t-1} + q(\theta_t)v_t, \tag{250}$$

$$y_t = a_t n_t^{\alpha}. \tag{251}$$

So, the firm wants to maximise profits subject to the law of motion of employment and its production technology.

Now, use the law of motion of employment (250) to solve for v_t and then substitute it and (251) into the firm's objective function (249) to get:

$$\arg \max_{\{n_t\}_{t=0}^{\infty}} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \lambda_{1,t+s} \left(a_{t+s} n_{t+s}^{\alpha} - w_{t+s} n_{t+s} - \frac{\kappa}{q(\theta_{t+s})} \left[n_{t+s} - (1-\delta) n_{t-1+s} \right] \right),$$

which gives us the following FOC:

$$\frac{\kappa}{q(\theta_t)} = \alpha \frac{y_t}{n_t} - w_t + (1-\delta)\beta \frac{\lambda_{1,t+1}}{\lambda_{1,t}} \frac{\kappa}{q(\theta_{t+1})}. \tag{252}$$

This equation is the JC condition for the decentralised economy. In words, it states that the expected cost of posting a vacancy (the LHS) is equal to the expected benefits that the additional vacancy takes into production (RHS). Note that no where in the solution has the firm considered the effects of changes in vacancies and unemployment on the labour market transition probabilities – the transition probabilities are entirely exogenous to the solution. Furthermore, note that the real wage w_t enters the JC condition for the firm (the Ramsey planner isn't affected by prices), and so the amount of vacancies that are posted depends on how the wage splits the surplus between workers and firms when they match.

As previously mentioned, workers and firms negotiate wages in each period according to Nash Bargaining. Recall that $\mathcal{J}_t$ is the firm's surplus and $\mathcal{W}_t$ is the worker's surplus. According to Nash bargaining, the workers take a fraction η of the total surplus:

$$\begin{aligned} \mathcal{W}_t &= \eta(\mathcal{J}_t + \mathcal{W}_t) = \eta \mathcal{S}_t \\ \implies (1-\eta)\mathcal{W}_t &= \eta \mathcal{J}_t, \end{aligned} \tag{253}$$

where (253) is called the Nash Bargaining sharing rule. This rule can be formally derived from:

$$w_t = \arg\max_{w_t} \mathcal{W}_t(w_t)^{\eta} \mathcal{J}_t(w_t)^{1-\eta},$$

with $\partial \mathcal{W}_t(w_t)/\partial w_t > 0$ and $\partial \mathcal{J}_t / \partial w_t < 0$. The solution is given by:

$$\begin{aligned} 0 &= \frac{\partial}{\partial w_t} \mathcal{W}_t(w_t)^{\eta} \mathcal{J}_t(w_t)^{1-\eta} \\ 0 &= \eta \mathcal{W}_t^{\eta-1} \mathcal{J}_t^{1-\eta} - (1-\eta)\mathcal{W}_t^{\eta} \mathcal{J}_t^{-\eta} \\ \implies (1-\eta)\mathcal{W}_t &= \eta \mathcal{J}_t. \end{aligned}$$

Define the worker's surplus from working as:

$$\mathcal{W}_t = w_t - b + (1-\delta)\beta \frac{\lambda_{1,t+1}}{\lambda_{1,t}} \left[1 - p(\theta_{t+1})\right] \mathcal{W}_{t+1}, \tag{254}$$

where b can be thought of as some unemployment benefit – or any form of value when the household is unemployed. In words this equation is a law of motion of sorts for worker surplus. It states that the surplus in period t is comprised of wages, unemployment benefits, and a discounted value of the surplus in $t+1$ discounted by the relevant probabilities (and the stochastic discount factor, of course).

The firm's surplus from a match is simply what we got from solving its problem (252):

$$\mathcal{J}_t = \frac{\kappa}{q(\theta_t)} = \alpha \frac{y_t}{n_t} - w_t + (1-\delta)\beta \frac{\lambda_{1,t+1}}{\lambda_{1,t}} \underbrace{\frac{\kappa}{q(\theta_{t+1})}}_{\mathcal{J}_{t+1}},$$

which we can also think of as a sort of arbitrage condition.

To derive the wage from Nash bargaining, recall that we had:

$$(1-\eta)\mathcal{W}_t = \eta\mathcal{J}_t,$$

which we also assumes holds for period $t+1$ too. Now, let's substitute in the expressions for $\mathcal{W}_t$ and $\mathcal{J}_t$ to get:

$$(1-\eta)\left[w_t - b + (1-\delta)\beta\frac{\lambda_{1,t+1}}{\lambda_{1,t}}\left[1-p(\theta_{t+1})\right]\mathcal{W}_{t+1}\right] = \eta\left[\alpha\frac{y_t}{n_t} - w_t + (1-\delta)\beta\frac{\lambda_{1,t+1}}{\lambda_{1,t}}\mathcal{J}_{t+1}\right], \tag{255}$$

and recall that:

$$\mathcal{J}_t = \frac{\kappa}{q(\theta_t)}, \tag{256}$$

and that the Nash Bargaining condition implies:

$$\begin{aligned} \mathcal{W}_t &= \frac{\eta}{1-\eta}\mathcal{J}_t \\ \implies \mathcal{W}_t &= \frac{\eta}{1-\eta}\frac{\kappa}{q(\theta_t)}. \end{aligned} \tag{257}$$

Substitute in (257) into (255) to get:

$$(1-\eta)\left[w_t - b + (1-\delta)M_{t,t+1}\left[1-p(\theta_{t+1})\right]\frac{\eta}{1-\eta}\frac{\kappa}{q(\theta_{t+1})}\right] = \eta\left[\alpha\frac{y_t}{n_t} - w_t + (1-\delta)M_{t,t+1}\frac{\kappa}{q(\theta_{t+1})}\right],$$

and rearrange:

$$(1-\eta)w_t - (1-\eta)b + (1-\delta)M_{t,t+1}\left[1-p(\theta_{t+1})\right]\eta\frac{\kappa}{q(\theta_{t+1})} = \eta\alpha\frac{y_t}{n_t} - \eta w_t + \eta(1-\delta)M_{t,t+1}\frac{\kappa}{q(\theta_{t+1})}$$

$$w_t = (1-\eta)b + \eta\left[\alpha\frac{y_t}{n_t} + (1-\delta)M_{t,t+1}\frac{\kappa}{q(\theta_{t+1})} - (1-\delta)M_{t,t+1}\left[1-p(\theta_{t+1})\right]\frac{\kappa}{q(\theta_{t+1})}\right]$$

$$w_t = (1-\eta)\underbrace{b}_{\text{Worker reservation wage}} + \eta\left[\underbrace{\alpha\frac{y_t}{n_t} + (1-\delta)M_{t,t+1}\frac{\kappa}{q(\theta_{t+1})}p(\theta_{t+1})}_{\text{Firm reservation wage}}\right], \tag{258}$$

where $M_{t,t+1} = \beta\frac{\mathbb{E}_t\lambda_{1,t+1}}{\lambda_{1,t}}$ is the stochastic discount factor/pricing kernel. As we can see, the wage splits the surplus according to the bargaining power of the worker, η. If the firm has all the bargaining power ($\eta \to 0$), the real wage is set equal to the minimum the worker would accept to work (b). Conversely, if the worker has all the bargaining power ($\eta \to 1$), the real wage is set equal to the outside option for the firm.

Note also that (258) contains the term $p(\theta_{t+1})/q(\theta_{t+1}) = \theta_{t+1}$ which is nothing but labour market tightness. This implies that the tighter the labour market, the higher the wage. Why? A higher θ means that there are several vacancies for a given number of unemployed workers. The labour market is tight, and a firm has to "compete" with other firms to attract a worker. But since recruiting costs are higher, this means that firms are prepared to pay a higher wage in order to find a match.

Also note that the bargained wage (258) depends on parameters which are influenced by fiscal and labour market policy: the job separation rate δ, the job creation rate $p(\theta_{t+1})$, and – perhaps most critically – the outside option of working b. The higher b is, the higher the wage.

Finally, to solve for the decentralised equilibrium, use the bargained wage (258) in the JC condition (252):

$$\frac{\kappa}{q(\theta_t)} = \alpha\frac{y_t}{n_t} - \left[(1-\eta)b + \eta\left(\alpha\frac{y_t}{n_t} + (1-\delta)M_{t,t+1}\kappa\frac{p(\theta_{t+1})}{q(\theta_{t+1})}\right)\right] + (1-\delta)M_{t,t+1}\frac{\kappa}{q(\theta_{t+1})}$$

$$\implies \frac{\kappa}{q(\theta_t)} = (1-\eta)\left[\alpha\frac{y_t}{n_t} - b\right] - (1-\delta)M_{t,t+1}\left[1-\eta p(\theta_{t+1})\right]\frac{\kappa}{q(\theta_{t+1})}. \tag{259}$$

8.4.9 The Ramsay planner equilibrium vs the market equilibrium

Are the allocations in the decentralised economy efficient? Compare the JC conditions in equilibrium for the Ramsey planner and the decentralised economy. The JC condition for the Ramsey planner

(248) is

$$\frac{\kappa}{q(\theta_t)} = (1-\xi)\alpha\frac{y_t}{n_t} + (1-\delta)M_{t,t+1}\left[1 - \xi p(\theta_{t+1})\right]\frac{\kappa}{q(\theta_{t+1})},$$

and for the market economy (assuming $b = 0$) (259),

$$\frac{\kappa}{q(\theta_t)} = (1-\eta)\alpha\frac{y_t}{n_t} - (1-\delta)M_{t,t+1}\left[1 - \eta p(\theta_{t+1})\right]\frac{\kappa}{q(\theta_{t+1})}.$$

Comparing the two JC conditions, the decentralised economy equilibrium is Pareto efficient if and only if

$$\eta = \xi.$$

In other words, when the worker's bargaining power is equal to the elasticity of the matching function with respect to unemployment (recall that $m(v_t, u_t) = v_t^{1-\xi}u_t^{\xi}$). This is known as the Hosios Condition.

8.4.10 The Hosios Condition

Is the Hosios Condition feasible? Yes, in principle, since $0 \leq \{\eta, \xi\} \leq 1$. However, it might not be satisfied in practice, as the bargaining power η might not be the same as ξ. These parameters are both very different objects. The parameter η is determined without reference to the matching function, while ξ can be interpreted as the relative weight of unemployment to matches (or, $1-\xi$ could be the relative contribution of vacancies to matches).

If ξ is high, then unemployment in the economy contributes more to generating matches than vacancies do. This implies that in equilibrium, high unemployment improves labour market efficiency, which in turn implies that the wage bargaining parameter η would be high (since high wages lead to high unemployment). This argument identifies a positive relation between ξ and η rather than absolute equality.

Recall that we talked about the insider-outsider dilemma. The wage bargaining process has two types of agents: i) Insiders: hired workers and firms that fill a vacancy, and ii) Outsiders: unemployed workers and firms with unfilled vacancies. The wage bargaining process reflects the interests of the insiders, and not the outsiders. For firms, their externality is that if hiring is too hire it makes the outside firms worse-off (tighter labour market, low vacancy filling probability). For workers, their externality is that if the labour market is loose, it makes unemployed workers outside the bargaining contract worse-off (slack labour market, low job finding probability).

In other words, the insiders ignore the interests of the outsiders, yet the wage decisions affect the allocations of the outsiders. If unemployed workers and firms without matches were given the chance to participate in the bargaining process, they would choose a wage rule that delivers the allocations chosen by the Ramsey planner.

We can use the JC condition and the Beveridge curve to visualise the efficient level of unemployment in (u, v) space. Recall that the decentralised JC condition (259):

$$\frac{\kappa}{q(\theta_t)} = (1-\eta)\alpha\frac{y_t}{n_t} - (1-\delta)M_{t,t+1}\left[1-\eta p(\theta_{t+1})\right]\frac{\kappa}{q(\theta_{t+1})},$$

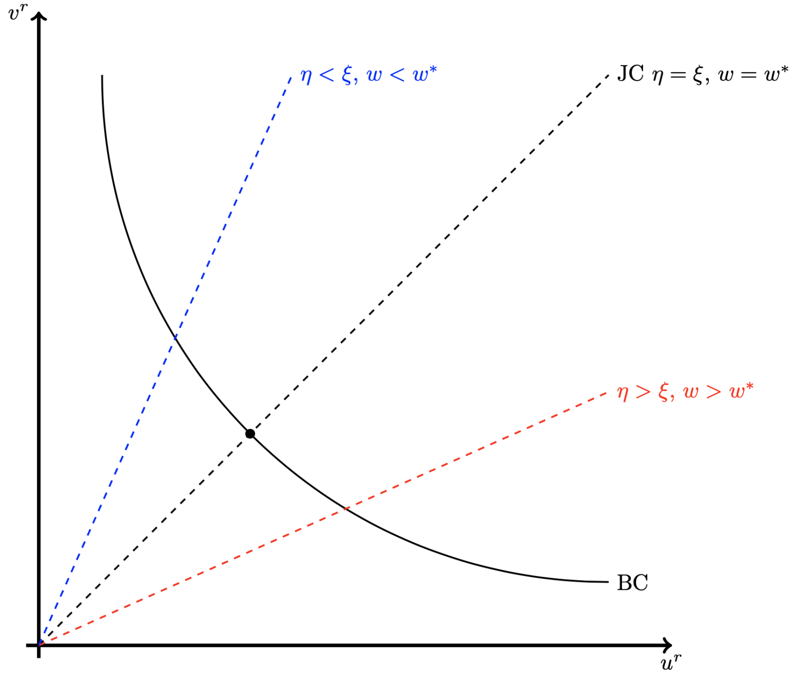
and at the steady state it becomes:

$$\frac{\kappa}{q(\theta)} = (1-\eta)\alpha\frac{y}{n} + \beta(1-\delta)\left[1-\eta p(\theta)\right]\frac{\kappa}{q(\theta)}.$$

Plotting the JC condition and Beveridge curve we get Figure 42. How do the plots of the Ramsey planner equilibrium and market equilibrium differ?

- If $\eta = \xi$: the market wage is equal to the Pareto optimum wage, and the amount of vacancies and unemployment are efficient;
- If $\eta < \xi$: the market wage is lower than the Pareto optimum wage, and therefore vacancy posting is suboptimally high and unemployment is suboptimally low; and
- If $\eta > \xi$: the market wage is suboptimally higher than the Pareto optimum wage, and therefore vacancy posting is suboptimally low and unemployment is suboptimally high.

Figure 42: Plot of the Beveridge Curve and Job Creation Condition



8.4.11 Labour market policies

Fiscal policy and labour market policies can offset the inefficient distortions of wage bargaining.

For example, consider the duration of filling a vacancy:

$$\frac{1}{q(\theta_t)} = \theta_t^{\xi}.$$

If the duration of filling a vacancy is high then firms are causing more congestion to other firms posting vacancies. In other words, the vacancy posting rate is too high. In this case, it may be efficient to "tax" the firm with a higher wage, implemented by increasing η. Suppose the government levies a wage tax, τ, on firms so that the effective wage bill becomes $(1+\tau)w_t$. The JC condition for the market economy (252) becomes:

$$\frac{\kappa}{q(\theta_t)} = \alpha\frac{y_t}{n_t} - (1+\tau)w_t + (1-\delta)\beta\frac{\lambda_{1,t+1}}{\lambda_{1,t}}\frac{\kappa}{q(\theta_{t+1})}, \tag{260}$$

and the wage equation (258) becomes:

$$(1-\eta)\left\{w_t - b + (1-\delta)M_{t,t+1}\left[1-p(\theta_{t+1})\right]\frac{\eta}{1-\eta}\frac{\kappa}{q(\theta_{t+1})}\right\} = \eta\left\{\begin{array}{c}\alpha\frac{y_t}{n_t} - (1+\tau)w_t \\ +(1-\delta)M_{t,t+1}\frac{\kappa}{q(\theta_{t+1})}\end{array}\right\}$$

$$(1-\eta)\left\{(1-\delta)M_{t,t+1}\left[1-p(\theta_{t+1})\right]\frac{\eta}{1-\eta}\frac{\kappa}{q(\theta_{t+1})} - b\right\} + w_t\left(1+\tau\eta\right) = \eta\left\{\begin{array}{c}\alpha\frac{y_t}{n_t} \\ +(1-\delta)M_{t,t+1}\frac{\kappa}{q(\theta_{t+1})}\end{array}\right\},$$

and with a bit cleaning up:

$$w_t = \frac{1-\eta}{1+\tau\eta}b + \frac{\eta}{1+\tau\eta}\left[\alpha\frac{y_t}{n_t} + (1-\delta)M_{t,t+1}\kappa\frac{p(\theta_{t+1})}{q(\theta_{t+1})}\right]. \tag{261}$$

Using (261) (and assuming that $b = 0$), and substituting it into the JC condition (260) yields:

$$\frac{\kappa}{q(\theta_t)} = \left(1 - \frac{1+\tau}{1+\eta\tau}\eta\right)\alpha\frac{y_t}{n_t} + (1-\delta)M_{t,t+1}\left[1 - \frac{1+\tau}{1+\tau\eta}\eta p(\theta_{t+1})\right]\frac{\kappa}{q(\theta_{t+1})}. \tag{262}$$

Recall that the Ramsey planner equilibrium (248) was:

$$\frac{\kappa}{q(\theta_t)} = (1-\xi)\alpha\frac{y_t}{n_t} + (1-\delta)\beta\mathbb{E}_t\frac{\lambda_{1,t+1}}{\lambda_{1,t}}\left[1-\xi p(\theta_{t+1})\right]\frac{\kappa}{q(\theta_{t+1})}.$$

Comparing the market equilibrium with taxes to the Ramsey planner equilibrium reveals that the efficiency condition is:

$$\xi = \frac{1+\tau}{1+\tau\eta}\eta,$$

which is associated with the tax equal to:

$$\tau = \frac{\xi-\eta}{\eta(1-\xi)}. \tag{263}$$

Equation (263) shows that if $\eta = \xi \implies \tau = 0$. In other words, the market equilibrium is efficient and there is no need to tax labour. But if $\eta < \xi$, then wages are too low which then implies a positive τ is optimal (a tax on labour) – there are too many job postings, so taxes should be put in place to disincentivise vacancy postings and increase unemployment. If $\eta > \xi$, then wages are set too high which implies $\tau < 0$ be optimal (subsidy on labour). A wage subsidy stimulates vacancy postings and decreases unemployment, achieving efficiency.

8.4.12 Endogenous job separation

Up until now our search and match model assumed the job separation rate δ to be exogenous. But we know that this parameter is in fact endogenous – it is determined by firms and it changes over the business cycle. In this section we allow firms to optimally choose the job separation rate to retain profitability. This endogenous job separation will be important to explain shifts in the Beveridge curve.

To illustrate the model with endogenous separation, we make the following assumptions: $\alpha = 1$, so that there is constant returns to scale for production in labour; there is no bargaining so $w_t = b$ (i.e., $\eta = 0$ in Nash Bargaining); and, there is no exogenous job separation $\delta = 0$.

Assume that the productivity of a firm i is comprised of aggregate productivity a and job specific productivity $\sigma\epsilon_i$:

$$y_i = a_t n_{i,t} \sigma_t \epsilon_{i,t}, \tag{264}$$

where $\sigma > 0$ and $\epsilon_i \sim F(\cdot)$ which is common for all firms and has support $\epsilon_i \in [\underline{\epsilon}, \overline{\epsilon}]$. In each period, there is a new draw of ϵ from $F(\cdot)$. Assume the first draw $\epsilon = \overline{\epsilon}$ to ensure that a new job is never destroyed, but it will not stay there.

The JC condition of the economy is similar to before:

$$\mathcal{J}(\overline{\epsilon}) = \frac{\kappa}{q(\theta)}, \tag{265}$$

where $\mathcal{J}(\overline{\epsilon})$ for the assumption $\epsilon = \overline{\epsilon}$. To characterise the JC condition in presence of endogenous separation, we need to further define $\mathcal{J}(\overline{\epsilon})$. Our assumptions deliver the steady-state firm's surplus for an active job with productivity ϵ equal to:

$$\mathcal{J}(\epsilon) = a\sigma\epsilon - b + \beta \int_{\underline{\epsilon}}^{\overline{\epsilon}} \mathcal{J}(x) dF(x), \tag{266}$$

where x is the draw of the new job-specific productivity shock from $F(\epsilon)$. Note that:

$$\mathbb{E}\left[\mathcal{J}(x)\right] = \int_{\underline{\epsilon}}^{\overline{\epsilon}} \mathcal{J}(x) dF(x) > 0, \tag{267}$$

which is the expected value of a job after a draw x from the distribution $F(\epsilon)$. $\mathbb{E}[\mathcal{J}(x)]$ is positive since if negative the firm destroys the job and posts a vacancy (i.e., ex-ante $\mathbb{E}[\mathcal{J}(x)]$ is never negative). To obtain $\mathcal{J}(\overline{\epsilon})$ in the JC condition (265) we need to determine $\mathbb{E}[\mathcal{J}(x)]$ and evaluate $\mathcal{J}(\epsilon)$ at $\overline{\epsilon}$.

$\mathcal{J}(\epsilon)$ is monotonic in ϵ, i.e.,

$$\frac{d\mathcal{J}(\epsilon)}{d\epsilon} = \frac{a\sigma}{1-\beta} > 0.$$

Job destruction is characterised by a cutoff rule:

$$\exists \epsilon_d : \epsilon < \epsilon_d \rightarrow \text{Separation}.$$

The cutoff threshold ϵ_d is determined by the condition:

$$\mathcal{J}(\epsilon_d) = 0, \tag{268}$$

and we can use this condition to determine our object of interest, $\mathbb{E}[\mathcal{J}(x)]$.

The value of the marginal job is:

$$\mathcal{J}(\epsilon_d) = a\sigma\epsilon_d - b + \beta\mathbb{E}[\mathcal{J}(x)].$$

From the job destruction condition (268) we have:

$$0 = a\sigma\epsilon_d - b + \beta\mathbb{E}[\mathcal{J}(x)],$$

and thus we derive:

$$\mathbb{E}[\mathcal{J}(x)] = \frac{b - a\sigma\epsilon_d}{\beta} > 0, \tag{269}$$

since ex-ante this is always positive. Therefore, the flow of profit of the marginal job, $a\sigma\epsilon_d - b$, must be negative for the job to be destroyed.

What determines $\mathcal{J}(\bar{\epsilon})$? Recall the steady state firm surplus with productivity ϵ (266):

$$\mathcal{J}(\epsilon) = a\sigma\epsilon - b + \beta\mathbb{E}[\mathcal{J}(x)],$$

and we just found $\mathbb{E}[\mathcal{J}(x)]$, so substitute that into (266) to get:

$$\mathcal{J}(\epsilon) = a\sigma(\epsilon - \epsilon_d),$$

and if $\epsilon = \bar{\epsilon}$, $\mathcal{J}(\bar{\epsilon})$ becomes:

$$\mathcal{J}(\bar{\epsilon}) = a\sigma(\bar{\epsilon} - \epsilon_d),$$

and substitute this into the job creation condition (265) to get:

$$\begin{aligned} a\sigma(\bar{\epsilon} - \epsilon_d) &= \frac{\kappa}{q(\theta)} \\ \Leftrightarrow q(\theta) &= \frac{\kappa}{a\sigma(\bar{\epsilon} - \epsilon_d)}. \end{aligned} \tag{270}$$

So job separation matters for job creation and is important for job turnover. If the job separation increases ($\epsilon_d \uparrow$), the probability of filling a vacancy increases (since there are unemployed workers) and therefore there is a larger market turnover (as we saw in the data).

Now, we turn to the Beveridge curve under endogenous job separation. The law of motion of employment becomes:

$$n_t = (1 - F(\epsilon_d))n_{t-1} + q(\theta_t)v_t, \tag{271}$$

where $F(\epsilon_d)$ is the endogenous separation rate. At the steady state (recall $n = 1-u$ and $q(\theta)v = p(\theta)u$)

we have:

$$\begin{aligned} n &= (1 - F(\epsilon_d))n + q(\theta)v \\ \implies 1 - u &= (1 - F(\epsilon_d))(1 - u) + p(\theta)u, \end{aligned} \tag{272}$$

which implies the steady state level of unemployment:

$$u = \frac{F(\epsilon_d)}{F(\epsilon_d) + p(\theta)}. \tag{273}$$

This equation is the Beveridge curve with endogenous job separation. An important implication of this Beveridge curve is that changes in the threshold of job-specific productivity shift it. For example, an increase in a decreases ϵ_d and therefore $F(\epsilon_d)$ falls. The Beveridge curve shifts inward, and unemployment is lower for any given level of job creation. Recent research shows that endogenous job separation generates significant non-linearities in the fluctuations of labour market variables over the business cycle.[55]

How does endogenous job separation affect efficiency? The Hosios condition continues to hold[56], although the details are beyond the scope of this course. The key intuition is that job separation is a structural feature of the economy, equally internalised by the market economy and the Ramsey planner. When the Hosios condition holds, the market economy sets the same job-specific productivity threshold and bargaining is efficient as in an economy with the Ramsey social planner.

8.5 Comments and key readings

There is still no consensus about how to model the labour market in a neoclassical framework. Increasing attention is being placed on search and match models, but this research strategy is still too recent to assess its performance. A number of studies which compare labour market models suggest that the key to understanding the cyclical behaviour of labour markets is to try and understand which margins a firm can adjust freely and which the firm has to treat as fixed. In the search and match

[55] See Pizzinelli et al. (2020).
[56] See Mortensen and Pissarides (1999)

models, the firm can choose vacancies freely but inherits an employment stock. Understanding these rigidities and their causes and the optimal responses of firms and workers to these rigidities is clearly a crucial issue.

The other important thing to notice about most of the models in this lecture is that they seek to improve the performance of the RBC model by introducing additional shocks: the most promising search and match models have aggregate and allocative shocks. This is increasingly how RBC models are being developed, with widespread opinion being that productivity shocks alone cannot explain business cycle fluctuations. Essentially, these models are trying to explain the zero correlation between wages and employment over the business cycle by letting both the labour supply and demand curves shift.

Key readings were mentioned throughout this chapter. For indivisible labour see Hansen's 1985 *JME* paper which introduced indivisible labour, Rogerson (1988) "Indivisible Labor, Lotteries and Equilibrium" is also good, and the textbook *ABCs of RBCs* by McCandless (2008) gives a very thorough treatment. McCandless also goes through linear quadratic dynamic programming which was used by both Hansen (1985) and Kydland and Prescott (1982).

Advanced Macroeconomics by Romer (2012) gives a good rundown of efficiency wages, including the Shapiro-Stiglitz model. These notes were based primarily on Romer's material.

The literature on search and match models have been increasing at a fairly rapid pace since Diamond, Mortensen, and Pissarides were awarded the Nobel prize in 2010. Specific papers are "The Beveridge Curve" Blanchard et al. (1989), "Aggregate Demand Management in Search Equilibrium" Diamond (1982*a*), "Wage Determination and Efficiency in Search Equilibrium" Diamond (1982*b*), "Job Creation and Job Destruction in the Theory of Unemployment" Mortensen and Pissarides (1994), and *Equilibrium Unemployment Theory* by Pissarides (2000).

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9 Real Dynamics in the RBC Model

9.1 Introduction

This is the final chapter in which we look at ways of amending and extending the baseline RBC model in order to improve its performance. We've looked at alternative ways of modelling the labour market (indivisible labour, efficiency wages, and search and match), and we saw how they significantly improve the performance of the RBC model. However, on their own, they are still not sufficient, so RBC models (and other current DSGE models) usually incorporate additional sources of dynamics to help them come close to the data. There is considerable controversy about this approach because each extra source of dynamics introduces additional free parameters into the model which can be criticised as being ad hoc. Lucas warned to "beware economists bearing free parameters," and there is some discomfort amongst macroeconomists that these additional sources of dynamics are not immune to the Lucas critique. In other words, there is inherent danger in adding mechanisms to match the data because these mechanisms may change once policy changes. With that said, in this section we will be looking at:

1. Habits: The assumption in these models is that consumers gain more utility from consuming in the current period if current consumption is close to that in the previous period. This means consumers tend to smooth consumption even more than before, which adds inertia and makes consumption smoothing an even stronger propagation mechanism.

2. Adjustment costs: Data from engineering studies suggests that it is not possible to adjust capital instantaneously – Rome wasn't build in a day. This is usually modelled as a cost of changing the capital stock or a cost of changing the level of investment. These two assumptions have different implications for the response of the economy to technology shocks.

3. Investment specific technological change: Recent work argues that aggregate technology shocks are not the prime impulse for business cycle dynamics. Instead, the key impulse is a shock to the rate at which output goods are converted into productive capital. It is argued that these shocks induce dynamics that are more consistent with observed data.

Some of the topics and concepts introduced in this chapter will be revisited when we look at macro-finance – the modifications we make to the RBC model here have important implications to explaining the equity premium puzzle. But without further ado, let's begin our last effort of patching up the RBC model.

9.2 Habits

We will first look at a general model of habits inspired by Abel (1990) and Galí (1994). It is assumed that the preferences of the representative agent are defined over consumption c_t relative to a preference parameter v_t:

$$\frac{1}{1-\sigma}\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left(\frac{c_{t+s}}{v_{t+s}}\right)^{1-\sigma},$$

where the preference parameter v_t represents the individual's habit. The most common habit model defines v_t by:

$$v_t = \left(c_{t-1}^D C_{t-1}^{1-D}\right)^\gamma,$$

with $\gamma > 0$ and $0 < D < 1$. c_{t-1} is individual consumption in the pervious period and C_{t-1} is aggregate consumption in the previous period. The idea is that preferences are a function of a weighted average of what happened in the previous period.[57] If $D = 1$ then utility is defined as consumption relative to what the individual did in the previous period, which is arguably the most natural interpretation of what it means for a consumer to have a habit. This is known as the internal habit and means that the individual prefers their consumption in the current period to be as close as possible relative to their consumption in the previous period. In contrast, when $D = 0$, utility is defined as consumption relative to what happed on aggregate in the previous period. This is an external habit and implies that an individual prefers their consumption to be close to average consumption in the previous period – a sort of "catching up with the Joneses" effect. Whether habits are internal or external has important implications for the consumption Euler equation and for consumption and aggregate dynamics. If

[57] We could write utility as:

$$U_t = \frac{c_t^{1-\sigma} C_t^\gamma C_{t-1}^\lambda}{1-\sigma},$$

which would show effects of both "catching up and keeping up with the Joneses".

habits are internal then the consumer internalises the effect of current period consumption decisions on their habit next period (i.e., the consumer knows that its current period decision affects habits in future periods); if habits are external this effect is absent.

As an alternative, it is possible to define the habit relative to individual or aggregate consumption in the current period:

$$v_t = \left(c_t^D C_t^{1-D}\right)^\gamma .$$

These models are known as "keeping up with the Joneses" as habits and decisions are formed contemporaneously. The remainder of these notes will focus on "catching up" habits.

The marginal utility of consumption in the general habit model can be calculated by substituting the expression for v_t into the utility maximisation problem, and then differentiating with respect to c_t:

$$\begin{aligned}
\frac{\partial U_t}{\partial c_t} &= \frac{\partial}{\partial c_t}\left\{\frac{1}{1-\sigma}\left[\frac{c_t}{\left(c_{t-1}^D C_{t-1}^{1-D}\right)^\gamma}\right]^{1-\sigma} + \frac{\beta}{1-\sigma}\mathbb{E}_t\left[\frac{c_{t+1}}{\left(c_t^D C_t^{1-D}\right)^\gamma}\right]^{1-\sigma}\right\} \\
&= \left(\frac{c_t}{\left(c_{t-1}^D C_{t-1}^{1-D}\right)^\gamma}\right)^{-\sigma}\frac{1}{\left(c_{t-1}^D C_{t-1}^{1-D}\right)^\gamma} + \beta\mathbb{E}_t\left[\frac{c_{t+1}}{\left(c_t^D C_t^{1-D}\right)^\gamma}\right]^{-\sigma}\left(-D\gamma\frac{c_{t+1}}{c_t^{1+D\gamma}C_t^{(1-D)\gamma}}\right) \\
&= \left(\frac{c_t}{v_t}\right)^{-\sigma}\frac{1}{v_t} - \beta D\gamma\mathbb{E}_t\left[\left(\frac{c_{t+1}}{v_{t+1}}\right)^{-\sigma}\frac{c_{t+1}}{v_{t+1}}\frac{1}{c_t}\right] \\
\Leftrightarrow \lambda_t &= \underbrace{\left(\frac{c_t}{v_t}\right)^{1-\sigma}\frac{1}{c_t}}_{U_{c,t}} - \beta D\gamma\mathbb{E}_t\left[\underbrace{\left(\frac{c_{t+1}}{v_{t+1}}\right)^{1-\sigma}\frac{1}{v_{t+1}}}_{U_{v,t+1}}\frac{v_{t+1}}{c_t}\right],
\end{aligned} \tag{274}$$

where to get from the third to the fourth line, I multiply the first term on the RHS by c_t/c_t and the second term by v_{t+1}/v_{t+1}. Note that (274) is nothing but the marginal utility of individual consumption. The marginal utility of consumption appears in the now-familiar intertemporal Euler equation:

$$\mathbb{E}_t\left[\beta R_{t+1}\frac{\lambda_{t+1}}{\lambda_t}\right] = 1,$$

which when log-linearised becomes:

$$\hat{\lambda}_t = \mathbb{E}_t \hat{\lambda}_{t+1} + \mathbb{E}\hat{R}_{t+1}. \tag{275}$$

In the time separable case $\gamma = 0$, $v_t = 1$, so $\lambda_t = c_t^{-\sigma}$ and $\hat{\lambda}_t = -\sigma \hat{c}_t$. As usual, the consumption Euler equation is:

$$\hat{c}_t = \mathbb{E}_t \hat{c}_{t+1} - \frac{1}{\sigma}\mathbb{E}_t \hat{R}_{t+1}.$$

When habits are external we have $\gamma > 0$, $D = 0$, $v_t = C_{t-1}^{\gamma}$, and so $\lambda_t = c_t^{-\sigma} C_{t-1}^{\gamma(\sigma-1)}$. This is the assumption in Smets and Wouters (2007). In equilibrium, $c_t = C_t$ so the marginal utility of consumption is

$$\lambda_t = c_t^{-\sigma} c_{t-1}^{\gamma(\sigma-1)},$$

and

$$\hat{\lambda}_t = -\sigma \hat{c}_t + \gamma(\sigma - 1)\hat{c}_{t-1}.$$

The consumption Euler equation is then:

$$\hat{c}_t = \frac{\gamma(\sigma-1)}{\gamma(\sigma-1)+\sigma}\hat{c}_{t-1} + \frac{\sigma}{\gamma(\sigma-1)+\sigma}\mathbb{E}_t \hat{c}_{t+1} - \frac{1}{\gamma(\sigma-1)+\sigma}\mathbb{E}_t \hat{R}_{t+1}. \tag{276}$$

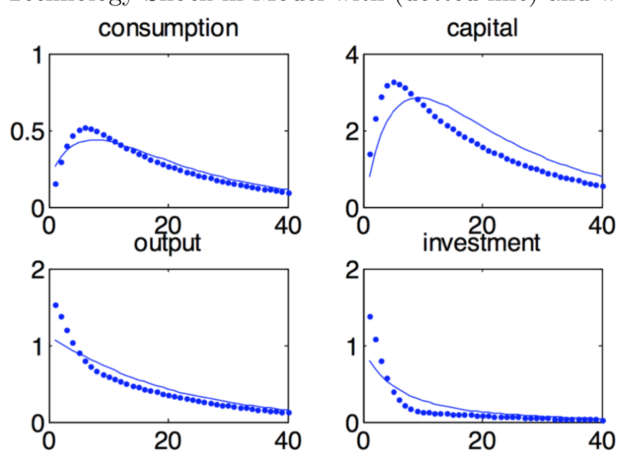
This is the consumption Euler equation under external habit formation. We have terms in $\hat{c}_{t-1}$ and $\mathbb{E}_t \hat{c}_{t+1}$ on the RHS so consumption decisions are both backward and forward looking. The backward-looking component is driven by the external habit, with the coefficients on the backward and forward looking terms summing to unity. Current consumption also reacts less to expectations of the real interest rate $\mathbb{E}_t \hat{R}_{t+1}$ under external habits – as the consumer prefers to keep consumption this period close to consumption in the previous period (the reference point for habits), and so the change in consumption for a given expected real interest rate will be less. Close examination of (276) shows that the habit terms drop out if preferences are logarithmic. This is a feature of our preferences specification, because with logarithmic preferences and $D = 0$, the per-period utility function becomes:

$$\log\left(\frac{c_t}{v_t}\right) = \log c - \gamma \log C_{t-1},$$

and individual and aggregate (lagged) consumption are additively separable.

Adding external habits to an RBC model potentially changes its dynamics significantly. There are externalities in this model (running from individual to aggregate consumption) so we need to solve for the decentralised equilibrium rather than the Ramsey planner's problem. Figure 43 shows IRFs for a technology shock in models with (the dotted line) and without (the solid line) habits. The most obvious difference in the responses is that consumption does not increase as quickly on impact in the model with habits. Intuitively, the consumer is initially reluctant to increase consumption from its steady state value because that would involve moving away from their habits. Once they do start to increase consumption, though, there is an additional incentive to keep consumption high (the consumer has developed new habits) so consumption rises to a higher level under habits than without.

Figure 43: Response to Technology Shock in Model with (dotted line) and without (solid line) Habits



When habits are internal we have $\gamma > 0$, $D = 1$, $v_t = c_{t-1}^{\gamma}$. This is the assumption in Christiano et al. (2005). We can use (274) to write:

$$\lambda_t = c_t^{-\sigma} c_{t-1}^{\gamma(\sigma-1)} - \beta\gamma \mathbb{E}_t \left[c_{t+1}^{1-\sigma} c_t^{\gamma(\sigma-1)-1} \right],$$

which when log-linearised implies:

$$\hat{\lambda}_t = \frac{\gamma(\sigma-1)}{1-\beta\gamma} \hat{c}_{t-1} - \frac{\sigma + \beta\gamma(\gamma(\sigma-1)-1)}{1-\beta\gamma} \hat{c}_t + \frac{\beta\gamma(\sigma-1)}{1-\beta\gamma} \hat{c}_{t+1}.$$

Substituting into the log-linearised intertemporal Euler equation (275) gives: the Euler equation under internal habits:

$$\begin{aligned} \hat{c}_t = & \frac{\gamma(\sigma-1)}{\sigma + \beta\gamma(\gamma(\sigma-1)-1) + \gamma(\sigma-1)} \hat{c}_{t-1} + \frac{\beta\gamma(\sigma-1) + \sigma + \beta\gamma(\gamma(\sigma-1)-1)}{\sigma + \beta\gamma(\gamma(\sigma-1)-1) + \gamma(\sigma-1)} \mathbb{E}_t \hat{c}_{t+1} \\ & - \frac{\beta\gamma(\sigma-1)}{\sigma + \beta\gamma(\gamma(\sigma-1)-1) + \gamma(\sigma-1)} \mathbb{E}_t \hat{c}_{t+2} - \frac{1-\beta\gamma}{\sigma + \beta\gamma(\gamma(\sigma-1)-1) + \gamma(\sigma-1)} \mathbb{E}_t \hat{R}_{t+1}. \end{aligned}$$

The dynamics under internal habits are richer than those under external habits. we now have terms in $\hat{c}_{t-1}$, $\mathbb{E}_t \hat{c}_{t+1}$, and $\mathbb{E}_t \hat{c}_{t+2}$ as the consumer recognises that i) this period's consumption choice needs to be close to last period's consumption choice because of the habit, and ii) next period's consumption choice will become the habit in the following period. Once again, the sum of the coefficients on the three terms sum to unity.

In all these models of habits, we have worked with a representative agent and a single consumption good. This is not entirely satisfactory as one might imagine habits to work best at the individual product level. For example, it is easier to imagine the consumer becoming addicted to cigarettes or fried green tomatoes than become addicted to consumption per se. The paper by Ravn et al. (2006) introduces "deep habits" that are defined at product rather than aggregate level. This is an obvious step forward, though for analytical tractability they are forced to rely on external habits at the product level. In other words, a consumers becomes addicted to cigarettes because everyone else consumes them. This is not ideal but at present there are no tractable models of internal habits at the product level.

9.3 Adjustment costs

The view that adjustment costs are important in explaining economic fluctuations goes back to Tobin's historical q paper (1977), with its focus on the relationship between the market value of installed capital and the replacement cost of capital. Indeed, in the simple RBC model, with output costless to transform between consumption goods and productive capital, there is no reason for Tobin's q to deviate from unity. Models with adjustment costs break this feature by assuming that it takes resources to transform output goods into productive capital. Exactly what form these costs take is essentially an engineering question and depends on the mechanics of production and management processes.

We begin by looking at a model where the firm finds it costly to change the level of investment from one period to another. This investment adjustment cost model is very popular and is used in Christiano et al. (2005) and Smets and Wouters (2007). One possible motivation is that the firm has an investment department which has a certain capacity to plan and implement investments. If actual investments deviate from this norm, then costs are incurred. To see this in a simple model, define $i_t = A_t k_t^{\alpha} - c_t$ as the quantity of output made available for transformation into productive capital. In a model without adjustment costs the increase in productive capital would simply be i_t. But in the investment adjustment cost model, it is assumed that only a fraction, $1 - s(i_t/i_{t-1})$, of the output made available actually gets transformed into productive capital.

The function $s(i_t/i_{t-1})$ is assumed to satisfy $s(1) = s'(1) = 0$ and $s''(1) = \kappa$. Under these restriction we have that the investment adjustment cost is an increasing convex function of how much this period's investment deviates from last period's investment. In the steady state $i_t = i_{t-1}$ so there are no steady state adjustment costs. There are no externalities in the model with adjustment costs so equilibrium can be solved as a Ramsey planner's problem:

$$\max \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \frac{c_{t+s}^{1-\sigma}}{1-\sigma},$$

subject to

$$k_{t+1} = (1-\delta)k_t + \left[1 - s\left(\frac{i_t}{i_{t-1}}\right)\right] i_t,$$
$$i_t = A_t k_t^{\alpha} - c_t.$$

We solve this problem the standard way. Use a Lagrangian with λ_t as the multiplier on the resource constraint and μ_t as the multiplier on the investment equation. The FOC with respect to i_t is:

$$\lambda_t \left[1 - s\left(\frac{i_t}{i_{t-1}}\right)\right] - \lambda_t s'\left(\frac{i_t}{i_{t-1}}\right)\frac{i_t}{i_{t-1}} + \beta \mathbb{E}_t \lambda_{t+1} s'\left(\frac{i_{t+1}}{i_t}\right)\left(\frac{i_{t+1}}{i_t}\right)^2 = \mu_t, \tag{277}$$

which is a non-linear equation in λ_t, λ_{t+1}, μ_t, i_{t-1}, i_t, and i_{t+1}. To take a log-linear approximation, it is useful to write:

$$\frac{\mu_t}{\lambda_t} = 1 - s\left(\frac{i_t}{i_{t-1}}\right) - s'\left(\frac{i_t}{i_{t-1}}\right)\frac{i_t}{i_{t-1}} + \beta \mathbb{E}_t \frac{\lambda_{t+1}}{\lambda_t} s'\left(\frac{i_{t+1}}{i_t}\right)\left(\frac{i_{t+1}}{i_t}\right)^2,$$

and use the properties of $s(i_t/i_{t-1})$ to give the log-linearised FOC for investment:

$$\hat{i}_t = \frac{1}{1+\beta}\hat{i}_{t-1} + \frac{\beta}{1+\beta}\mathbb{E}_t \hat{i}_{t+1} + \frac{1}{\kappa(1+\beta)}\hat{q}_t, \tag{278}$$

where q_t is Tobin's q and is the shadow price of installed capital, $\hat{q} = \hat{\lambda}_t - \hat{\mu}_t$. The presence of investment adjustment costs introduces inertia in investment, as reflected by the lagged investment term. The investment decision also becomes forward looking, as it is costly to change the level of investment. The larger the parameter κ, the less sensitive current investment is to the shadow value of installed capital.

An alternative specification is the capital adjustment costs model that assumes the fraction of output available for investment that is transformed into productive capital is $1 - s(i_t/k_t)$. The function $s(\cdot)$ is assumed to satisfy $s(\omega) = s'(\omega) = 0$ and $s''(\omega) = \epsilon$, where ω is the steady state investment to capital ratio. This capital adjustment cost model can be motivated by thinking that it is increasingly difficult for a firm to make large than small investments. The cost therefore depends on the quantity of investment each period rather than the change in investment. Dividing through by k_t provides a useful scaling of the function and ensures neat and intuitive FOCs. In steady state $\omega = i_t/k_t$ so there are no steady state adjustment costs. The Ramsey planner solves:

$$\max \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \frac{c_{t+s}^{1-\sigma}}{1-\sigma},$$

subject to:

$$\begin{aligned} k_{t+1} &= (1-\delta)k_t + \left[1 - s\left(\frac{i_t}{k_t}\right)\right] i_t, \\ i_t &= A_t k_t^{\alpha} - c_t. \end{aligned}$$

Setting up the Lagrangian, the FOC with respect to i is:

$$\lambda_t \left[1 - s\left(\frac{i_t}{k_t}\right)\right] - \lambda_t s'\left(\frac{i_t}{k_t}\right)\frac{i_t}{k_t} = \mu_t. \tag{279}$$

Tobin's q is again the shadow price of installed capital, $q_t = \lambda_t/\mu_t$, and so:

$$q_t \left[1 - s\left(\frac{i_t}{k_t}\right) - s'\left(\frac{i_t}{k_t}\right)\frac{i_t}{k_t}\right] = 1,$$

which when log-linearised becomes:

$$\hat{i}_t = \hat{k}_t + \frac{1}{\epsilon\omega^2}\hat{q}_t. \tag{280}$$

In contrast to investment adjustment costs, investment now responds immediately to movements in the current shadow price of capital, $\hat{q}_t$. Hence, capital adjustment costs are not in themselves able to generate the sort of inertia in investments observed in the data.

9.4 Investment specific productivity shocks

The final mechanism we examine to change the real dynamics of the model is investment specific productivity shocks. The productivity shocks in all the models we have examined so far have been neutral in the sense that they increase the productivity of resources when making either consumption or productive capital. This improvement in productivity has been behind the IRFs which show consumption, capital, labour, output, and investment all rising after a technology shock. However, there is evidence that productivity shocks are not as neutral as this. In particular, there may be shocks which affect the ability of firms to change output goods into productive capital. These investment specific technology shocks impact on the relative efficiency with which firms can transform output into consumption goods and productive capital. After a positive shock to investment specific technology we would expect to see the economy shift from production of consumption goods to production of productive capital. Investment specific technology shocks do not introduce externalities, so we can illustrate their effects by solving the Ramsey planner's problem:

$$\max \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s U(c_{t+s}, l_{t+s}),$$

subject to:

$$\begin{aligned} c_t + i_t &= A_t k_t^{\alpha} l_t^{1-\alpha}, \\ k_{t+1} &= (1-\delta)k_t + v_t i_t, \\ \log A_{t+1} &= \rho \log A_t + \epsilon_{A,t}, \\ \log v_{t+1} &= \tau \log v_t + \epsilon_{v,t}, \end{aligned}$$

where A_t is the neutral productivity shock and v_t is the investment specific productivity shock. The IRFs of this economy are shown below:

Figure 44: Response to Neutral Technology Shock ($\epsilon_{A,t}$)

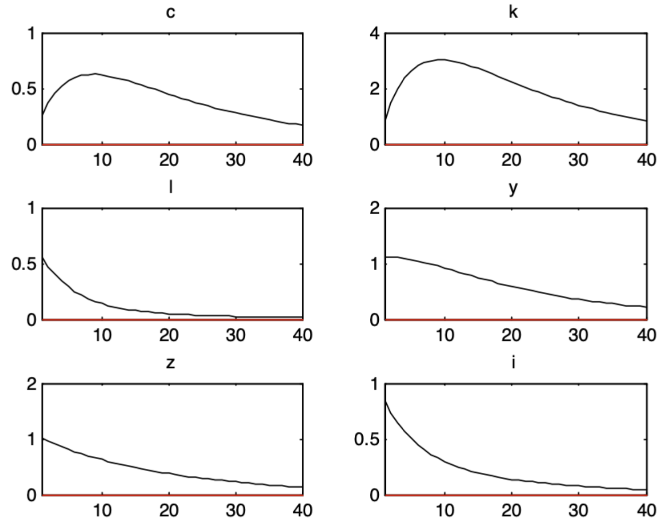
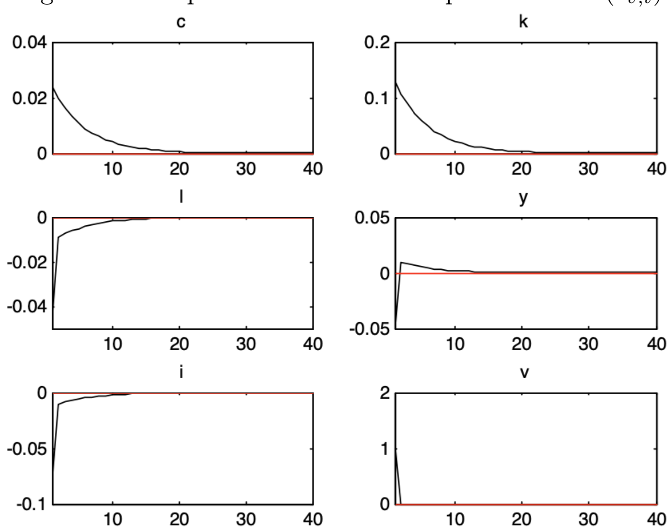


Figure 45: Response to Investment-Specific Shock ($\epsilon_{v,t}$)



The response to a neutral technology shock is as in the baseline RBC model, with labour, output, investment, capital, and consumption all rising after a positive shock.

The response of the model to an investment-specific technology shock is more nuanced: $v_t i_t$ rises but investment i_t falls as it becomes temporarily more efficient to turn output goods into productive capital than consumption goods. That consumption and investment move in opposite directions has the potential to better fit observed data. Indeed, the paper by Fisher (2006) shows that a combination of neutral and investment specific technology shocks can explain 73% of the variation in hours and 44% of the variation in output before 1982, and 38% of the variation in hours and 80% of the variation in output afterwards. These numbers are considerably higher than in corresponding models with only neutral technology shocks, and have reignited the debate on the relative role of supply and demand shocks in driving the business cycle. The paper by Justiniano and Primiceri (2008) further contributes to this discussion by estimating a DSGE model with stochastic volatility and finds that reduced investment specific technology shocks play a very important role in the Great Moderation of US business cycles seen from 1984 to 2007.

9.5 Comments and key readings

This concludes our focus on improving the performance of the baseline RBC model. The subsections in this chapter are quite self-contained, and have the relevant references within. However, for convenience, the main references are: "Asset Prices under Habit Formation and Catching up with the Joneses" by Abel (1990), "Keeping Up with the Joneses: Consumption Externalities, Portfolio Choice, and Asset Prices" by Galí (1994), and "Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy" by Christiano et al. (2005). *Advanced Macroeconomics* by Romer (2012) also offers a good treatment of the dynamics presented in this chapter.

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Part II

Monetary Economics

10 Building a Monetary DSGE Model

10.1 Introduction

During the years following the papers of Kydland and Prescott (1982), Hansen (1985), Prescott (1986), and Lucas (1987), RBC theory provided the main reference framework for the analysis of economic fluctuations. It ushered in a revolution in macroeconomics, which was both methodological and conceptual.

As we've seen, RBC models introduced macroeconomists to DSGE models – giving us techniques and toolkits from other much more quantitatively demanding disciplines such as engineering and computer science. This was made possible following the Rational Expectations revolution ushered in by macroeconomists such as Lucas, Prescott, Sargent, and Wallace. DSGE models featured optimally acting agents, whose behaviour was able to be aggregated in order to construct a system of equilibrium equations. These models allowed experiments to be conducted, essentially allowing macroeconomists to undertake counterfactual analysis of the economy subject to a variety of shocks. As Galí (2015) points out, the most striking dimension of the RBC revolution, however, was conceptual. It rested on three basic claims:

- The efficiency of business cycles. The bulk of economic fluctuations observed in industrialised countries could be interpreted as an equilibrium outcome resulting from the economy's response to exogenous variations in real forces (most importantly, technology), in an environment characterised by perfect competition and frictionless markets. According to that view, cyclical fluctuations did not necessarily signal an inefficient allocation of resources. In fact, the fluctuations generated by the standard RBC model were fully optimal. That view had an important corollary: Stabilisation policies may not be necessary or desirable, and they could even be counterproductive!

This was in stark contrast with the conventional interpretation, tracing back to Keynes (1936), of recessions as periods with an inefficiently low utilisation of resources that could be brought to an end by means of economic policies aimed at expanding aggregate demand.

- The importance of technology shocks as a source of economic fluctuations. This claim was derived from the ability of the baseline RBC model to generate "realistic" looking fluctuations in output and other macroeconomic variables, even when variations in total factor productivity – calibrated to match properties of the Solow residual – are assumed to be the only exogenous driving force of the model. Such an interpretation of economic fluctuations was in stark contrast with the traditional view of technological change as a source of long term growth, unrelated to business cycles.

- The limited role of monetary factors. Most importantly, RBC theory sought to explain economic fluctuations with no reference to monetary factors, even abstracting from the existence of money in the models.

We have examined each of these claims in the previous chapters, and called into question their validity. While the RBC model provided us with a good training ground to familiarise ourselves with DSGE models, the RBC model itself had many shortcomings. Most notably, and as stated by the Galí quote above, the RBC model had little relevance for the analysis of macroeconomic policy. We could add monetary policy – as was done by papers such as Cooley and Hansen (1989) – but, in the absence of any price or wage frictions, it would have no real effects. Monetary policy could not change the real interest rate or influence real output. This is concerning for us because if you recall our Kaldor stylised facts, and the characteristics of business cycles, some variables appear to be more "sticky" than others, and the RBC model failed to explain some key empirical findings. So, the first order of business is to incorporate frictions or rigidities, so that monetary policy has a role to play, and that by doing so, perhaps we can build a model which can better explain the business cycle and other empirical findings.

In other words, to allow for a realistic model of business cycles and monetary policy, we need a framework in which prices don't simply follow the money supply, and nominal interest rates and inflation don't just move together one-for-one. In this kind of "Keynesian" model, prices are sticky, so real interest rates can be influenced by the central bank. Real interest rates can affect the performance of the economy, which in turn influences inflation via a Phillips Curve relationship.

With that rather lengthy recap of the RBC model, it's now time to depart the RBC theory framework, and embark on a journey which will eventually lead us to the New Keynesian framework. Before formally developing a New Keynesian model, however, we will go over the Lucas Critique, build a couple classical models in which we incorporate money and inflation, and examine the empirical facts that our New Keynesian model has to match. Much like when we first started with the RBC model, we need to observe the data to give us some direction and motivation for what kind of model we want to build. Let's get started.

10.2 The Lucas Critique

The Lucas Critique (Lucas 1976) is an important philosophical point that forms the basis of much of modern macroeconomics. From Keynes until the mid-1970s, macroeconomics looked very different to what it does now. On the theoretical side, people used variants of a textbook IS-LM (investment-saving liquidity-money) model. That model did not take agent optimisation, dynamics, or expectations formation very seriously. On the empirical side, people used "large scale" macroeconometric models. These were essentially systems of simultaneous equations featuring aggregate variables – many of the larger models would feature hundreds of variables. The design of these macroeconometric models was based on fit and forecasting – regressions, essentially – with little attention paid to any underlying theory or actual economics. There was no microfoundation, and agents' behaviour was postulated to be based on adaptive expectations, which were essentially ad-hoc.

The essential gist of Lucas' Critique[58] is that it is fraught with hazard to try and predict the effects of a policy change based on correlations (regression coefficients) based on historical data. We say that a parameter is "structural" if it is invariant to the rest of the economic environment, and in particular the policy environment. A parameter is "reduced form" if it is not invariant to the environment, or more generally if that parameter cannot be mapped back into some economic primitive. We'll consider two examples to make this point.

[58]The Lucas Critique also has significant implications and ramifications for disciplines outside of economics.

10.2.1 Example: Simple consumption saving model

Consider a very simple two period consumption saving model with a fixed real interest rate and no uncertainty. The household takes income flows to be exogenous. It solves the following problem:

$$\max_{C_t, C_{t+1}} \frac{C_t^{1-\sigma} - 1}{1-\sigma} + \beta \frac{C_{t+1}^{1-\sigma} - 1}{1-\sigma},$$

subject to:

$$C_t + \frac{C_{t+1}}{1+r} = Y_t + \frac{Y_{t+1}}{1+r}.$$

The FOC, or Euler equation, is of course:

$$C_t^{-\sigma} = \beta(1+r)C_{t+1}^{-\sigma}.$$

There are two structural parameters here: β and σ, which govern how heavily you discount future utility flows and how much curvature there is in the utility function. Let's assume that $\sigma = 1$ (which means the utility function collapses to logarithmic utility via L'Hopital's Rule). We can then derive a consumption function that looks like:

$$C_t = \frac{1}{1+\beta}\left(Y_t + \frac{Y_{t+1}}{1+r}\right).$$

Here the marginal propensity to consume (MPC) is the partial derivative of C_t with respect to Y_t, which is $\frac{1}{1+\beta}$. This just a transformation of a structural parameter, and so we could consider the MPC itself to actually be structural.

Now, suppose an econometrician estimates a regression of consumption on income:

$$C_t = \alpha + \gamma Y_t + u_t.$$

This regression is misspecified in the sense that it omits Y_{t+1} – this is an error term. If current income is uncorrelated with future income, Y_t would be uncorrelated with the error term, and we could get

$\gamma = \frac{1}{1+\beta}$ (at least in a large enough sample). But what if current income is correlated with future income (i.e. income is persistent)? Then there is an omitted variable; Y_t will be positively correlated with the error term, which will mean that you will get an upward biased estimate of γ.

Suppose that in the past changes in come have been very persistent – meaning that when Y_t changes, Y_{t+1} changes by almost the same amount. The consumption function derived from the theory would suggest that consumption would then react roughly one-for-one with changes in income. Suppose an econometrician goes and estimates this equation and comes back with a large estimate of γ (close to 1, say). He then goes to a policy adviser and says reports that the MPC is close to 1. This implies that giving households more income (say, through a tax cut), will cause households to spend most of their additional income. Suppose the policy maker did give households an extra dollar of income through a tax cut. The economic theory tells us that raising household income will cause them increase their consumption by only $\frac{1}{1+\beta}$. If $\beta = 0.99$, say, then this means that the additional consumption will only be around $\frac{1}{2}$. This is smaller than the results estimated from the regression, which suggests that the MPC is close to 1. In this example, using the correlation between income and consumption estimated from past data (when income changes were very persistent) is not informative about what will happen if you consider a temporary change in income.

10.2.2 Example: The Phillips Curve

Consider another example, which was really the thing that Lucas was criticising. As we will see later in the course, it is possible to derive a "Phillips Curve" which shows some relationship between economic activity, inflation, and expected inflation:

$$\pi_t = \theta(u_t - u^N) + \beta\mathbb{E}_t\pi_{t+1}, \tag{281}$$

where θ is a coefficient, β is a discount factor, π_t is inflation, $\mathbb{E}_t\pi_{t+1}$ is expected inflation, u_t is the unemployment rate, and u^N is the "natural rate" of unemployment. θ and β are structural parameters.

Particularly before Rational Expectations, macroeconomists didn't know how to treat expectations seriously; and indeed, many models were static and so had no role for expectations of what was going to happen in the future. Suppose an econometrician estimated the following regression:

$$\pi_t = \xi(u_t - u^N) + \epsilon_t.$$

As in the above example, this regression is misspecified relative to the theory – the error term includes expected future inflation. But suppose that in historical data expected inflation was pretty stable. This would mean there wouldn't be much bias in the coefficient estimate, and we would expect that an estimate of ξ would be close to the true θ. Suppose that the true $\theta < 0$: there is a negative relationships between inflation and unemployment. One would be tempted to conclude that raising inflation would lead to a reduction in unemployment. So the econometrician goes to the policymaker and says "Let's raise inflation and this will result in lower unemployment!" But will it?

It will, but only to the extent to which higher inflation doesn't get incorporated into higher inflation expectations. If people are paying attention, they will expected more inflation – $\mathbb{E}_t\pi_{t+1}$ will rise, which means u_t won't fall by as much as the simple regression would have predicted. Again, using past correlations to predict the effects of a policy change may very well be misleading.

As an aside, the modern incarnation of the Phillips Curve is usually traced to a late 1958 study by the LSE's A.W. Phillips. Phillips showed that low unemployment was associated wth high inflation, presumably because tight labour markets stimulated wage inflation. A 1960 study by Solow and Samuelson replicated these findings for the US.

Figure 46: A.W. Phillips' Graph

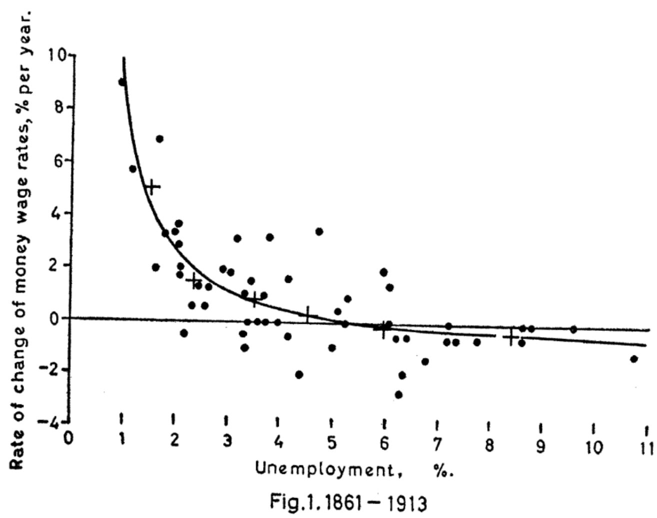


Figure 47: Solow and Samuelson's Description of the Phillips Curve

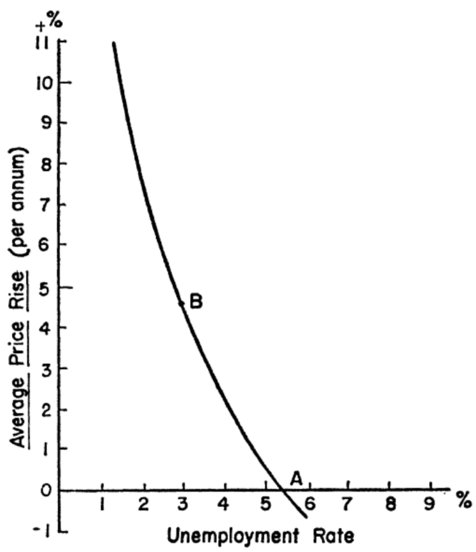


FIGURE 2

MODIFIED PHILLIPS CURVE FOR U.S.

This shows the menu of choice between different degrees of unemployment and price stability, as roughly estimated from last twenty-five years of American data.

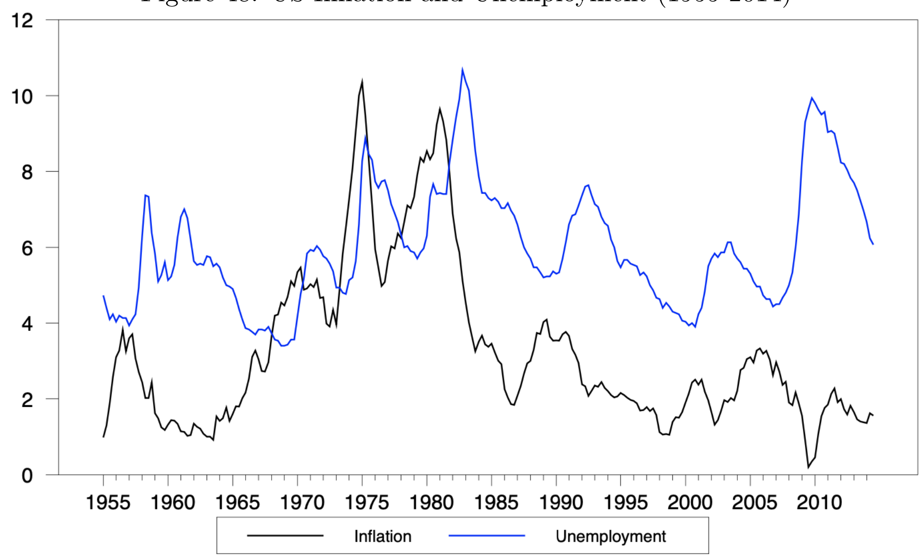
Interestingly, if one reads Friedman's 1967 AEA Presidential Address, there are references to agents forming expectations of future inflation values:

> [...]At any moment of time, there is some level of unemployment which has the property that it is consistent with equilibrium in the structure of real wage rates. At that level of unemployment, real wage rates are tending on the average to rise at a "normal" secular rate[...]
>
> [...]A lower level of unemployment is an indication that there is an excess demand for labor that will produce upward pressure on real wage rates. A higher level of unemployment is an indication that there is an excess supply of labor that will produce downward pressure on real wage rates.[...]
>
> [...]The "natural rate of unemployment" in other words, is the level that would be ground out by the Walrasian system of general equilibrium equations, provided there is imbedded in them the actual structural characteristics of the labor and commodity markets, including market imperfections[...]
>
> [...]You will recognise the close similarity between this statement and the celebrated Phillips Curve. The similarity is not coincidental. Phillips' analysis of the relation between unemployment and wage change is deservedly celebrated as an important and original contribution. But, unfortunately, it contains a basic defect—the failure to distinguish between nominal wages and real wages.[...]
>
> [...]Implicitly, Phillips wrote his article for a world in which everyone anticipated that nominal prices would be stable and in which that anticipation remained unshaken and immutable whatever happened to actual prices and wages. Suppose, by contrast, that everyone anticipates that prices will rise at a rate of more than 75 percent a year[...]
>
> [...]Then wages must rise at that rate simply to keep real wages unchanged. An excess supply of labor will be reflected in a less rapid rise in nominal wages than in anticipated prices, not in an absolute decline in wages.[...]
>
> [...]Restate Phillips' analysis in terms of the rate of change of real wages—and even more precisely, anticipated real wages—and it all falls into place.[...]

> [...]Income and spending will start to rise. To begin with, much or most of the rise in income will take the form of an increase in output and employment rather than in prices ..... Employees will start to reckon on rising prices of the things they buy and to demand higher nominal wages for the future.[...]
>
> [...]In order to keep unemployment at its target level [below the natural rate], the monetary authority would have to raise monetary growth still more ... the "market" rate can be kept below the "natural" rate only by inflation. And ... only by accelerating inflation.[...]

Essentially, Friedman predicted the Phillips Curve relationship would collapse. Turns out he was right:

Figure 48: US Inflation and Unemployment (1955-2014)



Source: Whelan (2016)

Friedman didn't use equations in his AEA address, but a rough model of his ideas were much like what we have in equation (281):

$$\pi_t = \pi_t^e - \theta(u_t - u^N).$$

Friedman pointed out that if policymakers tried to exploit an apparent Phillips Curve tradeoff, then the public would get used to high inflation and come to expect it: π_t^e would drift up and the tradeoff between inflation and output would worsen. In the long-run, you can't fool the public ($\pi_t^e \approx \pi_t$), so you can't keep unemployment away from its "natural rate" $u_t \approx u^N$.

10.2.3 Is econometrics useful?

The conclusion of the Lucas Critique is that we need to take economic theory seriously – correlations (or regression coefficients) estimated in the data may not be policy-invariant, and therefore may not be useful in thinking about "counterfactuals" where we think of what would happen under alternative policy regimes.

Some people (incorrectly) interpret the Lucas Critique as saying we shouldn't do econometrics at all in macro. This is too strong. The Lucas Critique tells us that we need to take theory seriously when doing econometrics; and when we do econometrics without theory (e.g. reduced form econometrics), be honest and open about the potential misgivings. In both of the examples we have above, we actually have regression specifications implied by the theory – it's just that in the regressions we considered running, there was an omitted variable. "Theory" doesn't tell us values of structural parameters like β or θ – that's what econometrics is for. But theory might tell us what kind of econometric models to run, what kind of restrictions we can impose, etc. Then once we have good estimates of the structural parameters, we can use the model to to consider the effects of different policies.

It is actually here where the implications of rational expectations can be useful. Consider the two period consumption model (this time, make it stochastic so that the point is clearer). The theory tells us to run a regression like:

$$C_t = \alpha_1 Y_t + \alpha_2 \mathbb{E}_t Y_{t+1} + \epsilon_t.$$

But the problem here is that we don't necessarily observe $\mathbb{E}_t Y_{t+1}$. Rational Expectations tells us how to get around this, however. In particular, Rational Expectations tells us that $\mathbb{E}_t Y_{t+1} = Y_{t+1} + u_{t+1}$, where u_{t+1} is i) mean zero, and ii) uncorrelated with anything known at date t or earlier. So Rational Expectations tells us that we can run the following regression:

$$C_t = \alpha_1 Y_t + \alpha_2 Y_{t+1} + v_t.$$

Now v_t is a composite error term, equal to $\epsilon_t + \alpha u_{t+1}$. Y_{t+1} is correlated with u_{t+1}, so OLS won't work here. But Rational Expectations tells us that we can instrument for Y_{t+1} with anything known at date t or earlier – that the forecast error, u_{t+1}, is uncorrelated with anything dated t or earlier, making anything dated t or earlier valid instruments. We could do a similar exercise for the Phillips Curve equation, including realised future inflation on the RHS and instrumenting for it with something known at time t or earlier. In other words, taking Rational Expectations seriously often gives us a "theory of the error term" in regression models and therefore guides us on how to deal with that error term.

10.3 Does money matter? Evidence and stylised facts

What are the basic empirical regularities that monetary economics must explain? Monetary economics focuses on the behaviour of prices, monetary aggregates, nominal and real interest rates, and output, so a useful starting point is to summarise briefly what macroeconomic data tells us about the relationships amongst these variables.

10.3.1 Long-run relationships

A nice summary of long-run monetary relationships is provided by McCandless and Weber (1995). They examined data by covering a 30-year period from 110 countries using several definitions of money. Based on their analysis, two primary conclusions emerge. The first is that the correlation between inflation and the growth rate of the money supply is almost 1, varying between 0.92 and 0.96. This strong positive relationship between inflation and money is consistent with many other studies based on smaller samples of countries and different time periods. This correlation is normally taken to support one of the basic tenets of the quantity theory of money: a change in the growth rate of money induces "an equal change in the rate of price inflation" (Lucas 1980). Using US data from 1955 and 1975, Lucas plotted annual inflation against the annual growth rates of money. While the scatter plot suggests only a loose but positive relationship between inflation and money growth, a much stronger relationship emerged when Lucas filtered the data to remove short-run volatility.

The high correlation between inflation and money growth does not, however, have any implication for causality. If countries followed policies under which money supply growth rates were exogenously determined, then the correlation could be taken as evidence that money growth causes inflation, with an almost one-to-one relationship between them. An alternative possibility, equally consistent with the high correlation, is that other factors generate inflation, and central banks allow the growth rate of money to adjust. But in any case, a sensible model of monetary economics would imply neutrality in the long-run.

McCandless and Weber's second general conclusion is that there is no correlation between either inflation or money growth and the growth rate of real output. Thus, there are countries with low output growth and high money growth and inflation – and countries with every other combination as well. This conclusion is not as robust as the money growth-inflation one; McCandless and Weber reported a positive correlation between real growth and money growth, but not inflation, for a subsample of OECD countries. Barro (1998; 2013) reported a negative correlation between inflation and growth in a cross-country sample. Bullard and Keating (1995) examined post-WWII data from 58 countries, concluding for the sample as a whole that the evidence that permanent shifts in inflation produce permanent effects on the level of output is weak, with some evidence of positive effects of inflation on output amongst low-inflation countries and zero or negative effects for higher-inflation countries.

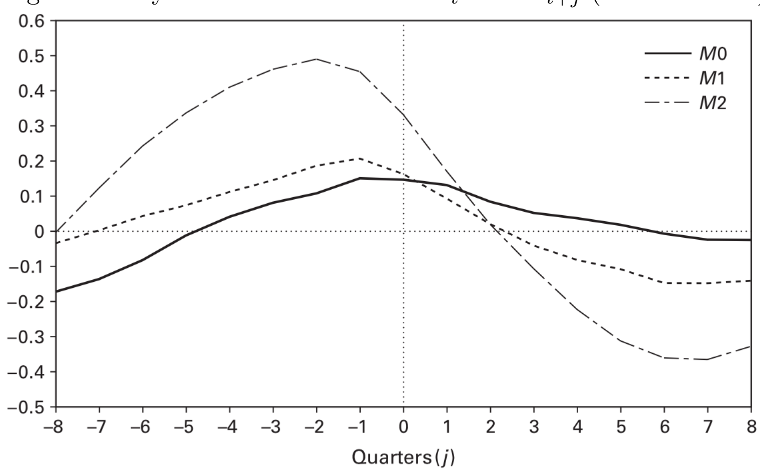
Bullard (1999) surveyed much of the existing empirical work on the long-run relationship between money growth and real output. His main finding was that while shocks to the level of the money supply do not appear to have long-run effects on real output, this was not the case with respect to shocks to money growth. Despite the diversity of empirical findings concerning the long-run relationship between inflation and real growth, and other measures of real economic activity such as unemployment, the general consensus was well summarised by the proposition, "[...]that there is no long-run tradeoff between the rate of inflation and the rate of unemployment" (Taylor 1996).

10.3.2 short-run relationships

The long-run empirical regularities of monetary economics are important for gauging how well the steady state properties of a theoretical model match the data. Much of our interest in monetary economics, however arises because of a need to understand how monetary phenomena in general and monetary policy in particular affect the behaviour of the macroeconomy over time periods of months or quarters. short-run dynamic relationships between money, inflation, and output reflect both the way in which private agents respond to economic disturbances and the way in which the monetary policy authority responds to those same disturbances.

Some evidence on short-run correlations for the US are provided in Figures 49 and 50. The figures show correlations between the detrended[59] log of real GDP and three different monetary aggregates, each also in detrended log form. Data are quarter from 1967:1 to 2008:2, and the figures plot, for the entire sample and for the subperiod 1984:1 to 2008:2., the correlation between real GDP, Y_t, and a monetary aggregate, M_t. The three aggregates are the monetary base (sometimes referred to as M_0 or M^B), $M1$, and $M2$.

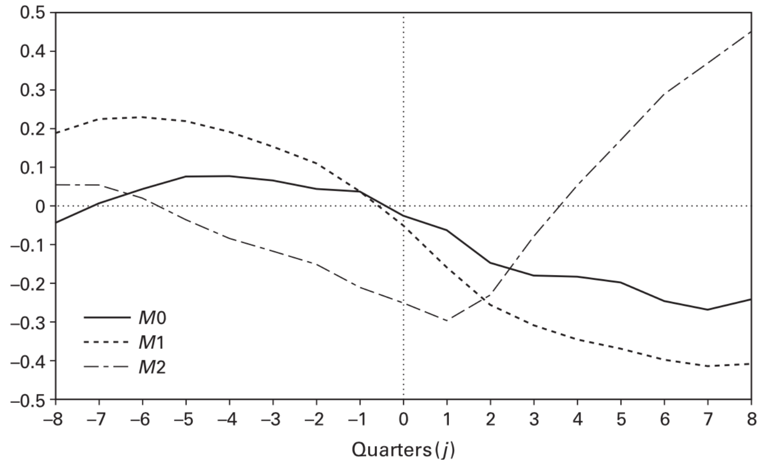
Figure 49: Dynamic Correlations for Y_t and M_{t+j} (1967:1-2008:2)



Source: Walsh (2010)

[59]Trends are estimated using a HP filter.

Figure 50: Dynamic Correlations for Y_t and M_{t+j} (1984:1-2008:2)



Source: Walsh (2010)

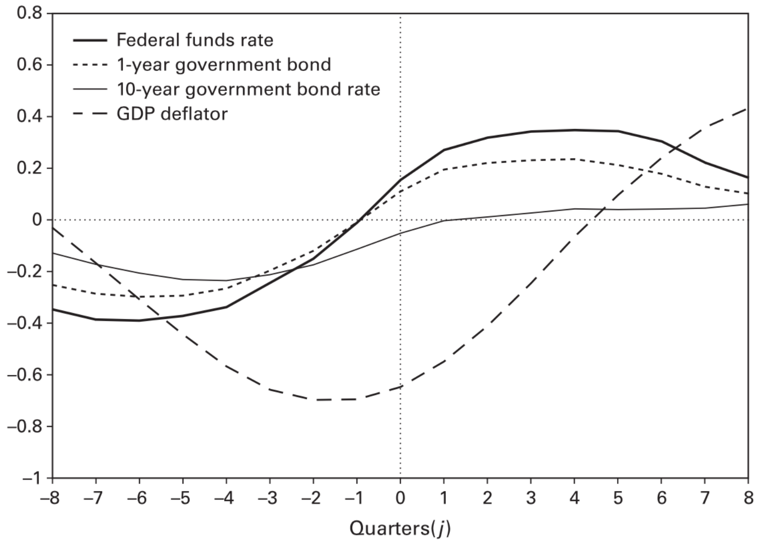
As Figure 49 shows, the correlations with real output change substantially as one moves from M_0 to $M2$. The narrow measure M_0 is positively correlated with real GDP at both leads and lags over the entire period, but future M_0 is negatively correlated with real GDP in the period since 1984. $M1$ and $M2$ are positively correlated at lags but negatively correlated at leads over the full sample. In other words, high GDP (relative to trend) tends to be preceded by high values of $M1$ and $M2$ but followed by low values. The positive correlation between Y_t and M_{t+j} for $j < 0$ indicates that movements in money lead movements in output. This timing pattern played an important role in **FriedmanSchwartz**'s classic and highly influential *A Monetary History of the United States*, 1867-1960. The larger correlations between GDP and $M2$ arise in part from the endogenous nature of an aggregate such as $M2$, depending as it does on banking sector behaviour as well as on that of the nonbank private sector.[60] However, these patterns for $M2$ are reversed in the later period, though $M1$ still leads GDP.

Figures 51 and 52 show the cross correlations between detrended real GDP and several interest rates and between detrended real GDP and the detrended GDP deflator. The interest rates range from the Federal Funds Rate, to the 1-year and 10-year rates on government bonds. The three interest rate

[60]See King and Plosser (1984) and Coleman (1996) for more.

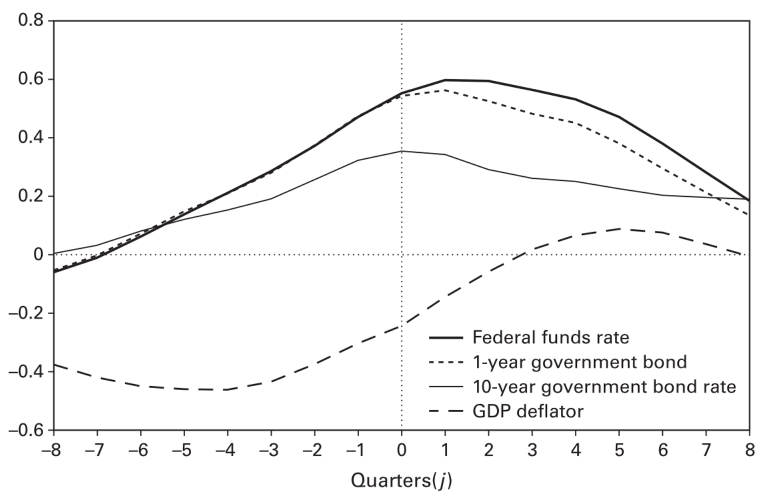
series display similar correlations with real output, although the correlations become smaller for the longer term rates.

Figure 51: Dynamic Correlations of Output, Prices, and Interest Rates (1967:1-2008:2)



Source: Walsh (2010)

Figure 52: Dynamic Correlations of Output, Prices, and Interest Rates (1984:1-2008:2)



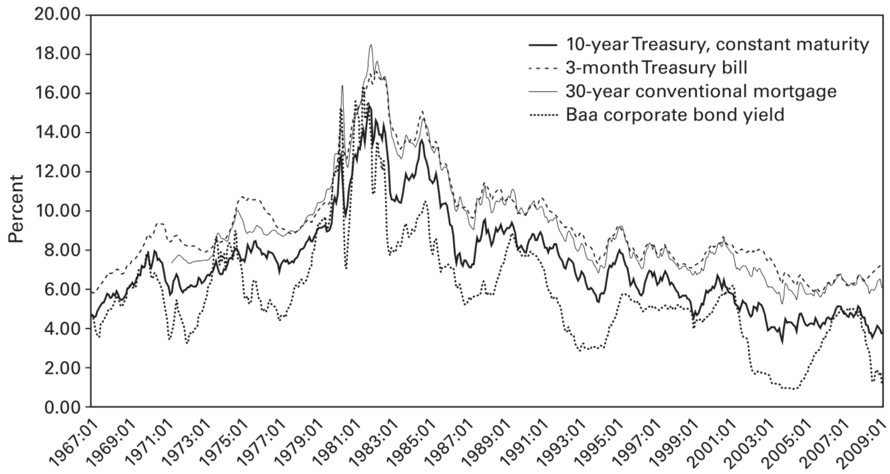
Source: Walsh (2010)

For the entire sample period (Figure 51), low interest rates tend to lead output, and a rise in output tends to be followed by higher interest rates. This pattern is less pronounced in the later period (Figure 52), and interest rates appear to rise prior to an increase in detrended GDP.

In contrast, the GDP deflator tends to be below trend when output is above trend, but increases in real output tend to be followed by increases in prices, though this effect is absent in the more recent period. Kydland and Prescott (1990) argued that the negative contemporaneous correlation between output and price series suggests that supply shocks, not demand shocks, must be responsible for business cycle fluctuations. Aggregate supply shocks would cause prices to be countercyclical, whereas demand shocks would be expected to make prices procyclical. However, if prices were sticky, a demand shock would initially raise output above trend, and prices would respond very little. If prices did eventually rise while output eventually returned to trend, prices could be rising as output was falling, producing a negative unconditional correlation between the two even though it was demand shocks generating the fluctuations (Ball and Mankiw 1994).

Most models used to address issues in monetary theory and policy contain only a single interest rate. Generally, this is interpreted as a short term rate of interest and is often viewed as an overnight market interest rate that the central bank can control. The assumption of a single interest rate is a useful simplification if all interest rates tend to move together. Figure 53 shows several longer term market rates of interest for the US. As the figure suggests, interest rates do tend to display similar behaviour, although the 3-month T-Bill rate, the shortest maturity shown, is more volatile than the other rates. There are periods, however, when rates at different maturities and riskiness move in opposite directions. For example, during 2008, the rate on corporate bonds rose while the rates on government debt, both at 3-month and 10-year maturities, were falling.

Figure 53: Interest Rates (1967:01-2008:09)



Source: Walsh (2010)

10.3.3 Estimating the effect of money on output

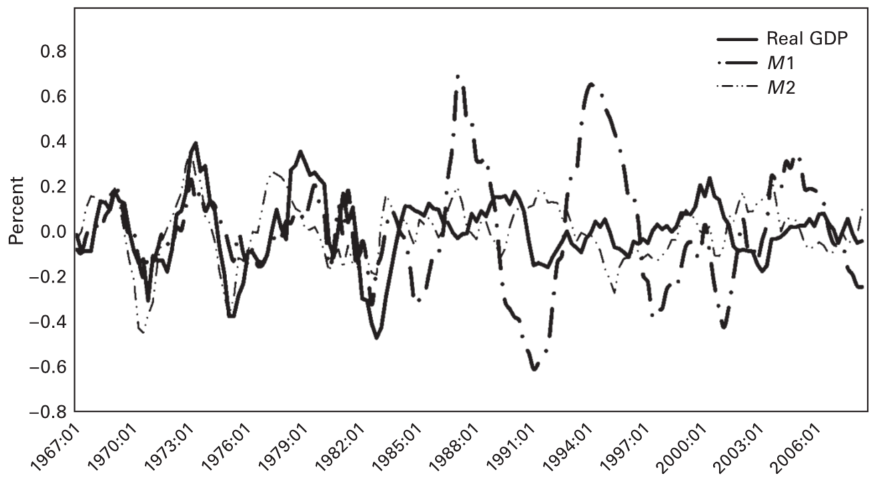
The tools that have been employed to estimate the impact of monetary have evolved over time as the result of developments in time series econometrics and changes in the specific questions posed by theoretical models. The literature of the empirical evidence on the relationship between monetary policy and US macroeconomic behaviour has focused on whether monetary policy disturbances actually have played an important role in US economic fluctuations. Equally important, the empirical evidence is useful in judging whether the predictions of different theories about the effects of monetary policy are consistent with the evidence. A key paper in this literature is Christiano et al. (1999).

10.3.4 The evidence of Friedman and Schwartz

Friedman and Schwartz's (1963) study of the relationship between money and business cycles still represents probably the most influential empirical evidence that money does matter for business cycle fluctuations. Their evidence, based on almost 100 years of data from the US, relies heavily on patterns timing; systematic evidence that money growth rate changes lead changes in real economic activity is taken to support a causal interpretation in which money causes output fluctuations.

The nature of this evidence is apparent in Figure 54, which shoes two detrended money supply measures and real GDP. The monetary aggregates in the figure, $M1$ and $M2$, are quarterly observations on the deviations of the actual series from trend. The sample period is 1967:1-2008:2, so it's after the Friedman and Schwartz period of study. The figure reveals slowdowns in money leading most business cycle downturns through the early 1980's. However, the pattern is not so apparent after 1982.

Figure 54: Detrended Money and Real GDP (1967:1-2008:2)



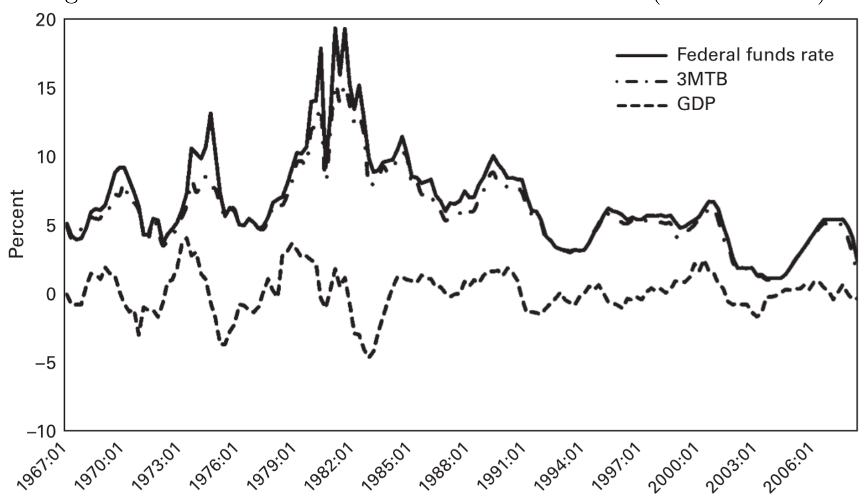
Source: Walsh (2010)

While it is suggestive, evidence based on timing patterns and simple correlations may not indicate the true causal role of money. Since the Fed and the banking sector respond to economic developments, movements in the monetary aggregates are not exogenous, and the correlation patterns need no reflect any causal effect of monetary policy on economic activity. If, for example, the central bank is implementing monetary policy by controlling the value of some short-term market interest rate, the nominal stock of money will be affected both by policy actions that change interest rates and by developments in the economy that are not related to policy actions. An economic expansion may lead banks to expand lending in ways that produce an increase in the stock of money, even if the central bank has not changed its policy. If the money stock is used to measure monetary policy, the relationship observed in the data between money and output may reflect the impact of output on money, nit the impact of money and monetary policy on output.

10.3.5 Reverse causation argument

Tobin (1970) was the first to model formally the idea that the positive correlation between money and output – the correlation that Friedman and Schwartz interpreted as providing evidence that money caused output movements – could in fact reflect just the opposite – output might be causing money. A more modern treatment of what is known as the reverse causation argument was provided by King and Plosser (1984) and Coleman (1996).

Figure 55: Interest Rates and Detrended Real GDP (1967:1-2008:2)



Source: Walsh (2010)

King and Plosser deduced that the correlation between broad aggregates such as $M1$ and $M2$ and output arises from the endogenous response of the banking sector economic disturbances that are not the result of monetary policy actions. The endogeneity problem is likely to be particularly severe if the monetary authority has employed a short-term interest rate as its main policy instrument, and this has generally been the case in the US. Changes in the money stock will then be endogenous and cannot be interpreted as representing policy actions. Figure 55 shows the behaviour of two short-term nominal interest rates, the 3-month T-Bill rate (3MTB) and the Federal Funds Rate, together with detrended GDP. Like Figure 54, Figure 55 provides some support for the notion that monetary policy actions have contributed to US business cycles. Interest rates have typically increased prior to economic downturns. But whether this is evidence that monetary policy has caused or contributed to cyclical fluctuations cannot be inferred from the figure; the movements in interest rates may simply reflect the Fed's response to the state of the economy.

10.3.6 Technical aside: Granger causality

One of the goals of economic analysis is to judge whether there is a causal relationship between economic variables. Generally, it is hard to detect the causal relationship from data, and we have to rely on economic theory. Recall our discussion in the "Primer to DSGE Models" section – one advantage of time series analysis is that it does not have to rely on economic theory. So what if there was a way to filter out potential causation flows between variables from the data? Granger (1969) introduced the concept of Granger causality based on forecasting techniques in time series econometrics. A variable X is said to Granger-cause Y if and only if lagged values of X have marginal predictive content in a forecasting equation for Y. In other words, having some information about the future of Y is not enough for Granger causality, and that past observations of X should have more information about the future of Y than the past Y observations.

Consider the simple bivariate VAR(2) model:

$$
\begin{aligned}
Y_{1,t} &= c_1 + \phi_{11}^{(1)} Y_{1,t-1} + \phi_{12}^{(1)} Y_{2,t-1} + \phi_{11}^{(2)} Y_{1,t-2} + \phi_{12}^{(2)} Y_{2,t-2} + \epsilon_{1,t} \\
Y_{2,t} &= c_2 + \phi_{21}^{(1)} Y_{1,t-1} + \phi_{22}^{(1)} Y_{2,t-1} + \phi_{21}^{(2)} Y_{1,t-2} + \phi_{22}^{(2)} Y_{2,t-2} + \epsilon_{2,t}.
\end{aligned}
$$

Y_2 does not Granger-cause Y_1 is analogous to saying that the coefficients of Y_2 in the Y_1 equation are all 0. In other words, $\phi_{12}^{(1)} = \phi_{12}^{(2)} = 0$. We can test for the Granger causality running from Y_2 to Y_1 by F-testing the null hypothesis that $\phi_{12}^{(1)} = \phi_{12}^{(2)} = 0$.

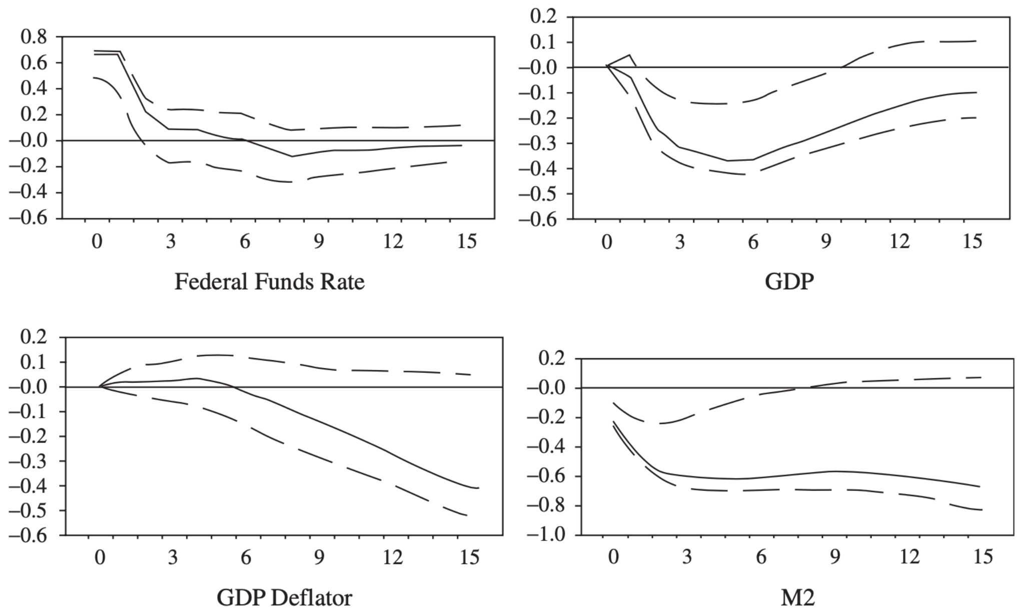
10.3.7 Model-based approach

In an important contribution, Sims (1972) introduced the notion of Granger causality into the debate over the real effects of money. Sims' original work used log levels of US nominal GNP and money (both $M1$ and the monetary base). He found evidence that money Granger-caused GNP. That is, the behaviour of money helped to predict future GNP. However, using the index of industrial protection to measure real output, Sims (1980) found that the fraction of output variation explained by money was greatly reduced when a nominal interest rate was added to the regression equation. Stock and Watson (1989) provided a systematic treatment of the trend specification in testing whether money Granger-causes real output. They concluded that money does help to predict future output even when prices and an interest rate are included.

A large literature has examined the value of monetary indicators in forecasting output. One interpretation of Sims' finding was that including an interest rate reduces the apparent role of money because, at least in the US, a short-term interest rate than the money supply provides a better measure of monetary policy actions (we will cover this soon).

As alluded to earlier, the contemporary seminal paper to understand effects of monetary policy shocks on output is Christiano et al. (1999) (CEE) and Stock and Watson (2001). Figure 56 shows the dynamic responses of the Federal Funds Rate, log GDP, log GDP deflator, and the money supply ($M2$) to an exogenous tightening of monetary policy from the CEE paper. Note that the path of the funds rate itself, depicted in the top left graph, shows an initial increase of about 75 basis points, followed by a gradual return to its original level. In response to that tightening of monetary policy, GDP declines with a characteristic hump-shaped pattern. It reaches a trough after five quarters at a level about 50 basis points below its original level, and then it slowly reverts back to its original level. That estimated response of GDP can be viewed as evidence of sizeable and persistent real effects of monetary policy shocks. On the other hand, the GDP deflator displays a flat response for over a year, after which it declines. That estimated sluggish response of prices to the policy tightening is generally interpreted as evidence of substantial price rigidities (more on this soon). Finally, note that $M2$ displays a persistent decline the face of the rise in the Federal Funds Rate, suggesting that the Fed needs to reduce the amount of money in circulation in order to bring about the increase in the nominal rate. The observed negative co-movement is between money supply and nominal interest rates is known as the "liquidity effect".

Figure 56: Estimated Dynamic Response to a Monetary Policy Shock (Quarterly)



Source: Christiano et al. (1999)

Also notice, however, that a contractionary monetary policy shock seems to lead to an initial small increase in the GDP deflator. This is referred to by macroeconomists as a "price puzzle". The effect is small and temporary (and barely statistically significant) but still puzzling. The most commonly accept explanation for the price puzzle is that it reflects the fact that the variables included in the VAR do not span the full information set available to the Fed. Suppose the Fed tends to raise the Funds Rate whenever it forecasts that inflation might rise in the future. To the extend that the Fed is unable to offset the factors that led it to forecast higher inflation, or to the extend that the Fed acts too late to prevent inflation from rising, the increase in the Funds Rate will be followed by a rise in prices. Sims (1992), using similar VARs as CEE, also showed that this price puzzle occurs for monetary policy shocks for France, Germany, Japan, and the UK.

One solution to resolving this puzzle would then be to include variables like commodity prices or other asset prices in the VAR (Sims 1992; Chari, Christiano, and Eichenbaum 1995; and Bernanke and Mihov 1998). An alternative interpretation of the price puzzle is provided by Barth and Ramey

(2001). They argued that contractionary monetary policy operates on aggregate supply as well as aggregate demand. For example, an increase in interest rates raises the cost of holding inventories and thus acts as a positive cost push shock. This negative supply effect raises prices and lowers output. Such an effect is called the cost channel of monetary policy.

10.3.8 Criticisms of the VAR approach

First, some of the impulse responses do not accord with most economists' priors. In particular, the price puzzle – the finding that a contractionary policy shock, as measured by a Federal Funds Rate shock, tends to be followed by a rise in the price level – is troublesome. As noted earlier, the price puzzle can be solved by including oil prices or commodity prices in the VAR system, and the generally accepted interpretation is that lacking these inflation-sensitive prices, a standard VAR misses important information that is available to policymakers. A related but more general point is that many of the VAR models used to assess monetary policy fail to incorporate forward-looking variables. Central banks look at a lot of information in setting policy. But because policy is likely to respond to forecasts of future economic conditions, VARs may attribute the subsequent movements input and inflation to the policy action.

At best, the VAR approach identifies only the effects of monetary policy shocks, shifts in policy unrelated to the endogenous response of policy to developments in the economy. Yet most, if not all, of what one thinks of in terms of policy and policy design represents the endogenous response of policy to the economy, and "most variation in monetary policy instruments is accounted for by responses of policy to the state of the economy, not by random disturbances to policy" (Sims and Zha 1998). So it is unfortunate that primary empirical tool – VAR analysis – used to assess the impact of monetary policy is uninformative about the role played by policy rules. If policy is completely characterised as a feedback rule on the economy, so that there are no exogenous policy shocks, then the VAR methodology would conclude that monetary policy doesn't matter. Yet while monetary policy is not causing output movements in this example, it does not follow that policy is unimportant; the response of the economy to non-policy shocks may depend importantly on the way monetary policy endogenous adjusts. This broadly echoes our discussion of the Lucas Critique and the role of econometrics in macroeconomic theory.

10.4 Money in the utility function

Putting money into a general equilibrium model is not easy. The main issue being is that money is a dominated asset – it does not have a return, and hence it has no value in equilibrium. We need to specify a role for money so that agents wish to hold positive quantities in equilibrium. There are broadly three approaches to integrating money into general equilibrium models:

- Money in the utility (MIU) function (Sidrauski 1967);
- Transaction costs:
  - Shoe-leather costs (Baumol 1952; Tobin 1956);
  - Cash-in-advance (CIA) constraint (Clower 1967; Lucas 1982; Svensson 1985); and
  - Transaction technologies (e.g. Schmitt-Grohé and Uribe (2010)).
- Search and match (Kiyotaki and Wright 1989; Williamson and Wright 2010).

All of the above methods – with the exception of search and match – are fairly ad-hoc and aren't microfounded.

In this section we will examine the MIU approach. The idea is that money provides some service to the economy and that the benefits of that service can be expressed in the utility function. If one assumes that having more real money balances means that one will be able to reduce the time and energy spent making transactions, for example, one might include real balances in the utility function as a way of representing these utility gains. We will use a utility function in which increased holdings of real balances directly increases welfare. Utility that individuals wish to maximise is still the present value of an infinite sequence of additively separable subutilities. The utility function of an individual in period t is:

$$u\left(c_t, \frac{M_t}{P_t}\right) = u\left(c_t, m_t\right),$$

where the only important change is that we now add real balances of the individual, M_t/P_t, as a variable, and to keep to intuition simple we abstract from labour supply decisions of the household.

The rationale for adding real balances to the utility function is the presumption that additional real balances reduce the cost of making transactions or reduce search (since they solve the noncoincidence-of-wants problem that arises in barter trade). One of the benefits of putting money in the utility function is that if there are other assets that individuals can hold, capital, for instance, the model will produce a real rate of return for money that is less than that of the other assets.

The benefit does not come without costs. As noted, money doesn't do anything, and so there is no clear use for it. Just keeping the money in your possession creates the utility. In economies with just one good in which all agents are identical, no trades ever take place and money really is not ever used for anything. Nevertheless, as a rough approximation of the gains from using money, and in particular, for giving money value when there are interest-earning assets available, MIU functions models are useful.

10.4.1 A simple Sidrauski MIU model

A unit mass of identical households each choose sequences of $\{c_t, M_t, k_t, B_t\}_{t=0}^{\infty}$ to maximise the infinite horizon discounted utility function:

$$\max_{\{c_t, M_t, k_t, B_t\}} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s u\left(c_{t+s}, m_{t+s}\right),$$

subject to the sequence of period t budget constraints:

$$P_t y_t + P_t \tau_t + R_{t-1} B_{t-1} + M_{t-1} = P_t c_t + P_t \underbrace{[k_t - (1-\delta) k_{t-1}]}_{I_t} + B_t + M_t,$$

where τ_t are lump sum transfers of money from the monetary authority to the household in period t, and $R_t = 1 + i_t$ is the gross nominal interest rate. We can rewrite the budget constraint in real terms as:

$$y_t + \tau_t + \frac{R_{t-1} b_{t-1} + m_{t-1}}{1+\pi_t} = c_t + k_t - (1-\delta) k_{t-1} + b_t + m_t \tag{282}$$

where net inflation π_t is defined as:

$$\pi_t = \frac{P_t - P_{t-1}}{P_{t-1}}.$$

Note that here we use the end of period timing convention for the capital stock and bond holdings.

Instead of proceeding with a Lagrange, we can use dynamic programming to solve. First, define ω_t as a composition of the state variables for the household, and use what we know about the economy's aggregate resources and its production technology to write:

$$\omega_t \equiv f(k_{t-1}) + \tau_t + \frac{R_{t-1}b_{t-1} + m_{t-1}}{1+\pi_t} + (1-\delta)k_{t-1} = c_t + k_t + b_t + m_t. \tag{283}$$

The household's problem is to choose paths for c_t, k_t, b_t, and m_t to maximise utility subject to Then we can write the value function as:

$$V(\omega_t) = \max_{\{c_t k_t, m_t, b_t\}} \left\{ u(c_t, m_t) + \beta V(\omega_{t+1}) \right\}, \tag{284}$$

subject to (283) and:

$$\omega_{t+1} = f(k_t) + \tau_{t+1} + \frac{R_t b_t + m_t}{1+\pi_{t+1}} + (1-\delta)k_t.$$

Using (283), we can write:

$$k_t = \omega_t - c_t - m_t - b_t,$$

and then we can write (284) as:

$$V(\omega_t) = \max_{c_t, b_t, m_t} \left\{ u(c_t, m_t) + \beta V\left(f(\omega_t - c_t - m_t - b_t) + \tau_{t+1} + \frac{R_t b_t + m_t}{1+\pi_{t+1}} + (1-\delta)(\omega_t - c_t - m_t - b_t) \right) \right\}.$$

So the maximisation problem is now an unconstrained problem over c_t, b_t, and m_t. The FOCs are:

$$\frac{\partial V(\omega_t)}{\partial c_t} = u_c(c_t, m_t) - \beta V_\omega(\omega_{t+1}) \left[f_k(k_t) + (1-\delta) \right] = 0, \tag{285}$$
$$\frac{\partial V(\omega_t)}{\partial b_t} = \beta V_\omega(\omega_{t+1}) \left[\frac{R_t}{1+\pi_{t+1}} - f_k(k_t) - (1-\delta) \right] = 0, \tag{286}$$
$$\frac{\partial V(\omega_t)}{\partial m_t} = u_m(c_t, m_t) - \beta V_\omega(\omega_{t+1}) \left[f(k_t) + (1-\delta) \right] + \beta V_\omega(\omega_{t+1}) \left[\frac{1}{1+\pi_{t+1}} \right] = 0, \tag{287}$$

together with the transversality conditions:

$$\lim_{t\to\infty} \beta^t \lambda_t x_t = 0, \;\; x = k, b, m,$$

where λ_t is the marginal utility of period t consumption. The envelope theorem implies:

$$V_\omega(\omega_t) = \beta V_\omega(\omega_{t+1}) \left[f_k(k_t) + (1-\delta) \right]. \tag{288}$$

Combine the envelope condition (288) with (285) to get:

$$\lambda_t = u_c(c_t, m_t) = V_\omega(\omega_t). \tag{289}$$

The FOCs have straightforward interpretations. Since initial resources ω_t must be divided between consumption, capital, bonds, and money balances, each use must yield the same marginal benefit at an optimum allocation. Using (285) and (289), we can write (287) as:

$$u_m(c_t, m_t) + \beta \frac{u_c(c_{t+1}, m_{t+1})}{1+\pi_{t+1}} = u_c(c_t, m_t), \tag{290}$$

which states that the marginal benefit of adding to money holdings at time t must equal the marginal utility of consumption at time t. The marginal benefit of additional money holdings has two components. First, money directly yields utility u_m. Second, real money balances at time t adds $1/(1+\pi_{t+1})$ to real resources at time $t+1$; this addition to ω_{t+1} is worth $V_\omega(\omega_{t+1})$ at $t+1$, or $\beta V_\omega(\omega_{t+1})$ at time t. This, the total marginal benefit of money at time t is $u_m(c_t, m_t) + \beta V_\omega(\omega_{t+1})/(1+\pi_{t+1})$. Equation

(290) is then obtained by noting that $V_\omega(\omega_{t+1}) = u_c(c_{t+1}, m_{t+1})$.

We can derive the value of money as an asset by rewriting (290) as:

$$\begin{aligned}
&\frac{u_c(c_t, m_t)}{P_t} = \frac{u_m(c_t, m_t)}{P_t} + \beta \frac{1}{P_t} \frac{u_c(c_{t+1}, m_{t+1})}{(1+\pi_{t+1})}, \\
\Leftrightarrow &\frac{u_c(c_t, m_t)}{P_t} = \frac{u_m(c_t, m_t)}{P_t} + \beta \frac{1}{P_t} \frac{u_c(c_{t+1}, m_{t+1})}{1 + \frac{P_{t+1} - P_t}{P_t}} \\
\Leftrightarrow &\frac{u_c(c_t, m_t)}{P_t} = \frac{u_m(c_t, m_t)}{P_t} + \beta \frac{1}{P_t} \frac{u_c(c_{t+1}, m_{t+1})}{\frac{P_t}{P_t} + \frac{P_{t+1} - P_t}{P_t}} \\
\Leftrightarrow &\frac{u_c(c_t, m_t)}{P_t} = \frac{u_m(c_t, m_t)}{P_t} + \beta \frac{1}{P_t} \frac{u_c(c_{t+1}, m_{t+1})}{\frac{P_{t+1}}{P_t}} \\
\Leftrightarrow &\underbrace{\frac{u_c(c_t, m_t)}{P_t}}_{\text{Price}} = \underbrace{\frac{u_m(c_t, m_t)}{P_t}}_{\text{Dividend}} + \underbrace{\beta \frac{u_c(c_{t+1}, m_{t+1})}{P_{t+1}}}_{\text{Price tomorrow}}.
\end{aligned}$$

Then, roll forward to get:

$$\begin{aligned}
\frac{u_c(c_t, m_t)}{P_t} &= \frac{u_m(c_t, m_t)}{P_t} + \beta \left(\frac{u_m(c_{t+1}, m_{t+1})}{P_{t+1}} + \beta \frac{u_c(c_{t+2}, m_{t+2})}{P_{t+2}} \right) \\
&= \frac{u_m(c_t, m_t)}{P_t} + \beta \frac{u_m(c_{t+1}, m_{t+1})}{P_{t+1}} + \beta^2 \frac{u_c(c_{t+2}, m_{t+2})}{P_{t+2}} + ...,
\end{aligned}$$

Hence, the value of money (in terms of utils) can be written as:

$$\frac{u_c(c_t, m_t)}{P_t} = \sum_{i=0}^{\infty} \beta^i \frac{u_m(c_{t+i}, m_{t+i})}{P_{t+i}}. \tag{291}$$

10.4.2 Opportunity cost of holding money and Fisher relation

Rewrite (286) as:

$$\begin{aligned}
f_k(k_t) &= \frac{R_t}{1+\pi_{t+1}} - (1-\delta) \\
\Leftrightarrow 1 + r_t &= f_k(k_t) + (1-\delta),
\end{aligned}$$

where $1 + r_t$ is the [gross] real interest rate. Then combine this with (285) and (290) to get the opportunity cost of holding money:

$$\begin{aligned}
1 &= \frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} + \beta \left[\frac{1}{1+\pi_{t+1}}\right] \frac{u_c(c_{t+1}, m_{t+1})}{u_c(c_t, m_t)} \\
\frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} &= 1 - \frac{1}{\underbrace{[f_k(k_t) + (1-\delta)]}_{1+r_t}(1+\pi_{t+1})} \\
\frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} &= \frac{i}{R_t} \equiv \Upsilon_t, \qquad (292)
\end{aligned}$$

where (285) implied:

$$\beta \frac{u_c(c_{t+1}, m_{t+1})}{u_c(c_t, m_t)} = \frac{1}{1+r_t}.$$

Note, from (292) can be written as:

$$\begin{aligned}
\frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} &= \frac{P_t i_t}{P_t(1+i_t)} \\
\Leftrightarrow \frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} &= \frac{P_m(t)}{P_c(t)} \\
\Leftrightarrow \text{MRS} &= \text{MRT}.
\end{aligned}$$

Equation (292) also makes use of (286), which links the nominal return on bonds, inflation, and the real interest rate (or, alternatively, the real return on capital). This latter equation can be written as:

$$\begin{aligned}
1 + i_t &= [f_k(k_t) + 1 - \delta]\,(1+\pi_{t+1}) \\
&= (1+r_t)(1+\pi_{t+1}).
\end{aligned}$$

This relationship between the real and nominal rates of interest is called the Fisher relationship, after Irving Fisher (1896). It expresses the gross nominal rate of interest as equal to the gross real return on capital times 1 plus the expected rate of inflation. If one notes that $(1+x)(1+y) \approx 1+x+y$ when

x and y are small, we can write the Fisher relationship as:

$$i_t = r_t + \pi_{t+1}. \tag{293}$$

To interpret (292), consider a very simply choice problem in which the agent must pick x and z to maximise $u(x, z)$ subject to a budget constraint of the form $x + pz = y$, where p is the relative price of z. The FOC conditions imply:

$$\frac{u_z}{u_x} = p,$$

in words, the marginal rate of substitution between z and x equals the relative price of z in terms of x. Comparing this to (292) shows that Υ can be interpreted as the relative price of real money balances in terms of the consumption good. The marginal rate of substitution between money and consumption is set equal to the price, or opportunity cost, of holding money. The opportunity cost of holding money is directly related to the nominal rate of interest. The household could hold one unit less of money, purchasing instead a bond yielding a nominal return of R (gross) or i (net); the real value of this payment is $i/(1+\pi)$, and since it is received in period $t+1$, its present value is:

$$\frac{i}{(1+r)(1+\pi)} = \frac{i}{(1+i)}.$$

Since money is assumed to pay no rate of interest, the opportunity cost of holding money is affected both by the real return on capital and the rate of inflation. If the price level is constant (so $\pi = 0$), then the foregone earnings from holding money rather than capital are determined by the real return to capital. If the price level is rising ($\pi > 0$), the real value of money in terms of consumption declines, and this adds to the opportunity cost of holding money.

In deriving the FOCs for the household's problem, it could have been equivalently assumed that the household leased its capital to firms, receiving a rental rate of r_k, and sold its labour services at a wage rate of w. Household income would then be $r_k k + w$. With competitive firms hiring capital and labour in perfectly competitive markets under constant returns to scale, $r_k = f'(k)$ and $w = f(k) - kf'(k)$, so household income would be (using Euler's Theorem):

$$r_k k + w = f_k(k)k + [f(k) - kf_k(k)] = f(k),$$

as in (283).[61]

10.4.3 Steady state equilibrium and money neutrality

Consider the properties of this economy when it is in a steady-state equilibrium and the nominal supply of money growing at the rate θ. We use our usual notation to denote steady state variables with a bar (e.g. $\bar{x}$). The steady state values of consumption, the capital stock, real money balances, inflation, and the nominal interest rate must satisfy the FOCs for the household's decision problem given by (285)-(287), the economy wide budget constraint, and the specification of the exogenous growth rate of M. Note that with real money balances constant in the steady state, it must be that the prices are growing at the same rate as the nominal stock of money, or $\bar{\pi} = \theta$. Use (289) to eliminate $V_\omega(\bar{\omega})$, the FOCs can be written as:

$$0 = u_c(\bar{c}, \bar{m}) - \beta u_c(\bar{c}, \bar{m}) \left[f_k(\bar{k}) + (1 - \delta) \right], \tag{294}$$

$$0 = \frac{1 + \bar{i}}{1 + \theta} - f_k(\bar{k}) - (1 - \delta), \tag{295}$$

$$0 = u_m(\bar{c}, \bar{m}) - \beta u_c(\bar{c}, \bar{m}) \left[f(\bar{k}) + (1 - \delta) \right] + \beta \frac{u_c(\bar{c}, \bar{m})}{1 + \theta}, \tag{296}$$

and the economy wide resource constraint is:

$$f(\bar{k}) + \bar{\tau} + (1 - \delta)\bar{k} + \frac{\bar{m}}{1 + \theta} = \bar{c} + \bar{k} + \bar{m}, \tag{297}$$

where $\bar{b} = 0$.

(295) is the steady state form of the Fisher relation, linking real and nominal interest rates. This

[61] For this approach, see McCandless (2008).

can be seen by noting that the real return on capital (net of depreciation) is:

$$\bar{r} \equiv f_k(\bar{k}) - \delta,$$

so (295) can be written as:

$$1 + \bar{i} = (1 + \bar{r})(1 + \bar{\pi}) = (1 + \bar{r})(1 + \theta).$$

Notice that in (294)-(297) money appears only in the form of real money balances. Thus, any change in the nominal quantity of money that is matched by a proportional change in the price level, leaving $\bar{m}$ unchanged, has no effect on the economy's real equilibrium. This is described by saying that the model exhibits neutrality of money. If prices do not adjust immediately in response to a change in M, the na model might display non-neutrality with respect to changes in M in the short-run but still exhibit monetary neutrality in the long-run, once all prices have adjusted.

Dividing (294) by $u_c(\bar{c}, \bar{m})$ yields:

$$\begin{aligned} 0 &= 1 - \beta \left[f_k(\bar{k}) + 1 - \delta \right] \\ \Leftrightarrow f_k(\bar{k}) &= \frac{1}{\beta} - 1 + \delta. \end{aligned} \tag{298}$$

This equation defines the steady state capital labour ratio $\bar{k}$ as a function of β and δ. If the production function is Cobb-Douglas, say $f(k) = k^\alpha$ for $0 < \alpha \leq 1$, then $f_k(k) = \alpha k^{\alpha - 1}$ and:

$$\bar{k} = \left[\frac{\alpha\beta}{1 + \beta(\delta - 1)} \right]^{\frac{1}{1-\alpha}}. \tag{299}$$

What is particularly relevant here is the implication that the steady state capital labour ratio is independent of i) all parameters of the utility function other than the subjective discount rate β, and ii) the steady state rate of inflation $\bar{\pi}$. In fact, $\bar{k}$ depends only on the production function, the depreciation rate, and the discount rate. It is independent of the rate of inflation and the growth of money.

Because changes in the nominal quantity of money are engineered in this model by making lump sum transfers to the public, the real value of these transfers must equal:

$$\begin{aligned}\frac{M_t - M_{t-1}}{P_t} &= \frac{\theta M_{t-1}}{P_t} \\ &= \frac{\theta m_{t-1}}{1+\pi_t}.\end{aligned}$$

Hence, steady state transfers are given by:

$$\bar{\tau} = \frac{\theta \bar{m}}{1+\bar{\pi}} = \frac{\theta \bar{m}}{1+\theta},$$

and the budget constraint (297) reduces to:

$$\bar{c} = f(\bar{k}) - \delta \bar{k}. \tag{300}$$

The steady state level of consumption is equal to output minus replacement investment and is completely determined once the level of steady state capital is known. Assuming that $f(k) = k^{\alpha}$, then $\bar{k}$ is given by (299) and:

$$\bar{c} = \left[\frac{\alpha\beta}{1+\beta(\delta-1)}\right]^{\frac{\alpha}{1-\alpha}} - \delta\left[\frac{\alpha\beta}{1+\beta(\delta-1)}\right]^{\frac{1}{1-\alpha}}.$$

Steady state consumption per capital depends on the parameters of the production function α, the rate of depreciation δ, and the subjective rate of time discount β.

The Sidrauski MIU model exhibits a property called the superneutrality of money; the steady state values of the capital stock, consumption, and output are all independent of the rate of growth of the nominal money stock. That is, not only is money neutral, so that proportional changes in the level of nominal money balances and prices have no real effects, but changes in the rate of growth of nominal money also have no effect on the steady state capital stock or, therefore, on output or consumption. Since the real rate of interest is equal to the marginal product of capital, it also is invariant across steady states that differ only in their rates of money growth. Thus, the Sidrauski MIU model possesses the properties of both neutrality and superneutrality.

10.4.4 The demand for money

Returning to the opportunity cost of money (292), this equation characterises the demand for real money balances as a function of the nominal rate of interest and real consumption. For example, suppose that the utility function in consumption and real balances is of the constant elasticity of substitution (CES) form:

$$u(c_t, m_t) = \left[ac_t^{1-b} + (1-a)m_t^{1-b}\right]^{\frac{1}{1-b}},$$

with $a \in (0,1)$ and $b > 0, b \neq 1$. Then:

$$\frac{u_m}{u_c} = \left(\frac{1-a}{a}\right)\left(\frac{c_t}{m_t}\right)^b,$$

and (292) can be written as (in the limit as $b \to \infty$):

$$m_t = \left(\frac{1-a}{a}\right)^{\frac{1}{b}} \left(\frac{i_t}{1+i_t}\right)^{-\frac{1}{b}} c_t. \tag{301}$$

In terms of the more common log specification used to model empirical money demand equations, we have:

$$\log\frac{M_t}{P_t} = \frac{1}{b}\log\left(\frac{1-a}{a}\right) + \log c - \frac{1}{b}\log\frac{i_t}{1+i_t}, \tag{302}$$

which gives the real demand for money as a negative function of the nominal rate of interest and a positive function of consumption. The consumption (income) elasticity of money demand is equal to 1 in this specification. The elasticity of money demand with respect to the opportunity cost variable $\Upsilon_t = i_t/(1+i_t)$ is $1/b$. For simplicity, this will often be referred to as the interest elasticity of demand.

For $b = 1$, the CES specification becomes $u(c_t, m_t) = c_t^a m_t^{1-a}$. Note from (302) that in this case, the consumption (income) elasticity of money demand and the elasticity with respect to the opportunity cost measure Υ_t are both equal to 1.

While the parameter b governs the interest elasticity of demand, the steady state level of money holdings depends on the value of a. From (301), the ratio of real money balances to consumption in the steady state will be:

$$
\begin{aligned}
\frac{\bar{m}}{\bar{c}} &= \left(\frac{1-a}{a}\right)^{\frac{1}{b}} \left(\frac{\bar{i}}{1+\bar{i}}\right)^{-\frac{1}{b}} \\
&= \left(\frac{1-a}{a}\right)^{\frac{1}{b}} \left(\frac{\bar{r}+\bar{\pi}}{(1+\bar{r})(1+\bar{\pi})}\right)^{-\frac{1}{b}} \\
&= \left(\frac{1-a}{a}\right)^{\frac{1}{b}} \left(\frac{1+\beta^{-1}+\bar{\pi}}{\beta^{-1}(1+\bar{\pi})}\right)^{-\frac{1}{b}} \\
\therefore \frac{\bar{m}}{\bar{c}} &= \left(\frac{1-a}{a}\right)^{\frac{1}{b}} \left(\frac{\beta+1+\bar{\pi}}{1+\bar{\pi}}\right)^{-\frac{1}{b}}.
\end{aligned} \tag{303}
$$

The ratio of $\bar{m}$ to $\bar{c}$ is decreasing in a; an increase in a reduces the weight given to real money balances in the utility function and results in smaller holdings of money (relative to consumption) in the steady state. Increases in inflation also reduce the ratio of money holdings to consumption by increasing the opportunity cost of holding money!

10.4.5 The welfare cost of inflation

Because money holdings yield direct utility and higher inflation reduces real money balances, inflation generates a welfare loss. This raises two questions: How large is the welfare cost of inflation? Is there an optimal rate of inflation that maximises the steady state welfare of the representative household?

The second question – the optimal rate of inflation – was originally addressed by Bailey (1956) and Friedman (1969). Their basic intuition was the following. The private opportunity cost of holding money depends on the nominal rate of interest. The social marginal cost of producing money, that is, running the printing presses, is essentially zero. The wedge that arises between the private marginal cost and the social marginal cost when the nominal rate of interest is positive generates an inefficiency. This inefficiency would be eliminated if the private opportunity cost were also equal to zero, and this will be the case if the nominal rate of interest equals zero. Formally, in the MIU model, this condition can be represented by:

$$\frac{\partial u(\bar{c},\bar{m})}{\partial \theta} = 0 = u_m \frac{\partial \bar{m}}{\partial \theta},$$

because $\bar{c}$ is independent of money growth, θ. Then:

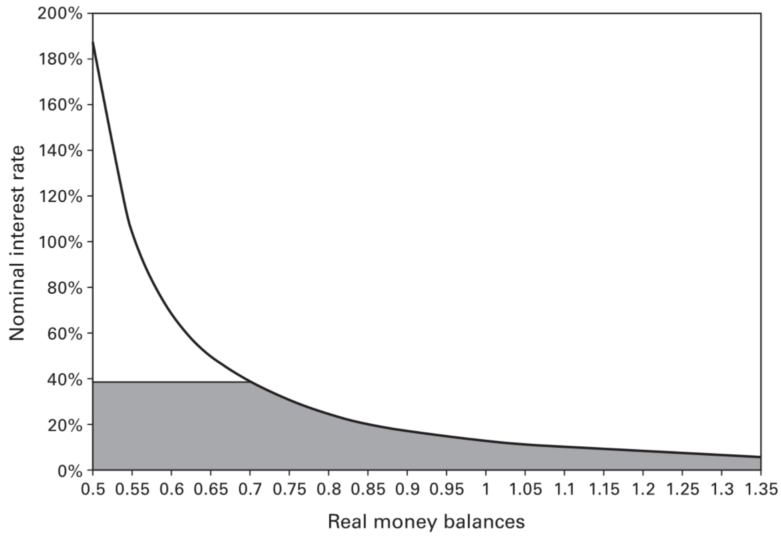
$$\begin{aligned} u_m &= 0, \\ \implies \frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} &= \frac{i}{1+i_t} = 0 \\ \implies i_t &= 0. \end{aligned}$$

But $i = 0$ requires that $\pi = -r/(1+r) \approx -r$. So the optimal rate of inflation is a rate of deflation approximately equal to the real return on capital.

In the steady state, real money balances are directly related to the inflation rate, so the optimal rate of inflation is also frequently discussed under the heading of the optimal quantity of money (Friedman 1969). With utility depending direct on m, one can think of the government choosing its policy instrument θ (and therefore π) to achieve the steady state optimal value of m.

The major criticism of this result is due to Phelps (1973), who pointed out that money growth generates revenue for the government – the inflation tax. The implicit assumption so far has been that variations in money growth are engineered via lump sum transfers. Any effects on government revenue can be offset by a suitable adjustment in these lump sum transfers. But if governments only have distortionary taxes available for financing expenditures, then reducing inflation tax revenues to achieve the Friedman rule of a zero nominal interest rate requires that the lost revenue be replaced through increases in other distortionary taxes. Reducing the nominal rate of interest to zero would increase the inefficiencies generated by the higher level of other taxes that would be needed to replace the lost inflation tax revenues. To minimise the total distortions associated with raising a given amount of revenue, it may be optimal to rely on the inflation tax to some degree. Other work on this issue is by Chari, Christiano, and Kehoe (1991; 1996) and Correia and Teles (1996).

Figure 57: Welfare Costs of Inflation as Measured by the Area under the Demand Curve



Source: Walsh (2010)

As for the first question – what is the welfare cost of inflation? Beginning with Bailey (1956), this welfare cost has been calculated from the area under the money demand curve (showing money demand as a function of the nominal interest rate) because this provides a measure of the consumer surplus lost as a result of having a positive nominal rate of interest. Figure 57 is based on the money demand function given by 301 with $a = 0.9$ and Chari, Kehoe, et al. (2000)'s implied value for b of 2.56. At a nominal interest rate of i^*, the deadweight loss is measured by the shaded area under the money demand curve.

Lucas (1994) provided estimates of the welfare costs of inflation by starting with the following specification of the instantaneous utility function:

$$u(c, m) = \frac{1}{1-\sigma}\left\{\left[c\varphi\left(\frac{m}{c}\right)\right]^{1-\sigma} - 1\right\}.$$

With this utility function 292 becomes:

$$\frac{u_m}{u_c} = \frac{\varphi'(x)}{\varphi(x) - x\varphi'(x)} = \frac{i}{1+i} = \Upsilon, \tag{304}$$

where $x \equiv m/c$. Normalising so that the steady state consumption equations 1, $u(1, m)$ will be maximised when $\Upsilon = 0$, implying that the optimal x is defined by $\varphi'(m^*) = 0$. Lucas proposed to measure the costs of inflation by the percentage increase in steady state consumption necessary to make the household indifferent between a nominal interest rate of i and a nominal rate of 0. If this cost is denoted $w(\Upsilon)$ it is defined by:

$$u(1 + w(\Upsilon), m(\Upsilon)) \equiv u(1, m^*),$$

where $m(\Upsilon)$ denotes the solution of (304) for real money balances evaluated at steady state consumption $c = 1$.

Suppose, following Lucas, that $\varphi(m) = (1 + Bm^{-1})^{-1}$, where B is a positive constant. Solving (304), one obtains $m(i) = B^{0.5}\Upsilon^{-0.5}$. Note that $\varphi' = 0$ requires that $m^* = \infty$. But $\varphi(\infty) = 1$ and $u(1, \infty) = 0$, so $w(\Upsilon)$ is the solution to $u(1 + w(\Upsilon), B^{0.5}\Upsilon^{-0.5}) = u(1, \infty) = 0$. Using the definition of the utility function, one obtains $1 + w(\Upsilon) = 1 + \sqrt{B\Upsilon}$, or:

$$w(\Upsilon) = \sqrt{B\Upsilon}.$$

Based on US annual data from 1900 to 1985, Lucas reported an estimate of 0.0018 for B. Hence, the welfare loss arising from a nominal interest rate of 10 percent would be $\sqrt{(0.0018)(0.1/1.1)} = 0.0013$, or just over 1 percent of aggregate consumption.

10.4.6 Breaking superneutrality and model dynamics

Suppose that labour was endogenously determined, so that the household's utility function is:

$$u(c_t, m_t, l_t) = u(c_t, m_t, 1 - h_t),$$

and that we get a new optimality condition:

$$\frac{u_l(c_t, m_t, l_t)}{u_c(c_t, m_t, l_t)} = f_h(k_{t-1}, h_t).$$

This states that an optimum, the marginal rate of substitution between consumption and leisure must equal the marginal product of labour.

The full details are in McCandless (2008) and Walsh (2010), which we won't delve into now, but what we find is that so long as the household's preferences are separable, superneutrality holds – changes in the steady state rate of inflation will alter nominal interest rates and the demand for real money balances, but different inflation rates have no effect on the steady state values of the capital stock, labour supply, or consumption.

If utility is not separable, so that either u_l or u_c (or both) depend on m, then money is not superneutral. Variations in average inflation that affect the opportunity cost of holding money will affect the steady state level of m. Different levels of $\bar{m}$ will change the value of $\bar{h}$. In other words, the steady state effect of money growth on real variables will depend on "strange" cross elasticities: u_{cm} and u_{lm}.

But does it make sense for the effects of monetary growth be channelled through effects of m on labour supply? What's even more troubling for us that the model's dynamics have unpalatable implications. For example, Walsh (2010) demonstrates that a positive money shock increases the nominal rate of interest; if there is persistence in the process for money growth, money growth rate shocks increase expected inflation and raise the nominal interest rate, while the real quantity of money actually falls.

This doesn't quite line up with the empirical findings we discussed earlier in this section, and so our search for a model which accommodates money continues.

10.5 Cash in advance constraints

A direct approach to generating a role for money proposed by Clower (1967) and developed by Lucas (1980; 1982), Stockman (1981), Svensson (1985), Lucas and Stokey (1987), and Cooley and Hansen (1989), captures the role of money as a medium of exchange by requiring explicitly that money be used to purchase goods.

Timing assumptions are important in cash in advance (CIA) models. In Lucas (1982), agents are able to allocate their portfolios between cash and other assets at the start if each period, after observing any current shocks but prior to purchasing goods. This timing is often described by saying that the asset market opens first and then the goods market opens. If there is a positive opportunity cost of holding money and the asset market opens first, agents will only hold an amount of money that is sufficient to finance their desired level of consumption. In Svensson (1985), the goods market opens first. this implies that agents have available for spending only the cash carried over from the previous period, and so cash balances must be chosen before agents know how much spending they will wish to undertake. For example, if uncertainty is resolved after money balances are chosen, an agent may find that he is holding cash balances that are too low to finance his desired spending level. Or he may be left with more cash than he needs, thereby forgoing interest income.

To understand the structure of CIA models, this section reviews a simplified version of a model due to Svensson (1985). Although, the alternative timing used by Lucas (1982) is also briefly discussed. After the model and its equilibrium conditions are set out, the steady state is examined and the welfare costs of inflation a CIA model are discussed.

10.5.1 A simple Svensson CIA model

Consider the following representative agent model. The agent's objective is to chose a path for consumption and asset holdings to maximise:

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \ \beta \in (0,1),$$

where $u(\cdot)$ is bounded, continuously differentiable strictly increasing, and strictly concave, and the maximisation is subject to a sequence of CIA and budget constraints. The agent enters the period with money holdings M_{t-1} and receives a lump-sum transfer T_t (in nominal terms). If goods markets open first, the CIA constraint takes the form:

$$P_t c_t \leq M_{t-1} + T_t,$$

where c_t is real consumption, P_t is the aggregate price level, and T_t are nominal lump sum transfers in period t. In real terms this is:

$$c_t \leq \frac{M_{t-1}}{P_t} + \frac{T_t}{P_t} = \frac{m_{t-1}}{1+\pi_t} + \tau_t, \tag{305}$$

where $m_{t-1} = M_{t-1}/P_{t-1}$, $\pi_t = (P_t/P_{t-1}) - 1$ is the net inflation rate, and $\tau_t = T_t/P_t$. Note the timing: M_{t-1} refers to nominal money balances chosen by the agent in period $t-1$ and carried into period t. The real value of these balances is determined by the period t price level P_t. Since we have assumed away any uncertainty, the agent knows P_t at the time M_{t-1} is chosen. This specification of the CIA constraint assumes that income from production during period t will not be available for consumption purchases until period $t+1$.

The budget constraint, nominal terms, is:

$$P_t\omega_t \equiv P_t f(k_{t-1}) + (1-\delta)P_t k_{t-1} + M_{t-1} + T_t + R_{t-1}B_{t-1} \geq P_t c_t + P_t k_t + M_t + B_t,$$

where ω_t is the agent's time t real resources, consisting of income generated during period t, $f(k_{t-1})$, the undepreciated capital stock $(1-\delta)k_{t-1}$, money holdings, the transfer from the government, and gross nominal interest earnings on the agent's $t-1$ holdings of nominal one-period bonds, B_{t-1}. Physical capital depreciates at the rate δ. These resources are used to purchase consumption, capital, bonds, and nominal money holdings that are then carried into period $t+1$. Dividing through by the time t price level, the budget constraint can be rewritten in real terms as:

$$\omega_t \equiv f(k_{t-1}) + (1-\delta)k_{t-1} + \tau_t + \frac{m_{t-1} + R_{t-1}b_{t-1}}{1+\pi_t} \geq c_t + m_t + b_t + k_t. \tag{306}$$

Note that real resources available to the representative agent in period $t+1$ are given by:

$$\omega_{t+1} = f(k_t) + (1-\delta)k_t + \tau_{t+1} + \frac{m_t + R_t b_t}{1+\pi_{t+1}}. \tag{307}$$

The period t gross nominal interest rate R_t can be divided by $1+\pi_{t+1}$, the gross inflation rate in $t+1$, to yield the gross real rate of return from period t to $t+1$, and can be denoted by:

$$1 + r_t = \frac{R_t}{1 + \pi_{t+1}} = \frac{1 + i_t}{1 + \pi_{t+1}}. \tag{308}$$

With this notation (307) can be written as:

$$\omega_{t+1} = f(k_t) + (1 - \delta)k_t + \tau_{t+1} + (1 + r_t)a_t - \left(\frac{i_t}{1 + \pi_{t+1}}\right) m_t, \tag{309}$$

where $a_t \equiv m_t + b_t$ is the agent's holding of nominal financial assets (money and bonds). Writing it this way shows that there is a cost to holding money when the nominal interest rate is positive. This cost is $i_t/(1 + \pi_{t+1})$; since this is the cost in terms of period $t+1$ real resources, the discounted cost at time t of holding an additional unit of money is:

$$\frac{i_t}{(1 + r_t)(1 + \pi_{t+1})} = \frac{i_t}{1 + i_t} = \Upsilon_t.$$

This is the same expression for the opportunity cost of money in the MIU model.

10.5.2 Lucas' alternate timing convention

Equation (305) is based on the timing convention that goods markets open before asset markets. The model of Lucas (1982) assumed the reverse, and individuals can engage in asset transactions at the start of each period before the goods market has opened. In the present model, this would mean that the agent enters period t with financial wealth that can be used to purchase nominal bonds B_t or carried as cash into the goods market to purchase consumption goods. The CIA constraint would then take the form:

$$c_t \leq \frac{m_{t-1}}{1 + \pi_t} + \tau_t - b_t. \tag{310}$$

In this case, the household is able to adjust its portfolio between money and bonds before entering the goods market to purchase consumption goods.

To understand the implications of this alternative timing, suppose there is a positive opportunity cost of holding money. Then, if the asset market opens first, the agent will only hold an amount of money that is just sufficient to finance the desired level of consumption. Since the opportunity cost of holding m is positive whenever the nominal interest rate is greater than zero, (310) will always hold with equality as long as the nominal rate of interest is positive. When uncertainty is introduced, the CIA constraint may not bind when (305) is used and the goods market opens before the asset market. For example, if period t's income is uncertain and is realised after M_{t-1} has been chosen, a bad income realisation may cause the agent to reduce consumption to a point where the CIA constraint is no longer binding. Or a disturbance that causes an unexpected price decline might, by increasing the real value of the agent's money holdings, result in a nonbinding constraint. Since a nonstochastic environment holds in this section, the CIA constraint will bind under either timing assumption if the opportunity cost of holding money is positive.

10.5.3 Equilibrium

The choice variables at time t are c_t, m_t, b_t, and k_t. An individual agent's state at time t can be characterised by her resources ω_t and her real cash holdings m_{t-1}; both are relevant because the consumption choice is constrained by the agent's resources and by cash holdings. To analyse the agent's decision problem, one can define the value function:

$$V(\omega_t, m_{t-1}) = \max_{c_t, k_t, b_t, m_t} \left\{ u(c_t) + \beta V(\omega_{t+1}, m_t) \right\}, \tag{311}$$

subject to (306), (305), and (309):

$$\begin{aligned}
\omega_t &\geq c_t + m_t + b_t + k_t, \\
c_t &= \frac{m_{t-1}}{1+\pi_t} + \tau_t, \\
\omega_{t+1} &= f(k_t) + (1-\delta)k_t + \tau_{t+1} + (1+r_t)a_t - \left(\frac{i_t}{1+\pi_{t+1}} \right) m_t,
\end{aligned}$$

Using the expression for ω_{t+1} and letting λ_t and μ_t denote the Lagrangian/Kuhn-Tucker multipliers for the budget constraint and CIA constraint, respectively, the FOCs take the form:

$$\frac{\partial V(\omega_t, m_{t-1})}{\partial c_t} = u'(c_t) - \lambda_t - \mu_t = 0, \tag{312}$$

$$\frac{\partial V(\omega_t, m_{t-1})}{\partial k_t} = \beta(f'(k_t) + 1 - \delta)V_\omega(\omega_{t+1}, m_t) - \lambda_t = 0, \tag{313}$$

$$\frac{\partial V(\omega_t, m_{t-1})}{\partial m_t} = \beta\left(1 + r_t - \frac{i_t}{1+\pi_{t+1}}\right)V_\omega(\omega_{t+1}, m_t) + \beta V_m(\omega_{t+1}, m_t) - \lambda_t = 0, \tag{314}$$

$$\frac{\partial V(\omega_t, m_{t-1})}{\partial b_t} = \beta\left(1 + r_t\right)V_\omega(\omega_{t+1}, m_t) - \lambda_t = 0. \tag{315}$$

By the envelope theorem:

$$\frac{\partial V(\omega_t, m_{t-1})}{\partial \omega_t} = \lambda_t, \tag{316}$$

$$\frac{\partial V(\omega_t, m_{t-1})}{\partial m_{t-1}} = \left(\frac{1}{1+\pi_t}\right)\mu_t. \tag{317}$$

From (316), λ_t is equal to the marginal utility of wealth. According to (312), the marginal utility of consumption exceeds the marginal utility of wealth by the value of liquidity services, μ_t. The individual must hold money in order to purchase consumption, so the “cost”, to which the marginal utility of consumption is set equal, is the marginal utility of wealth plus the cost of the liquidity services needed to finance the transaction.

In terms of λ, (315) becomes:

$$\lambda_t = \beta(1 + r_t)\lambda_{t+1}, \tag{318}$$

which is a standard asset pricing equal and is a familiar condition problems involving intertemporal optimisation. Along the optimal path, the marginal cost (in terms of today’s utility) from reducing wealth slightly, λ_t, must equal the utility value of carrying that wealth forward one period, earning a gross real return $1 + r_t$, where tomorrow’s utility is discounted back to today at the rate β; that is, $\lambda_t = \beta(1 + r_t)\lambda_{t+1}$ along the optimal path.

Using (316) and (317), the FOC (314) can be expressed as:

$$\lambda_t = \beta \left(\frac{\lambda_{t+1} + \mu_{t+1}}{1 + \pi_{t+1}} \right).$$

This equation can be interpreted as an asset pricing equation for money. The price of a unit of money in terms of goods is just $1/P_t$ at time t; its value in utils is λ_t / P_t. Now, by dividing the above equation by P_t, it can be rewritten as:

$$\frac{\lambda_t}{P_t} = \beta \left(\frac{\lambda_{t+1}}{P_{t+1}} + \frac{\mu_{t+1}}{P_{t+1}} \right).$$

Solving this equation forward implies that:

$$\frac{\lambda_t}{P_t} = \sum_{i=1}^{\infty} \beta^i \left(\frac{\mu_{t+i}}{P_{t+i}} \right).$$

From (317), μ_{t+i}/P_{t+i} is equal to $V_m(\omega_{t+i}, m_{t+i-1})/P_{t+i-1}$. This last expression though is just the partial of the value function with respect to time $t+i-1$ nominal money balances:

$$\begin{aligned} \frac{\partial V(\omega_{t+i}, m_{t+i-1})}{\partial M_{t+i-1}} &= V_m(\omega_{t+i}, m_{t+i-1}) \left(\frac{\partial m_{t+i-1}}{\partial M_{t+i-1}} \right) \\ &= \frac{V_m(\omega_{t+i}, m_{t+i-1})}{P_{t+i-1}} \\ &= \left(\frac{\mu_{t+i}}{P_{t+i}} \right). \end{aligned}$$

This means that we can write:

$$\frac{\lambda_t}{P_t} = \sum_{i=1}^{\infty} \beta^i \frac{\partial V(\omega_{t+i}, m_{t+i-1})}{\partial M_{t+i-1}}. \tag{319}$$

In words, the current value of money in terms of utility is equal to the present value of the marginal utility of money in all future periods. This is an interesting result; it says that money is just like any other asset in the sense that its value can be thought of as equal to the present discounted value of the stream of returns generated by the asset. In the case of money, these returns take the form of liquidity services. If the CIA constraint were not binding, these liquidity services would have no value ($\mu = V_m = 0$) and nor would money. But if the constraint is binding, then money has value because it yields valued liquidity services.

The result that the value of money, λ/P, satisfies an asset pricing relationship is not unique to the CIA approach. For example, a similar relationship is implied by the MIU approach. The model employed in the analysis of the MIU approach implied that:

$$\frac{\lambda_t}{P_t} = \beta\left(\frac{\lambda_{t+1}}{P_{t+1}}\right) + \frac{u_m(c_t, m_t)}{P_t},$$

which can be solved forward to yield:

$$\frac{\lambda_t}{P_t} = \sum_{i=0}^{\infty} \beta^i \left[\frac{u_m(c_{t+i}, m_{t+i})}{P_{t+i}}\right].$$

Here, the marginal utility of money u_m plays a role exactly analogous to that played by the Lagrangian on the CIA constraint μ. The one difference is that in the MIU approach, m_t yields utility at time t, whereas in the CIA approach, the value of money accumulated at time t is measured by μ_{t+1} because the cash cannot be used to purchase consumption goods until period $t+1$.

An expression for nominal rate of interest can be obtained by using our results for λ_t to obtain:

$$\begin{aligned}
\lambda_t &= \beta(1+r_t)\lambda_{t+1} \\
\lambda_t &= \beta\left[\frac{\lambda_{t+1} + \mu_{t+1}}{1+\pi_{t+1}}\right] \\
\Leftrightarrow (1+r_t)(1+\pi_{t+1})\lambda_{t+1} &= \lambda_{t+1} + \mu_{t+1}.
\end{aligned}$$

Since $1 + i_t = (1+r_t)(1+\pi_{t+1})$, the nominal interest rate is given by:

$$i_t = \left(\frac{\lambda_{t+1} + \mu_{t+1}}{\lambda_{t+1}}\right) - 1 = \frac{\mu_{t+1}}{\lambda_{t+1}}. \tag{320}$$

Thus, the nominal rate of interest is positive if and only if money yields liquidity services ($\mu_{t+1} > 0$). In particular, if the nominal interest rate is positive, the CIA constraint is binding ($\mu > 0$).

10.5.4 The steady state

In the steady state, (318) implies that:

$$1 + \bar{r} = \frac{1}{\beta},$$

and:

$$\begin{aligned} i &= \frac{1 + \bar{\pi}}{\beta} - 1 \\ &\approx \frac{1}{\beta} - 1 + \bar{\pi}. \end{aligned}$$

In addition (313) gives the steady state capital stock as the solution to:

$$f_k(\bar{k}) = \bar{r} + \delta = \frac{1}{\beta} - 1 + \delta.$$

So this CIA model, like the Sidrauski MIU model, exhibits superneutrality. The steady state capital stock depends only on the time preferences parameter β, the rate of depreciation δ, and the production function. It is independent of the rate of inflation. Since steady state consumption is equal to $f(\bar{k}) - \delta\bar{k}$, it too is independent of the rate of inflation.

It has been shown that the marginal utility of consumption could be written as the marginality of wealth (λ) times 1 plus the nominal rate of interest, reflecting the opportunity cost of holding the money required to purchase goods for consumption. Using (320), the ratio of the liquidity value of money, measured by the Lagrangian multiplier μ, to the marginal utility of consumption is:

$$\frac{\mu_t}{u_c(c_t)} = \frac{\mu_t}{\lambda_t(1 + i_t)} = \frac{i_t}{1 + i_t}.$$

This expression is exactly parallel to the result in the MIU framework, where the ratio of the marginal utility of money to the marginal utility of consumption was equal to the nominal interest rate divided by 1 plus the nominal rate, that is, the relative price of money in terms of consumption.

With the CIA constraint binding, real consumption is equal to real money balances. In the steady state, constant consumption implies that the stock of nominal money balances and the price level must be changing at the same rate. Define θ as the growth rate of the nominal quantity of money (so that $T_t = \theta M_{t-1}$); then:

$$\bar{\pi} = \theta.$$

The steady state inflation rate is, as usual, determined by the rate of growth of the nominal money stock.

One difference between the CIA model and the MIU model is that with $\bar{c}$ independent of inflation and the CIA constraint binding, the fact that $\bar{c} = \bar{m}$ in the CIA model implies that the steady state money holdings are also independent of inflation.

10.5.5 Welfare costs of inflation and model dynamics

The CIA model, because it is based explicitly on behavioural relationships consistent with utility maximisation, can be used to assess the welfare costs of inflation and to determine the optimal rate of inflation. The MIU approach had very strong implications for the optimal inflation rate. Steady state utility of the representative household was maximised when the nominal rate of interest equalled zero. It has already been suggested that this conclusion continues to hold when money produces transaction services.

In the basic CIA model, however, there is no optimal rate of inflation that maximises the steady state welfare of the representative household. The reason follows directly from the specification of utility as a function only of consumption and the result that consumption is independent of the rate of inflation (superneutrality). Steady state welfare is equal to:

$$\sum_{t=0}^{\infty} \beta^t u(\bar{c}) = \frac{u(\bar{c})}{1-\beta},$$

and is invariant to the inflation rate. Comparing across steady states, any inflation rate is as good as any other!

This finding is not robust to modifications in the basic CIA model. In particular, once the model is extended to incorporate a labour-leisure choice, consumption will no longer be independent of the inflation rate, and there will be a well defined optimal rate of inflation. Because leisure can be "purchased" without the use of money (i.e. leisure is not subject to the CIA constraint), variations in the rate of inflation will affect the marginal rate of substitution between consumption and leisure. With different inflation rates leading to different levels of steady state consumption and leisure, steady state utility will be a function of inflation. This type of substitution plays an important role in the model of Cooley and Hansen (1989).

In other words, in the Cooley and Hansen (1989) model (which is essentially an RBC model with CIA constraints), including leisure breaks superneutrality. An increase in π will lead to an increase in the relative price of consumption verses leisure, causing a substitution and eventually a decline in employment and consumption. This effect may be ambiguous as in the MIU model, but if the cross elasticity $u_{cm} > 0$, which implies that money and consumption are complements, then an increase in πwill lead to a decline in m, a decline in u_c, and a reduction in the supply of labour.

Furthermore, like the MIU model, the results of the Cooley and Hansen model suggest that an increase in money supply lead to an increase in the nominal interest rate (the other effects are qualitatively much like the MIU model, but greater in magnitude).

10.6 A classical monetary model

In this section, we move toward a full-scale DSGE model by introducing a simple model of a classical monetary economy, featuring perfect competition and fully flexible prices in all markets. Many of the predictions from this simple monetary model will not aline with the empirical evidence we reviewed in an early section. But nevertheless, the analysis of the simple monetary model provides a benchmark that will be useful later on. For this section we will adopt the notation of Galí (2015). To remotivate why we're doing this, the following quote from Lucas is fitting:

> "Nominal variables - the quantity of money, the general price level, and nominal rates of interest - play no role in the Kydland-Prescott model [...] One consequence of this omission is that these theories cannot shed light on the problem of inflation or on the observed associations between movements in money and prices and real economic activity. [...] What I would like to do next, then, is to introduce money into a neoclassical dynamic

framework in such a way as to restate in modern terms the quantity theory of money, inflation, and interest."

Lucas (1987)

Without further ado, let's get started.

10.6.1 Households

The representative household seeks to maximise the objective function:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s u(C_{t+s}, N_{t+s}),$$

where C_t is consumption of the single good, and N_t denotes hours of work or employment, and is subject to the following flow budget constraints:

$$P_t C_t + Q_t B_t \leq B_{t-1} + W_t N_t - D_t,$$

where P_t is the price of the consumption good, W_t denotes the nominal wage, B_t are holdings of one-period nominally riskless discount bonds purchased in period t and maturing in $t+1$, Q_t is the bond price[62], and D_t represents dividends to the household expressed in nominal terms. We also assume that the household is subject to a solvency constraint that prevents it from engaging in Ponzi type schemes:

$$\lim_{T\to\infty} \mathbb{E}_t \Lambda_{t,T} \frac{B_T}{P_T} \geq 0, \ \forall t,$$

where $\Lambda_{t,T}$ is the stochastic discount factor:

$$\Lambda_{t,T} = \beta^{T-t} \frac{u_{c,T}}{u_{c,t}}.$$

[62] The yield on the one period bond is defined by $Q_t \equiv (1+\text{yield})^{-1}$. Note that $i_t \equiv -\log Q_t = \log(1+\text{yield}_t) \approx \text{yield}_t$, where the latter approximation will be accurate as long as the nominal yield is "small".

The optimality conditions implied by the household maximisation problem are given by:

$$-\frac{u_{n,t}}{u_{c,t}} = \frac{W_t}{P_t}, \tag{321}$$

$$Q_t = \beta \mathbb{E}_t \frac{u_{c,t+1}}{u_{c,t}} \frac{P_t}{P_{t+1}}. \tag{322}$$

If we assume CRRA utility:

$$u(C_t, N_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi},$$

then the optimality conditions become:

$$\frac{W_t}{P_t} = C_t^{\sigma} N_t^{\varphi}, \tag{323}$$

$$Q_t = \beta \mathbb{E}_t \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{P_t}{P_{t+1}}. \tag{324}$$

Note, for future reference, (323), which is our familiar intratemporal Euler equation (or labour supply schedule) can be written in log linear form as:

$$\hat{W}_t - \hat{P}_t = \sigma \hat{C}_t + \varphi \hat{N}_t, \tag{325}$$

where hat variables denote log deviations from steady state. A log linearised equation for (324) with constant rates of inflation and consumption growth is given by:

$$\hat{C}_t = \mathbb{E}_t \hat{C}_{t+1} - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}\hat{\pi}_{t+1} - \rho), \tag{326}$$

where $i_t = -\log Q_t$, $\rho = -\log \beta$, and $\hat{\pi}_{t+1} = \hat{P}_{t+1} - \hat{P}_t$ is the rate of inflation between t and $t+1$. Notice that $\hat{i}_t$ corresponds to the log of the gross yield on the one period bond; henceforth, it is referred to as the nominal interest rate. Similarly, ρ can be interpreted as the household's discount rate.

We explored motivations for holding real money balances previously in this chapter (the MIU and CIA models). We didn't, however, postulate a demand function for real money balances. In log linear form, let's say that the demand for real money balances is given by:

$$\hat{M}_t - \hat{P}_t = \hat{Y}_t - \eta \hat{i}_t, \tag{327}$$

where $\eta \geq 0$ denotes the interest semi-elasticity of money demand.

10.6.2 Firms

A representative firm is assumed whose technology is described by a production function given by:

$$Y_t = A_t N_t^{1-\alpha},$$

where A_t represents the level of technology, and $\hat{A} = \log A_t$ evolves exogenously according to some AR(1) stochastic process:

$$\hat{A}_t = \rho_a \hat{A}_{t-1} + \epsilon_{a,t},$$

where $\rho_a \in (0, 1)$.

Each period the firm maximises profits:

$$P_t Y_t - W_t N_t,$$

subject to the production technology, taking prices and wages as given. The firm's maximisation problem yields the optimality condition:

$$\frac{W_t}{P_t} = (1 - \alpha) A_t N_t^{-\alpha}. \tag{328}$$

In words, the firm hires labour up to the point where its marginal product equals the real wage. Equivalently, the price of output must equal the marginal cost:

$$P_t = \frac{W_t}{(1 - \alpha) A_t N_t^{-\alpha}}.$$

In log linear terms, this is:

$$\hat{W}_t - \hat{P}_t = \hat{A}_t - \alpha \hat{N}_t \tag{329}$$

This equation can be interpreted as a labour demand schedule, mapping the real wage into the quantity of labour demanded, given the level of technology.

10.6.3 Equilibrium

The baseline model abstracts from sources of goods demand other than consumption (like investment, government purchases, or net exports). Accordingly, the goods market clearing condition is given by:

$$\hat{Y}_t = \hat{C}_t, \tag{330}$$

that is, all output must be consumed.

By combining the optimality conditions of households and firms with the goods market clearing condition and the log-linear aggregate production relationship:

$$\hat{Y}_t = \hat{A} + (1-\alpha)\hat{N}_t, \tag{331}$$

one can determine the equilibrium levels of employment and output:

$$\hat{N}_t = \psi_{na}\hat{A}_t, \tag{332}$$

$$\hat{Y}_t = \psi_{ya}\hat{A}_t, \tag{333}$$

where:

$$\psi_{na} = \frac{1-\sigma}{\sigma(1-\alpha)+\varphi+\alpha},$$
$$\psi_{ya} = \frac{1+\varphi}{\sigma(1-\alpha)+\varphi+\alpha},$$

Furthermore, given the equilibrium process for output, (326) can be used to determine the implied real interest rate $\hat{r}_t \equiv \hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1}$, as:

$$\begin{aligned} \hat{r}_t &= \rho + \sigma \mathbb{E}_t \Delta \hat{Y}_{t+1} \\ &= \rho - \sigma(1 - \rho_a)\psi_{ya}\hat{A}_t. \end{aligned} \tag{334}$$

Finally, the equilibrium real wage $\hat{\omega}_t \equiv \hat{W}_t - \hat{P}_t$, is given by:

$$\hat{\omega}_t = \psi_{\omega a}\hat{A}_t, \tag{335}$$

where:

$$\psi_{\omega a} = \frac{\sigma + \varphi}{\sigma(1-\alpha) + \varphi + \alpha}.$$

Notice that the equilibrium levels of employment, output, and the real interest rate are determined independently of monetary policy. In other words, monetary policy is neutral with respect to those real variables (i.e. money is neutral in the short-run). In the simple model, output, employment, and the real wage fluctuate in response to variations in technology. In particular, output always rises in the face of a productivity increase, with the size of the increase being given by $\psi_{ya} > 0$. The same is true for the real wage. On the other hand, the sign of the employment is ambiguous, depending on whether σ (which measures the strength of the income effect on labour supply) is larger or small than 1. When $\sigma < 1$, the substitution effect on labour supply resulting from a higher real wage dominates the negative effect caused by a smaller marginal utility of consumption, leading to an increase in employment. The converse is true whenever $\sigma > 1$. When the utility of consumption is logarithmic ($\sigma = 1$), employment remains unchanged in the face of technology variations, for substitution and income effects cancel one another. Finally, and under the assumption on the process of technology, the real interest rate goes down in response to a positive technology shock.

What about nominal variables, like inflation or the nominal interest rate? Not surprisingly, and in contrast with real variables, their equilibrium values cannot be determined independently of monetary policy. To illustrate how nominal variables are influenced by the way monetary policy is conducted, their equilibrium behaviour under alternative monetary policy rules will be considered next.

10.6.4 Monetary policy and price level determination

Let us start by examining the implications of some interest rate rules. We will need these rules in order to close our model. Rules that involve monetary aggregates will be introduced later. Recall the Fisher relation:

$$\hat{i}_t = \mathbb{E}_t \hat{\pi}_{t+1} + \hat{r}_t,$$

which implies that the nominal rate adjusts one-for-one with expected inflation, given a real interest rate that is determined exclusively by real factors. Equation (334) implies that in the steady state without growth $\bar{r} = \rho$, that is, the real interest rate is equal corresponds to the household's discount rate. Thus it follows from the Fisher relation that:

$$\bar{i} = \rho + \bar{\pi}.$$

In what follows the analysis is restricted to nonexplosive equilibrium paths for inflation and the nominal interest rate, that is, equilibrium paths that remain within a bounded neighbourhood of the steady state, for sufficiently small fluctuations in the exogenous driving forces.

10.6.5 An exogenous path for the nominal interest rate

Le us first consider the case of a monetary policy that implies an exogenous path for the nominal interest rate. For concreteness, let us assume the rule:

$$\hat{i}_t = 0, \tag{336}$$

A particular case of this rule corresponds to a constant interest rate, that is, $i_t = \bar{i}$ for all t. Note that $\bar{\pi} = \bar{i} - \rho$ is the steady state inflation (or the implicit long-run inflation target) associated with the rule above. So from the Fisher relation we have:

$$\begin{aligned}\mathbb{E}_t\hat{\pi}_{t+1} &= \hat{i}_t - \hat{r}_t \\ &= -\hat{r}_t.\end{aligned}$$

While this pins down expected inflation $\mathbb{E}\hat{\pi}_{t+1}$ because $\hat{r}_t$ is a function of $\hat{A}_t$ and parameters, it does not pin down actual inflation. Why? Because any process which satisfies the following:

$$\mathbb{E}_t\hat{P}_{t+1} - \hat{P}_t = -\hat{r}_t - \xi_{t+1},$$

where $\mathbb{E}_t\xi_{t+1} = 0$ is consistent with the equilibrium equations. Shocks such as $\mathbb{E}_t\xi_{t+1}$ are referred to in the literature as sunspot shocks. An equilibrium in which such non-fundamental factors may cause fluctuations in one or more variables is referred to as an indeterminate equilibrium. The example above shows how an exogenous nominal interest rate leads to price level indeterminacy.

10.6.6 A simple interest rate rule

Suppose that the central bank adjusts the nominal interest rate in response to deviations of inflation from a target, $\bar{\pi}$, according to the interest rate rule:

$$\hat{i}_t = \phi_\pi\hat{\pi}_t, \tag{337}$$

where $\phi_\pi \geq 0$ is a coefficient determining the strength of the endogenous response of monetary policy. Combining this rule with the Fisher relation, we have:

$$\hat{\pi}_t = \frac{1}{\phi_\pi}\mathbb{E}_t\hat{\pi}_{t+1} + \frac{1}{\phi_\pi}\hat{r}_t, \tag{338}$$

and if we add in "monetary policy shocks" denoted by v_t, where the v_t is given by:

$$v_t = \rho_v v_{t-1} + \epsilon_{v,t},$$

we get:

$$\hat{\pi}_t = \frac{1}{\phi_\pi}\mathbb{E}_t\hat{\pi}_{t+1} + \frac{1}{\phi_\pi}\hat{r}_t + \frac{1}{\phi_\pi}\hat{v}_t. \tag{339}$$

If $\phi_\pi > 1$, the previous difference equation has only one nonexplosive solution. that solution can be obtained by solving (339) forward which yields:

$$\hat{\pi}_t = \sum_{k=0}^{\infty}\phi_\pi^{-(k+1)}\mathbb{E}_t\left[\hat{r}_{t+k} - v_{t+k}\right].$$

The previous equation fully determines inflation (and, hence, the price level) as a function of the path of the real interest rate, which in turn is a function of exogenous real forces, as shown in (334). In particular, under the assumed driving processes for technology and preference parameters, inflation can be written as:

$$\hat{\pi}_t = -\frac{\sigma(1-\rho_a)\psi_{ya}}{\phi_\pi - \rho_a}\hat{A}_t - \frac{1}{\phi_\pi - \rho_v}\hat{v}_t. \tag{340}$$

Note that a central bank following a rule of the form considered here can influence the degree of inflation volatility through its choice of ϕ_π. The larger that coefficient is, the smaller will be the impact of real shocks on inflation. Monetary policy shocks, v_t, are seen to generate "unnecessary" fluctuations in inflation. Given equilibrium path for inflation, the price and nominal wage levels are uniquely determined by the identities $\hat{P}_t = \hat{P}_{t-1} + \hat{\pi}_t$ and $\hat{W}_t = \hat{\omega}_t - \hat{P}_t$.

On the other hand, if $\phi_\pi < 1$, the forward solution of (339) does not converge. Instead, the stationary solution takes the form:

$$\hat{\pi}_{t+1} = \phi_\pi\hat{\pi}_t - \hat{r}_t + \hat{v}_t + \xi_t, \tag{341}$$

where $\mathbb{E}_t\xi_{t+1} = 0$ is an arbitrary sequence of sunspot shocks.

Accordingly, any process for π_t satisfying (341) is consistent with equilibrium, while remaining in a neighbourhood of the steady state (for sufficiently small shocks). So, as in the case of an exogenous nominal rate, the price level (and, hence, inflation) are not determined uniquely when the interest rate rule implies a weak response of the nominal rate to deviations of inflation from target.

More specifically, the condition for a determinate price level, $\phi_\pi > 1$, requires that the central bank adjusts nominal interest rates more than one-for-one in response to any change in inflation, a property known as the Taylor principle. The previous result can be viewed as a particular instance of the need to satisfy the Taylor principle in order for an interest rate rule to bring about a determinate equilibrium.

10.6.7 An exogenous path for the money supply

What about a money supply rule? To eliminate $\hat{i}_t$ from our model and close it, we need to specify a money demand (which we did in (327)):

$$\hat{M}_t - \hat{P}_t = \hat{Y}_t - \eta \hat{i}_t,$$

and combine it with the Fisher relation to get:

$$\hat{P}_t = \frac{\eta}{1+\eta} \mathbb{E}_t \hat{P}_{t+1} + \frac{1}{1+\eta} \hat{M}_t + u_t, \tag{342}$$

where:

$$u_t = \frac{1}{1+\eta} (\eta \hat{r}_t - \hat{Y}_t).$$

Assuming that $\eta > 0$ and solving (342) forward, we get:

$$\begin{aligned}
\hat{P}_t &= \frac{1}{1+\eta} \sum_{i=0}^{\infty} \left(\frac{\eta}{1+\eta} \right)^i \mathbb{E}_t \hat{M}_{t+i} + \sum_{i=0}^{\infty} \left(\frac{\eta}{1+\eta} \right)^i \mathbb{E}_t u_{t+i} \\
&= \left(1 - \frac{\eta}{1+\eta} \right) \sum_{i=0}^{\infty} \left(\frac{\eta}{1+\eta} \right)^i \mathbb{E}_t \hat{M}_{t+i} + \sum_{i=0}^{\infty} \left(\frac{\eta}{1+\eta} \right)^i \mathbb{E}_t u_{t+i} \\
&= \sum_{i=0}^{\infty} \left(\frac{\eta}{1+\eta} \right)^i \mathbb{E}_t \hat{M}_{t+i} - \sum_{i=0}^{\infty} \left(\frac{\eta}{1+\eta} \right)^{i+1} \mathbb{E}_t \hat{M}_{t+i} + \sum_{i=0}^{\infty} \left(\frac{\eta}{1+\eta} \right)^i \mathbb{E}_t u_{t+i} \\
&= \hat{M}_t + \sum_{i=1}^{\infty} \left(\frac{\eta}{1+\eta} \right)^i \mathbb{E}_t \Delta \hat{M}_{t+i} + \sum_{i=0}^{\infty} \left(\frac{\eta}{1+\eta} \right)^i \mathbb{E}_t u_{t+i}.
\end{aligned} \tag{343}$$

Thus, when the monetary policy rule takes the form of an exogenous path for the money supply, the equilibrium price level is always determined uniquely.

Given our forward iterated solution, the money demand equation (327) can be used to solve for the nominal interest rate:

$$\begin{aligned}\hat{i}_t &= \frac{1}{\eta}\hat{Y}_t - \frac{1}{\eta}(\hat{M}_t - \hat{P}_t) \\ &= \frac{1}{\eta}\sum_{i=1}^{\infty}\left(\frac{\eta}{1+\eta}\right)^i \mathbb{E}_t \Delta \hat{M}_{t+i} + \frac{1}{\eta}\left[\hat{Y}_t + \sum_{i=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^i \mathbb{E}_t u_{t+i}\right]. \end{aligned} \tag{344}$$

Consider the case of an AR(1) process for the money supply:

$$\Delta \hat{M}_t = \rho_m \Delta \hat{M}_{t-1} + \epsilon_{m,t},$$

then we have:

$$\mathbb{E}_t \Delta \hat{M}_{t+i} = \rho_m^i \Delta \hat{M}_t.$$

This would yield a solution for the price level as:

$$\hat{P}_t = \hat{M}_t + \frac{\eta \rho_m}{1+\eta(1-\rho_m)}\Delta \hat{M}_t + \sum_{i=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^i \mathbb{E}_t u_{t+i},$$

which would imply that the price level response more than one-to-one with respect to an increase in the money supply.

The nominal interest rate is in turn given by:

$$\hat{i}_t = \frac{\rho_m}{1+\eta(1-\rho_m)}\Delta \hat{M}_t + \frac{1}{\eta}\left[\hat{Y}_t + \sum_{i=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^i \mathbb{E}_t u_{t+i}\right].$$

That is, in response to an expansion of the money supply, and as long as $\rho_m > 0$, the nominal interest rate is predicted to go up. In other words, the model implies the absence of the liquidity effect, in stark contrast with the evidence discussed earlier in the chapter.

10.7 Comments and key readings

This was quite a long and in depth chapter, where we motivated some of the deficiencies of an RBC model, as well as the empirical facts a model with nominal variables should achieve.

We started by looking at the long-run relationship between inflation and the growth of money, where for additional reading refer to McCandless and Weber (1995) and Lucas (1980). Then we looked at the relationship between inflation and growth, where additional recommended readings are Barro (1995; 1996) and Bullard (1999).

Then we moved onto short term responses of output to monetary policy: Key papers are Leeper et al. (1996) and Christiano et al. (1999), which focus on the role of identified VARs in estimating the effects of monetary policy, and King and Watson (1996), where the focus is on using empirical evidence to distinguish among competing business cycle models. King and Plosser (1984) examined the reverse causation argument – that changes in output lead changes in money. Coleman (1996) confirmed King and Plosser's results and shows that money should be highly correlated with lagged output than future output. Sims (1972; 1980) introduced Granger causality – that is the notion that changes in money Granger-caused changes in GNP.

Critically, we learnt about the price puzzle – that monetary policy shocks lead to an initial small increase in the price level before they have a prolonged decline – and we also discovered the liquidity effect (an increase in money should be followed by a decline in interest rates). Key papers were Christiano, Eichenbaum, and Evans (1995; 1999), Sims (1992), and Stock and Watson (2001).

We then looked at three models which essentially just added money to an RBC model: MIU model by Sidrauski (1967), a model with CIA constraints (Lucas 1980; Stockman 1981; and Svensson 1985), and a classical monetary model with some interest rules (as in Galí (2015)). None of these models seem to say much about either the pricing puzzle, nor – more critically – about the liquidity effect. So, we now move onto the New Keynesian model.

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11 The New Keynesian DSGE Model

11.1 Introduction

In the previous sections, we discussed critiques during the 1970s of Keynesian ideas from economists who favoured the use of rational expectations as a modelling device. Recall that following the Lucas Critique and the RBC boom, Keynesian economists had to head back to the drawing board to derive a model in which parameters were independent of shocks, and in which the behaviour of agents were fully rational and microfounded. Furthermore, these models needed to be dynamic. This was in stark contrast to past Keynesian static models such as in the Mundell-Fleming based IS-LM-AD-AS-BoP[63] framework, in which agents' behaviour was mostly ad-hoc for the purposes of fitting macro data.

Many different mechanisms were invoked, but most common was sticky prices. If prices didn't jump in line with money, then central banks could control real money supply and hence real interest rates. In a nutshell, the core focus of the Keynesian school of economics was that policy could affect real economic variables. So in the 1980's, after throwing out their static models, "New Keynesians" brought about their models showing some key theoretical points. See for example *New Keynesian Economics* by Mankiw and Romer (1991), "Monopolistic Competition and the Effects of Aggregate Demand" by Blanchard and Kiyotaki (1987), and Akerlof and Yellen's idea of bounded rationality of firms leading to sticky price dynamics. Then, in the 1990's, the New Keynesian school of economics had an important breakthrough: they developed the New Keynesian Phillips Curve (NKPC) (Roberts 1995). While it looked a lot like the old Phillips Curve, it featured future expected inflation based on rational expectations. An immense amount of research and literature surrounding New Keynesian economics and the NKPC boomed. The New Keynesian DSGE revolution flourished, essentially kicking the RBC and neoclassical economists to the curb when it came to mainstream macroeconomic theory.

In this section we derive the canonical New Keynesian (NK) model. For more information on the derivation, see textbooks by Galí, Walsh, and Woodford, as well as the seminal paper "The Science of Monetary Policy: A New Keynesian Perspective" by Clarida et al. (1999). For the sake of sanity, we will initially omit capital and investment in this model (doing so will allow us to derive nice analytical

[63]Investment-Saving-Liquidity-Money-Aggregate Demand-Aggregate Supply-Balance of Payments.

solutions).

The NK model takes an RBC model as its backbone and adds some nominal rigidities – here, we will add sticky prices – which allows shocks to affect the real economy. Namely, it will allow monetary policy shocks to affect real variables. This is different to the simple monetary models or RBC models we looked at before where money was neutral. To get price stickiness in the model, we will assume that some firms are price-setters. As such, we need to move away from perfect competition toward monopolistic competition, where we have a continuum of firms all of which produce a slightly differentiated product and a downward sloping demand curve. To keep things conceptual and digestible, we will split production into two sectors: final goods which are produced by perfectly competitive firms (or a single representative firm), and intermediate goods which are produced by the monopolistically competitive firms. So the intermediate firms produce their differentiated output which is then aggregated and combined into a final good for consumption.

All the key mechanisms of the canonical NK model happen with intermediate firms. There are two main ways to model the market power and price stickiness that these monopolistically competitive firms induce: Calvo pricing and Rotemberg pricing. Up to a first order Taylor approximation, these two pricing schemes produce identical results (they result in the same NKPC). We will see that with either Calvo or Rotemberg pricing, we get nice aggregation of the behaviour of the intermediate firms, and it will allow us to derive the NKPC. Without any further ado, let's begin.

11.2 Motivation: Nominal rigidities and the Phillips Curve

We've gone over non-neutrality of money in the previous section, so there's a couple more pieces to go over to motivate our New Keynesian model: nominal rigidities ("sticky prices") and going over the old-school Keynesian favourite: the Phillips Curve.

11.2.1 Evidence of nominal rigidities

Most attempts to uncover evidence on the existence and importance of price rigidities have generally relied on the analysis of micro data, that is, data on the prices of individual goods and services. In an early survey of that research, Taylor (1999) concludes that there is ample evidence of price rigidities, with the average frequency of price adjustment being about one year. In addition, he points to the very limited evidence of synchronisation of price adjustments, thus providing some justification for the assumption of staggered price setting commonly found in the New Keynesian model. The study of Bils and Klenow (2004), based on the analysis of the average frequencies of price changes for 350 product categories underlying the US CPI called into question that conventional wisdom by uncovering a median duration of prices between 4 and 6 months. Nevertheless, more recent evidence by Nakamura and Steinsson (2008), using data on the individual prices underlying the US CPI and excluding price changes associated with sales, has led to a reconsideration of the Bils-Klenow evidence, with an upward adjustment of the estimated median duration to a range between 8 and 11 months. Dhyne et al. (2006) found similar evidence for the Euro area. It is worth mentioning that, in addition to evidence of substantial price rigidities, most studies find a large amount of heterogeneity in price durations across sectors/types of goods.

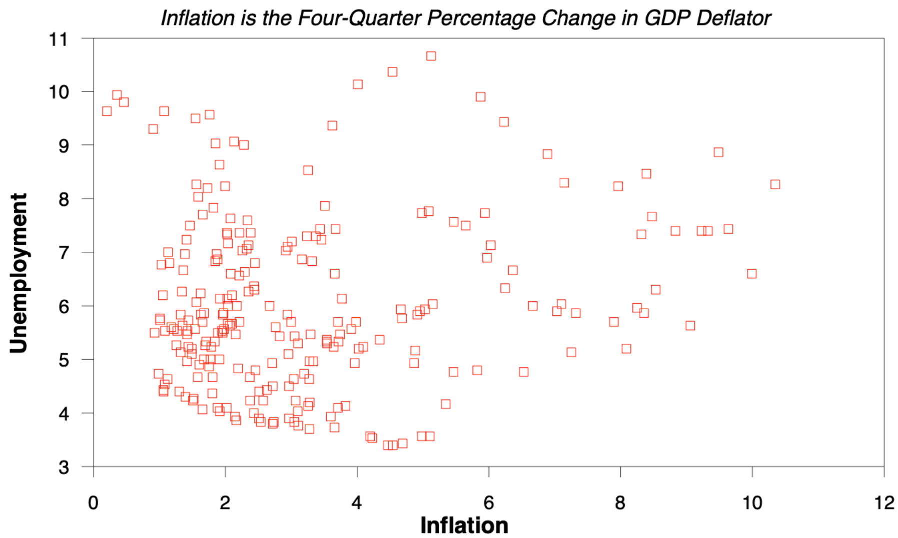
The literature also contains several studies using micro data that provides analogous evidence on nominal rigidities for wages. Taylor (1999) contains an early survey of that literature and suggests an estimate of the average frequency of wage changes of about one year, the same as for prices. A significant branch of the literature on wage rigidities has focused on the possible existence of asymmetries that make wage cuts very rare or unlikely. Bewley (1999) detailed study of firms' wage policies finds ample evidence of downward nominal wage rigidities.

11.2.2 The Phillips Curve (again)

Recall the discussion we had about the Phillips Curve when we explored the Lucas Critique. In a nutshell, to allow for a realistic model of monetary policy, we needed a framework in which prices didn't simply follow the money supply and nominal interest rates don't just move together one-for-one. In this kind of Keynesian model, prices are sticky, so real interest rates can be influenced by the central bank. Real interest rates can affect the performance of the economy, which in turn influences inflation via Phillips Curve relationship (see Figures 46 and 47).

Recall however that the Phillips Curve has failed empirically:

Figure 58: US Inflation and Unemployment (1955-2014)



Source: Whelan (2016)

We discussed briefly some potential reasons for why the Phillips Curve failed – which essentially followed the gist of Lucas' critique of macroeconomics at the time, and followed the speech given by Friedman in his 1967 AEA address – in the long-run, you can't fool the public ($\pi_t^e \approx \pi_t$) so you can't keep unemployment away from its "natural rate" ($U_t \approx U^*$).

Friedman thought that inflation expectations were determined adaptively. For instance, people use last year's inflation rate as a guide to what to expect this year (a rule of thumb approach). If we set $\pi_t^e = \pi_{t-1}$ then the expectations augmented Phillips Curve:

$$\pi_t = \pi_t^e - \gamma(U_t - U^*),$$

becomes:

$$\pi_t = \pi_{t-1} - \gamma(U_t - U^*). \tag{345}$$

This relates the change in inflation to the gap between unemployment and its natural rate. When unemployment is below its natural rate, inflation will be increasing; when it is above it, it will be decreasing. Unemployment below the natural rate implies an accelerating price level. The relationship

(345) is known as the accelerationist Phillips Curve.

In practice, there are a few complications. Inflation expectations are likely to be better captured by a weighted average of past inflation rates rather than just a single lag, implying:

$$\pi_t = \sum_{i=1}^{N} \beta_i \pi_{t-i} - \gamma(U_t - U^*),$$

where $\sum_{i=1}^{N} \beta_i = 1$.

Further, we don't know what the natural rate is, but we can estimate it from:

$$\pi_t = \alpha - \gamma U_t + \sum_{i=1}^{N} \beta_i \pi_{t-i},$$

where if we set:

$$\begin{aligned} \alpha - \gamma U^* &= 0 \\ \implies U^* &= \frac{\alpha}{\gamma}, \end{aligned}$$

which is known as the NAIRU (Non Accelerating Inflation Rate of Unemployment).

But this kind of estimation was precisely in the firing line of the Lucas Critique – which implied that econometric NAIRU estimates were not useful. Furthermore, consider the expectations-augmented Phillips Curve again:

$$\pi_t = \pi_t^e - \gamma(U_t - U^*).$$

We can only have $U_t \neq U^*$ when there is unexpected inflation so $\pi_t \neq \pi_t^e$. If expectations are rational, then these must be random and unpredictable based on publicly available information, so there's no room for systematic predictable (Keynesian) stabilisation.

The Rational Expectations school pioneered a different approach with models based on individual agents pursuing optimising behaviour, and over time many advocates of Rational Expectations came to believe that monetary policy had little to do with business cycles. Many turned to RBC theory. But – as we now well know – the RBC school hit a brick wall, and Keynesians came back with a new model...

11.3 Households

We assume a representative agent household[64] that consumes supplies labour, accumulates bonds, holds shares in firms, and accumulates money. It gets utility from holding real money balances and disutility from working. Its problem is:

$$\max_{C_t,N_t,B_t,M_t} \mathbb{E}_t \left[\sum_{s=0}^{\infty} \beta^s \left(\frac{C_{t+s}^{1-\sigma}}{1-\sigma} - \psi \frac{N_{t+s}^{1+\eta}}{1+\eta} + \theta \ln \left(\frac{M_{t+s}}{P_{t+s}} \right) \right) \right], \tag{346}$$

where we assume that the household has familiar CRRA preferences with σ being the Arrow-Pratt coefficient of relative risk aversion. We've been hand wavy and assumed that utility from holding real money balances is logarithmic. So long as real money balances are additively separable, they don't affect our results. In fact, if we were to assume that the central bank targets the interest rate rather than money supply, then we could ignore real money in utility altogether (which we will do later on). The nominal flow budget constraint is:

$$P_t C_t + B_t + M_t - M_{t-1} \leq W_t N_t + D_t - P_t T_t + R_{t-1} B_{t-1}, \tag{347}$$

where money is the numeraire, P_t is the price goods in terms of money, B_{t-1} is the stock of nominal bonds a household enters the period with[65], and they pay out a gross interest rate of $R_{t-1} = 1 + i_{t-1}$. Households enter period t with nominal money balances of M_{t-1}, earn a nominal wage of W_t on the labour they supply, earn nominal profits D_t remitted to them by firms, and T_t is a lump sum tax paid to the government. Using our timing trick from when we solved the RBC model, the Lagrangian for

[64] So we are working with a representative agent NK (RANK) model, as opposed to the newer heterogeneous agent NK (HANK) models.

[65] Note that I am using end of period notation.

the household is:

$$
\begin{aligned}
\mathcal{L} = \mathbb{E}_t & \left[\sum_{s=0}^{\infty} \beta^s \left(\frac{C_{t+s}^{1-\sigma}}{1-\sigma} - \psi \frac{N_{t+s}^{1+\eta}}{1+\eta} + \theta \ln \left(\frac{M_{t+s}}{P_{t+s}} \right) \right) \right] \\
& + \lambda_t \left(W_t N_t + D_t - P_t T_t + R_{t-1} B_{t-1} - P_t C_t - B_t - M_t + M_{t-1} \right) \\
& + \beta \mathbb{E}_t \left[W_{t+1} N_{t+1} + D_{t+1} - P_{t+1} T_{t+1} + R_t B_t - P_{t+1} C_{t+1} - B_{t+1} - M_{t+1} + M_t \right].
\end{aligned}
$$

The FOCs are:

$$
\frac{\partial \mathcal{L}}{\partial C_t} = C_t^{-\sigma} - \lambda_t P_t = 0, \tag{348}
$$

$$
\frac{\partial \mathcal{L}}{\partial N_t} = -\psi N_t^{\eta} + \lambda_t W_t = 0, \tag{349}
$$

$$
\frac{\partial \mathcal{L}}{\partial B_t} = -\lambda_t + \beta \mathbb{E}_t \left[\lambda_{t+1} R_t \right] = 0, \tag{350}
$$

$$
\frac{\partial \mathcal{L}}{\partial M_t} = \theta \frac{1}{M_t} - \lambda_t + \beta \mathbb{E}_t \left[\lambda_{t+1} \right] = 0. \tag{351}
$$

From (350) we know that $\lambda_t = \beta \mathbb{E}_t[\lambda_{t+1} R_t]$, so we can write the FOCs as:

$$
C_t^{-\sigma} = P_t \beta \mathbb{E}_t[\lambda_{t+1} R_t], \tag{352}
$$

$$
\psi N_t^{\eta} = \beta \mathbb{E}_t[\lambda_{t+1} R_t] W_t, \tag{353}
$$

$$
\frac{\theta}{M_t} = \lambda_t - \beta \mathbb{E}_t[\lambda_{t+1}]. \tag{354}
$$

Then, from (352), like we do in the RBC models, we have:

$$
\frac{C_t^{-\sigma}}{P_t} = \beta \mathbb{E}_t[\lambda_{t+1} R_t] = \lambda_t,
$$

so we can roll one period ahead to get:

$$
\frac{C_{t+1}^{-\sigma}}{P_{t+1}} = \lambda_{t+1},
$$

and combining (352) and (353) we can get rid of λ_t from our FOCs:

$$\psi N_t^\eta = \frac{C_t^{-\sigma}}{P_t} W_t = C_t^{-\sigma} w_t, \tag{355}$$

$$C_t^{-\sigma} = P_t \beta \mathbb{E}_t \left[\frac{C_{t+1}^{-\sigma}}{P_{t+1}} R_t \right] = \beta \mathbb{E}_t \left[C_{t+1}^{-\sigma} \frac{P_t}{P_{t+1}} R_t \right], \tag{356}$$

and for our third FOC:

$$\begin{aligned}
\theta \frac{1}{M_t} &= \lambda_t - \beta \mathbb{E}_t[\lambda_{t+1}] \\
&= \frac{C_t^{-\sigma}}{P_t} - \beta \mathbb{E}_t \left[\frac{C_{t+1}^{-\sigma}}{P_{t+1}} \right] \\
\theta \frac{P_t}{M_t} &= C_t^{-\sigma} - \beta \mathbb{E}_t \left[\frac{C_{t+1}^{-\sigma} P_t}{P_{t+1}} \right],
\end{aligned}$$

and since $\frac{C_t^{-\sigma}}{R_t} = \beta \mathbb{E}_t \left[\frac{C_{t+1}^{-\sigma} P_t}{P_{t+1}} \right]$:

$$\begin{aligned}
\theta \frac{P_t}{M_t} &= C_t^{-\sigma} - \frac{C_t^{-\sigma}}{R_t} \\
&= \frac{C_t^{-\sigma} R_t - C_t^{-\sigma}}{R_t} \\
&= \frac{C_t^{-\sigma} + i_t C_t^{-\sigma} - C_t^{-\sigma}}{R_t} \\
\therefore \theta \left(\frac{M_t}{P_t} \right)^{-1} &= \frac{i_t C_t^{-\sigma}}{R_t}.
\end{aligned} \tag{357}$$

These FOCs characterise the optimising behaviour of households in the canonical NK model.

11.4 Firms and production

As previously mentioned, we split production into two. There is a representative competitive final goods firm which aggregates intermediate inputs according to a CES technology. But these intermediate goods are imperfect substitutes, which causes the demand for these goods to be downward sloping. Hence, intermediate firms have a degree of market power. Intermediate firms are large in number (we assume a continuum of them), and so they behave as in monopolistic competition. They control their price, but treat other prices as given. These firms produce output using labour and are subject to an aggregate productivity shock. They are not freely able to adjust prices each period, however.

11.4.1 Final goods producer

The final output good is a CES aggregate, utilising the Dixit-Stiglitz aggregator, of a continuum of intermediate goods:

$$Y_t = \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}, \ \epsilon > 0, \tag{358}$$

so final good firms maximise their profits by selecting how much of each intermediate good to purchase, and so their problem is:

$$\max_{Y_t(j)} \ P_t Y_t - \int_0^1 P_t Y_t(j) dj.$$

The FOC for a typical intermediate good j is:

$$\begin{aligned}
0 &= P_t \frac{\epsilon}{\epsilon-1} \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}-1} \left(\frac{\epsilon-1}{\epsilon} \right) Y_t(j)^{\frac{\epsilon-1}{\epsilon}-1} - P_t(j) \\
P_t(j) &= P_t \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{1}{\epsilon-1}} Y_t(j)^{-\frac{1}{\epsilon}} \\
\frac{P_t(j)}{P_t} &= \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{1}{\epsilon-1}} Y_t(j)^{-\frac{1}{\epsilon}} \\
\left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} &= \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{-\frac{\epsilon}{\epsilon-1}} Y_t(j) \\
Y_t(j) &= \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}} \\
\implies Y_t(j) &= \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t. \qquad (359)
\end{aligned}$$

The relative demand for intermediate good j is dependent of j's relative price, with ϵ the price elasticity of demand, and is proportional to aggregate output, Y_t. So, for example, demand for j scales with aggregate economy size.

From Kiyotaki and Blanchard (1987), we can derive a price index for the aggregate economy:

$$P_t Y_t \equiv \int_0^1 P_t(j) Y_t(j) dj.$$

Then, plugging in the demand for good j from (359) we have:

$$\begin{aligned}
P_t Y_t &= \int_0^1 P_t(j) \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t dj \\
&= \int_0^1 P_t(j) P_t(j)^{-\epsilon} P_t^{\epsilon} Y_t dj \\
&= P_t^{\epsilon} Y_t \int_0^1 P_t(j)^{1-\epsilon} dj \\
P_t^{1-\epsilon} &= \int_0^1 P_t(j)^{1-\epsilon} dj \\
\implies P_t &= \left(\int_0^1 P_t(j)^{1-\epsilon} dj \right)^{\frac{1}{1-\epsilon}}. \qquad (360)
\end{aligned}$$

11.4.2 Intermediate producers

A typical intermediate firm produces output according a constant returns to scale technology in labour, with a common productivity shock, A_t:

$$Y_t(j) = A_t N_t(j). \qquad (361)$$

Intermediate firms pay a common wage. They are not freely able to adjust price so as to maximise profit each period, but will always act to minimise cost. The cost minimisation problem is to minimise total cost subject to the constraint producing enough to meet demand (again, see Blanchard and Kiyotaki (1987) for the derivation of this problem):

$$\min_{N_t(j)} \; W_t N_t(j),$$

subject to

$$A_t N_t(j) \geq Y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t.$$

The Lagrangian for an intermediate firm j's problem is:

$$\mathcal{L} = W_t N_t(j) - \varphi_t(j) \left(A_t N_t(j) - \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t \right),$$

and the FOC is:

$$\frac{\partial \mathcal{L}}{\partial N_t(j)} = W_t - \varphi_t(j) A_t = 0,$$

which then implies:

$$\varphi_t(j) = \frac{W_t}{A_t}. \tag{362}$$

But notice that neither W_t nor A_t are firm j specific, so in fact we can write $\varphi_t(j)$ as simply φ_t. Now, what is φ_t? It is an intermediate firm's nominal marginal cost – how much will costs change if you are forced to produce an extra unit of output. Now, formulate the intermediate firm's real flow profit as:

$$\frac{D_t(j)}{P_t} = \frac{P_t(j)}{P_t} Y_t(j) - \frac{W_t}{P_t} N_t(j),$$

and substitute in the nominal marginal cost from (362):

$$\begin{aligned} \frac{\Pi_t(j)}{P_t} &= \frac{P_t(j)}{P_t} Y_t(j) - \frac{\varphi_t A_t}{P_t} N_t(j) \\ &= \frac{P_t(j)}{P_t} Y_t(j) - mc_t A_t N_t(j), \end{aligned} \tag{363}$$

where $mc_t = \frac{\varphi_t}{P_t}$ is the real marginal cost for an intermediate firm. Now, buckle up because this is where the fun begins...

11.4.3 Monopolistic competition with Calvo pricing

Firms are not freely able to adjust price each period, In particular, each period there is a fixed probability of $1 - \phi$ that a firm can adjust its price. This means that ϕ firms are unable to change their price each period. Since we assume unit mass of intermediate firms, the probability that a firm is stuck with its price in any given period is ϕ, ϕ^2 for two periods, and so on. Since there is a chance that the firm will get stuck with its price for multiple periods, the pricing problem becomes dynamic.

After all, suppose if you were an intermediate firm with the chance to change your price[66]: You would want to optimise your pricing decision taking into account all future periods where you are potentially unable to price your output optimally. Thus, firms will need to be smart and discount future profits by some discount factor that is dynamic: Firms will discount s periods into the future by:

$$\tilde{M}_{t+s}\phi^s,$$

where

$$\tilde{M}_{t+s} = \beta^s \frac{u'(C_{t+s})}{u'(C_t)},$$

is the stochastic discount factor. Note that discounting is by both the usual stochastic discount factor as well as by the probability that a price chosen in period t will still be in use in period $t+s$. The dynamic problem of an updating firm can be written as

$$\max_{P_t(j)} \mathbb{E}_t \left[\sum_{s=0}^{\infty} \tilde{M}_{t+s}\phi^s \left(\frac{P_t(j)}{P_{t+s}} \underbrace{\left(\frac{P_t(j)}{P_{t+s}}\right)^{-\epsilon} Y_t}_{Y_{t+s}(j)} - mc_{t+s} \underbrace{\left(\frac{P_t(j)}{P_{t+s}}\right)^{-\epsilon} Y_{t+s}}_{Y_{t+s}(j)} \right) \right], \tag{364}$$

where we assume that output will equal demand. Multiplying this term out we get

$$\begin{aligned}
&\max_{P_t(j)} \mathbb{E}_t \left[\sum_{s=0}^{\infty} \tilde{M}_{t+s}\phi^s \left(P_t(j)^{1-\epsilon} P_{t+s}^{\epsilon-1} Y_t - mc_{t+s} P_t(j)^{-\epsilon} P_{t+s}^{\epsilon} Y_{t+s} \right) \right] \\
\Leftrightarrow &\max_{P_t(j)} \mathbb{E}_t \left[\sum_{s=0}^{\infty} \left(\tilde{M}_{t+s}\phi^s P_t(j)^{1-\epsilon} P_{t+s}^{\epsilon-1} Y_t - \tilde{M}_{t+s}\phi^s mc_{t+s} P_t(j)^{-\epsilon} P_{t+s}^{\epsilon} Y_{t+s} \right) \right],
\end{aligned}$$

and so the FOC is:

$$\begin{aligned}
0 &= \mathbb{E}_t \left[\sum_{s=0}^{\infty} \left((1-\epsilon)\tilde{M}_{t+s}\phi^s P_t(j)^{-\epsilon} P_{t+s}^{\epsilon-1} Y_t + \epsilon \tilde{M}_{t+s}\phi^s mc_{t+s} P_t(j)^{-\epsilon-1} P_{t+s}^{\epsilon} Y_{t+s} \right) \right] \\
&= \mathbb{E}_t \left[(1-\epsilon) P_t(j)^{-\epsilon} \sum_{s=0}^{\infty} \tilde{M}_{t+s}\phi^s P_{t+s}^{\epsilon-1} Y_t + \epsilon P_t(j)^{-\epsilon-1} \sum_{s=0}^{\infty} \tilde{M}_{t+s}\phi^s mc_{t+s} P_{t+s}^{\epsilon} Y_{t+s} \right],
\end{aligned}$$

[66] i.e. You get a visit from the "Calvo fairy", giving you the chance to change your price.

and moving the first term in the expectations operator to the LHS, we get:

$$
\begin{aligned}
(\epsilon-1)P_t(j)^{-\epsilon}\mathbb{E}_t\left[\sum_{s=0}^{\infty}\tilde{M}_{t+s}\phi^s P_{t+s}^{\epsilon-1}Y_t\right] &= \epsilon P_t(j)^{-\epsilon-1}\mathbb{E}_t\left[\sum_{s=0}^{\infty}\tilde{M}_{t+s}\phi^s mc_{t+s}P_{t+s}^{\epsilon}Y_{t+s}\right] \\
P_t(j) &= \frac{\epsilon}{\epsilon-1}\frac{\mathbb{E}_t\left[\sum_{s=0}^{\infty}\tilde{M}_{t+s}\phi^s mc_{t+s}P_{t+s}^{\epsilon}Y_{t+s}\right]}{\mathbb{E}_t\left[\sum_{s=0}^{\infty}\tilde{M}_{t+s}\phi^s P_{t+s}^{\epsilon-1}Y_t\right]} \\
P_t(j) &= \frac{\epsilon}{\epsilon-1}\frac{\mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s\frac{u'(C_{t+s})}{u'(C_t)}\phi^s mc_{t+s}P_{t+s}^{\epsilon}Y_{t+s}\right]}{\mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s\frac{u'(C_{t+s})}{u'(C_t)}\phi^s P_{t+s}^{\epsilon-1}Y_t\right]} \\
P_t(j) &= \frac{\epsilon}{\epsilon-1}\frac{\mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s u'(C_{t+s})\phi^s mc_{t+s}P_{t+s}^{\epsilon}Y_{t+s}\right]}{\mathbb{E}_t\left[\sum_{s=0}^{\infty}\beta^s u'(C_{t+s})\phi^s P_{t+s}^{\epsilon-1}Y_t\right]} \qquad (365)
\end{aligned}
$$

It's worth noting that none of the variables on the RHS of the above equation depend on j. This means that any firm able to update their prices will update their prices to the same reset price, say, $P_t^{\#}$. We can write $P_t^{\#}$ compactly as:

$$P_t^{\#} = \frac{\epsilon}{\epsilon-1}\frac{X_{1,t}}{X_{2,t}}, \qquad (366)$$

where $\frac{\epsilon}{\epsilon-1}$ is the markup charged by monopolistically competitive firms[67], and our auxiliary variables $X_{1,t}$ and $X_{2,t}$ are:

$$X_{1,t} = u'(C_t)mc_t P_t^{\epsilon}Y_t + \phi\beta\mathbb{E}_t X_{1,t+1}, \qquad (367)$$

$$X_{2,t} = u'(C_t)P_t^{\epsilon-1}Y_t + \phi\beta\mathbb{E}_t X_{2,t+1}. \qquad (368)$$

Notice that the second terms of our auxiliary variables are equal to 0 when $\phi = 0$. i.e., If all firms are able to change their prices freely, then prices are flexible which means that $mc_t P_t = \varphi_t$ and $P_t^{\#} = \frac{\epsilon}{\epsilon-1}\varphi_t$. This is important to note down.

11.5 Equilibrium and aggregation

To close the model, we need to specify an exogenous process for our technology shocks A_t, some kind of monetary policy rule to determine M_t, and a fiscal rule to determine T_t. Let the aggregate productivity

[67] Galí refers to the markup as $\mathcal{M}$ in his textbook.

term follow an AR(1) process such as:

$$\ln A_t = \rho_a \ln A_{t-1} + \epsilon_{a,t}. \tag{369}$$

Then, for money, let's suppose that nominal money supply also follows an AR(1) process in the growth rate:

$$\Delta \ln M_t = (1 - \rho_m)\bar{\pi} + \rho_m \Delta \ln M_{t-1} + \epsilon_{m,t}, \tag{370}$$

where $\Delta \ln M_t = \ln M_t - \ln M_{t-1}$. Writing the growth of money in this way implies that the mean growth rate of money is equal to the steady state inflation rate $\bar{\pi}$, as we want real money balances to be stationary so M_t and P_t grow at the same rate in the steady state. For both the law of motion for technology and nominal money, I assume that they contain white noise shock terms such that $\epsilon_{a,t} \sim N(0, \sigma_a^2)$ and $\epsilon_{m,t} \sim N(0, \sigma_m^2)$.

In this economy, the government prints money, so it earns seignorage. Right now, we assume that the government does not consume, and that it does not take part in bond markets. The nominal government budget budget constraint is:

$$0 \leq P_t T_t + M_t - M_{t-1}.$$

In words, the change in the nominal money supply, $M_t - M_{t-1}$ is nominal revenue for the government. Since it does no spending, at equality lump sum taxes must satisfy:

$$T_t = -\frac{M_t - M_{t-1}}{P_t}.$$

So if money growth is positive, e.g. $M_t > M_{t-1}$, then lump sum taxes will be negative – the government will be rebating its seignorage revenue to the households via lump sum transfers.

In equilibrium, bond-holding is always zero in all periods: $B_t = 0$. Using this, plus the relationship between the lump sum tax and money growth derived above, the household budget constrain can be written in real terms:

$$
\begin{aligned}
P_tC_t + B_t + M_t - M_{t-1} &\leq W_tN_t + D_t + P_tT_t + R_{t-1}B_{t-1} \\
\Leftrightarrow P_tC_t + M_t - M_{t-1} &\leq W_tN_t + D_t + P_t\left(-\frac{M_t - M_{t-1}}{P_t}\right) \\
\Leftrightarrow C_t &= w_tN_t + \frac{D_t}{P_t}.
\end{aligned}
$$

Real dividends received by the household are just the sum of real profits from intermediate goods firms (since the final good firm is competitive and earns no economic profit):

$$
\begin{aligned}
\frac{D_t}{P_t} &= \int_0^1 \left(\frac{P_t(j)}{P_t}Y_t(j) - \frac{W_t}{P_t}N_t(j)\right) dj \\
&= \int_0^1 \left(\frac{P_t(j)}{P_t}Y_t(j) - w_tN_t(j)\right) dj \\
&= \int_0^1 \frac{P_t(j)}{P_t}Y_t(j)dj - w_t\int_0^1 N_t(j)dj \\
&= \int_0^1 \frac{P_t(j)}{P_t}Y_t(j)dj - w_tN_t.
\end{aligned}
$$

So, the household budget constraint becomes:

$$
\begin{aligned}
C_t &= w_tN_t + \int_0^1 \frac{P_t(j)}{P_t}Y_t(j)dj - w_tN_t \\
\implies C_t &= \int_0^1 \frac{P_t(j)}{P_t}Y_t(j)dj,
\end{aligned}
$$

and since:

$$
Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t,
$$

we have:

$$\begin{aligned} C_t &= \int_0^1 \frac{P_t(j)}{P_t} \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t dj \\ &= \int_0^1 P_t(j)^{1-\epsilon} P_t^{\epsilon-1} Y_t dj \\ &= P_t^{\epsilon-1} Y_t \int_0^1 P_t(j)^{1-\epsilon} dj, \end{aligned}$$

but $\int_0^1 P_t(j)^{1-\epsilon} dj = P_t^{1-\epsilon}$ from (360), so the P_t terms drop out and we have the market clearing condition:

$$C_t = Y_t \tag{371}$$

Now, we need to solve for Y_t. From the demand for intermediate goods we have:

$$Y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t,$$

and using the production for each intermediate firm, this is:

$$A_t N_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t.$$

Integrating over all goods, we have:

$$\int_0^1 A_t N_t(j) dj = \int_0^1 \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t dj,$$

and with a bit of rearranging we have:

$$\begin{aligned} A_t \int_0^1 N_t(j) &= Y_t \underbrace{\int_0^1 \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} dj}_{v_t^P} \\ \implies Y_t &= \frac{A_t N_t}{v_t^P}. \end{aligned} \tag{372}$$

The new variable we have defined, v_t^P, is a measure of price dispersion. If there were no pricing frictions, all firms would charge the same price, and $v_t^P = 1$. If prices are different, one can show that this expression is bound from below by unity. Since $v_t^P \geq 1$, price dispersion leads to lower output. The economy produces less than it otherwise would given A_t and aggregate labour input if prices are disperse. This is the gist for why price stability is a good thing.

Our full set of equilibrium conditions are:

$$C_t^{-\sigma} = \beta \mathbb{E}_t \left[\frac{C_{t+1}^{-\sigma} R_t P_t}{P_{t+1}} \right], \tag{373}$$

$$\psi N_t^{\eta} = C_t^{-\sigma} w_t, \tag{374}$$

$$\frac{M_t}{P_t} = \theta \frac{R_t}{i_t} C_t^{\sigma}, \tag{375}$$

$$mc_t = \frac{w_t}{A_t}, \tag{376}$$

$$C_t = Y_t, \tag{377}$$

$$Y_t = \frac{A_t N_t}{v_t^P}, \tag{378}$$

$$v_t^P = \int_0^1 \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} dj, \tag{379}$$

$$P_t^{1-\epsilon} = \int_0^1 P_t(j)^{1-\epsilon} dj, \tag{380}$$

$$P_t = \frac{\epsilon}{\epsilon - 1} \frac{X_{1,t}}{X_{2,t}}, \tag{381}$$

$$X_{1,t} = C_t^{-\sigma} mc_t P_t^{\epsilon} Y_t + \phi \beta \mathbb{E}_t X_{1,t+1}, \tag{382}$$

$$X_{2,t} = C_t^{-\sigma} P_t^{\epsilon-1} Y_t + \phi \beta \mathbb{E}_t X_{2,t+2}, \tag{383}$$

$$\ln A_t = \rho_a \ln A_{t-1} + \epsilon_{a,t}, \tag{384}$$

$$\Delta \ln M_t = (1 - \rho_m)\bar{\pi} + \rho_m \Delta \ln M_{t-1} + \epsilon_{m,t}, \tag{385}$$

$$\Delta \ln M_t = \ln M_t - \ln M_{t-1}. \tag{386}$$

This is 14 equations in 14 aggregate variables. But there are three issues with the way we have written up this system of equations: 1) we have heterogeneity (j shows up); 2) the price level shows up, which may not be stationary; and, 3) nominal money growth shows up, which isn't stationary. So we will rewrite these conditions using Calvo pricing to get rid of the j terms, using inflation instead of price levels, and replace M_t by real balances, $m_t = M_t/P_t$.

11.5.1 Re-writing the equilibrium conditions

Begin by rewriting gross inflation as $1+\pi_t = \frac{P_t}{P_{t-1}}$. The consumption Euler equation can be re-written as:

$$C_t^{-\sigma} = \beta \mathbb{E}_t \left[C_{t+1}^{-\sigma} R_t (1+\pi_t) \right]. \tag{387}$$

The demand for money equation is already written in terms of real money balances, so it's fine:

$$m_t = \frac{\theta R_t C_t^{-\sigma}}{i_t}. \tag{388}$$

Now we need to get rid of the j terms in the price level and price dispersion expansions. The expression for the price level is:

$$P_t^{1-\epsilon} = \int_0^1 P_t(j)^{1-\epsilon} dj.$$

Now, recall that a fraction $(1-\phi)$ of these firms will update their price – after a visit from the Calvo fairy – to the same reset price, $P_t^{\#}$. The other fraction ϕ will charge the price they charged in the previous period. Since it doesn't matter how we "order" these firms along the unit interval, this means we can break up the integral on the RHS above as:

$$\begin{aligned} P_t^{1-\epsilon} &= \int_0^{1-\phi} \left(P_t^{\#}\right)^{1-\epsilon} dj + \int_{1-\phi}^1 P_{t-1}(j)^{1-\epsilon} dj \\ \Leftrightarrow P_t^{1-\epsilon} &= (1-\phi)\left(P_t^{\#}\right)^{1-\epsilon} + \int_{1-\phi}^1 P_{t-1}(j)^{1-\epsilon} dj. \end{aligned}$$

Now, watch the Calvo magic. Because the firms who get to update are randomly chosen, and because there are a large number of firms, the integral of individual prices over some subset of the unit interval will simply be proportional to the integral over the entire unit interval, where the proportion is equal to the subset of the unit interval over which the integral is taken. This means:

$$\int_{1-\phi}^{1} P_t(j)^{1-\epsilon} dj = \phi \int_0^1 P_{t-1}(j)^{1-\epsilon} dj = \phi P_{t-1}^{1-\epsilon}.$$

Therefore we have

$$P_t^{1-\epsilon} = (1-\phi)\left(P_t^{\#}\right)^{1-\epsilon} + \phi P_{t-1}^{1-\epsilon}.$$

Tada! We've gotten rid of the heterogeneity. The Calvo assumption allows us to integrate out the heterogeneity and not worry about keeping track of what each firm is doing from the perspective of looking at the behaviour of aggregates. Now, we want to write things in terms of inflation, so divide both sides by $P_{t-1}^{1-\epsilon}$, and define $1+\pi_t^{\#} = \frac{P_t^{\#}}{P_{t-1}}$ as reset price inflation:

$$\begin{aligned}
\frac{P_t^{1-\epsilon}}{P_{t-1}^{1-\epsilon}} &= (1-\phi)\frac{\left(P_t^{\#}\right)^{1-\epsilon}}{P_{t-1}^{1-\epsilon}} + \phi\frac{P_{t-1}^{1-\epsilon}}{P_{t-1}^{1-\epsilon}} \\
\Leftrightarrow (1+\pi_t)^{1-\epsilon} &= (1-\phi)(1+\pi_t^{\#})^{1-\epsilon} + \phi. \qquad (389)
\end{aligned}$$

Now, look at the price dispersion term. Notice we can use the same Calvo trick we used above here:

$$\begin{aligned}
v_t^P &= \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} dj \\
&= \int_0^{1-\phi} \left(\frac{P_t(j)}{P_t}\right)^{\epsilon} dj + \int_{1-\phi}^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} dj \\
&= \int_0^{1-\phi} \left(\frac{P_t^{\#}}{P_t}\right)^{-\epsilon} dj + \int_{1-\phi}^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} dj \\
&= \int_0^{1-\phi} \left(\frac{P_t^{\#}}{P_{t-1}}\right)^{-\epsilon} \left(\frac{P_{t-1}}{P_t}\right)^{-\epsilon} dj + \int_{1-\phi}^1 \left(\frac{P_t(j)}{P_{t-1}}\right)^{-\epsilon} \left(\frac{P_t}{P_{t-1}}\right)^{-\epsilon} dj \\
&= (1-\phi)\left(\frac{P_t^{\#}}{P_{t-1}}\right)^{-\epsilon} \left(\frac{P_{t-1}}{P_t}\right)^{-\epsilon} + \left(\frac{P_t}{P_{t-1}}\right)^{-\epsilon} \int_{1-\phi}^1 \left(\frac{P_t(j)}{P_{t-1}}\right)^{-\epsilon} dj \\
&= (1-\phi)(1+\pi_t^{\#})^{-\epsilon}(1+\pi_t)^{\epsilon} + (1+\pi_t)^{\epsilon} \int_{1-\phi}^1 \left(\frac{P_t(j)}{P_{t-1}}\right)^{-\epsilon} dj,
\end{aligned}$$

and use the Calvo trick on the last term of the RHS to get:

$$v_t^P = (1-\phi)(1+\pi_t^{\#})^{-\epsilon}(1+\pi_t)^{\epsilon} + (1+\pi_t)^{\epsilon}\phi v_{t-1}^P. \tag{390}$$

Now, adjust the reset price expression. Define two new auxiliary variables:

$$\begin{aligned} x_{1,t} &= \frac{X_{1,t}}{P_t^{\epsilon}}, \\ x_{2,t} &= \frac{X_{2,t}}{P_t^{\epsilon-1}}, \end{aligned}$$

so:

$$\begin{aligned} x_{1,t} &= \frac{C_t^{-\sigma} mc_t P_t^{\epsilon} Y_t}{P_t^{\epsilon}} + \frac{\phi\beta\mathbb{E}_t X_{1,t+1}}{P_t^{\epsilon}}, \\ x_{2.t} &= \frac{C_t^{-\sigma} P_t^{\epsilon-1} Y_t}{P_t^{\epsilon-1}} + \frac{\phi\beta\mathbb{E}_t X_{2,t+1}}{P_t^{\epsilon-1}}. \end{aligned}$$

Multiplying and divide by P_{t+1} and the right powers to get:

$$\begin{aligned} x_{1,t} &= C_t^{-\sigma} mc_t Y_t + \frac{\phi\beta\mathbb{E}_t X_{1,t+1}}{P_{t+1}^{\epsilon}}\left(\frac{P_{t+1}}{P_t}\right)^{\epsilon} \\ &= C_t^{-\sigma} mc_t Y_t + \phi\beta\mathbb{E}_t\left[(1+\pi_{t+1})^{\epsilon} x_{1,t+1}\right], \qquad (391) \\ x_{2.t} &= C_t^{-\sigma} Y_t + \frac{\phi\beta\mathbb{E}_t X_{2,t+1}}{P_{t+1}^{\epsilon-1}}\left(\frac{P_{t+1}}{P_t}\right)^{\epsilon-1} \\ &= C_t^{-\sigma} Y_t + \phi\beta\mathbb{E}_t\left[(1+\pi_{t+1})^{\epsilon-1} x_{2,t+1}\right]. \qquad (392) \end{aligned}$$

Note, we can write $\frac{X_{1,t}}{X_{2,t}} = P_t \frac{x_{1,t}}{x_{2,t}}$, so the reset price expression can now be written as:

$$P_t^{\#} = \frac{\epsilon}{\epsilon-1} P_t \frac{x_{1,t}}{x_{2,t}},$$

and by dividing both sides by P_{t-1} we can write this in terms of inflation:

$$1 + \pi_t^{\#} = \frac{\epsilon}{\epsilon - 1} \frac{x_{1,t}}{x_{2,t}} (1 + \pi_t). \tag{393}$$

The process for real money balances can be converted into real terms quite easily:

$$\begin{aligned}
&\Delta \ln m_t = \ln m_t - \ln m_{t-1} \\
\Leftrightarrow &\Delta \ln m_t = \ln M_t - \ln P_t - \ln M_{t-1} + \ln P_{t-1} \\
\therefore &\Delta \ln M_t = \Delta \ln m_t + \pi_t.
\end{aligned}$$

So we can write the process for money growth in terms of real balance growth as:

$$\Delta \ln m_t = (1 - \rho_m)\bar{\pi} - \pi_t + \rho_m \Delta \ln m_{t-1} + \rho_m \pi_{t-1} + \epsilon_{m,t}. \tag{394}$$

The full set of rewritten equilibrium conditions is:

$$C_t^{-\sigma} = \beta \mathbb{E}_t \left[\frac{R_t}{C_{t+1}^{\sigma}(1+\pi_{t+1})} \right], \tag{395}$$

$$\psi N_t^{\eta} = \frac{w}{C_t^{\sigma}}, \tag{396}$$

$$m_t = \theta \frac{R_t}{i_t} C_t^{\sigma}, \tag{397}$$

$$mc_t = \frac{w_t}{A_t}, \tag{398}$$

$$C_t = Y_t, \tag{399}$$

$$Y_t = \frac{A_t N_t}{v_t^P}, \tag{400}$$

$$v_t^P = (1-\phi)(1+\pi_t^{\#})^{-\epsilon}(1+\pi_t)^{\epsilon} + (1+\pi_t)^{\epsilon}\phi v_{t-1}^P, \tag{401}$$

$$(1+\pi_t)^{1-\epsilon} = (1-\phi)(1+\pi_t^{\#})^{1-\epsilon} + \phi, \tag{402}$$

$$1+\pi_t^{\#} = \frac{\epsilon}{\epsilon-1}\frac{x_{1,t}}{x_{2,t}}(1+\pi_t), \tag{403}$$

$$x_{1,t} = \frac{mc_t Y_t}{C_t^{\sigma}} + \phi\beta\mathbb{E}_t\left[(1+\pi_{t+1})^{\epsilon} x_{1,t+1}\right], \tag{404}$$

$$x_{2,t} = \frac{Y_t}{C_t^{\sigma}} + \phi\beta\mathbb{E}_t\left[(1+\pi_{t+1})^{\epsilon-1} x_{2,t+1}\right], \tag{405}$$

$$\ln A_t = \rho_a \ln A_{t-1} + \epsilon_{a,t}, \tag{406}$$

$$\Delta \ln m_t = (1-\rho_m)\bar{\pi} + \pi_t + \rho_m \Delta \ln m_{t-1} + \rho_m \pi_{t-1} + \epsilon_{m,t}, \tag{407}$$

$$\Delta \ln m_t = \ln m_t - \ln m_{t-1}. \tag{408}$$

11.6 The steady state

We now solve for the non-stochastic steady state of the model. Let bar variables (e.g. $\bar{x}$) denote steady state values.

We have $\bar{A} = 1$, and since output and consumption are always equal, it must be that $\bar{Y} = \bar{C}$. Steady state inflation is equal to the exogenous target, $\bar{\pi}$. From the growth rate of real money balances, in the steady state we have:

$$\Delta \ln \bar{m} = (1 - \rho_m)\bar{\pi} - (1 - \rho_m)\bar{\pi} + \rho_m \Delta \ln \bar{m},$$

and this implies:

$$\begin{aligned} \Delta \ln \bar{m} - \rho_m \Delta \ln \bar{m} &= (1 - \rho_m)\bar{\pi} - (1 - \rho_m)\bar{\pi} \\ (1 - \rho_m)\Delta \ln \bar{m} &= 0 \\ \implies \ln \bar{m} &= 0, \end{aligned} \tag{409}$$

which means that real money balances are stationary in the steady state.

Next, from the Euler equation, we have:

$$\begin{aligned} \bar{C}^{-\sigma} &= \frac{\beta \bar{R}}{\bar{C}^{\sigma}(1 + \bar{\pi})} \\ \implies \bar{R} &= \frac{1 + \bar{\pi}}{\beta} \\ \Leftrightarrow 1 + \bar{i} &= \frac{1 + \bar{\pi}}{\beta} \\ \implies i &\approx \bar{r} + \bar{\pi}, \end{aligned} \tag{410}$$

where

$$\beta = \frac{1}{1 + \bar{r}}.$$

(410) is the familiar Fisher equation, and $\bar{r}$ in the expression for β is the discount rate (whereas β is the discount factor), and is also referred to as the net real interest rate.

From the price evolution equation, we can derive the steady state expression for reset price inflation:

$$
\begin{aligned}
(1+\bar{\pi})^{1-\epsilon} &= (1-\phi)(1+\bar{\pi}^{\#})^{1-\epsilon} + \phi \\
\frac{(1+\bar{\pi})^{1-\epsilon} - \phi}{1-\phi} &= (1+\bar{\pi}^{\#})^{1-\epsilon} \\
\implies 1+\bar{\pi}^{\#} &= \left(\frac{(1+\bar{\pi})^{1-\epsilon} - \phi}{1-\phi}\right)^{\frac{1}{1-\epsilon}}. \qquad (411)
\end{aligned}
$$

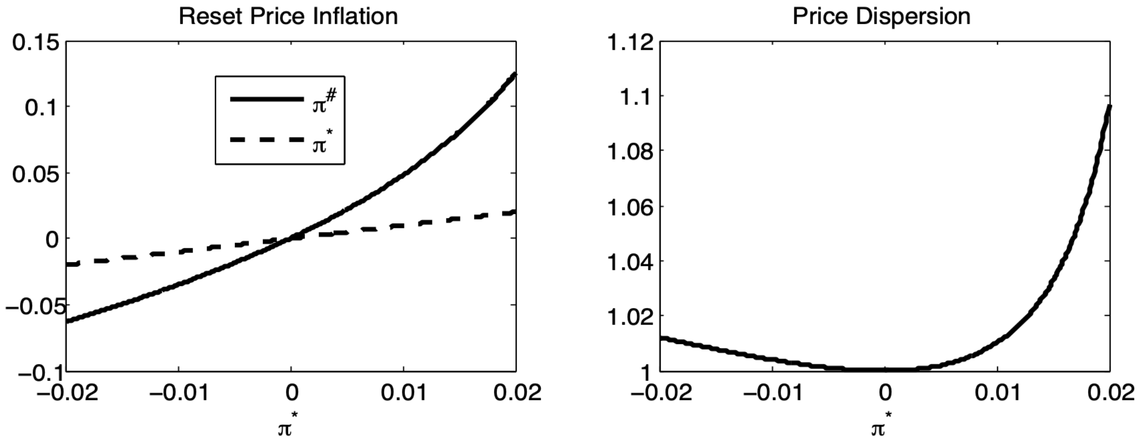
If $\bar{\pi} = 0$, then $\bar{\pi}^{\#} = \bar{\pi}$, since the RHS of the above expression is equal to 1. If $\bar{\pi} > 0 \implies \bar{\pi}^{\#} > \bar{\pi}$, and if $\bar{\pi} < 0 \implies \bar{\pi}^{\#} < \bar{\pi}$. With this in hand, we can solve for steady state price dispersion:

$$
\begin{aligned}
\bar{v}^P &= (1-\phi)(1+\bar{\pi}^{\#})^{-\epsilon}(1+\bar{\pi})^{\epsilon} + (1+\bar{\pi})^{\epsilon}\phi\bar{v}^P \\
(1-(1+\bar{\pi})^{\epsilon}\phi)\bar{v}^P &= \frac{(1-\phi)(1+\bar{\pi})^{\epsilon}}{(1+\bar{\pi}^{\#})^{\epsilon}}. \qquad (412)
\end{aligned}
$$

If $\bar{\pi} = 0$, then $\bar{v}^P = 1$. If $\bar{\pi} \neq 0$, then $v^P > 1$. Figure 59 contains two plots. The left plot maps steady state inflation (denoted as π^*) and steady state reset price inflation ($\pi^{\#}$). We can see that steady state reset price inflation is less than steady state inflation for negative steady state inflation, and greater than steady state inflation for positive steady state inflation[68]. The right plot maps steady state inflation against steady state price dispersion. We see that steady state price dispersion is equal to unity for when steady state inflation is equal to zero, and is rising for any value of steady state inflation not equal to zero.

[68]This is an awful sentence. Essentially we are confirming what we found in the equation for the steady set reset price inflation.

Figure 59: Steady State Inflation and Price Dispersion



$\epsilon = 10, \phi = 0.75$ (sourced from Sims (2017))

Now, we can solve for the steady state ratio of $\bar{x}_1/\bar{x}_2$:

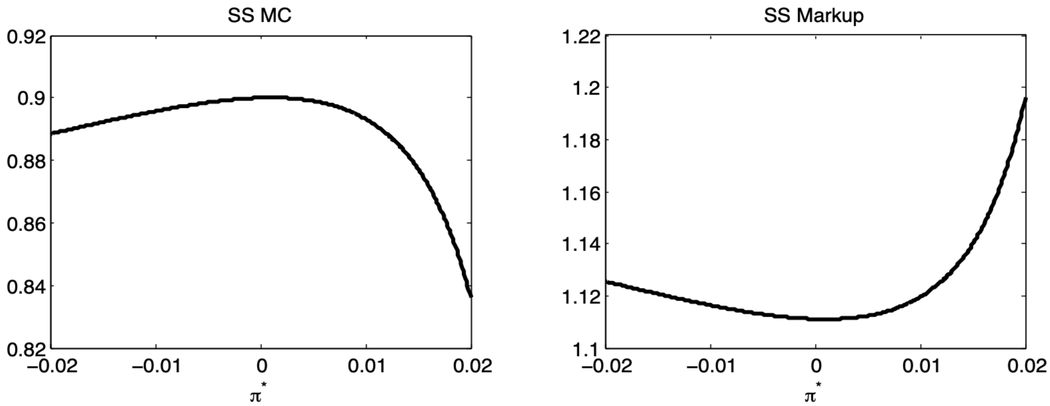
$$\frac{\bar{x}_1}{\bar{x}_2} = \frac{1+\bar{\pi}^{\#}}{1+\bar{\pi}}\frac{\epsilon-1}{\epsilon}, \tag{413}$$

and we also know that:

$$\begin{aligned}\frac{\bar{x}_1}{\bar{x}_2} &= \bar{mc}\frac{1-\phi\beta(1+\bar{\pi})^{\epsilon-1}}{1-\phi\beta(1+\bar{\pi})^{\epsilon}} \\ \implies \bar{mc} &= \frac{1+\bar{\pi}^{\#}}{1+\bar{\pi}}\frac{\epsilon-1}{\epsilon}\frac{1-\phi\beta(1+\bar{\pi})^{\epsilon}}{1-\phi\beta(1+\bar{\pi})^{\epsilon-1}}.\end{aligned} \tag{414}$$

In words, the steady state real marginal cost is inverse to the price markup. If $\bar{\pi} = 0$, then $\bar{mc} = \frac{\epsilon-1}{\epsilon}$. In other words, if steady state inflation is zero, then the steady state markup will be what it would be if prices were flexible. If $\bar{\pi} \neq 0$, then $\bar{mc} < \frac{\epsilon-1}{\epsilon}$, which means that the steady state markup will be higher than it would if inflation were zero.

Figure 60: Steady State Marginal Cost and Markup



$\epsilon = 10, \phi = 0.75, \beta = 0.99$ (sourced from Sims (2017))

With the steady steady state marginal cost in hand, we now look at the labour supply condition. We already know that $\bar{w} = \bar{mc}$ (since $\bar{A} = 1$). The lower is marginal cost, the bigger is the wedge between the wage and the marginal product of labour (i.e., the more distorted the economy is). Then we have:

$$\begin{aligned} \psi \bar{N}^{\eta} &= \frac{\bar{w}}{\bar{C}^{\sigma}} \\ \Leftrightarrow \psi \bar{N}^{\eta} &= \frac{\bar{w}}{\bar{Y}^{\sigma}}, \end{aligned}$$

and since $\bar{Y} = \frac{\bar{A}\bar{N}}{\bar{v}^P}$, we have:

$$\begin{aligned} \psi \bar{N}^{\eta} &= \bar{w} \left(\frac{\bar{N}}{\bar{v}^P} \right)^{-\sigma} \\ &= \bar{w} \bar{N}^{-\sigma} \left(\bar{v}^P \right)^{\sigma} \\ &= \bar{mc} \bar{N}^{-\sigma} \left(\bar{v}^P \right)^{\sigma} \\ \psi \bar{N}^{\eta+\sigma} &= \bar{mc} \left(\bar{v}^P \right)^{\sigma} \\ \therefore \bar{N} &= \left(\frac{\bar{mc} \left(\bar{v}^P \right)^{\sigma}}{\psi} \right)^{\frac{1}{\eta+\sigma}}. \end{aligned} \tag{415}$$

Finally, since we have $\bar{Y}$, steady state $\bar{m}$ is easy:

$$\bar{m} = \frac{\theta \bar{R}}{\bar{i}} \bar{Y}^{\sigma}. \tag{416}$$

11.7 The flexible price equilibrium

We now consider the hypothetical equilibrium case where all prices are flexible (i.e., when $\phi = 0$). But, even under flexible prices, we still have monopolistic competition. Since we have no endogenous state variables when prices are flexible, we can solve for the flex price equilibrium by hand. In this section superscript f denotes the hypothetical flex price allocation.

When $\phi = 0$, we have $\bar{\pi}^{\#} = \bar{\pi}$ regardless of what $\bar{\pi}$ is. Then, from the price dispersion equation we have:

$$v_t^{P,f} = \left(\frac{1 + \bar{\pi}^{\#}}{1 + \bar{\pi}} \right)^{-\epsilon} = 1, \tag{417}$$

and combining this result with the auxiliary variables $x_{1,t}$ and $x_{2,t}$ we have:

$$mc_t^f = \frac{\epsilon - 1}{\epsilon}. \tag{418}$$

In words, if prices are flexible, all firms charge the same prices, and price dispersion is at its lower bound of 1 and marginal costs are constant. Since marginal cost is the inverse of the price markup, we can say that under a flex price equilibrium, firms will set prices equal to a fixed markup over marginal cost. This therefore implies:

$$w_t^f = \frac{\epsilon - 1}{\epsilon} A_t, \tag{419}$$

and if we plug this into the labour supply condition we have:

$$\psi \left(N_t^f \right)^{\eta} = \left(Y_t^f \right)^{-\sigma} \frac{\epsilon - 1}{\epsilon} A_t,$$

and suppose $Y_t^f = A_t N_t^f$, we see:

$$\psi \left(N_t^f\right)^{\eta} = A_t^{-\sigma} \left(N_t^f\right)^{-\sigma} \frac{\epsilon - 1}{\epsilon} A_t$$
$$\therefore N_t^f \left(\frac{1}{\psi} \frac{\epsilon - 1}{\epsilon} A_t^{1-\sigma}\right)^{\frac{1}{\sigma+\eta}}. \tag{420}$$

This implies that the flexible price output is:

$$Y_t^f = \left(\frac{1}{\psi} \frac{\epsilon - 1}{\epsilon}\right)^{\frac{1}{\sigma+\eta}} A_t^{\frac{1+\eta}{\sigma+\eta}}. \tag{421}$$

Note that if $\sigma = 1$, then N_t^f is a constant and not a function of A_t. In other words, if prices are flexible and $\sigma = 1$ (meaning we have log utility), labour hours would not react to technology shocks A_t. What is driving this is that, if $\sigma = 1$, then preferences are consistent with King et al. (1988) preferences[69], in which the income and substitution effects of changes in A_t exactly offset. When there is capital in the model, this offset only occurs in the long-run, so that labour hours are constant in the long-run, but not in the short-run as capital adjusts to steady state. Without capital, the cancellation of income and substitution effects holds at all times.

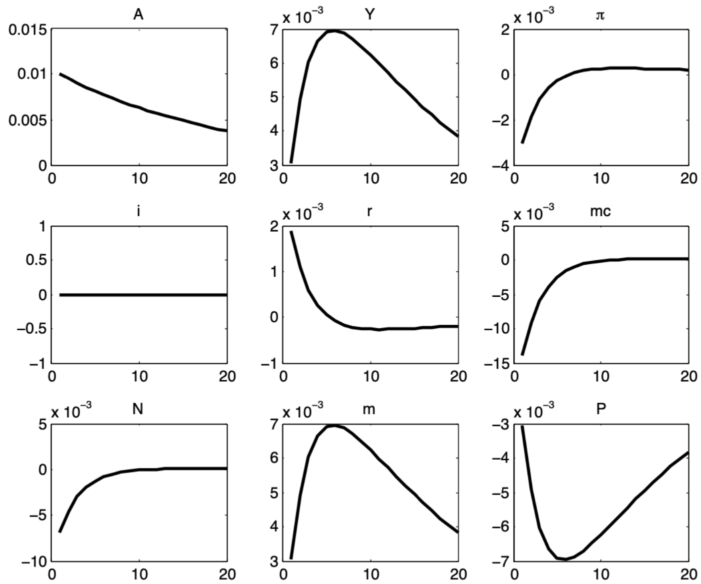
Also note that in the flex price equilibrium, nominal shocks have no real effects. This makes sense as we no longer have any nominal rigidities or stickiness. We still have monopolistic competition, but all the monopolistically competitive firms (the intermediate firms) are able to optimise their prices each and every period without any cost.

11.8 Quantitative analysis

We can solve the model quantitatively in Dynare using a first order approximation about the steady state. Using the parameter values as in Sims (2017), we have: $\phi = 0.75$, $\sigma = 1$, $\eta = 1$, $\psi = 1$, $\epsilon = 10$, $\theta = 1$, $\rho_a = 0.95$, $\rho_m = 0.0$, and $\bar{\pi} = 0$. We assume that the standard deviation of both shocks are 0.01. IRFs to the productivity shock are shown below.

[69]See King et al. (1988) and King et al. (2002).

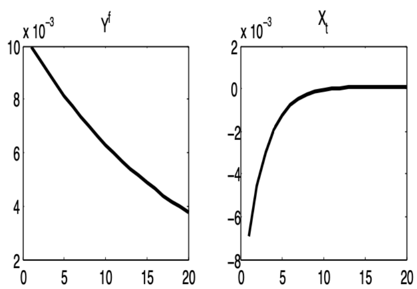
Figure 61: IRFs to Productivity Shock



Source: Sims (2017)

There are a couple if interesting things to point out here. Output responds very little on impact, and significantly less than the increase in A_t. Indeed, we actually see a fairly large decline in N_t when A_t goes up. Inflation falls. The response of the price level (which we compute by cumulating the response of inflation) is roughly the mirror image of the output response. The nominal interest rate does not move at all at any horizon, though the real interest rate increases. Real marginal cost falls, which suggests that the real wage rises by less than A_t (effectively, firms charge bigger markups).

Figure 62: IRFs to Productivity Shock



Source: Sims (2017)

Above, we plot the IRFs of the flexible price level of output and a variable we call the "output gap" (which we will define soon), defined as $\ln X_t = \ln Y_t - \ln Y_t^f$. because output responds significantly less than the flexible price level of output to the productivity shock, we see a large negative output gap opening up following the positive productivity shock.

What's going on here? If $\phi = 0$, we see that output would respond significantly more to the productivity shock than in the baseline case where we used $\phi = 0.75$. Why? When prices are sticky, output becomes [partially] "demand-determined", and with exogenous money supply the way we have it here, price rigidity prevents demand from rising sufficiently when "supply" increases, so output rises by "too little" relative to what would happen with flexible prices. An easy way to see this is to look at the money demand relationship. in logs, we have:

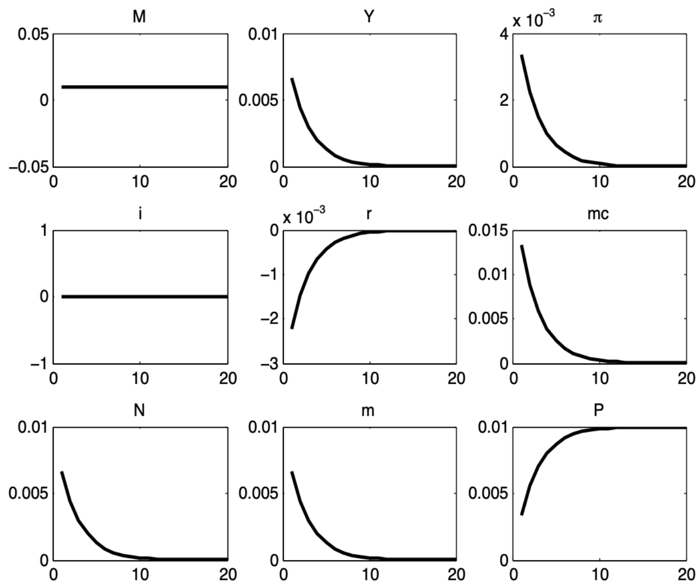
$$\ln m_t = \ln \theta + \ln(1 + i_t) - \ln i_t + \sigma \ln Y_t.$$

To the extent to which the nominal interest rate doesn't move (which it doesn't here), the movement in output must be proportional to the movement in real balances. Since we've assumed that M_t is set exogenously, the only way m_t can move is through changes in P_t. Hence, as we can see in the IRFs, the output movement ends up just being the mirror image of the movement in P_t. And since prices are sticky, P_t can't move enough relative to what it would do under price flexibility. Hence, m_t fails to increase sufficiently, and Y_t can't rise as much as it would if prices were flexible.

There is another way to see how price rigidity effectively limits the demand increase, resulting in a response of output that is too small relative to what would happen in the absence of price rigidity. If prices were flexible, in the period of the shock, P_t would immediately fall (so m_t could rise), but would then start to rise. This means that expected inflation would actually rise. Given a fixed nominal interest rate (via the logic above), this means that the real interest rate would fall if prices were flexible. With price stickiness, in contrast, inflation falls, and stays persistently low (basically ,waves of firms come each period and cut their prices, so inflation stays low for a while). this means that expected inflation falls, not uses as it would would if prices were flexible. This means that the real interest rate rises when A_t increases, which works to choke off demand.

Next, consider a shock to the money supply. Since we have assumed that $\rho_m = 0$, nominal money follows a random walk, so the shock results in a one time permanent level shift in M_t. Here, we observe that Y_t, N_t, and π_t all rise. There is a temporary rise in m_t. mc_t rises, which means that w_t rises (since A_t is fixed): this is necessary to get workers to work more. The real interest rate falls, though again the nominal interest rate doesn't move. Evidently, having sticky prices allows the nominal monetary shock to have real effects.

Table 5: IRFs to Monetary Shock

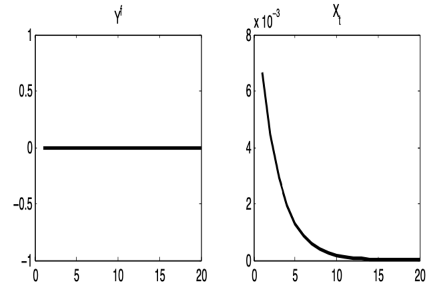


Source: Sims (2017)

What is going on here? There are again a couple of ways to see this. Focusing on the money demand relationship, we again have the result that, for a fixed nominal interest rate, real balances and real GDP move together. When M_t increases, if prices were flexible P_t would increase by the same amount, so real balances wouldn't change, and hence Y_t wouldn't change. But with sticky prices, P_t can't increase sufficiently, so m_t rises, and therefore so too does output. Another way to see what is going on is by focusing on the real interest rate. If prices were flexible, the one time increase in M_t would be met by a one time permanent increase in P_t, so $\mathbb{E}_t P_{t+1} = P_t$, and therefore expected inflation would not react. With expected inflation fixed, and the nominal rate fixed, there would be no effect on the real interest rate. But with price stickiness, because not all firms can immediately adjust their prices, the aggregate price level adjusts slowly, and in particular $\mathbb{E}_t P_{t+1} > P_t$, so expected inflation rises. Higher expected inflation with a fixed nominal rate means a lower real interest rate, which stimulates expenditure and results in the output increase.

Below we show the IRFs of the flexible price level of output and the output gap to the monetary policy shock. Since the flexible price level of output does not react, the response of the gap is identical to the response of output.

Figure 63: IRFs to Monetary Shock



Source: Sims (2017)

11.9 Log-linearising the canonical New Keynesian model

The vast majority of the macroeconomic literature presents the New Keynesian model in log-linear form. This will (on top of some other minor tweaks) allow us to write the model very compactly. The log linearisation is a massive pain in the neck, but the final product is well worth it. For sanity, we will also assume that we log-linearise about a steady state with $\bar{\pi} = 0$.

Start with Euler equation for consumption:

$$\begin{aligned} C_t^{-\sigma} &= \beta \mathbb{E}_t \left[\frac{C_{t+1}^{-\sigma} R_t P_t}{P_{t+1}} \right] \\ \Leftrightarrow Y_t^{-\sigma} &= \beta \mathbb{E}_t \left[\frac{Y_{t+1}^{-\sigma} R_t P_t}{P_{t+1}} \right] \\ \Leftrightarrow Y_t^{-\sigma} &= \beta \mathbb{E}_t \left[\frac{Y_{t+1}^{-\sigma} R_t}{1 + \pi_{t+1}} \right], \end{aligned}$$

and then take logs:

$$
\begin{aligned}
-\sigma \ln Y_t &= \ln \beta - \sigma \mathbb{E}_t \ln Y_{t+1} + \ln R_t - \mathbb{E}_t \ln(1+\pi_{t+1}) \\
\Leftrightarrow -\sigma \ln Y_t &= \ln \beta - \sigma \mathbb{E}_t \ln Y_{t+1} + \ln i_t - \mathbb{E}_t \ln \pi_{t+1} \\
\Leftrightarrow -\sigma \ln Y_t &= \ln \beta - \sigma \mathbb{E}_t \ln Y_{t+1} + i_t - \mathbb{E}_t \pi_{t+1},
\end{aligned}
$$

then use our log-linear rules (I use a Taylor series expansion about the steady state):

$$
-\sigma \ln \bar{Y} - \frac{\sigma}{\bar{Y}} \left(Y_t - \bar{Y}\right) = \ln \beta - \sigma \ln \bar{Y} - \frac{\sigma}{\bar{Y}} \left(\mathbb{E}_t Y_{t+1} - \bar{Y}\right) + \bar{i} + (i_t - \bar{i}) - \bar{\pi} - \left(\mathbb{E}_t \pi_{t+1} - \bar{\pi}\right),
$$

where the $-\sigma \ln \bar{Y}$ terms cancel out:

$$
-\frac{\sigma}{\bar{Y}} \left(Y_t - \bar{Y}\right) = \ln \beta - \frac{\sigma}{\bar{Y}} \left(\mathbb{E}_t Y_{t+1} - \bar{Y}\right) + \bar{i} + (i_t - \bar{i}) - \bar{\pi} - \left(\mathbb{E}_t \pi_{t+1} - \bar{\pi}\right),
$$

and then we know that in the steady state, $\bar{\pi} = 0$, which would imply that by the Fisher equation $\bar{i} = \bar{r}$. Since we're taking logs $\ln R_t = i_t$, and we know $1 + \bar{r} = \frac{1}{\beta}$, hence $\ln(1+\bar{r}) = -\ln \beta$, and so we can write $-\ln \beta = \bar{i}$, and so we have

$$
\begin{aligned}
-\frac{\sigma}{\bar{Y}} \left(Y_t - \bar{Y}\right) &= -\frac{\sigma}{\bar{Y}} \left(\mathbb{E}_t Y_{t+1} - \bar{Y}\right) + (i_t - \bar{i}) + \left(\mathbb{E}_t \pi_{t+1} - \bar{\pi}\right) \\
\Leftrightarrow -\sigma \hat{Y}_t &= -\sigma \mathbb{E}_t \hat{Y}_{t+1} + \hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1},
\end{aligned}
$$

where $\hat{Y}_t = \frac{Y_t - \bar{Y}}{\bar{Y}}$ denotes percent (log) deviations from steady state for Y_t, and variables already in rate form are expressed as absolute deviations (e.g. $\hat{\pi}_t = \pi_t - \bar{\pi}$ and $\hat{i}_t = i_t - \bar{i}$). We can rewrite the above equation as:

$$
\hat{Y}_t = \mathbb{E}_t \hat{Y}_{t+1} - \sigma^{-1} \left(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1}\right), \tag{422}
$$

which is called the "New Keynesian IS Curve" or "Dynamic IS Equation" (DISE). The naming is a bit odd: in old Keynesian models, IS stood for "Investment = Saving", but here we don't have any investment. But if you stare at (422) long enough you can see that it states an inverse relationship between consumption today and the real interest rate.

Now, let's look at the labour supply equation:

$$\psi N_t^{\eta} = \frac{w_t}{C_t^{\sigma}},$$
$$\implies \eta \hat{N}_t = \hat{w}_t - \sigma \hat{C}_t,$$

but we know that:

$$\hat{w}_t = \hat{m}c_t + \hat{A}_t, \tag{423}$$

and $\hat{C}_t = \hat{Y}_t$, so:

$$\eta \hat{N}_t = \hat{m}c_t + \hat{A}_t - \sigma \hat{Y}_t. \tag{424}$$

Next, we log-linearise the production function:

$$\hat{Y}_t = \hat{A}_t + \hat{N}_t - \hat{v}_t^P.$$

What is $\hat{v}_t^P$? This is going to be messy... First, start by taking logs of the price dispersion equation:

$$\ln v_t^P = \ln\left((1-\phi)(1+\pi_t^{\#})^{-\epsilon}(1+\pi_t)^{\epsilon} + (1+\pi_t)^{\epsilon}\phi v_{t-1}^P\right),$$

Now, totally differentiate the above to get:

$$\frac{v_t^P - \bar{v}^P}{\bar{v}^P} = \frac{1}{\bar{v}^P}\left\{\begin{array}{c} -\epsilon(1-\phi)(1+\bar{\pi}^{\#})^{-\epsilon-1}(1+\bar{\pi})^{\epsilon}(\pi_t^{\#} - \bar{\pi}^{\#}) + \epsilon(1-\phi)(1+\bar{\pi}^{\#})^{-\epsilon}(1+\bar{\pi})^{\epsilon-1}(\pi_t - \bar{\pi}) \\ +\epsilon(1+\bar{\pi})^{\epsilon-1}\phi\bar{v}^P(\pi_t - \bar{\pi}) + (1+\bar{\pi})^{\epsilon}\phi(v_{t-1}^P - \bar{v}^P) \end{array}\right\},$$

simplify things by using our facts about $\bar{v}^P = 1$ when $\bar{\pi} = 0 = \bar{\pi}^{\#}$:

$$\begin{aligned} \hat{v}_t^P &= -\epsilon(1-\phi)(1+\bar{\pi}^{\#})^{-\epsilon-1}(1+\bar{\pi})^{\epsilon}\hat{\pi}_t^{\#} + \epsilon(1-\phi)(1+\bar{\pi}^{\#})^{-\epsilon}(1+\bar{\pi})^{\epsilon-1}\hat{\pi}_t \\ &\quad + \epsilon(1+\bar{\pi})^{\epsilon}\phi\bar{v}^P\hat{\pi}_t + (1+\bar{\pi})\phi(v_{t-1}^P - \bar{v}^P), \\ \implies \hat{v}_t^P &= -\epsilon(1+\phi)\hat{\pi}_t^{\#} + \epsilon(1-\phi)\hat{\pi}_t + \epsilon\phi\hat{\pi}_t + \phi\hat{v}_{t-1}^P, \end{aligned}$$

and this can be written as:

$$\hat{v}_t^P = -\epsilon(1-\phi)\hat{\pi}_t^{\#} + \epsilon\hat{\pi}_t + \phi\hat{v}_{t-1}^P.$$

Next, log-linearise the equation for the evolution of inflation:

$$\begin{aligned}
(1+\pi_t)^{1-\epsilon} &= (1-\phi)(1+\pi_t^{\#})^{1-\epsilon} + \phi \\
\implies (1-\epsilon)\ln(1+\pi_t) &= \ln\left((1-\phi)(1+\pi_t^{\#})^{1-\epsilon} + \phi\right) \\
\therefore (1-\epsilon)\pi_t &= \ln\left((1-\phi)(1+\pi_t^{\#})^{1-\epsilon} + \phi\right),
\end{aligned}$$

and then totally differentiate:

$$(1-\epsilon)(\pi_t - \bar{\pi}) = (1+\bar{\pi})^{\epsilon-1}\left((1-\epsilon)(1-\phi)(1+\bar{\pi}^{\#})^{-\epsilon}(\pi_t^{\#} - \bar{\pi}^{\#})\right),$$

where $(1+\bar{\pi})^{\epsilon-1}$ shows up because the term inside the large parentheses is equal to $(1+\bar{\pi})^{1-\epsilon}$ in the steady state, and when we take the derivative of the log this term gets inverted at the steady state. We can use what we know about the zero inflation steady state to write:

$$\begin{aligned}
(1-\epsilon)\hat{\pi}_t &= (1-\epsilon)(1-\phi)\hat{\pi}_t^{\#} \\
\Leftrightarrow \hat{\pi}_t &= (1-\phi)\hat{\pi}_t^{\#}.
\end{aligned} \tag{425}$$

In words, actual inflation is just proportional to reset price inflation, where the constant is equal to the fraction of firms that are updating their prices. Now use this by substituting it into the expression for price dispersion to get:

$$\begin{aligned}
\hat{v}_t^P &= -\epsilon(1-\phi)\hat{\pi}_t^{\#} + \epsilon\left[(1-\phi)\hat{\pi}_t^{\#}\right] + \phi\hat{v}_{t-1}^P \\
\therefore \hat{v}_t^P &= \phi\hat{v}_{t-1}^P.
\end{aligned} \tag{426}$$

This is a fairly important equation to note. If we are approximating about the zero inflation steady state where $\bar{v}^P = 1$, then we're starting from a position in which $\hat{v}_{t-1}^P = 0$, which means that $\hat{v}_t^P = 0, \forall t$.

In other words, about a zero inflation steady state, price dispersion is a second order phenomenon, and we can just ignore it in a first order approximation about a zero inflation steady state.[70]

We now move onto log-linearising the production function:

$$\hat{Y}_t = \hat{A}_t + \hat{N}_t. \tag{427}$$

Then, substitute $\hat{N}_t = \hat{Y}_t - \hat{A}_t$ into the labour supply condition (424) to get

$$\begin{aligned}
\eta(\hat{Y}_t - \hat{A}_t) &= \hat{mc}_t + \hat{A}_t - \sigma\hat{Y}_t \\
\implies \hat{mc}_t &= \eta(\hat{Y}_t - \hat{A}_t) - \hat{A}_t + \sigma\hat{Y}_t \\
&= (\sigma + \eta)\hat{Y}_t - (1 + \eta)\hat{A}_t.
\end{aligned}$$

Now, we know that

$$Y_t^f = \left(\frac{1}{\psi}\frac{\epsilon - 1}{\epsilon}\right)^{\frac{1}{\sigma+\eta}} A_t^{\frac{1+\eta}{\sigma+\eta}},$$

and log-linearising this we have

$$\begin{aligned}
\hat{Y}_t^f &= \frac{1+\eta}{\sigma+\eta}\hat{A}_t \qquad (428) \\
\Leftrightarrow \hat{A}_t &= \frac{\sigma+\eta}{1+\eta}\hat{Y}_t^f,
\end{aligned}$$

and so we can substitute this into our expression for $\hat{mc}_t$ from above to get:

$$\begin{aligned}
\hat{mc}_t &= (\sigma + \eta)\hat{Y}_t - (1 + \eta)\frac{\sigma+\eta}{1+\eta}\hat{Y}_t^f \\
&= (\sigma + \eta)(\hat{Y}_t - \hat{Y}_t^f) \\
\hat{mc}_t &= (\sigma + \eta)\hat{X}_t \qquad (429)
\end{aligned}$$

In words, deviations of real marginal cost are proportional to the output gap, $\hat{X}_t = \hat{Y}_t - \hat{Y}_t^f$. Recall

[70] This is why we can equate Calvo pricing to Rotemberg pricing up to a first order. Remember that there is no price dispersion in the Rotemberg model as all firms price identically.

that real marginal cost is the inverse of the price markup. So if the gap is zero, then markups are equal to the desired fixed steady state markup of $\frac{\epsilon}{\epsilon-1}$. If the output gap is positive, then real marginal cost is above its steady state, so markups are lower than desired (equivalently, the economy is less distorted). The converse is true when the gap is negative.

Next we log-linearise the reset price expression:

$$\begin{aligned} 1+\pi_t^{\#} &= \frac{\epsilon}{\epsilon-1}\frac{x_{1,t}}{x_{2,t}}(1+\pi_t), \\ \implies \ln(1+\pi_t^{\#}) &= \ln\left(\frac{\epsilon}{\epsilon-1}\right) + \ln x_{1,t} - \ln x_{2,t} + \ln(1+\pi_t) \\ \Leftrightarrow \pi_t^{\#} &= \ln\left(\frac{\epsilon}{\epsilon-1}\right) + \ln x_{1,t} - \ln x_{2,t} + \pi_t, \end{aligned}$$

which gives us

$$\hat{\pi}_t^{\#} = \hat{x}_{1,t} - \hat{x}_{2,t} + \hat{\pi}_t. \tag{430}$$

We now need to log-linearise the auxiliary variables:

$$\begin{aligned} x_{1,t} &= \frac{mc_t Y_t}{C_t^{\sigma}} + \phi\beta\mathbb{E}_t\left[(1+\pi_{t+1})^{\epsilon} x_{1,t+1}\right], \\ x_{2,t} &= \frac{Y_t}{C_t^{\sigma}} + \phi\beta\mathbb{E}_t\left[(1+\pi_{t+1})^{\epsilon-1} x_{2,t+1}\right]. \end{aligned}$$

This is going to be messy... Let's start with $x_{1,t}$. Because we have $Y_t = C_t$, we can write:

$$\ln x_{1,t} = \ln\left(mc_t Y_t^{1-\sigma} + \phi\beta\mathbb{E}_t\left[(1+\pi_{t+1})^{\epsilon} x_{1,t+1}\right]\right),$$

and totally differentiating we have:

$$
\begin{aligned}
\frac{x_{1,t}-\bar{x}_1}{\bar{x}_1} &= \frac{1}{\bar{x}_1}\left[\bar{Y}^{1-\sigma}(mc_t-\bar{mc})+(1-\sigma)\bar{mc}\bar{Y}^{-\sigma}(Y_t-\bar{Y})\right.\\
&\quad\left.+\epsilon\phi\beta(1+\bar{\pi})^{\epsilon-1}\bar{x}_1(\mathbb{E}_t\pi_{t+1}-\bar{\pi})+\phi\beta\mathbb{E}_t(1+\bar{\pi})^{\epsilon}(\mathbb{E}_t x_{1,t+1}-\bar{x}_1)\right]\\
\hat{x}_{1,t} &= \frac{\bar{Y}^{1-\sigma}}{\bar{x}_1}(mc_t-\bar{mc})+\frac{(1-\sigma)\bar{mc}\bar{Y}^{-\sigma}}{\bar{x}_1}(Y_t-\bar{Y})\\
&\quad+\frac{\epsilon\phi\beta(1+\bar{\pi})^{\epsilon-1}\bar{x}_1}{\bar{x}_1}(\mathbb{E}_t\pi_{t+1}-\bar{\pi})+\frac{\phi\beta(1+\bar{\pi})^{\epsilon}}{\bar{x}_1}(\mathbb{E}_t x_{1,t+1}-\bar{x}_1)\\
&= \frac{\bar{Y}^{1-\sigma}\bar{mc}}{\bar{x}_1}\widehat{mc}_t+\frac{(1-\sigma)\bar{mc}\bar{Y}^{1-\sigma}}{\bar{x}_1}\hat{Y}_t+\epsilon\phi\beta\mathbb{E}_t\hat{\pi}_{t+1}+\phi\beta\mathbb{E}_t\hat{x}_{1,t+1},
\end{aligned}
$$

and in the steady state we know that $x_1=\frac{\bar{Y}^{1-\sigma}\bar{mc}}{1-\phi\beta}$, which yields

$$
\hat{x}_{1,t}=(1-\sigma)(1-\phi\beta)\hat{Y}_t+(1-\phi\beta)\hat{mc}_t+\epsilon\phi\beta\mathbb{E}_t\hat{\pi}_{t+1}+\phi\beta\mathbb{E}_t\hat{x}_{1,t+1}. \tag{431}
$$

Now we can deal with $x_{2,t}$. Start by log-linearising:

$$
\ln x_{2,t}=\ln\left(Y_t^{1-\sigma}+\phi\beta\mathbb{E}_t\left[(1+\pi_{t+1})^{\epsilon-1}x_{2,t+1}\right]\right),
$$

and then totally differentiate:

$$
\begin{aligned}
\frac{x_{2,t}-\bar{x}_2}{\bar{x}_2} &= \frac{1}{\bar{x}_2}\left[(1-\sigma)\bar{Y}^{-\sigma}(Y_t-\bar{Y})+(\epsilon-1)\phi\beta(1+\bar{\pi})^{\epsilon-2}\bar{x}_2(\mathbb{E}_t\pi_{t+1}-\bar{\pi})+\phi\beta(1+\bar{\pi})^{\epsilon-1}(\mathbb{E}_t x_{2,t+1}-\bar{x})\right]\\
\hat{x}_{2,t} &= \frac{(1-\sigma)\bar{Y}^{-\sigma}(Y_t-\bar{Y})}{\bar{x}_2}+\frac{(\epsilon-1)\phi\beta(1+\bar{\pi})^{\epsilon-2}\bar{x}_2(\mathbb{E}_t\pi_{t+1}-\bar{\pi})}{\bar{x}_2}+\frac{\phi\beta(1+\bar{\pi})^{\epsilon-1}(\mathbb{E}_t x_{2,t+1}-\bar{x})}{\bar{x}_2}\\
&= \frac{(1-\sigma)\bar{Y}^{1-\sigma}}{\bar{x}_2}\hat{Y}_t+(\epsilon-1)\phi\beta\hat{\pi}_t+\phi\beta\mathbb{E}_t\hat{x}_{2,t+1},
\end{aligned}
$$

and we know $\bar{x}_2=\frac{Y^{1-\sigma}}{1-\phi\beta}$, so we have :

$$
\hat{x}_{2,t}=(1-\sigma)(1-\phi\beta)\hat{Y}_t+(\epsilon-1)\phi\beta\mathbb{E}_t\hat{\pi}_{t+1}+\phi\beta\mathbb{E}_t\hat{x}_{2,t+1}. \tag{432}
$$

Now, subtracting $\hat{x}_{2,t}$ from $\hat{x}_{1,t}$, yields:

$$\begin{aligned}\hat{x}_{1,t} - \hat{x}_{2,t} &= (1-\sigma)(1-\phi\beta)\hat{Y}_t + (1-\phi\beta)\widehat{mc}_t + \epsilon\phi\beta\mathbb{E}_t\hat{\pi}_{t+1} + \phi\beta\mathbb{E}_t\hat{x}_{1,t+1} \\ &\quad - (1-\sigma)(1-\phi\beta)\hat{Y}_t + (\epsilon-1)\phi\beta\mathbb{E}_t\hat{\pi}_{t+1} + \phi\beta\mathbb{E}_t\hat{x}_{2,t+1} \\ &= (1-\phi\beta)\widehat{mc}_t + \phi\beta\mathbb{E}_t\hat{\pi}_{t+1} + \phi\beta\mathbb{E}_t\left(\hat{x}_{1,t+1} - \hat{x}_{2,t+1}\right),\end{aligned}$$

and we know that:

$$\hat{x}_{1,t} - \hat{x}_{2,t} = \hat{\pi}_t^{\#} - \hat{\pi}_t,$$

but:

$$\hat{\pi}_t^{\#} = \frac{1}{1-\phi}\hat{\pi}_t,$$

so we must have:

$$\hat{x}_{1,t} - \hat{x}_{2,t} = \frac{\phi}{1-\phi}\hat{\pi}_t,$$

so:

$$\begin{aligned}\frac{\phi}{1-\phi}\hat{\pi}_t &= (1-\phi\beta)\widehat{mc}_t + \phi\beta\mathbb{E}_t\hat{\pi}_{t+1} + \phi\beta\mathbb{E}_t\left(\hat{x}_{1,t+1} - \hat{x}_{2,t+1}\right) \\ \hat{\pi}_t &= \frac{(1-\phi)(1-\phi\beta)}{\phi}\hat{m}c_t + (1-\phi)\beta\mathbb{E}_t\hat{\pi}_{t+1} + (1-\phi)\beta\mathbb{E}\hat{\pi}_{t+1},\end{aligned}$$

and with a bit of cleaning up:

$$\hat{\pi}_t = \frac{(1-\phi)(1-\phi\beta)}{\phi}\widehat{mc}_t + \beta\mathbb{E}_t\hat{\pi}_{t+1}. \tag{433}$$

This expression is called the "New Keynesian Phillips Curve" (NKPC). It is "new" because it is forward-looking unlike classic Phillips Curves, but it's a Phillips Curve in the sense that it captures a relationship between inflation and some real measure. We can re-write the NKPC in terms of the output gap by using (429):

$$\begin{aligned}\hat{\pi}_t &= \frac{(1-\phi)(1-\phi\beta)(\sigma+\eta)}{\phi}(\hat{Y}_t - \hat{Y}_t^f) + \beta\mathbb{E}_t\hat{\pi}_{t+1} \\ \Leftrightarrow \hat{\pi}_t &= \kappa\hat{X}_t + \beta\mathbb{E}_t\hat{\pi}_{t+1}, \end{aligned} \tag{434}$$

where $\kappa = \frac{(1-\phi)(1-\phi\beta)(\sigma+\eta)}{\phi}$ and is often referred to as the slope of the NKPC. Using the terminal condition that inflation will return to steady state eventually, using (433) and the law of iterated expectations we can solve the NKPC forward to get:

$$\begin{aligned}\hat{\pi}_t &= \frac{(1-\phi)(1-\phi\beta)}{\phi}\hat{m}c_t + \beta\mathbb{E}_t\begin{bmatrix} \frac{(1-\phi)(1-\phi\beta)}{\phi}\hat{m}c_{t+1} \\ +\beta\mathbb{E}_{t+1}\left[\frac{(1-\phi)(1-\phi\beta)}{\phi}\hat{m}c_{t+2} + \beta\mathbb{E}_{t+2}\hat{\pi}_{t+3}\right]\end{bmatrix} \\ &= \frac{(1-\phi)(1-\phi\beta)}{\phi}\sum_{j=0}^{\infty}\beta^j\widehat{mc}_{t+j}.\end{aligned}$$

In words, current inflation is proportional to the present discounted value of expected real marginal cost. Real marginal cost is the inverse of the price markup. In the model without price rigidity, firms desire constant markups. If expected future marginal cost is high, then firms will have low markups. Firms given the option of updating prices today will try to increase price today (since they may be stuck with that price in the future) to hit their desired price markup (and vice-versa), putting upward pressure on current inflation (and vice-versa). Thus, the slope of the NKPC is decreasing in ϕ: when ϕ is large, the coefficient on marginal cost (or the gap) is small, suggesting that real movements put little upward pressure on inflation. In the limiting case as $\phi \to +\infty$, the NKPC becomes vertical, implying that $\widehat{mc}_t = 0$ and $\hat{Y}_t = \hat{Y}_t^f$.

The rest of the equations are fairly straightforward. The expressions for A_t and money growth are already log-linear so we have:

$$\hat{A}_t = \rho_a \hat{A}_{t-1} + \epsilon_{a,t}, \tag{435}$$

$$\Delta\hat{m}_t = -\hat{\pi}_t + \rho_m \hat{\pi}_{t-1} + \rho_m \Delta\hat{m}_{t-1} + \epsilon_{m,t}, \tag{436}$$

$$\Delta\hat{m}_t = \hat{m}_t - \hat{m}_{t-1}. \tag{437}$$

Finally, log-linearise the money demand function:

$$\begin{aligned} m_t &= \frac{\theta R_t C_t^\sigma}{i_t} \\ \implies \ln m_t &= \ln\theta + \ln R_t + \sigma \ln C_t - \ln i_t, \end{aligned}$$

and using what we know about R_t and C_t:

$$\begin{aligned} \ln m_t &= \ln\theta + i_t + \sigma \ln Y_t - \ln i_t \\ \ln\bar{m} + \frac{m_t - \bar{m}}{\bar{m}} &= \ln\theta + (i_t - \bar{i}) + \sigma\ln\bar{Y} + \frac{Y_t - \bar{Y}}{\bar{Y}} - \ln\bar{i} - \frac{i_t - \bar{i}}{\bar{i}} \\ \frac{m_t - \bar{m}}{\bar{m}} &= (i_t - \bar{i}) + \frac{Y_t - \bar{Y}}{\bar{Y}} - \frac{i_t - \bar{i}}{\bar{i}} \\ \hat{m}_t &= \hat{i}_t + \hat{Y}_t - \frac{1}{\bar{i}}\hat{i}_t. \end{aligned}$$

Recall that $\hat{i} = i_t - \bar{i}$, where $i = \frac{1}{\beta} - 1$, since $\bar{\pi} = 0$. Hence we can write the above equation as:

$$\hat{m}_t = \hat{i}_t - \frac{\beta}{1-\beta}\hat{i}_t + \sigma\hat{Y}_t,$$

or equivalently as:

$$\hat{m}_t = \left(1 - \frac{\beta}{1-\beta}\right)\hat{i}_t + \sigma\hat{Y}_t. \tag{438}$$

Which is pretty straightforward: Demand for real money balances is decreasing in the real interest rate and increasing in Y_t (think of the LM curve).

The complete log-linearised system of equations is:

$$\hat{Y}_t = \mathbb{E}_t \hat{Y}_{t+1} - \sigma^{-1} \left(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1} \right), \tag{439}$$

$$\hat{\pi}_t = \kappa \hat{X}_t + \beta \mathbb{E}_t \pi_{t+1}, \tag{440}$$

$$\hat{Y}_t^f = \frac{1+\eta}{\sigma+\eta} \hat{A}_t. \tag{441}$$

$$\hat{A}_t = \rho_a \hat{A}_{t-1} + \epsilon_{a,t}, \tag{442}$$

$$\Delta \hat{m}_t = -\hat{\pi}_t + \rho_m \hat{\pi}_{t-1} + \rho_m \Delta \hat{m}_{t-1} + \epsilon_{m,t}, \tag{443}$$

$$\Delta \hat{m}_t = \hat{m}_t - \hat{m}_{t-1}, \tag{444}$$

$$\hat{m}_t = \left(1 - \frac{\beta}{1-\beta} \right) \hat{i}_t + \sigma \hat{Y}_t. \tag{445}$$

This is seven equations in seven variables. In true Keynesian fashion, "aggregate demand" is given by the DISE, "aggregate supply" is given by the NKPC, and we have equations for productivity shocks, money supply, money demand, and the flex price equilibrium. But we can go simpler...

11.10 The canonical New Keynesian model with a Taylor Rule and Calvo pricing

So far our model has contained an exogenous rule for money growth. But this doesn't seem to match how monetary policy is conducted. We want monetary policy to focus on changing the interest rate in response to endogenous changes in inflation and output.[71] A popular interest rate rule is the Taylor Rule:[72]

$$i_t = (1-\rho_i)\bar{i} + \rho_i i_{t-1} + (1+\rho_i) \left[\phi_\pi (\pi_t - \bar{\pi}) + \phi_y (\ln X_t - \ln \bar{X}) \right] + \epsilon_{i,t}, \tag{446}$$

where $\bar{X}$ is the steady state output gap, $\epsilon_{i,t}$ is a monetary policy shock, ϕ_π, ϕ_y, and ρ_i are coefficients, with $\phi_\pi > 1$[73]. Notice that money does not enter the interest rate rule. We can replace the money

[71] We shall later see that exogenous rules for the interest rate may lead to indeterminacy.

[72] Taylor's original rule was

$$i_t^F = 4 + 1.5(\bar{\pi}_t - 2) + 0.5(y_t - y_t^*),$$

where i_t^F is the Federal Funds Rate, $\bar{\pi}_t$ is annual inflation, and y_t^* was trend (log) GDP.

[73] This is crucial for determinacy.

growth process with the interest rate rule and assume that the central bank provides sufficient money at all times to meet money demand at the interest rate. Given the preferences households have of money – that they are additively separable – we could actually ignore money altogether and assume a cashless economy.

The full set of equilibrium conditions for the cashless economy are:

$$C_t^{-\sigma} = \beta \mathbb{E}_t \left[\frac{C_{t+1}^{-\sigma} R_t}{1+\pi_{t+1}} \right], \tag{447}$$

$$\psi N_t^{\eta} = C_t^{-\sigma}, \tag{448}$$

$$m_t = \theta \frac{R_t}{i_t} C_t^{\sigma}, \tag{449}$$

$$mc_t = \frac{w_t}{A_t}, \tag{450}$$

$$C_t = Y_t, \tag{451}$$

$$Y_t = \frac{A_t N_t}{v_t^P}, \tag{452}$$

$$v_t^P = (1-\phi)(1+\pi_t^{\#})^{-\epsilon}(1+\pi_t)^{\epsilon} + (1+\pi_t)^{\epsilon} \phi v_{t-1}^P, \tag{453}$$

$$(1+\pi_t)^{1-\epsilon} = (1-\phi)(1+\pi_t^{\#})^{1-\epsilon} + \phi, \tag{454}$$

$$1+\pi_t^{\#} = \frac{\epsilon}{\epsilon-1} \frac{x_{1,t}}{x_{2,t}}(1+\pi_t), \tag{455}$$

$$x_{1,t} = C_t^{-\sigma} mc_t Y_t + \phi \beta \mathbb{E}_t \left(1+\pi_{t+1}\right)^{\epsilon} x_{1,t+1}, \tag{456}$$

$$x_{2,t} = C_t^{-\sigma} Y_t + \phi \beta \mathbb{E}_t \left(1+\pi_{t+1}\right)^{\epsilon-1} x_{2,t+1}, \tag{457}$$

$$\ln A_t = \rho \ln A_{t-1} + \epsilon_{a,t}, \tag{458}$$

$$i_t = (1-\rho_i)\bar{i} + \rho_i i_{t-1} + (1+\rho_i)\left[\phi_\pi(\pi_t - \bar{\pi}) + \phi_y(\ln X_t - \ln \bar{X})\right] + \epsilon_{i,t}. \tag{459}$$

We got rid of the law of motion for money, and thus we can get rid of $\Delta \ln m_t$. So that's one less variable and one less equation. We could even get rid of m_t entirely from the model, but keeping it will prove to be useful – we want to see how real money balances are moving in the background.

One caveat worth pointing out: we may be tempted to also think that the steady state output gap is zero, i.e. $\bar{Y} = \bar{Y}^f$. This will only be the case if $\bar{\pi} = 0$, otherwise $\bar{Y} < \bar{Y}^f$. From before we know that:

$$\bar{Y}^f = \left(\frac{1}{\psi}\frac{\epsilon - 1}{\epsilon}\right)^{\frac{1}{\sigma+\eta}}, \tag{460}$$

and for the sticky price economy we have:

$$\bar{N} = \left(\frac{1}{\psi}\left(\bar{v}^P\right)^{\sigma} \bar{mc}\right)^{\frac{1}{\eta+\sigma}}. \tag{461}$$

We know that

$$\bar{Y} = \frac{\bar{N}}{\bar{v}^P},$$

which implies that steady state output is

$$\bar{Y} = \left(\frac{1}{\psi}\right)^{\frac{1}{\sigma+\eta}} \left(\bar{v}^P\right)^{-\frac{\eta}{\eta+\sigma}} \bar{mc}^{\frac{1}{\eta+\sigma}}, \tag{462}$$

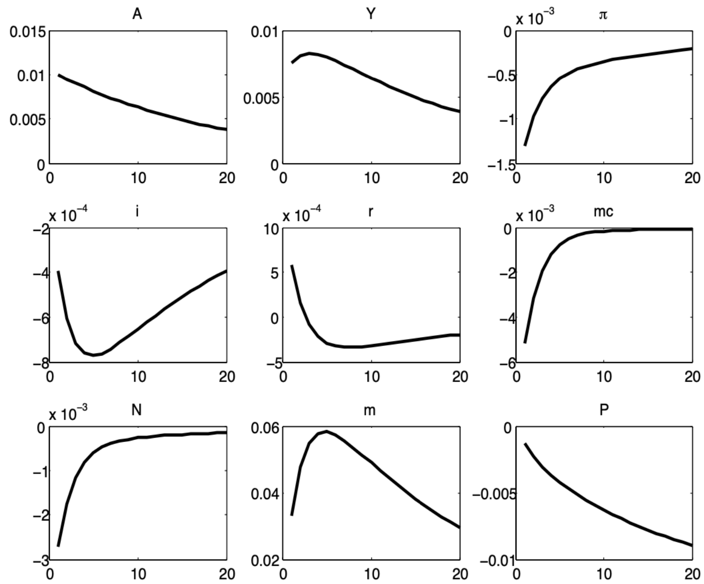
and we also know

$$\bar{mc} = \frac{1 - \phi\beta(1+\bar{\pi})^{\epsilon}}{1 - \phi\beta(1+\bar{\pi})^{\epsilon-1}} \frac{1+\bar{\pi}^{\#}}{1+\bar{\pi}} \frac{\epsilon - 1}{\epsilon}.$$

If $\bar{\pi} = 0$, then $\bar{mc} = \frac{\epsilon-1}{\epsilon}$ and $\bar{v}^P = 1$, so this reduces to the same expression as $\bar{Y}^f$, so we'll have $\bar{Y} = \bar{Y}^f$. But if $\bar{\pi} > 0$, you can show that $\bar{mc} < \frac{\epsilon-1}{\epsilon}$, and we know that $\bar{v}^P > 1$. Since the exponent on $\bar{mc}$ is positive, and the exponent on $\bar{v}^P$ negative, this means that $\bar{\pi} > 0$ will mean that $\bar{Y} < \bar{Y}^f$, which means that the steady state output gap will be negative, $\ln \bar{X} = \ln \bar{Y} - \ln \bar{Y}^f < 0$.

If you actually simulate the model with the Taylor Rule against the model with the money supply rule, you will find some pretty stark differences. Consider the case of a productivity shock.

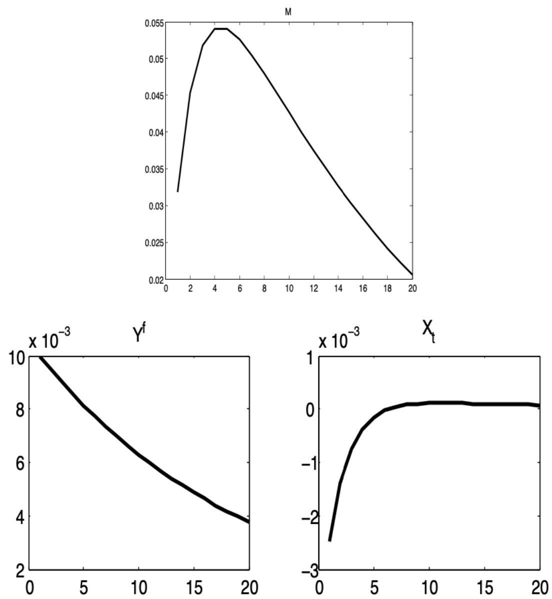
Figure 64: IRFs of Productivity Shock (Taylor Rule)



Source: Sims (2017)

Under the Taylor Rule, output increases significantly more than under the money supply rule; there is a smaller drop in hours worked on impact; smaller increase in the real interest rate; smaller drop in inflation; response of the price level seems to be more or less permanent, whereas under a money supply rule it was mean reverting; and, nominal money supply increases significantly. In other words, under the Taylor Rule, money supply is basically endogenous. An increase in output and transactional demand sees the central bank substantially increase nominal money balances, which also leads to an increase in real money balances. We don't have to rely on just the price level falling to get increases in real balances.

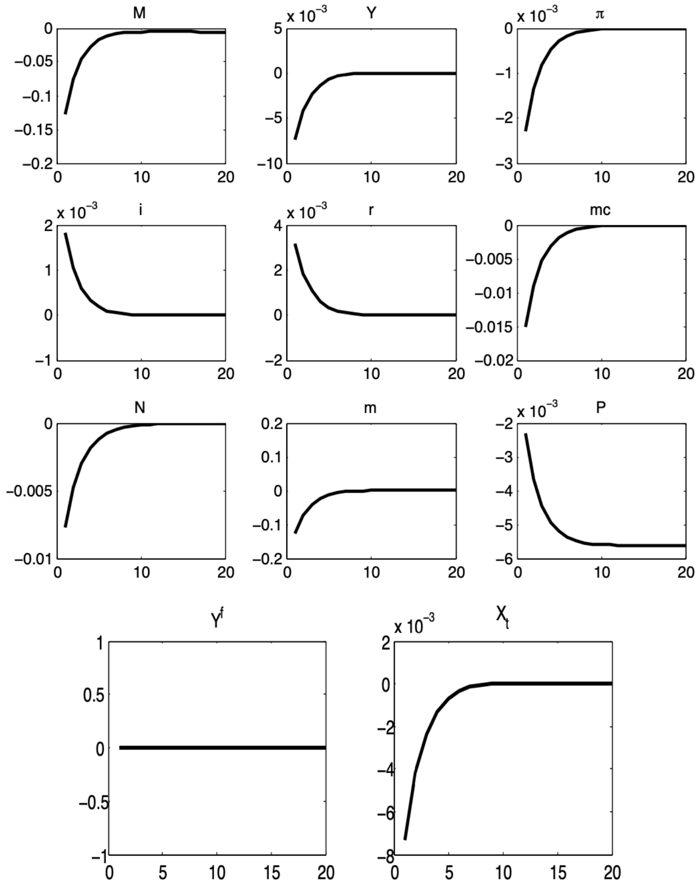
Figure 65: IRFs of Productivity Shock (Taylor Rule)



Source: Sims (2017)

Next, consider a positive shock to the Taylor Rule, which raises the nominal interest rate. This coincides with a decline in the money supply, an increase in the real interest rate, and a decline in economic activity. The channels at play for why this nominal shock has real effects are the same as above when we thought about the nominal shock in terms of the money supply. There are two ways to think about. First, the decrease in the money supply is matched by a less than proportional decrease in the price level because of price stickiness; this means that real balances decline, which via the basic logic above necessitates a decline in output. It also has effect of raising the real interest rate. The nominal rate rises, and because of price stickiness expected inflation does not rise enough, so the real rate rises, which leads to a reduction in demand.

Figure 66: IRFs of Monetary Policy Shock (Taylor Rule)



Source: Sims (2017)

We can write the model under the Taylor Rule in log-linear form:

$$\hat{Y}_t = \mathbb{E}_t \hat{Y}_{t+1} - \sigma^{-1} \left(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1} \right), \tag{463}$$

$$\hat{\pi}_t = \kappa \hat{X}_t + \beta \mathbb{E}_t \hat{\pi}_{t+1}, \tag{464}$$

$$\hat{Y}_t^f = \frac{1+\eta}{\sigma+\eta} \hat{A}_t, \tag{465}$$

$$\hat{A}_t = \rho_a \hat{A}_{t-1} + \epsilon_{a,t}, \tag{466}$$

$$\hat{i}_t = \rho_i \hat{i}_{t-1} + (1-\rho_i) \left(\phi_\pi \hat{\pi}_t + \phi_y \hat{X}_t \right) + \epsilon_{i,t}. \tag{467}$$

We're now down to five equations! But, we can go even simpler by eliminating $\hat{A}_t$ and just writing the model in terms of $\hat{Y}_t^f$. Let $\omega = \frac{1+\eta}{\sigma+\eta}$:

$$\begin{aligned}\hat{Y}_t^f &= \omega \hat{A}_t \\ &= \omega \left(\rho_a \hat{A}_{t-1} + \epsilon_{a,t} \right) \\ &= \omega \left(\rho_a \frac{1}{\omega} \hat{Y}_{t-1}^f + \epsilon_{a,t} \right),\end{aligned}$$

or

$$\hat{Y}_t^f = \rho_a \hat{Y}_{t-1}^f + \omega \epsilon_{a,t}. \tag{468}$$

Now, let's rewrite the DISE in terms of the output gap, $\hat{X}_t$, instead of output. We can do this by subtracting $\hat{Y}_t^f$ and $\mathbb{E}_t \hat{Y}_{t+1}^f$ from both sides:

$$\begin{aligned}\hat{Y}_t - \hat{Y}_t^f - \mathbb{E}_t \hat{Y}_{t+1}^f &= \mathbb{E}_t \hat{Y}_{t+1} - \hat{Y}_t^f - \mathbb{E}_t \hat{Y}_{t+1}^f - \sigma^{-1} \left(\hat{i} - \mathbb{E}_t \hat{\pi}_{t+1} \right) \\ \Leftrightarrow \hat{X}_t &= \mathbb{E}_t \hat{X}_{t+1} + \mathbb{E}_t \hat{Y}_{t+1}^f - \hat{Y}_t^f - \sigma^{-1} \left(\hat{i} - \mathbb{E}_t \hat{\pi}_{t+1} \right).\end{aligned}$$

We know from the Fisher equation that

$$\hat{r}_t = \hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1},$$

and consider the case where $\phi = 0$ (so we have a flex price equilibrium). Then, we know that $\hat{X}_t = 0$. We can then solve for a hypothetical flex price real interest rate – known as the "Wicksellian natural rate of interest":[74]

$$\begin{aligned}0 &= \mathbb{E}_t \hat{Y}_{t+1}^f - \hat{Y}_t^f - \sigma^{-1} \hat{r}_t^f \\ \implies \hat{r}_t^f &= \sigma \left(\mathbb{E}_t \hat{Y}_{t+1}^f - \hat{Y}_t^f \right).\end{aligned} \tag{469}$$

[74]Wicksell's most influential contribution was his theory of interest, originally published in German as *Geldzins und Güterpreise*, in 1898. The English translation *Interest and Prices* became available in 1936; a literal translation of the original title would read *Money Interest and Commodity Prices.* Wicksell invented the key term natural rate of interest and defined it at that interest rate which is compatible with a stable price level.

In words, the "natural"/flex price interest rate is proportional to the expected growth rate of the flex price level of output. We can use this to write the Euler equation/DISE as:

$$\hat{X}_t = \mathbb{E}_t \hat{X}_{t+1} - \sigma^1 \left(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1} - \hat{r}_t^f \right). \tag{470}$$

Note also that since $\mathbb{E}_t \hat{Y}_{t+1}^f = \rho_a \hat{Y}_t^f$, we can write (469) as:

$$\hat{r}_t^f = \sigma \hat{Y}_t^f (\rho_a - 1). \tag{471}$$

Plug in the AR(1) process (468) we derived for $\hat{Y}_t^f$:

$$\begin{aligned}
\hat{r}_t^f &= \sigma \left[\rho_a \hat{Y}_{t-1}^f + \omega \epsilon_{a,t} \right] (\rho_a - 1) \\
&= \sigma \left[\rho_a \left[\frac{\hat{r}_{t-1}^f}{\sigma(\rho_a - 1)} \right] + \omega \epsilon_{a,t} \right] (\rho_a - 1) \\
\therefore \hat{r}_t^f &= \rho_a \hat{r}_{t-1}^f + \sigma(\rho_a - 1)\omega \epsilon_{a,t},
\end{aligned} \tag{472}$$

recalling that $\omega = \frac{1+\eta}{\sigma+\eta}$. We now have all we need to write down the full log-linearised New Keynesian model with a Taylor Rule and Calvo pricing as:

$$\text{DISE: } \hat{X}_t = \mathbb{E}_t \hat{X}_{t+1} - \sigma^{-1} \left(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1} - \hat{r}_t^f \right), \tag{473}$$

$$\text{NKPC: } \hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \kappa \hat{X}_t, \tag{474}$$

$$\text{Taylor Rule: } \hat{i}_t = \rho_i \hat{i}_{t-1} + (1 - \rho_i) \left(\phi_\pi \hat{\pi}_t + \phi_y \hat{X}_t \right) + \epsilon_{i,t}, \tag{475}$$

$$\hat{r}_t^f = \rho_a \hat{r}_{t-1}^f + \sigma(\rho_a - 1)\omega \epsilon_{a,t}. \tag{476}$$

This is called referred to in the macroeconomic literature as the "Canonical New Keynesian Model", often written down with just the first three equations, with the law of motion for $\hat{r}_t^f$ in the background (since it is driven by exogenous technology shocks).

11.11 The method of undetermined coefficients and the Rational Expectations solution

The three equation New Keynesian (NK) model has two jump variables/control variables $(\hat{\pi}_t, \hat{X}_t)$ and two state variables $(\hat{i}_t, \hat{r}_t^f)$, where $\hat{i}_t$ is an endogenous state variable and $\hat{r}_t^f$ is an exogenous state variable. We could solve for the policy functions mapping the states into the jump variables – Dynare will do this very easily for us. Or, since we don't capital and investment in this model, we can use the method of undetermined coefficients.

The method of undetermined coefficients involves us guessing a policy function function (linear in this case since the system is log-linear), imposing that, and the solving a system of equations for the policy rule coefficients. In a small scale model without capital, this is pretty easy to do, and gives us nice analytical solutions.

11.11.1 A simple example: technology shock with no persistence

Consider the three equation NK model with the exogenous process for $\hat{r}_t^f$. Let's assume that $\rho_i = 0$ and turn off $\epsilon_{i,t}$ to make things even simpler:

$$\begin{aligned}
\hat{X}_t &= \mathbb{E}_t \hat{X}_{t+1} - \sigma^{-1} \left(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1} - \hat{r}_t^f \right), \\
\hat{\pi}_t &= \beta \mathbb{E}_t \hat{\pi}_{t+1} + \kappa \hat{X}_t, \\
\hat{i}_t &= \phi_\pi \hat{\pi}_t + \phi_y \hat{X}_t, \\
\hat{r}_t^f &= \rho_a \hat{r}_{t-1}^f + \sigma(\rho_a - 1)\omega \epsilon_{a,t}.
\end{aligned}$$

Since $\hat{i}_t$ is no longer a state variable, we can substitute it out and insert it into the dynamic IS curve:

$$\begin{aligned}
\hat{X}_t &= \mathbb{E}_t \hat{X}_{t+1} - \sigma^{-1} \left(\phi_\pi \hat{\pi}_t + \phi_y \hat{X}_t - \mathbb{E}_t \hat{\pi}_{t+1} - \hat{r}_t^f \right), \\
\hat{\pi}_t &= \beta \mathbb{E}_t \hat{\pi}_{t+1} + \kappa \hat{X}_t.
\end{aligned}$$

Now, we want to look at how our jump variables react to a technology shock $\epsilon_{a,t}$, or rather $\hat{r}_t^f$. First, guess that the policy functions look like:

$$\hat{X}_t = A\hat{r}_t^f,$$
$$\hat{\pi}_t = B\hat{r}_t^f,$$

and then plug them into the DISE and the NKPC:

$$\text{Guessed DISE: } A\hat{r}_t^f = \mathbb{E}_t A\hat{r}_{t+1}^f - \sigma^{-1}\left(\phi_\pi B\hat{r}_t^f + \phi_y A\hat{r}_t^f - \mathbb{E}_t B\hat{r}_{t+1}^f - \hat{r}_t^f\right), \tag{477}$$
$$\text{Guessed NKPC: } B\hat{r}_t^f = \beta\mathbb{E}_t B\hat{r}_{t+1}^f + \kappa A\hat{r}_t^f. \tag{478}$$

Now, start with the guessed DISE (477) by writing in $\hat{r}_{t+1}^f$, and then doing some rearranging:

$$A\hat{r}_t^f = A\rho_a\hat{r}_t^f - \sigma^{-1}\left(\phi_\pi B\hat{r}_t^f + \phi_y A\hat{r}_t^f - B\rho_a\hat{r}_t^f - \hat{r}_t^f\right)$$
$$0 = \sigma A\hat{r}_t^f - \sigma A\rho_a\hat{r}_t^f + \phi_\pi B\hat{r}_t^f + \phi_y A\hat{r}_t^f - B\rho_a\hat{r}_t^f - \hat{r}_t^f, \tag{479}$$

and then do the same for the NKPC:

$$B\hat{r}_t^f = \kappa A\hat{r}_t^f + \beta B\rho_a\hat{r}_{t+1}^f$$
$$0 = \kappa A\hat{r}_t^f + \beta B\rho_a\hat{r}_t^f - B\hat{r}_t^f \tag{480}$$

So we now have two equations in two unknowns. Now, we need to do a lot of tedious algebra... From the NKPC, we have:

$$\kappa A = B - \beta B\rho_a$$
$$\therefore A = \frac{(1-\beta\rho_a)}{\kappa}B,$$

and plug this into the guessed DISE:

$$
\begin{aligned}
0 &= \sigma \left[\frac{B(1-\beta\rho_a)}{\kappa}\right] - \sigma\rho_a \left[\frac{B(1-\beta\rho_a)}{\kappa}\right] + \phi_\pi B + \phi_y \left[\frac{B(1-\beta\rho_a)}{\kappa}\right] - B\rho_a - 1 \\
\therefore B &= \frac{1}{\frac{\sigma(1-\beta\rho_a)}{\kappa} - \frac{\sigma\rho_a(1-\beta\rho_a)}{\kappa} + \phi_\pi + \frac{\phi_y(1-\beta\rho_a)}{\kappa} - \rho_a} \\
&= \frac{\kappa}{(1-\beta\rho_a)(\phi_y - \sigma - \sigma\rho_a) + \kappa(\phi_\pi - \rho_a)}, \qquad (481) \\
\implies A &= \frac{(1-\beta\rho_a)}{\kappa} \frac{\kappa}{(1-\beta\rho_a)(\phi_y - \sigma - \sigma\rho_a) + \kappa(\phi_\pi - \rho_a)} \\
&= \frac{1-\beta\rho_a}{(1-\beta\rho_a)(\phi_y - \sigma - \sigma\rho_a) + \kappa(\phi_\pi - \rho_a)}. \qquad (482)
\end{aligned}
$$

So, our policy functions are

$$
\hat{X}_t = \frac{1-\beta\rho_a}{(1-\beta\rho_a)(\phi_y - \sigma - \sigma\rho_a) + \kappa(\phi_\pi - \rho_a)} \hat{r}_t^f, \qquad (483)
$$

$$
\hat{\pi}_t = \frac{\kappa}{(1-\beta\rho_a)(\phi_y - \sigma - \sigma\rho_a) + \kappa(\phi_\pi - \rho_a)} \hat{r}_t^f. \qquad (484)
$$

11.11.2 A Rational Expectations solution: shocks with persistence

Suppose we have the following model:

$$
\begin{aligned}
\hat{X}_t &= \mathbb{E}_t \hat{X}_{t+1} - \sigma^{-1}\left(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1} - \hat{r}_t^f\right), \\
\hat{\pi}_t &= \kappa \hat{X}_t + \beta \mathbb{E}_t \hat{\pi}_{t+1}, \\
\hat{i}_t &= \phi_\pi \hat{\pi}_t + \phi_y \hat{X}_t + v_t, \\
\hat{r}_t^f &= \sigma^2 \omega \mathbb{E}_t \Delta a_{t+1},
\end{aligned}
$$

where:

$$
\begin{aligned}
v_t &= \rho_v v_{t-1} + \epsilon_{v,t}, \ \epsilon_{v,t} \sim N(0, \sigma_v^2), \\
a_t &= \rho_a a_{t-1} + \epsilon_{a,t}, \ \epsilon_{a,t} \sim N(0, \sigma_a^2).
\end{aligned}
$$

Again, propose a guess for the policy function (linear since our model is log-linear). But let's do some rearranging first. Here, our shocks are our state variables, and since neither $\hat{i}_t$ nor $\hat{r}_t^f$ are state variables, let's substitute them into our jump variables:

$$\hat{X}_t = \mathbb{E}_t \hat{X}_{t+1} - \sigma^{-1} \left(\underbrace{\phi_\pi \hat{\pi}_t + \phi_y \hat{X}_t + v_t}_{\hat{i}_t} - \mathbb{E}_t \hat{\pi}_{t+1} - \underbrace{\sigma^2 \omega \mathbb{E}_t \Delta a_{t+1}}_{\hat{r}_t^f} \right),$$

$$\hat{\pi}_t = \kappa \hat{X}_t + \beta \mathbb{E}_t \hat{\pi}_{t+1}.$$

Now, take a guess for the form of the policy functions:

$$\hat{X}_t = A v_t + B a_t, \tag{485}$$

$$\hat{\pi}_t = C v_t + D a_t, \tag{486}$$

and also, since our shocks have persistence and will have some leftover term in period $t+1$, consider the Rational Expectations (RE) guessed solution:

$$\mathbb{E}_t \hat{X}_{t+1} = A \rho_v v_t + B \rho_a a_t, \tag{487}$$

$$\mathbb{E}_t \hat{\pi}_{t+1} = C \rho_v v_t + D \rho_a a_t. \tag{488}$$

So that we have four equations in four unknowns. Notice that if we didn't consider the RE guessed solutions, we would only have two equations in four unknowns.

Now, use our guessed solutions and sub them into the NKPC:

$$C v_t + D a_t = \beta (C \rho_v v_t + D \rho_a a_t) + \kappa (A v_t + B a_t).$$

As a sanity check, let's ignore the technology shock a_t for now and just focus on the monetary policy shock, v_t. This will yield (after a massive amount of painful algebra):

$$\hat{X}_t = -(1-\beta\rho_v)\Lambda_v v_t, \tag{489}$$

$$\hat{\pi}_t = -\kappa\Lambda_v v_t, \tag{490}$$

$$\hat{i}_t = \left[\sigma(1-\rho_v)(1-\beta\rho_v) - \rho_v\kappa\right]\Lambda_v v_t, \tag{491}$$

$$\hat{r}_t = \sigma(1-\rho_v)(1-\beta\rho_v)\Lambda_v v_t, \tag{492}$$

where:

$$\Lambda_v = \frac{1}{(1-\beta\rho_v)\left[\sigma(1-\rho_v)+\phi_y\right] + \kappa(\phi_\pi - \rho_v)}, \tag{493}$$

and where $\Lambda_v > 0$ if the Taylor Principle[75] holds.

11.12 The New Keynesian model with Rotemberg (1982) pricing

Under Rotemberg pricing all intermediate goods firms are able to adjust their pricing, but with a quadratic adjustment cost. In equilibrium they all behave identically which makes aggregation work out nicely. Whether a NKPC is derived via Calvo or Rotemberg pricing makes little difference up to a first order approximation about a zero inflation steady state.

Intermediate firms still face the same demands from final goods firms under Rotemberg pricing and they produce output according to:

$$Y_t(j) = A_t N_t(j).$$

Cost minimisation implies that real marginal cost $mc_t = \frac{w_t}{A_t}$, where $w_t = \frac{W_t}{P_t}$ is the real wage common to all firms. So, some firm j's problem is:

$$\min_{N_t(j)} \; W_t N_t(j),$$

subject to :

$$A_t N_t(j) \geq Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} Y_t.$$

[75] We will discuss this soon.

So the Lagrangian form the firm problem is:

$$\mathcal{L} = W_t N_t(j) - \varphi_t(j) \left(A_t N_t(j) - \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t \right),$$

which yields the following FOC:

$$\begin{aligned} \mathcal{L}_{N_t(j)} :\ & W_t = \varphi_t(j) A_t \\ \implies & \varphi_t(j) = \frac{W_t}{A_t} = \varphi_t,\ \forall j. \end{aligned} \tag{494}$$

The nominal flow profit for producer j is given by:

$$\Pi_t(j) = P_t(j) Y_t(j) - W_t N_t(j) - \frac{\psi}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 P_t Y_t, \tag{495}$$

where ψ is the Rotemberg cost of price adjustment parameter, and is measured in units of the final good. Next, write the profit function in real terms:

$$\frac{\Pi_t(j)}{P_t} = \frac{P_t(j)}{P_t} Y_t(j) - \underbrace{\frac{W_t}{P_t}}_{w_t} N_t(j) - \frac{\psi}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 Y_t,$$

and since $w_t = A_t mc_t$:

$$\frac{\Pi_t(j)}{P_t} = \frac{P_t(j)}{P_t} Y_t(j) - mc_t \underbrace{A_t N_t(j)}_{Y_t(j)} - \frac{\psi}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 Y_t,$$

and then sub in for $Y_t(j)$:

$$\begin{aligned} \frac{\Pi_t(j)}{P_t} &= \frac{P_t(j)}{P_t} \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t - mc_t \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t - \frac{\psi}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 Y_t \\ &= \left(\frac{P_t(j)}{P_t} \right)^{1-\epsilon} Y_t - mc_t \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon} Y_t - \frac{\psi}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 Y_t. \end{aligned}$$

So firms choose $P_t(j)$ to max expected present discounted value of flow profit each periods, where discounting is done by the household's stochastic discount factor:

$$\begin{aligned}\frac{\partial\left(\frac{\Pi_t(j)}{P_t}\right)}{\partial P_t(j)} = (1-\epsilon)\left(\frac{P_t(j)}{P_t}\right)^{-\epsilon}\frac{Y_t}{P_t} + \epsilon mc_t\left(\frac{P_t(j)}{P_t}\right)^{\epsilon-1}\frac{Y_t}{P_t} - \psi\left(\frac{P_t(j)}{P_{t-1}(j)} - 1\right)\frac{Y_t}{P_{t-1}(j)} \\ + \beta\psi\mathbb{E}_t\left[\frac{u_C(C_{t+1}, N_{t-1})}{u_C(C_t, N_t)}\left(\frac{P_{t+1}(j)}{P_t(j)} - 1\right)\left(\frac{P_{t+1}(j)}{P_t(j)}\right)\left(\frac{Y_{t+1}}{P_t(j)}\right)\right] = 0, \end{aligned} \tag{496}$$

and rearrange:

$$\begin{aligned}(\epsilon-1)\left(\frac{P_t(j)}{P_t}\right)^{-\epsilon}\frac{Y_t}{P_t} = \epsilon mc_t\left(\frac{P_t(j)}{P_t}\right)^{\epsilon-1}\frac{Y_t}{P_t} - \psi\left(\frac{P_t(j)}{P_{t-1}(j)} - 1\right)\frac{Y_t}{P_{t-1}(j)} \\ + \beta\psi\mathbb{E}_t\left[\frac{u_C(C_{t+1}, N_{t-1})}{u_C(C_t, N_t)}\left(\frac{P_{t+1}(j)}{P_t(j)} - 1\right)\left(\frac{P_{t+1}(j)}{P_t(j)}\right)\left(\frac{Y_{t+1}}{P_t(j)}\right)\right],\end{aligned}$$

then divide both the LHS and RHS by Y_t, multiply both the LHS and RHS by P_t, and note that gross inflation $1 + \pi_t = \frac{P_t}{P_{t-1}}$, and since all firms behave identically, $P_t(j) = P_t$:

$$\begin{aligned}\epsilon - 1 &= \epsilon mc_t - \psi\left(\frac{P_t}{P_{t-1}} - 1\right)\frac{P_t}{P_{t-1}} + \beta\psi\mathbb{E}_t\left[\frac{u_C(C_{t+1}, N_{t-1})}{u_C(C_t, N_t)}\left(\frac{P_{t+1}}{P_t} - 1\right)\left(\frac{P_{t+1}}{P_t}\right)\left(\frac{Y_{t+1}}{P_t}\right)\right]\frac{P_t}{Y_t} \\ &= \epsilon mc_t - \psi\pi_t(1+\pi_t) + \beta\psi\mathbb{E}_t\left[\frac{u_C(C_{t+1}, N_{t-1})}{u_C(C_t, N_t)}\pi_{t+1}\left(1+\pi_{t+1}\right)\left(\frac{Y_{t+1}}{Y_t}\right)\right],\end{aligned}$$

and then we can make the simplifying assumption that $u_C = C_t^{-1}$ (the case of log utility), and we know that $Y_t = C_t$, so the consumption terms in the parentheses cancel out:

$$\epsilon - 1 = \epsilon mc_t - \psi\pi_t(1+\pi_t) + \beta\psi\mathbb{E}_t\left[\pi_{t+1}\left(1+\pi_{t+1}\right)\right]. \tag{497}$$

Phew! Now, we need to log-linearise!

$$\ln(\epsilon - 1) = \ln\left\{\epsilon mc_t - \psi\pi_t(1+\pi_t) + \beta\psi\mathbb{E}_t\left[\pi_{t+1}\left(1+\pi_{t+1}\right)\right]\right\},$$

then totally differentiate:

$$0 = \frac{1}{\epsilon - 1}\left\{\epsilon dmc_t - \psi d\pi_t(1+\bar{\pi}) - \psi\bar{\pi}d\pi_t + \beta\psi\mathbb{E}_t\left[d\pi_{t+1}(1+\bar{\pi})\right] + \beta\psi\mathbb{E}_t\left[\bar{\pi}d\pi_{t+1}\right]\right\},$$

and since we know that in the steady state $\bar{\pi} = 0$, and that $d\pi_t = \hat{\pi}_t$:

$$0 = \frac{\epsilon}{\epsilon - 1} dmc_t - \frac{\psi}{\epsilon - 1}\hat{\pi}_t + \frac{\beta\psi}{\epsilon - 1}\mathbb{E}_t\hat{\pi}_{t+1}.$$

Now, we need to use a little trick. We know that $\frac{\epsilon}{\epsilon-1}$ is nothing but $\bar{mc}^{-1}$, so the first term on the RHS is:

$$\frac{dmc_t}{\bar{mc}} = \frac{mc_t - \bar{mc}}{\bar{mc}} = \hat{mc}_t,$$

so we have:

$$\begin{aligned} 0 &= \hat{mc}_t - \frac{\psi}{\epsilon - 1}\hat{\pi}_t + \frac{\beta\psi}{\epsilon - 1}\mathbb{E}_t\hat{\pi}_{t+1}, \\ \implies \frac{\psi}{\epsilon - 1}\hat{\pi}_t &= \hat{mc}_t + \frac{\beta\psi}{\epsilon - 1}\mathbb{E}_t\hat{\pi}_{t+1} \\ \therefore \hat{\pi}_t &= \frac{\epsilon - 1}{\psi}\hat{mc}_t + \beta\mathbb{E}_t\hat{\pi}_{t+1}, \end{aligned} \tag{498}$$

which is nothing but the NKPC, and it will match the Calvo pricing-based NKPC if:

$$\psi = \frac{(\epsilon - 1)\phi}{(1 - \phi)(1 - \phi\beta)}.$$

Finally, under Rotemberg pricing, the aggregate resource constraint comes out to:

$$Y_t = C_t + \frac{\psi}{2}\pi_t^2 Y_t, \tag{499}$$

and log-linearising this yields:

$$\begin{aligned} \ln Y_t &= \ln\left[C_t + \frac{\psi}{2}\pi_t^2 Y_t\right] \\ dY_t &= dC_t + \frac{\psi}{2}2\bar{\pi}\bar{Y}d\pi_t + \frac{\psi}{2}\bar{\pi}^2 dY_t \\ \frac{dY_t}{\bar{Y}} &= \frac{dC_t}{\bar{Y}} + \psi\bar{\pi}d\pi_t + \frac{\frac{\psi}{2}\bar{\pi}^2 dY_t}{\bar{Y}}, \\ \hat{Y}_t &= \frac{dC_t}{\bar{Y}}, \end{aligned}$$

but $\bar{Y} = \bar{C}$, so:

$$\therefore \hat{Y}_t = \hat{C}_t. \tag{500}$$

11.13 Empirical evidence on the canonical New Keynesian model

11.13.1 The wrong sign

The NKPC is perhaps the central relationship in the modern approach to monetary policy analysis (as the Euler equation was known long before the NKPC became popular in the 1990s). Despite this success, there are some well known problems with it as an empirical model of inflation. A practical problem is out to measure the output gap X_t. A reasonable approach would be to assume that, on average, output tends to return to its natural rate, so the natural rate can be proxied by a simple trend (as measured, for instance, by the HP filter). So instead of $\hat{X}_t = \hat{Y}_t - \hat{Y}_t^f$, we could use $\tilde{Y}_t = \hat{Y}_t - Y_t^{tr}$, and estimate the NKPC with data:

$$\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \kappa \tilde{Y}_t, \tag{501}$$

where Y_t^{tr} is estimated trend output. Since we can't observe $\mathbb{E}_t \hat{\pi}_{t+1}$ we can substitute the realised $\hat{\pi}_{t+1}$ and use an estimation technique such as instrumental variables to deal with the fact this is a noisy estimator of what we really want (i.e. we're facing a classical measurement error problem).

The problem is that when we estimate (501), the sign of κ usually comes out negative. This is shocking to some but actually not so surprising once you work through it. We already know the "accelerationist" fact that $\Delta\hat{\pi}_t$ is negatively correlated with unemployment rate. This means that it is positively correlated with the output gap. Because $\beta \approx 1$, we can proxy $\hat{\pi}_t - \beta \mathbb{E}_t \hat{\pi}_{t+1}$ with $\hat{\pi}_t - \hat{\pi}_{t+1} = -\Delta\hat{\pi}_{t+1}$. Looked at this way, it's not too surprising that the estimated output gap coefficient is negative – another reminder that, despite their apparent similarity, the new and older Phillips Curves are very different.

There are two possible responses to this failure: Either the model is wrong the output gap measure is wrong. In a famous paper, Galí and Gertler (1999) argued the latter. They suggest that deterministic trends do a bad job in capturing movements in the natural rate of output and suggested an alternative approach.

Remember that the "correct" variable driving inflation is the ratio of marginal cost to the price level. Galí and Gertler argue for proxying marginal cost with unit labour costs $W_t L_t / Y_t$ so that the proxy for real marginal cost is the labour share of income:

$$S_t = \frac{W_t L_t}{P_t Y_t}.$$

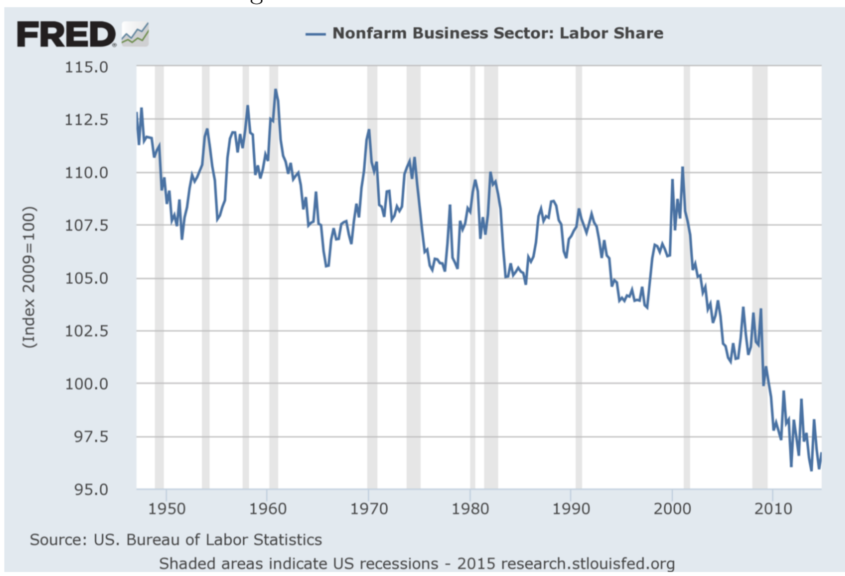
Galí and Gertler showed that estimating:

$$\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \gamma \hat{S}_t,$$

finds a positive γ. This is a very popular, widely-cited result – seen as putting the NKPC back on sound empirical footing.

However, Rudd and Whelan (2007) were not quite convinced of this result. They show that updating Galí and Gertler's estimates, the estimated labour share coefficient is no longer statistically significant. Also, real marginal cost should be procylical, rising when output is above potential (due to overtime compensation, production bottlenecks, and so on). Labour's share, however, has generally moved in countercyclical fashion – it has generally spiked upwards in recessions. Maybe output is actually above potential during recessions (negative technology shocks) but this seems unlikely. Furthermore, in many countries, there has been a downward trend in the labour share. This is now evident in the US data for the period after the Galí and Gertler (1999) study. Naive detrending methods may have problems, but they seem to give a better proxy for output's deviation from potential than the labour.

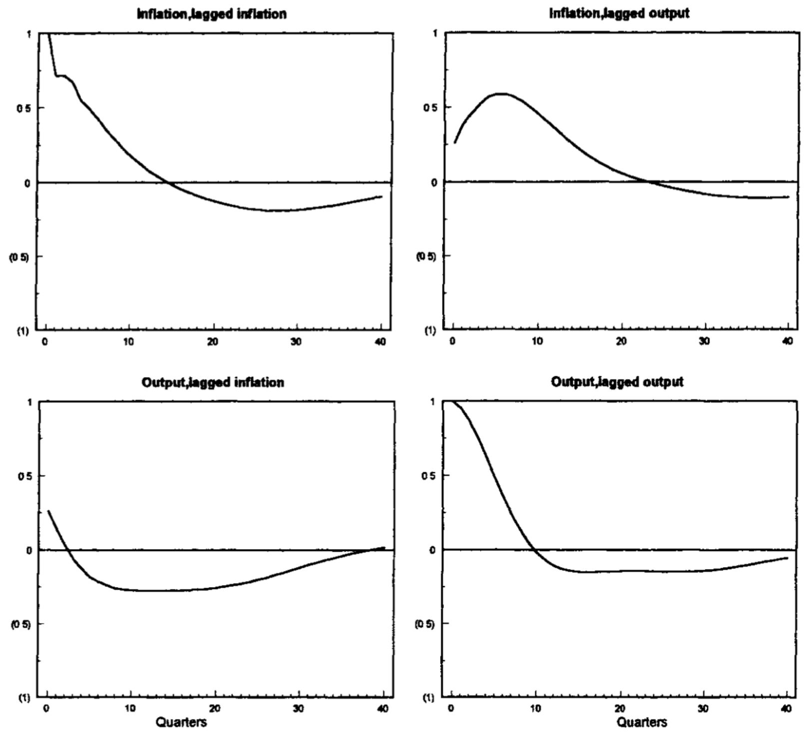
Figure 67: Labour Share in the US



11.13.2 The inflation persistence problem

Fuhrer and Moore (1995) explored inflation persistence with a statistical VAR (unconstrained VAR in output and inflation), where they found that inflation was very inertial – its autocorrelation remains positive for about 4 years:

Figure 68: Autocorrelation Function, Vector Autoregression



Source: Fuhrer and Moore (1995)

The NKPC can be rewritten as:

$$\hat{\pi}_t = \kappa \sum_{i=0}^{\infty} \beta^i \mathbb{E}_t \hat{X}_{t+i}$$

So, according the NKPC, inflation is purely a forward-looking jump variable. There is absolutely no inflation inertia whatsoever. The canonical New Keynesian model is a model of sluggish price adjustment – not sluggish inflation adjustment. In fact, of the Phillips Curves that we have seen, the only one which implied inflation inertia was the accelerationist Phillips Curve.

This issue may be better illustrated with an example. Let's assume that $\hat{X}_t$ follows an AR(1) process:

$$\hat{X}_t = \rho \hat{X}_{t-1} + e_t,$$

which would imply a solution for inflation of:

$$\hat{\pi}_t = A\hat{X}_t,$$

where A is some unknown constant. Period $t+1$ inflation would of course then be given by:

$$\mathbb{E}_t\hat{\pi}_{t+1} = \mathbb{E}_t A\hat{X}_{t+1} = \rho A\hat{X}_t,$$

subbing this into the NKPC would give:

$$\begin{aligned}
\hat{\pi}_t &= \beta\mathbb{E}_t\hat{\pi}_{t+1} + \kappa\hat{X}_t \\
&= \beta\rho A\hat{X}_t + \kappa\hat{X}_t \\
&= (\beta\rho A + \kappa)\hat{X}_t, \\
\implies A &= \frac{\kappa}{1-\beta\rho}.
\end{aligned}$$

So we can use A and $\hat{\pi}_t$ to rewrite our AR(1) process as:

$$\hat{\pi}_t = \rho\hat{\pi}_{t-1} + Ae_t,$$

where we can see that inflation dynamics depends only on the serial correlation of $\hat{X}_t$. There are no other endogenous mechanisms in the model to general inflation dynamics.

One more point to consider is from Estrella and Fuhrer (2002). The NKPC implies:

$$\beta\mathbb{E}_t\hat{\pi}_{t+1} - \hat{\pi}_t = -\kappa\hat{X}_t,$$

and since $\hat{\pi}_{t+1} - \mathbb{E}_t\hat{\pi}_{t+1} = \epsilon_{t+1}$ (the forecast error), we have:

$$\beta\hat{\pi}_{t+1} - \hat{\pi}_t = -\kappa\hat{X}_t + \beta(\underbrace{\hat{\pi}_{t+1} - \mathbb{E}_t\hat{\pi}_{t+1}}_{\epsilon_{t+1}}),$$

and since $\beta \approx 1$ in quarterly data, we have that:

$$\hat{\pi}_{t+1} - \hat{\pi}_t \approx -\kappa \hat{X}_t + \epsilon_{t+1}.$$

An increase in the output gap should lead to a fall in future inflation. In other words, an increase in unemployment should be associated with an increase in future/expected inflation (see the discussion above about the "wrong sign" of the NKPC).

Galí and Gertler (1999) also sought to address this in their paper using an econometric approach. Their main interest is in testing between the accelerationist and New Keynesian views. They begin by positing a "Hybrid Phillips Curve" (HPC) with backward looking and forward looking elements:

$$\hat{\pi}_t = \gamma_b \hat{\pi}_{t-1} + \gamma_f \mathbb{E}_t \hat{\pi}_{t+1} + \kappa \widehat{mc}_t + e_t. \tag{502}$$

When output is above normal, marginal costs are high, which increases desired relative prices. In the model, for example, desired relative prices rise when output rises because the real wage increases. To reiterate what we said in the previous section, Galí and Gertler therefore try a more direct approach to estimating marginal costs. Real marginal cost equals the real wage divided by the marginal product of labour. If the production function is Cobb-Douglas, so that $Y_t = A_t K_t^\alpha L_t^{1-\alpha}$, the marginal product of labour is $(1-\alpha)Y_t/L_t$. Thus, real marginal cost is $wL/[(1-\alpha)Y_t]$, where w_t is the real wage. That is, marginal cost is proportional to the share of income going to labour.[76] Galí and Gertler therefore focus on the equation:

$$\hat{\pi}_t = \gamma_b \hat{\pi}_{t-1} + \gamma_f \hat{\pi}_{t+1} + \lambda \hat{S}_t + e_t,$$

where, as before, S_t is labour's share. Typical estimates for Galí and Gertler's methodology using quarterly US data for the period 1960-1997 are:

$$\hat{\pi}_t = \underset{(0.020)}{0.378}\,\hat{\pi}_{t-1} + \underset{(0.016)}{0.591}\,\mathbb{E}_t \hat{\pi}_{t+1} + \underset{(0.004)}{0.015}\,\hat{S}_t + e_t,$$

[76] See also Sbordone (2002).

where the numbers in parentheses are standard errors. Thus their results appear to provide strong support for the importance of forward looking expectations. In a series of papers, however, Rudd and Whelan show that in fact the data provides little evidence for the NKPC and HPCs (see especially Rudd and Whelan (2005; 2006). They make two key points. The first, as previously mentioned, Rudd and Whelan dispute the inclusion of labour share to capture the rise in firms' marginal costs when output rises, finding that labour's share is low in booms and high in recessions. In Galí and Gertler's framework, this would mean that booms are times when the economy's flexible price level of output has risen even more than actual output, and when marginal costs are therefore unusually low. A much more plausible possibility, however, is that there are forces other than those considered by Galí and Gertler moving labour's share over the business cycle, and that labour's share is therefore a poor proxy for marginal costs.

Since labour's share is countercyclical, the finding of a large coefficient on expected future inflation and a positive coefficient on the share means that inflation tends to be above future inflation in recessions and below future inflation in booms. That is, inflation tends to fall in recessions and rise in booms, consistent with the accelerationist Phillips Curve and not with the NKPC.

Rudd and Whelan's second concern has to do with the information content of current inflation. Replacing $\hat{X}_t$ with a generic cost variable, $\widehat{mc}_t$, and then iterating the NKPC forward implies:

$$
\begin{aligned}
\hat{\pi}_t &= \kappa \widehat{mc}_t + \beta \mathbb{E}_t \hat{\pi}_{t+1} \\
&= \kappa \widehat{mc}_t + \beta \left[\kappa \mathbb{E}_t \widehat{mc}_{t+1} + \beta \mathbb{E}_t \hat{\pi}_{t+2} \right] \\
&= \ldots \\
&= \kappa \sum_{i=0}^{\infty} \beta^i \mathbb{E}_t \widehat{mc}_{t+i}.
\end{aligned} \tag{503}
$$

Thus the model implies that inflation should be a function expectations of future marginal costs, and thus that it should help predict marginal costs. Rudd and Whelan (2005) show, however, that the evidence for this hypothesis is minimal. When marginal costs are proxied by an estimate $\hat{X}_t$, inflation's predictive power is small – and goes in the wrong direction as we previously discussed.

The bottom line of this analysis is twofold. First, the evidence we have on the correct form of the

Phillips Curve is limited. The debate between Galí and Gertler and Rudd and Whelan, along with further analysis of the econometrics of the NKPC,[77] does not lead to clear conclusions on the basis of formal econometric studies. Second, although the evidence is not definitive, it points in the direction of inflation inertia and provides little support the NKPC. So, despite its popularity in applications, the theoretical foundations for the various HPCs are weak. These models are just as open to the Lucas Critique as traditional ones. Nevertheless, we will explore two extremely popular approaches to the HPC.

11.14 Models of staggered price adjustment with inflation inertia

Despite the empirical issues with HPC, they remain extremely popular in the literature as they introduce inflation inertia. We thus introduce two models in this section: Christiano, Eichenbaum, and Evans (2005) (CEE) and Mankiw and Reis (2002). CEE assume that between reviews, prices are adjusted for past inflation. This creates a direct role for past inflation in price behaviour. But whether this reasonable captures important microeconomic phenomena is not clear. Mankiw and Reis return to Fischer's assumption of prices that are predetermined but not fixed. This causes past beliefs about what inflation would be to affect price changes, and so creates behaviour similar to inflation inertia. In contrast to Fischer, however, they make assumptions that imply that some intervals between reviews of prices are quite long, which has important quantitative implications. Again, however, the strength of the microeconomic case for the realism of their key assumption is not clear.

11.14.1 Christiano, Eichenbaum, and Evans (2005): NKPC with indexation

CEE begin with Calvo's assumption that opportunities for firms to review their prices follow a Poisson process. As in the basic Calvo model let $1-\phi$ denote the fraction of firms that review their prices in a given period. Where CEE depart from Calvo is in their assumption about what happens between reviews. Rather than assuming that prices are fixed, they assume they are indexed to the previous period's inflation rate. This assumption captures the fact that even in the absence of a full-fledged reconsideration of their prices, firms can account for the overall inflationary environment. The as-

[77] For example Mavroeidis (2005, *JMCB*) and King and Plosser (2005, *JME*).

sumption that the indexing is to lagged rather than current inflation reflects the fact that firms do not continually obtain and use all available information.

Our analysis the model is similar to the analysis of the Calvo price-setting model. Since the firms that review their prices in a given period are chosen at random, the average [log] price in period t of the firms that do not review their prices is $P_{t-1} + \pi_{t-1}$. The average price in t is therefore:

$$P_t = \phi(P_{t-1} + \pi_{t-1}) + (1 - \phi)P_t^{\#}, \tag{504}$$

where $P_t^{\#}$ is the reset price set by firms that review their prices. Equation (504) implies:

$$\begin{aligned} P_t^{\#} - P_t &= P_t^{\#} - \left[\phi(P_{t-1} + \pi_{t-1}) + (1 - \phi)P_t^{\#}\right] \\ &= \phi P_t^{\#} - \phi(P_{t-1} + \pi_{t-1}) \\ &= \phi(P_t^{\#} - P_t) - \phi(P_{t-1} + \pi_{t-1} - P_t) \\ &= \phi(P_t^{\#} - P_t) + \phi(\pi_t - \pi_{t-1}). \end{aligned} \tag{505}$$

Thus:

$$P_t^{\#} - P_t = \frac{\phi}{1 - \phi}(\pi_t - \pi_{t-1}). \tag{506}$$

In words, this shows that to find the dynamics of inflation, we need to find $P_t^{\#} - P_t$. That is, we need to determine how firms that review their prices set their relative prices in period t. As in the Calvo model, a firm wants to set its price to minimise the expected discounted sum of the squared differences between its optimal and actual prices during the period before it is next able to review its price. Suppose a firm sets a price of $P_t^{\#}$ in period t and that it does not have an opportunity to review its price before period $t + j$. Then, because of the lagged indexation, its price in $t + j$ is:

$$P_{t+j} = P_t^{\#} + \sum_{\tau=0}^{j-1} \pi_{t+\tau}.$$

The profit maximising price in $t+j$ is:

$$P^*_{t+j} = P_{t+j} + (P^*_{t+j} - P_{t+j}) = P_t + \sum_{\tau=1}^{j} \pi_{t+\tau} + (P^*_{t+j} - P_{t+j}),$$

where P^*_t is the profit maximising price in period t. So the difference between P^*_{t+j} and P_{t+j} and in $t+j$, denoted as $e_{t,t+j}$ is:

$$\begin{aligned} P^*_{t+j} - P_{t+j} &= P_t + \sum_{\tau=1}^{j} \pi_{t+\tau} + (P^*_{t+j} - P_{t+j}) - P^{\#}_t + \sum_{\tau=0}^{j-1} \pi_{t+\tau} \\ e_{t,t+j} &= (P_t - P^{\#}_t) + (\pi_{t+j} - \pi_t) + (P^*_{t+j} - P_{t+j}), \end{aligned} \tag{507}$$

which holds for all $j \geq 0$. The discount factor is β, and the probability of non-adjustment each period is ϕ. Thus, similarly to the case without indexation, the firm sets:

$$P^{\#}_t - P_t = (1-\beta\phi) \sum_{j=0}^{\infty} \beta^j \phi^j \left[(\mathbb{E}_t \pi_{t+j} - \pi_t) + \mathbb{E}_t (P^*_{t+j} - P_{t+j}) \right]. \tag{508}$$

As in the derivation of the NKPC, it is helpful to rewrite this expression in terms of period t variables and the expectation of $P^{\#}_{t+1} - P_{t+1}$.:

$$P^{\#}_{t+1} - P_{t+1} = (1-\beta\phi) \sum_{j=0}^{\infty} \beta^j \phi^j \left[(\mathbb{E}_{t+1} \pi_{t+1+j} - \pi_{t+1}) + \mathbb{E}_{t+1} (P^*_{t+1+j} - P_{t+1+j}) \right].$$

Rewriting the π_{t+1} term as $\pi_t(\pi_{t+1} - \pi_t)$ and making use of the law of iterated expectations by taking expectations at t gives us:

$$\mathbb{E}_t \left[P^{\#}_{t+1} - P_{t+1} \right] = -\mathbb{E}_t[\pi_{t+1} - \pi_t] + (1-\beta\phi) \sum_{j=0}^{\infty} \beta^j \phi^j \left[(\mathbb{E}_t \pi_{t+1+j} - \pi_t) + \mathbb{E}_t (P^*_{t+1+j} - P_{t+1+j}) \right].$$

We can therefore rewrite (508) we:

$$P^{\#}_t - P_t = (1-\beta\phi)(P^*_t - P_t) + \beta\phi \left\{ \mathbb{E}_t \left[P^{\#}_{t+1} - P_{t+1} \right] + \mathbb{E}_t \left[\pi_{t+1} - \pi_t \right] \right\}.$$

The final step is to use (506) applied to both periods t and $t+1$:

$$\begin{aligned} P_t^{\#} - P_t &= \frac{\phi}{1-\phi}(\pi_t - \pi_{t-1}), \\ \mathbb{E}_t \left[P_{t+1}^{\#} - P_{t+1} \right] &= \frac{\phi}{1-\phi} \mathbb{E}_t \left[\pi_{t+1} - \pi_t \right], \end{aligned}$$

and doing the appropriate substitutions above yields:

$$\begin{aligned} \pi_t &= \frac{1}{1+\beta}\pi_{t-1} + \frac{\beta}{1+\beta}\mathbb{E}_t\pi_{t+1} + \frac{1}{1+\beta}\frac{1-\phi}{\phi}(1-\beta\phi)(P_t^* - P_t) \\ \hat{\pi}_t &\equiv \frac{1}{1+\beta}\hat{\pi}_{t-1} + \frac{\beta}{1+\beta}\mathbb{E}_t\hat{\pi}_{t+1} + \chi\widehat{mc}_t, \end{aligned} \tag{509}$$

which is known as the NKPC with indexation – as you can see it's a type HPC. It resembles the NKPC except that instead of a weight of β on expected future inflation and no role for past inflation, there is a weight of $\beta/(1+\beta)$ on expected future inflation and a weight of $1/(1+\beta)$ on lagged inflation. If $\beta \approx 1$, the weights are both close to one-half. An obvious generalisation of (509) is:

$$\hat{\pi}_t = \gamma\hat{\pi}_{t-1} + (1-\gamma)\mathbb{E}_t\hat{\pi}_{t+1} + \chi\widehat{mc}_t, \ \gamma \in [0,1]. \tag{510}$$

The that CEE's NKPC with indexation implies inflation inertia does not mean that model can account for the apparent output costs of deflation. To see this, consider the case of $\beta = 1$, so that (509) becomes:

$$\hat{\pi}_t = \frac{\hat{\pi}_{t-1}}{2} + \frac{\mathbb{E}_t\hat{\pi}_{t+1}}{2} + \chi\widehat{mc}_t.$$

The important limitation of the model is that its key microeconomic assumption appears unrealistic – we do not observe actual prices rising mechanically with lagged inflation. At the same time, however, it could be that price-setters behave in ways that cause their average prices to rise roughly with lagged inflation between the times that they seriously rethink their pricing policies in light of macroeconomic conditions, and that this aver- age adjustment is masked by the fact that individual nominal prices are not continually adjusted.

11.14.2 Mankiw and Reis (2002): NKPC with sticky information

Mankiw and Reis take a somewhat different approach to obtaining inflation inertia. Like CEE, they assume some adjustment of prices between the times that firms review their pricing policies. Their assumption, however, is that each time a firm reviews its price, it sets a path that the price will follow until the next review. That is, they reintroduce the idea from the Fischer model that prices are predetermined but not fixed.

Although the mechanics of the Mankiw–Reis model involve predetermined prices, their argument for predetermination differs from Fischer's. Fischer motivates his analysis in terms of labor contracts that specify a different wage for each period of the contract; prices are then determined as markups over wages. But such contracts do not appear sufficiently wide- spread to be a plausible source of substantial aggregate nominal rigidity. Mankiw and Reis appeal instead to what they call "sticky information." It is costly for price-setters to obtain and process information. Mankiw and Reis argue that as a result, they may choose not to continually update their prices, but to periodically choose a path for their prices that they follow until they next gather information and adjust their path.

As in the CEE model, in the Mankiw-Reis models opportunities to adopt new price paths follow a Poisson process, which yields the following NKPC:

$$\hat{\pi}_t = \left(\frac{(1-\phi)\mu}{\phi} \right) \widehat{mc}_t + (1-\phi) \sum_{j=0}^{\infty} \phi^j \mathbb{E}_{t-1-j} \left[\hat{\pi}_t + \mu \left(\widehat{mc}_t - \widehat{mc}_{t-1} \right) \right], \tag{511}$$

whereby inflation depends on past expectations of current variables, rather than on current expectations of future variables.

As with the CEE model, its assumptions about price-setting do not match what we observe at the microeconomic level: many prices and wages are fixed for extended periods, and there is little evidence that many price-setters or wage-setters set price or wage paths of the sort that are central to the model. And some phenomena, such as aggregate demand disturbances, appear to have smaller and less persistent real effects in higher-inflation economies, seem hard to explain without fixed prices. It is possible that to fully capture the major features of fluctuations, our microeconomic model will need to incorporate important elements both of adjustments between formal reviews, as in the models of this section, and of fixed prices.

Another limitation of the CEE and Mankiw-Reis models, like all models of pure time-dependence, is that the assumption of an exogenous and unchanging frequency of changes in firms' pricing plans is clearly too strong. The frequency of adjustment is surely the result of some type of optimising calculation, not an exogenous parameter. Perhaps more importantly, it could change in response to policy changes, and this in turn could alter the effects of the policy changes. That is, a successful model may need to incorporate elements of both time-dependence and state-dependence.

This leaves us in an unsatisfactory position. It appears that any model of price behaviour that does not include elements of both fixed prices and mechanical price adjustments, and elements of both time-dependence and state-dependence, will fail to capture important macroeconomic phenomena. Yet the hope that a single model could incorporate all these features and still be tractable seems far-fetched. The search for a single workhorse model of pricing behaviour – or for a small number of workhorse models together with an understanding of when each is appropriate – continues.

11.15 Estimating DSGE models

Because DSGE models are relatively complex, early researchers did not attempt to use econometrics to estimate their parameters. Instead, the early models were "calibrated" by picking parameter values that matched certain steady state values (labour share of income, capital-output ratio, and so on) with historical average values or else by using estimates of parameters from microeconomic studies (coefficient of relative risk aversion, labour supply elasticities, depreciation rates, and so on). A more formal approach was "indirect inference" – choosing parameters to match certain moments of the data. For example, Rotemberg and Woodford (1997) chose parameters that delivered impulse responses to monetary policy shocks that came closest to matching the data.

This approach has been developed to be considerably more sophisticated than the Rotemberg-Woodford paper, but it still falls short of using all the information in data. For example, monetary policy shocks typically only account for a small percentage of the variation in the sample, so why focus only on this?

Most state-of-the-art papers estimating DSGE models now use Bayesian econometric techniques that are similar to (but not the same as) the methods used for estimating VARs that we discussed earlier. To understand these techniques, we will need to cover a few issues:

- Breaking our model into observable and unobservable variables;
- The role played by the number of shocks in DSGE models;
- Kalman filter estimation of state-space models; and
- Bayesian methods for DSGE models.

11.15.1 State space representation

One starting point is to consider a solved model, and recall that DSGE models can be expressed in state space form. The modern approach to estimation starts with the solved version of the log-linearised model. Suppose we have a model described by:

$$\mathbf{K}\mathbf{Z}_t = \mathbf{A}\mathbf{Z}_{t-1} + \mathbf{B}\mathbb{E}_t\mathbf{Z}_{t+1} + \mathbf{H}\mathbf{X}_t,$$

where $\mathbf{Z}_t$ is a set of n endogenous variables and $\mathbf{X}_t$ is a set of k exogenous variables that evolve according to:

$$\mathbf{X}_t = \mathbf{D}\mathbf{X}_{t-1} + \boldsymbol{\epsilon}_t. \tag{512}$$

We previously showed before that the model has a solution of the form:

$$\mathbf{Z}_t = \mathbf{C}\mathbf{Z}_{t-1} + \mathbf{P}\mathbf{X}_t, \tag{513}$$

where $\mathbf{C}$ depends on the coefficients in $\mathbf{A}$ and $\mathbf{B}$, and $\mathbf{P}$ depends on the coefficients in $\mathbf{A}$, $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{D}$. This can be simulated to establish properties of the model. But how do we go from observable data back to obtain the "best"estimates of the coefficients in $\mathbf{A}$, $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{D}$? How this works depends on the kind of model and the kind of data that we have.

Suppose that all variables in $\mathbf{X}_t$ and $\mathbf{Z}_t$ are observable. Then the model makes a clear prediction that, given any set of structural parameters, $\mathbf{A}$, $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{D}$, the data will be given by (513). The "cross-equation restrictions" in DSGE models tend to be very limiting. In other words, given any values for the $\mathbf{A}$, $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{D}$ matrices, there are very particular patterns that must be obeyed by the $\mathbf{C}$ and $\mathbf{P}$ matrices. Most likely, there is no set of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{D}$ matrices that will allow (513) to perfectly fit the data.

In this case, maximum likelihood methods do not work. These methods ask "how likely" it is that a model might be able to explain the data. But here we know for sure that model does not fit the data. One way to address this issue is to add error terms, $\mathbf{u}_t$, and then apply maximum likelihood to estimate $\mathbf{A}$, $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{D}$ as those matrices that give the best fitting model of the form:

$$\mathbf{Z}_t = \mathbf{C}\mathbf{Z}_{t-1} + \mathbf{P}\mathbf{X}_t + \mathbf{u}_t.$$

Though the $\mathbf{u}_t$ don't have a microeconomic foundation, the size of the error terms for the best-fitting model gives us a sense of how well this model fits reality.

11.15.2 Maximum likelihood estimation

We can use maximum likelihood to estimate the $\mathbf{A}$, $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{D}$ coefficients that deliver the best-fitting joint model:

$$\begin{aligned} \mathbf{Z}_t &= \mathbf{C}\mathbf{Z}_{t-1} + \mathbf{P}\mathbf{X}_t + \mathbf{u}_t, \\ \mathbf{X}_t &= \mathbf{D}\mathbf{X}_{t-1} + \boldsymbol{\epsilon}_t, \end{aligned}$$

where it is assumed that $\mathbf{u}_t \sim N(\mathbf{0}, \mathbf{\Sigma}_u)$ and $\boldsymbol{\epsilon}_t \sim N(\mathbf{0}, \mathbf{\Sigma}_\epsilon)$.

Suppose we observe data $\mathbf{Z}_1, \mathbf{Z}_2, ..., \mathbf{Z}_T$ for our endogenous variables and $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_T$ for our exogenous variables. The log-likelihood function for the $\mathbf{X}$ data is:

$$l_X = -\frac{T}{2}\ln 2\pi - T\ln\left|\mathbf{\Sigma}_\epsilon^{-1}\right| - \frac{1}{2}\sum_{i=1}^{T}(\mathbf{X}_i - \mathbf{D}\mathbf{X}_{i-1})^\top \mathbf{\Sigma}_\epsilon^{-1}(\mathbf{X}_i - \mathbf{D}\mathbf{X}_{i-1}), \tag{514}$$

and the log-likelihood function for the $\mathbf{Z}$ data is:

$$l_Z = -\frac{T}{2}\ln 2\pi - T\ln\left|\mathbf{\Sigma}_u^{-1}\right| - \frac{1}{2}\sum_{i=1}^{T}(\mathbf{Z}_i - \mathbf{C}\mathbf{Z}_{i-1} - \mathbf{P}\mathbf{X}_i)^\top \mathbf{\Sigma}_u^{-1}(\mathbf{Z}_i - \mathbf{C}\mathbf{Z}_{i-1} - \mathbf{P}\mathbf{X}_i). \tag{515}$$

The likelihood for the full model multiplies the likelihood of the $\mathbf{X}$ data and the likelihood of the $\mathbf{Z}$ data, so the combined log-likelihood is the sum of the two likelihoods. So the maximum likelihood estimates of $\mathbf{A}$, $\mathbf{B}$, $\mathbf{H}$, $\mathbf{D}$, $\mathbf{\Sigma}_\epsilon$, and $\mathbf{\Sigma}_u$ are those that maximise the log-likelihood:

$$\begin{aligned} l_X + l_Z = &-T\ln 2\pi - T\left(\ln\left|\mathbf{\Sigma}_\epsilon^{-1}\right| + \ln\left|\mathbf{\Sigma}_u^{-1}\right|\right) \\ &- \frac{1}{2}\sum_{i=1}^{T}(\mathbf{X}_i - \mathbf{D}\mathbf{X}_{i-1})^\top \mathbf{\Sigma}_\epsilon^{-1}(\mathbf{X}_i - \mathbf{D}\mathbf{X}_{i-1}) \\ &- \frac{1}{2}\sum_{i=1}^{T}(\mathbf{Z}_i - \mathbf{C}\mathbf{Z}_{i-1} - \mathbf{P}\mathbf{X}_i)^\top \mathbf{\Sigma}_u^{-1}(\mathbf{Z}_i - \mathbf{C}\mathbf{Z}_{i-1} - \mathbf{P}\mathbf{X}_i), \end{aligned} \tag{516}$$

subject to the restrictions that map $\mathbf{A}$ and $\mathbf{B}$ into $\mathbf{C}$, and map $\mathbf{A}$, $\mathbf{B}$, $\mathbf{H}$, and $\mathbf{D}$ into $\mathbf{P}$.

This discussion has been about the case in which we can see all of the variables – both endogenous and exogenous – in our DSGE model. But, in fact, most DSGE models are not like this. Instead, these models tend to mix observable and unobservable variables.

Consider again the log-linearised RBC model we solved earlier in the course. For convenience, here is the model again:

$$\begin{aligned} \hat{Y}_t &= \left[1 - \frac{\alpha\delta}{\beta^{-1} + \delta - 1}\right]\hat{C}_t + \left[\frac{\alpha\delta}{\beta^{-1} + \delta - 1}\right]\hat{I}_t, \\ \hat{Y}_t &= \hat{A}_t + \alpha\hat{K}_{t-1} + (1-\alpha)\hat{N}_t, \\ \hat{K}_t &= \left[\frac{\alpha\delta}{\beta^{-1} + \delta - 1}\right]\hat{I}_t + (1-\delta)\hat{K}_{t-1}, \\ \hat{R}_t &= [1 - \beta(1-\delta)]\left[\hat{Y}_t - \hat{K}_{t-1}\right], \\ \hat{C}_t &= \mathbb{E}_t\hat{C}_{t+1} - \frac{1}{\sigma}\mathbb{E}_t\hat{R}_{t+1}, \\ \hat{N}_t &= \hat{Y}_t - \sigma\hat{C}_t, \\ \hat{A}_t &= \rho\hat{A}_{t-1} + \epsilon_t. \end{aligned}$$

This model features seven equations in six endogenous variables: $\hat{Y}_t$, $\hat{C}_t$, $\hat{I}_t$, $\hat{K}_t$, $\hat{N}_t$, $\hat{R}_t$, and one exogenous variable, $\hat{A}_t$. We can observe $\hat{Y}_t$, $\hat{C}_t$, $\hat{I}_t$, and $\hat{N}_t$ (or at least the HP-filtered version of them that we are likely to use to estimate the model). But we don't observe $\hat{A}_t$, and since we don't really know depreciation rates, this means we don't observe $\hat{K}_t$ or $\hat{R}_t$. So this model mixes four observable variables with three unobservable variables. Estimation of these kinds of models requires special techniques to handle the unobservable variables.

Models like the linearised RBC model provide a microfoundation for why we cannot find a perfect fitting model with the observed data: There is an unobservable technology series and all of the observed series depend on this. However, it is still not possible to estimate this joint model by maximum likelihood. This is because the same unobserved series shows up in all the reduced-form solution equations. So while the model features stochastic shocks, it has a feature that is known as stochastic singularity – the shocks in all the equations are just multiplies of each other.

The model thus predicts that certain ratios of the observed variables (e.g. current and lagged consumption, current and lagged investment) will be constant. In practice, these predictions will not hold in the data so there is no chance that this model can fit the data. In general, for a model to have a well-defined econometric estimates, it is necessary that for every observable variable there be at least one unobservable shock. This can either take the form of a "measurement error" or else involve a shock in each equation with a clear structural interpretation.

Log-linearised DSGE models with a mix of observable and unobservable variables are an example of state-space models. Recall that these models can be described using two equations. The first, known as the state or transition equation, describes how a set of unobservable state variables, $\mathbf{S}_t$, evolves over time as follows:

$$\mathbf{S}_t = \mathbf{F}\mathbf{S}_{t-1} + \mathbf{u}_t, \tag{517}$$

the term $\mathbf{u}_t$ can include either normally-distributed errors or perhaps zeroes if the equation being described is an identity. We will write this as $\mathbf{u}_t \sim N(\mathbf{0}, \mathbf{\Sigma}_u)$ though $\mathbf{\Sigma}_u$ may not have a full matrix rank. The second equation in a state-space model, which is known as the measurement equation, relates a set of observable variables, $\mathbf{Z}_t$, to the unobservable state variables:

$$\mathbf{Z}_t = \mathbf{H}\mathbf{S}_t + \mathbf{v}_t, \tag{518}$$

again, the term $\mathbf{v}_t$ can include either normally-distributed errors or perhaps zeroes if the equation being described is an identity. We will write this as $\mathbf{v}_t \sim N(\mathbf{0}, \mathbf{\Sigma}_v)$, though $\mathbf{\Sigma}_v$ may not have full rank.

11.15.3 Example: The RBC model

The solution the basic RBC model without labour input can be summarised as:

$$\begin{aligned}
\hat{K}_t &= A_{kk}\hat{K}_{t-1} + A_{kz}z_t, \\
\hat{C}_t &= A_{ck}\hat{K}_{t-1} + A_{cz}z_t, \\
z_t &= \rho z_{t-1} + \epsilon_t.
\end{aligned}$$

Not, let's assume that consumption and capital are only observed with error so that the two observable variables are:

$$\begin{aligned}
\hat{K}_t^* &= A_{kk}\hat{K}_{t-1} + A_{kz}z_t + u_t^k, \\
\hat{C}_t^* &= A_{ck}\hat{K}_{t-1} + A_{cz}z_t + u_t^c.
\end{aligned}$$

This can be written in state-space form as follows. The transition equation is:

$$\underbrace{\begin{bmatrix} \hat{K}_{t-1} \\ z_t \end{bmatrix}}_{\mathbf{S}_t} = \underbrace{\begin{bmatrix} A_{kk} & A_{kz} \\ 0 & \rho \end{bmatrix}}_{\mathbf{F}} \underbrace{\begin{bmatrix} \hat{K}_{t-2} \\ z_{t-1} \end{bmatrix}}_{\mathbf{S}_{t-1}} + \underbrace{\begin{bmatrix} 0 \\ \epsilon_t \end{bmatrix}}_{\mathbf{u}_t}, \tag{519}$$

and the measurement equation is:

$$\underbrace{\begin{bmatrix} \hat{K}^*_{t-1} \\ \hat{C}^*_t \end{bmatrix}}_{\mathbf{Z}_t} = \underbrace{\begin{bmatrix} 1 & 0 \\ A_{ck} & A_{cz} \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} \hat{K}_{t-1} \\ z_t \end{bmatrix}}_{\mathbf{S}_t} + \underbrace{\begin{bmatrix} u^k_{t-1} \\ u^c_t \end{bmatrix}}_{\mathbf{v}_t}. \tag{520}$$

Note that we had to do some manipulation to get the model in state-space form and the timing conventions associated with this representation are note quite the same as in the original model. Still, all standard DSGE models can be re-arranged to be put in this format.

11.15.4 MLE for DSGE models via Kalman filter

The Kalman filter provides a way to do maximum likelihood estimation of DSGE models that mix observable and unobservable variables. What is the Kalman filter? The basic idea of a Kalman filter is as its name suggests: to extract a signal from a bunch of noise.

Suppose we see a big increase in output in the latest quarterly data that is not accompanied by a burst of inflation. Does this mean we should assume there has been a big change in potential output? Probably not. Potential output probably doesn't move around a lot from quarter to quarter, and it is likely that there is a lot of fairly random noise in the quarterly fluctuations in inflation. But, there is also probably a useful signal in the data as well. So we are dealing with a type of signal extraction problem: What's the best way to extract a useful signal from information that also contains useless noise?

Suppose we are interested in getting an estimate of the value of a variable X. We don't observe X but instead we observe a variable Z that we know to be correlated with X. Specifically, let's assume that X and Z are jointly normally distributed so that:

$$\begin{bmatrix} X \\ Z \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_X \\ \mu_Z \end{bmatrix}, \begin{bmatrix} \sigma^2_X & \sigma_{XZ} \\ \sigma_{ZX} & \sigma^2_Z \end{bmatrix} \right).$$

In this case, the expected value of X conditional on observing Z is:

$$\mathbb{E}[X|Z] = \mu_X + \frac{\sigma_{XZ}}{\sigma_Z^2}(Z - \mu_Z).$$

Alternatively, if ρ is the correlation between X and Z, $\rho = \frac{\sigma_{XZ}}{\sigma_X \sigma_Z}$, then we can write:

$$\mathbb{E}[X|Z] = \mu_X + \rho\frac{\sigma_X}{\sigma_Z}(Z - \mu_Z). \tag{521}$$

The amount of weight you put on the information in Z when formulating an expectation for X depends on how correlated Z is with X and on their relative standard deviation. If Z has a high standard deviation (so it's a poor signal) then you don't place much weight on it.

For the vector case where $\mathbf{X}$ is an n-vector of variables and $\mathbf{Z}$ is an m-vector, there is a straightforward generalisation of the formula just presented. Denote the covariance of the variables in $\mathbf{X}$ as $\mathbf{\Sigma}_{XX}$, the covariance matrix of the variables in $\mathbf{Z}$ as $\mathbf{\Sigma}_{ZZ}$, and the matrix of covariances between the entires in $\mathbf{X}$ and $\boldsymbol{Z}$ as $\mathbf{\Sigma}_{XZ}$. If all the variables are jointly normally distributed then this can be written as:

$$\begin{bmatrix} \mathbf{X} \\ \mathbf{Z} \end{bmatrix} \sim N\left(\begin{bmatrix} \boldsymbol{\mu}_X \\ \boldsymbol{\mu}_Z \end{bmatrix}, \begin{bmatrix} \mathbf{\Sigma}_{XX} & \mathbf{\Sigma}_{XZ} \\ \mathbf{\Sigma}_{XZ}^\top & \mathbf{\Sigma}_{ZZ} \end{bmatrix}\right).$$

In this case, the expected value of $\mathbf{X}$ conditional on observing $\mathbf{Z}$ is:

$$\mathbb{E}[\mathbf{X}|\boldsymbol{Z}] = \boldsymbol{\mu}_X + \mathbf{\Sigma}_{XY}\mathbf{\Sigma}_{ZZ}^{-1}(\mathbf{Z} - \boldsymbol{\mu}_Z). \tag{522}$$

This formula will play an important role in our explanation of the Kalman filter.

Now, we know that the observed data are described by the transition equation (518):

$$\mathbf{Z}_t = \mathbf{H}\mathbf{S}_t + \mathbf{v}_t,$$

where we can't observe $\mathbf{S}_t$ but suppose we could replace it by an observable unbiased guess based on information available up to time $t-1$. Call this guess $\mathbf{S}_{t|t-1}$ and suppose its errors are normally distributed with a known covariance matrix:

$$\mathbf{S}_t - \mathbf{S}_{t|t-1} \sim N(\mathbf{0}, \mathbf{\Sigma}^S_{t|t-1}),$$

then the observed variables could be written as:

$$\mathbf{Z}_t = \mathbf{H}\mathbf{S}_{t|t-1} + \mathbf{v}_t + \mathbf{H}(\mathbf{S}_t - \mathbf{S}_{t|t-1}).$$

Because $\mathbf{S}_{t|t-1}$ is observable and the unobservable elements ($\mathbf{v}_t$ and $\mathbf{S}_t - \mathbf{S}_{t|t-1}$) are normally distributed, this model can be estimated via maximum likelihood methods. The variance of the error term after conditioning on period $t-1$'s estimate of the state variables is given by:

$$\mathbf{v}_t + \mathbf{H}(\mathbf{S}_t - \mathbf{S}_{t|t-1}) \sim N(\mathbf{0}, \mathbf{\Omega}_t),$$

where:

$$\mathbf{\Omega}_t = \mathbf{\Sigma}^v + \mathbf{H}\mathbf{\Sigma}^v_{t|t-1}\mathbf{H}^\top.$$

Let $\boldsymbol{\theta}$ represent the parameters of the model (i.e. $\boldsymbol{\theta} = (\mathbf{F}, \mathbf{H}, \mathbf{\Sigma}^v, \mathbf{\Sigma}^u)$. We will show later that $\mathbf{\Sigma}^S_{t|t-1}$ depends on these parameters. The log likelihood function for $\mathbf{Z}_t$ given the observables at time $t-1$ is:

$$\ln f(\mathbf{Z}_t|\mathbf{Z}_{t-1}, \boldsymbol{\theta}) = 2\ln 2\pi - \ln|\mathbf{\Omega}_t| - \frac{1}{2}(\mathbf{Z}_t - \mathbf{H}\mathbf{S}_{t|t-1})^\top \mathbf{\Omega}_t^{-1}(\mathbf{Z}_t - \mathbf{H}\mathbf{S}_{t|t-1}).$$

Given the initial estimates of the first-period unobservable state $\mathbf{S}_{1|0}$, the combined likelihood for all the observed data is the product of all the period-by-period likelihoods:

$$f(\mathbf{Z}_1, \mathbf{Z}_2, ..., \mathbf{Z}_T|\mathbf{S}_{1|0}, \boldsymbol{\theta}) = f(\mathbf{Z}_1|\mathbf{S}_{1|0}, \boldsymbol{\theta}) \prod_{i=2}^{T} f(\mathbf{Z}_i|\mathbf{Z}_{i-1}, \boldsymbol{\theta}).$$

So the combined log-likelihood function for the observed dataset is given by:

$$\ln f(\mathbf{Z}_1, \mathbf{Z}_2, ..., \mathbf{Z}_T|\mathbf{S}_{1|0}, \boldsymbol{\theta}) = -T\ln 2\pi - \sum_{i=1}^{T} \ln|\mathbf{\Omega}_i| - \frac{1}{2}\sum_{i=i}^{T}(\mathbf{Z}_i - \mathbf{H}\mathbf{S}_{i|i-1})^\top \mathbf{\Omega}_i^{-1}(\mathbf{Z}_i - \mathbf{H}\mathbf{S}_{i|i-1}).$$

The maximum likelihood parameter estimates are the set of matrices $\boldsymbol{\theta} = (\mathbf{F}, \mathbf{H}, \boldsymbol{\Sigma}^v, \boldsymbol{\Sigma}^u)$ that provide the largest value for this function.

We have described how to estimate the model's parameters via MLE provided we have an unbiased guess based on information available up to time $t-1$, which we called $\mathbf{S}_{t|t-1}$, with normally distributed errors. Here we describe a method for generating these unbiased guesses known as the Kalman filter. It is an iterative method. Starting from one period's estimates of the state variables, it uses the observable data for the next period to update these estimates.

Let's start with formulating an estimate of the state variable at time t given information at time $t-1$. This is easy enough:

$$\begin{aligned} \mathbf{S}_t &= \mathbf{F}\mathbf{S}_{t-1} + \mathbf{u}_t \\ \implies \mathbf{S}_{t|t-1} &= \mathbf{F}\mathbf{S}_{t-1|t-1}. \end{aligned}$$

This means that in period $t-1$, the expected value for the observables in period t are:

$$\begin{aligned} \mathbf{Z}_{t|t-1} &= \mathbf{H}\mathbf{S}_{t|t-1} \\ &= \mathbf{H}\mathbf{F}\mathbf{S}_{t-1|t-1}. \end{aligned}$$

Then, in period t, when we observe $\mathbf{Z}_t$ the question is how do we update our guesses for the state variable in light if the "news" in $\mathbf{Z}_t - \mathbf{H}\mathbf{F}\mathbf{S}_{t-1|t-1}$?

The assumptions of the model imply that:

$$\begin{bmatrix} \mathbf{S}_t \\ \mathbf{Z}_t \end{bmatrix} \sim N\left(\begin{bmatrix} \mathbf{F}\mathbf{S}_{t-1|t-1} \\ \mathbf{H}\mathbf{F}\mathbf{S}_{t-1|t-1} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}^S_{t|t-1} & (\mathbf{H}\boldsymbol{\Sigma}^S_{t|t-1})^\top \\ \mathbf{H}\boldsymbol{\Sigma}^S_{t|t-1} & \boldsymbol{\Sigma}^v + \mathbf{H}\boldsymbol{\Sigma}^S_{t|t-1}\mathbf{H}^\top \end{bmatrix} \right),$$

now we can use our earlier result about conditional expectations (522) to state that the minimum variance unbiased estimate of $\mathbf{S}_t$ given the observed $\mathbf{Z}_t$ is:

$$\mathbb{E}[\mathbf{S}_t|\mathbf{Z}_t] = \mathbf{S}_{t|t} = \mathbf{F}\mathbf{S}_{t-1|t-1} + \mathbf{K}_t(\mathbf{Z}_t - \mathbf{H}\mathbf{F}\mathbf{S}_{t-1|t-1}), \tag{523}$$

where:

$$\mathbf{K}_t = \left(\mathbf{H}\mathbf{\Sigma}^S_{t|t-1}\right)^\top \left(\mathbf{\Sigma}^v + \mathbf{H}\mathbf{\Sigma}^S_{t|t-1}\mathbf{H}^\top\right)^{-1}. \tag{524}$$

The covariance matrices required to compute this $\mathbf{K}_t$ matrix (known as the Kalman gain matrix) are updated by the formulae:

$$\mathbf{\Sigma}^S_{t|t-1} = \mathbf{F}\mathbf{\Sigma}^S_{t-1|t-1}\mathbf{F}^\top + \mathbf{\Sigma}^u,$$
$$\mathbf{\Sigma}^S_{t|t} = (\mathbf{I} - \mathbf{K}_t\mathbf{H})\mathbf{\Sigma}^S_{t|t-1}.$$

We still need an initial estimate $\mathbf{S}_{1|0}$ as well as its covariance matrix to start the filter process. In many macroeconomic models, the state variable can be assumed to have a zero mean without losing any generality, so that can work as first guess for the state. we can estimate the unconditional variance by estimating what the variance of an estimate of the state variable would be fro ma large sample of data. Recall that:

$$\mathbf{\Sigma}^S_{t|t-1} = \mathbf{F}\mathbf{\Sigma}^S_{t-1|t-1}\mathbf{F}^\top + \mathbf{\Sigma}^u.$$

The values of the covariance matrix generated by this equation will generally converge, so for our unconditional covariance matrix we can use a value of $\mathbf{\Sigma}$ that solves:

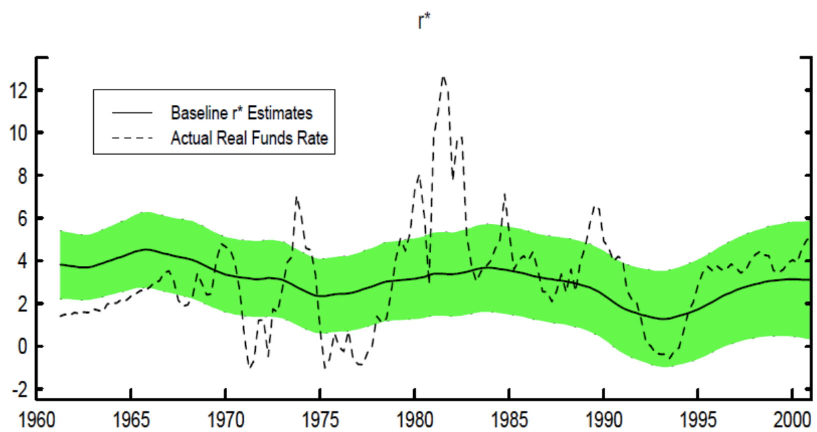
$$\mathbf{\Sigma} = \mathbf{F}\mathbf{\Sigma}\boldsymbol{F}^\top + \mathbf{\Sigma}^u.$$

The Kalman filter is what is known as a one-sided filter: The estimates of states at time t are based solely on information available at time t. No data after period t is used to calculate estimates of the unobserved state variables. This is a reasonable model for how someone might behave if they were learning about the state variables in real time. But researchers have access to the full history of the data set, including all observations after time t. For this reason, economists generally time-varying models using a method known as the Kalman smoother. This is a two-sided filter that uses data before and after time t to compute expected values of the state variables at time t.

We've already digested too much about the Kalman filter at this stage – far beyond what is required for a first year course in macroeconomics. So we won't jump into Kalman smoothing. The key idea is that you do a Kalman filter first and then work backwards from the final estimates further exploiting joint distribution properties.

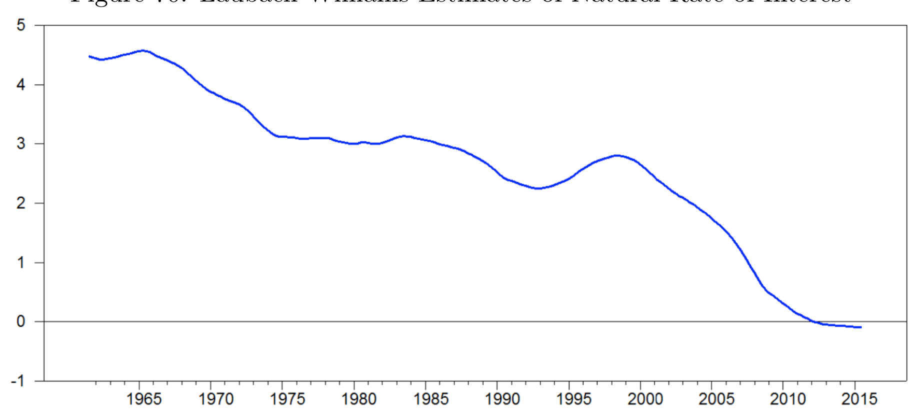
Below are some Kalman filter estimates from a paper by Laubach and Williams (2003).

Figure 69: Actual vs Natural Real Interest Rates



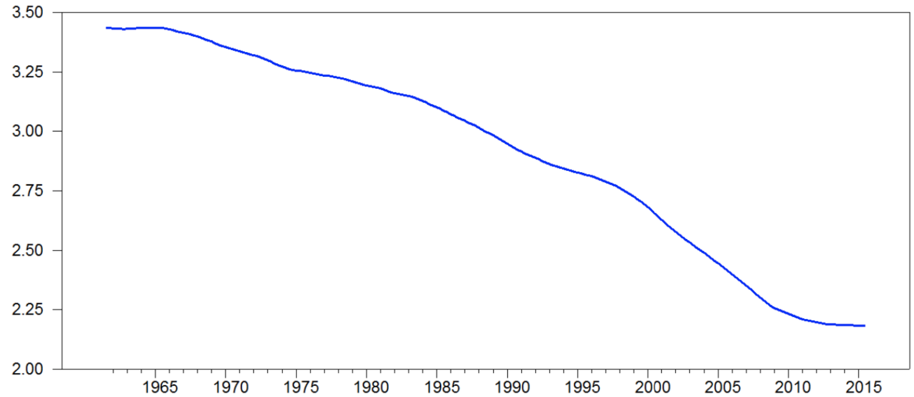
Source: Laubach and Williams (2003)

Figure 70: Laubach-Williams Estimates of Natural Rate of Interest



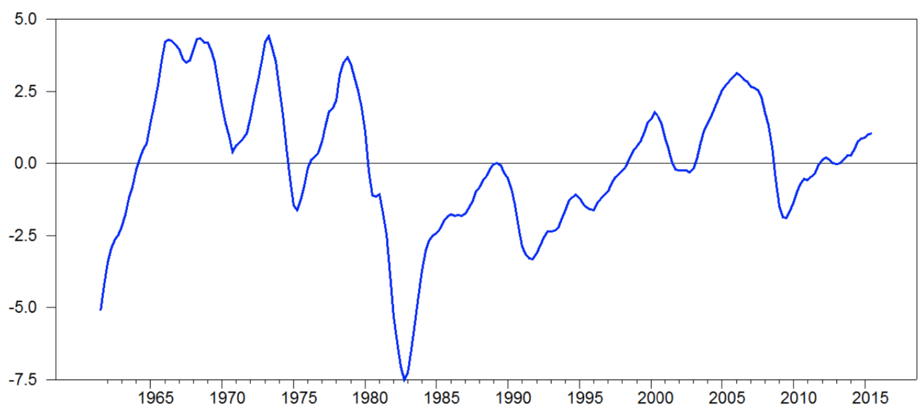
Source: Whelan (2016)

Figure 71: Laubach-Williams Estimates of Potential Output Growth



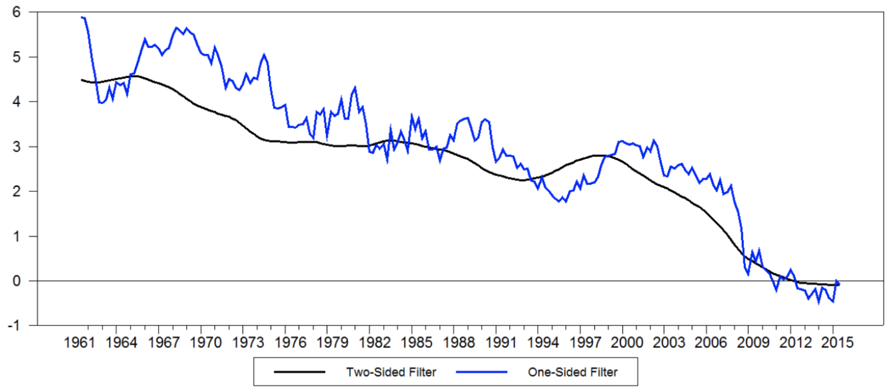
Source: Whelan (2016)

Figure 72: Laubach-Williams Estimates of Output Gap



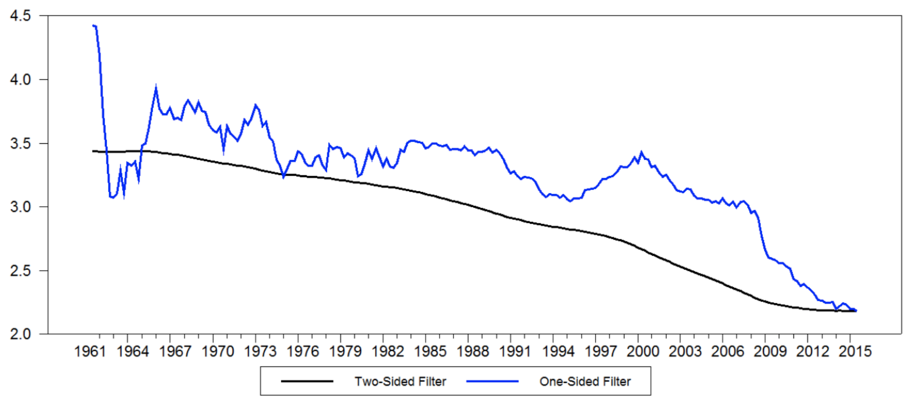
Source: Whelan (2016)

Figure 73: One-Sided and Two-Sided Estimates of Natural Rate of Interest



Source: Whelan (2016)

Figure 74: One-Sided and Two-Sided Estimates of Potential Output Growth



Source: Whelan (2016)

Oh, and you may be wondering if the HP filter and Kalman filter are related – they are. Recall that the HP filter is a way of choosing a trend for a series that had a time varying trend Y_t^* picked to minimise:

$$\sum_{t=1}^{N} \left[(Y_t - Y_t^*)^2 + \lambda(\Delta Y_t^* - \Delta Y_{t-1}^*) \right].$$

It may seems fairly ad-hoc but it can be viewed as an example of the Kalman filter. Consider the following state-space model:

$$
\begin{aligned}
Y_t &= Y_t^* + C_t, \\
\Delta Y_t^* &= \Delta Y_{t-1}^* + \epsilon_t^g, \\
C_t &= \epsilon_t^c,
\end{aligned}
$$

where $\text{Var}(\epsilon_t^g) = \sigma_g^2$ and $\text{Var}(\epsilon_t^c) = \sigma_c^2$. It can be shown that for large samples the HP filter technique is the same as Kalman filter estimation of this model when we set $\lambda = \sigma_c^2/\sigma_g^2$. Hodrick and Prescott assumed C_t had a standard deviation of 5 percentage points while ϵ_t^g had a standard deviation of one-eighth of a percentage point. Hence they chose:

$$
\lambda = \frac{5^2}{\left(\frac{1}{8}\right)^2} = (25)(64) = 1600.
$$

All this seems pretty complicated – it kind of is – but the good news is that software packages such as Dynare can do this for you with minimum effort from you once you have specified the model. In other words, computer packages can now:

- Sort your model into state-space methods;
- Search across a wide range possible parameter values;
- For each of these, apply the Kalman filter/smoother;
- Then, for each possible set of parameters, it can sum up each of the period-by-period likelihoods; and
- Then it can decide what the best parameters are and use standard MLE-related methods to calculate asymptotically valid standard errors.

If you think that this is a complicated process where things might go wrong, then you'd be right. Read the paper "The Econometrics of DGSE Models" by Fernández-Villaverde (2010). He discusses some of the problems associated with MLE for DSGE models and explains why a Bayesian approach of calculating the full posterior distribution may be preferable:

> "[...]maximising a complicated, highly dimensional function like the likelihood of a DSGE model is actually much harder than it is to integrate it, which is what we do in a Bayesian exercise. First, the likelihood of DSGE models is, as I have just mentioned, a highly dimensional object, with a dozen or so parameters in the simplest cases to close to a hundred in some of the richest models in the literature. Any search in a high dimensional function is fraught with peril. More pointedly, likelihoods of DSGE models are full of local maxima and minima and of nearly flat surfaces. This is due both to the sparsity of the data (quarterly data do not give us the luxury of many observations that micro panels provide) and to the flexibility of DSGE models in generating similar behaviour with relatively different combination of parameter values [...] Moreover, the standard errors of the estimates are notoriously difficult to compute and their asymptotic distribution a poor approximation to the small sample one."

11.15.5 Bayesian methods for DSGE

For the reasons stated above, Bayesian approaches to estimating DSGE models have become the standard approach in recent years. A prior distribution for the parameters is specified and then this is combined with the full likelihood function to produce an estimate of the posterior distribution. This posterior distribution can be integrated using numerical methods to produce means and confidence intervals of various sorts. Importantly, because you are using an estimate of the full likelihood function, you are less likely to fall victim to the major errors that can occur from using an incorrect MLE, which uses only one point of the function.

Dynare allows you to specify priors and to estimate DSGE model directly. Researchers generally specify prior means for parameters using values considered "reasonable" from other studies with the form of the distributions usually being of a form that fits with a "common sense" view of the potential range of outcomes. The estimation results are generally reported by comparing the posterior means with the prior means as well as reporting the "confidence intervals" from the posterior distributions.

Figure 75: Choosing Priors: Normal Distribution

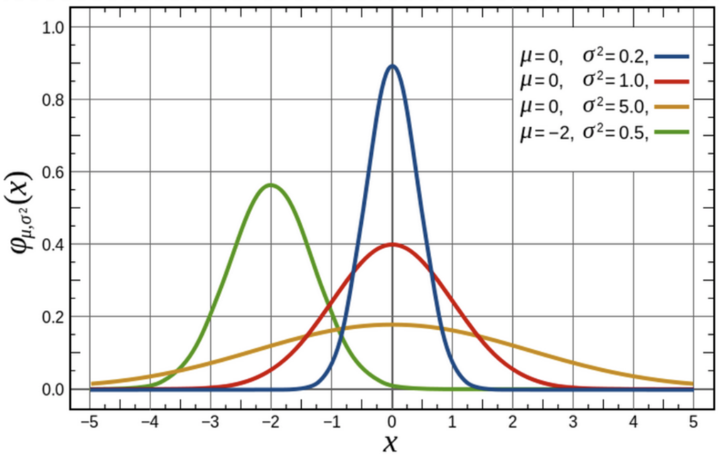


Figure 76: Choosing Priors: Gamma Distribution

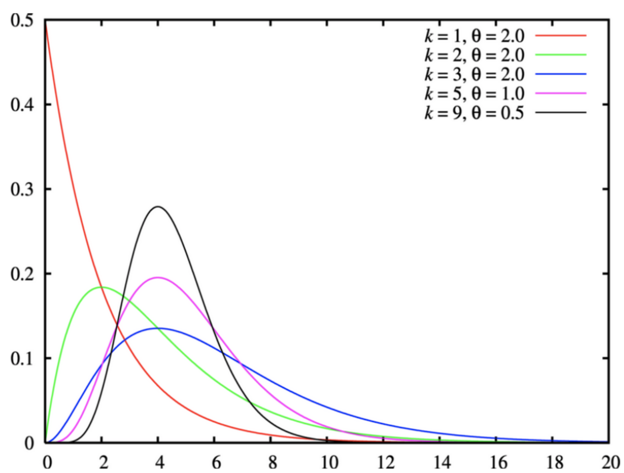
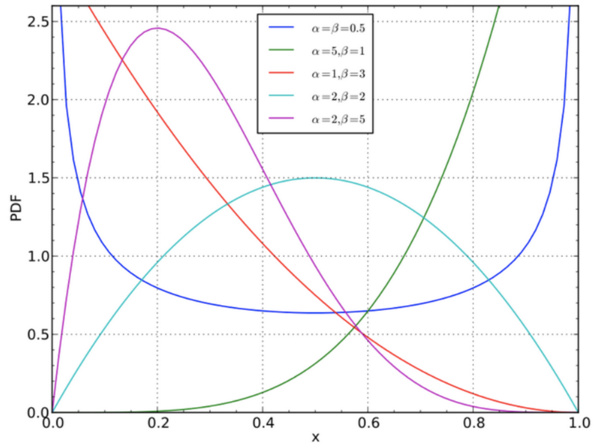


Figure 77: Choosing Priors: Beta Distribution



11.16 Medium-scale New Keynesian DSGE models: The Smets-Wouters model

Now we will discuss a paper presenting a modern DSGE model that has a number of New Keynesian features and which has been estimated with Bayesian methods. The paper is "Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach" Smets and Wouters (2007). Smets is an economist with the ECB and Wouters works for the National Bank of Belgium and the model was first developed for the Euro area. Models like this have been used for policy analysis at the ECB and other central banks. Smets and Wouters have two papers, one for the Euro area (Smets and Wouters 2003) and one for the US. Both papers have been among the most cited papers in economics in recent years.

11.16.1 The supply side

The aggregate production function is:

$$\hat{Y}_t = \phi_p(\alpha \hat{K}_t^s + (1-\alpha)\hat{L}_t + \epsilon_t^a),$$

where Y_t is GDP, L_t is labour input, ϵ_t^a is total factor productivity, and K_t^s is capital in use, which is determined by the amount of capital installed in the previous period and a capacity utilisation variable:

$$\hat{K}_t^s = \hat{K}_{t-1} + z_t.$$

There are costs of adjusting the amount of capital in use so optimisation conditions for producers imply the rate of capacity utilisation is linked to the marginal productivity of capital:

$$z_t = z_1 \hat{R}_t^k.$$

The marginal productivity of capital is a function of the capital-labour ratio and the real wage:

$$\hat{R}_t^k = -(\hat{K}_t - \hat{L}_t) + \hat{w}_t.$$

Total factor productivity evolves over time according to:

$$\epsilon_t^a = \rho_a \epsilon_{t-1}^a + \eta_t^a.$$

11.16.2 The demand side

The expenditure formulation of the aggregate resource constraint is:

$$\hat{Y}_t = C_y \hat{C}_t + I_y \hat{I}_t + z_y z_t + \epsilon_t^g,$$

where C_t is consumption, I_t is investment, and ϵ_t^g is exogenous spending. Terms like C_y and I_y are just constant parameters. The variable z_t features here too because we are assuming there are costs associated with having high rates of capacity utilisation. Exogenous spending is assumed to have two components: Government spending and an element related to productivity because "net exports may be affected by domestic productivity developments".

Taken together, exogenous spending changes over time according to:

$$\epsilon_t^g = \rho \epsilon_{t-1}^g + \eta_t^g + \rho_{ga} \eta_t^a.$$

Consumption is determined by:

$$\hat{C}_t = C_1 \hat{C}_{t-1} + (1 - C_1) \mathbb{E}_t \hat{C}_{t+1} + C_2 (\hat{L}_t - \mathbb{E}_t \hat{L}_{t+1}) - C_3 (\hat{R}_t - \mathbb{E}_t \hat{\pi}_{t+1} + \epsilon_t^b),$$

where C_1, C_2, C_3 are constant parameters, R_t is the interest rate paid on a one-period safe bond, and ϵ_t^b evolves according to:

$$\epsilon_t^b = \rho_b \epsilon_{t-1}^b + \eta_t^b.$$

There are a number of aspects to this equation. First, it is a consumption Euler equation with a backward-looking element added to it. This represents "habit formation" so that a term of the form $C_t - \lambda C_{t-1}$ replaces C_t in the utility function. Second, the term involving labour input allows for some substitution between consumption and labour input. Third, coefficients C_1, C_2, C_3 are themselves functions of deeper structural parameters. Fourth, Smets and Wouters describe the ϵ_b term as a "risk-premium" shock determining the willingness of households to hold the one-period bond. It can also be seen as a type of preference shock that influences the short-term consumption-saving decision.

Investment is determined by:

$$\hat{I}_t = I_1 \hat{I}_{t-1} + (1 - I_1) \mathbb{E}_t \hat{I}_{t+1} + I_2 \hat{Q}_t + \epsilon_t^i,$$

where:

$$\hat{Q}_t = Q_1 \mathbb{E}_t \hat{Q}_{t+1} + (1 - Q_1) \hat{R}_{t+1}^k - (\hat{R}_t - \mathbb{E}_t \hat{\pi}_{t+1} + \epsilon_t^b),$$

and:

$$\hat{K}_t = K_1 \hat{K}_{t-1} + (1 - K_1) \hat{I}_t + K_2 \epsilon_t^i.$$

Again, there's a lot going on here: Investment depends on lagged investment because there is an adjustment cost function that limits that amount of new investment that can come "on line" immediately; the main driving force behind investment is Q_t, which itself is determined by a forward-looking stochastic difference equation; solving the Q_t equation would show that Q_t depends positively on expected future marginal productivities of capital and negatively on future real interest rates (and "risk premia"); and, the future positive shock to investment also boosts the capital stock (representing "more productive" capital).

11.16.3 Prices

The mark-up of price over marginal cost is determined by:

$$\hat{\mu}_t^p = \alpha(\hat{K}_t - \hat{L}_t) + \epsilon_t^a - \hat{w}_t,$$

which factors in diminishing marginal productivity of capital, the effects of the productivity shock on costs and the real wage.

Price inflation is then determined by:

$$\hat{\pi}_t = \pi_1 \hat{\pi}_{t-1} + \pi_2 \mathbb{E}_t \hat{\pi}_{t+} - \pi_3 \hat{\mu}_t^p + \epsilon_t^p,$$

where ϵ_t^p is a price mark-up disturbance that evolves according to:

$$\epsilon_t^p = \rho_p \epsilon_{t-1}^p + \eta_t^p - \mu_p \eta_{t-1}^p.$$

This is an NKPC amended to provide a role for lagged inflation – so it's a type of HPC. This modelled in the paper via the assumption that most firms index their price to past inflation and only occasionally get to set an optimal price. Also, the mark-up shock affects both current and lagged inflation in an attempt to get at temporary price level shocks.

11.16.4 Wages

The model treats wages similarly to prices, with sticky wages that gradually adjust so that real wages move to equate the marginal costs and benefits of working. Specifically, wages move over time to equate real wages with the marginal rate of substitution between working and consuming. The gap between these is the "wage mark-up" defined as:

$$\begin{aligned}\hat{\mu}_t^w &= \hat{w}_t - \widehat{mrs}_t \\ &= \hat{w}_t - \left(\sigma \hat{L}_t - \frac{1}{1-\lambda/\gamma}(\hat{C}_t - \lambda \hat{C}_{t-1})\right).\end{aligned}$$

Wages are then given by:

$$\hat{w}_t = W_1 \hat{w}_{t-1} + (1-W_1)\mathbb{E}_t(\hat{w}_{t+1} + \hat{\pi}_{t+1}) - W_2 \hat{\pi}_t + W_3 \hat{\pi}_{t-1} - \hat{w}_t \hat{\mu}_t^w + \epsilon_t^w,$$

where:

$$\epsilon_t^w = \rho_w \epsilon_{t-1}^w + \eta_t^w - \mu_w \eta_{t-1}^w.$$

11.16.5 Monetary policy

The final element of the model is a rule for monetary policy. It is assumed that the central bank sets short term interest rates according to:

$$\hat{R}_t = \rho \hat{R}_{t-1} + (1-\rho)(R_\pi \hat{\pi}_t + R_y(\hat{Y}_t - \hat{Y}_t^f) + R_{\Delta y}\left[(\hat{Y}_t - \hat{Y}_t^f) - (\hat{Y}_{t-1} - \hat{Y}_{t-1}^f)\right] + \epsilon_t^r,$$

where:

$$\epsilon_t^r = \rho_r \epsilon_{t-1}^r + \eta_t^r.$$

Here the interest rate depends on last period's interest rate while gradually adjusting toward a target interest rate $(R_\pi \hat{\pi}_t + R_y(\hat{Y}_t - \hat{Y}_t^f))$ that depends on inflation and the gap between output and its potential level $(\hat{Y}_t - \hat{Y}_t^f)$. It also depends on the growth rate of this output gap. Potential output is defined as the level of output that would prevail if prices and wages were fully flexible. This means the model effectively needs to be "expanded" to add a "shadow" flexible-price economy.

11.16.6 Calibration and results

Relative to the pure RBC or New Keynesian models we saw before, this model has a tonne of additional features:

- Adjust costs for investment;
- Capacity utilisation costs;
- Habit persistence;
- Price indexation;
- Wage indexation; and
- Lots of new autocorrelated disturbance terms.

Essentially, every single tweak and wrinkle that we looked at in this entire course is in this model. These help the model to address the weaknesses of the previous models. Adjustment costs, utilisation costs, and habit persistence all help to "throw sand in wheels" of the model, making variables more sluggish and giving random shocks a more long-lasting effect. This was a weakness of the RBC model. Indexation deals with the New Keynesian model's failure to match inflation persistence.

The observable VAR system of the Smets-Wouters model is:

$$Y_t = \begin{bmatrix} dlGDP_t \\ dlCONS_t \\ dlINV_t \\ dlWAG_t \\ lHOURS_t \\ dlP_t \\ FEDFUNDS_t \end{bmatrix} = \begin{bmatrix} \bar{\gamma} \\ \bar{\gamma} \\ \bar{\gamma} \\ \bar{\gamma} \\ \bar{l} \\ \bar{\pi} \\ \bar{r} \end{bmatrix} + \begin{bmatrix} y_t - y_{t-1} \\ c_t - c_{t-1} \\ i_t - i_{t-1} \\ w_t - w_{t-1} \\ l_t \\ \pi_t \\ r_t \end{bmatrix}.$$

Below is a summary of calibration, estimation, and performance results of the model.

Figure 78: Priors and Posteriors: Structural Parameters

TABLE 1A—PRIOR AND POSTERIOR DISTRIBUTION OF STRUCTURAL PARAMETERS

	Prior distribution			Posterior distribution			
	Distr.	Mean	St. Dev.	Mode	Mean	5 percent	95 percent
φ	Normal	4.00	1.50	5.48	5.74	3.97	7.42
σ_c	Normal	1.50	0.37	1.39	1.38	1.16	1.59
h	Beta	0.70	0.10	0.71	0.71	0.64	0.78
ξ_w	Beta	0.50	0.10	0.73	0.70	0.60	0.81
σ_l	Normal	2.00	0.75	1.92	1.83	0.91	2.78
ξ_p	Beta	0.50	0.10	0.65	0.66	0.56	0.74
ι_w	Beta	0.50	0.15	0.59	0.58	0.38	0.78
ι_p	Beta	0.50	0.15	0.22	0.24	0.10	0.38
ψ	Beta	0.50	0.15	0.54	0.54	0.36	0.72
Φ	Normal	1.25	0.12	1.61	1.60	1.48	1.73
r_π	Normal	1.50	0.25	2.03	2.04	1.74	2.33
ρ	Beta	0.75	0.10	0.81	0.81	0.77	0.85
r_y	Normal	0.12	0.05	0.08	0.08	0.05	0.12
$r_{\Delta y}$	Normal	0.12	0.05	0.22	0.22	0.18	0.27
$\bar{\pi}$	Gamma	0.62	0.10	0.81	0.78	0.61	0.96
$100(\beta^{-1} - 1)$	Gamma	0.25	0.10	0.16	0.16	0.07	0.26
$\bar{l}$	Normal	0.00	2.00	−0.1	0.53	−1.3	2.32
$\bar{\gamma}$	Normal	0.40	0.10	0.43	0.43	0.40	0.45
α	Normal	0.30	0.05	0.19	0.19	0.16	0.21

Figure 79: Priors and Posteriors: Shock Processes

TABLE 1B—PRIOR AND POSTERIOR DISTRIBUTION OF SHOCK PROCESSES

	Prior distribution			Posterior distribution			
	Distr.	Mean	St. Dev.	Mode	Mean	95 percent	5 percent
σ_a	Invgamma	0.10	2.00	0.45	0.45	0.41	0.50
σ_b	Invgamma	0.10	2.00	0.24	0.23	0.19	0.27
σ_g	Invgamma	0.10	2.00	0.52	0.53	0.48	0.58
σ_I	Invgamma	0.10	2.00	0.45	0.45	0.37	0.53
σ_r	Invgamma	0.10	2.00	0.24	0.24	0.22	0.27
σ_p	Invgamma	0.10	2.00	0.14	0.14	0.11	0.16
σ_w	Invgamma	0.10	2.00	0.24	0.24	0.20	0.28
ρ_a	Beta	0.50	0.20	0.95	0.95	0.94	0.97
ρ_b	Beta	0.50	0.20	0.18	0.22	0.07	0.36
ρ_g	Beta	0.50	0.20	0.97	0.97	0.96	0.99
ρ_I	Beta	0.50	0.20	0.71	0.71	0.61	0.80
ρ_r	Beta	0.50	0.20	0.12	0.15	0.04	0.24
ρ_p	Beta	0.50	0.20	0.90	0.89	0.80	0.96
ρ_w	Beta	0.50	0.20	0.97	0.96	0.94	0.99
μ_p	Beta	0.50	0.20	0.74	0.69	0.54	0.85
μ_w	Beta	0.50	0.20	0.88	0.84	0.75	0.93
ρ_{ga}	Beta	0.50	0.20	0.52	0.52	0.37	0.66

Figure 80: Out-of-Sample Forecasting Beats VAR Models

TABLE 3—OUT-OF-SAMPLE PREDICTION PERFORMANCE

	GDP	dP	Fedfunds	Hours	Wage	CONS	INV	Overall
VAR(1)	*RMSE-statistic for different forecast horizons*							
1q	0.60	0.25	0.10	0.46	0.64	0.60	1.62	−12.87
2q	0.94	0.27	0.18	0.78	1.02	0.95	2.96	−8.19
4q	1.64	0.34	0.36	1.45	1.67	1.54	5.67	−3.25
8q	2.40	0.53	0.64	2.13	2.88	2.27	8.91	1.47
12q	2.78	0.63	0.79	2.41	4.09	2.74	10.97	2.36
BVAR(4)	*Percentage gains* (+) *or losses* (−) *relative to VAR*(1) *model*							
1q	2.05	14.14	−1.37	−3.43	2.69	12.12	2.54	3.25
2q	−2.12	15.15	−16.38	−7.32	−0.29	10.07	2.42	0.17
4q	−7.21	31.42	−12.61	−8.58	−3.82	1.42	0.43	0.51
8q	−15.82	33.36	−13.26	−13.94	−8.98	−8.19	−11.58	−4.10
12q	−15.55	37.59	−13.56	−4.66	−15.87	−3.10	−23.49	−9.84
DSG	*Percentage gains* (+) *or losses* (−) *relative to VAR*(1) *model*							
1q	5.68	2.05	−8.24	0.68	5.99	20.16	9.22	3.06
2q	14.93	10.62	−17.22	10.34	6.20	25.85	16.79	2.82
4q	20.17	46.21	1.59	19.52	9.21	26.18	21.42	6.82
8q	22.55	68.15	28.33	22.34	15.72	21.82	25.95	11.50
12q	32.17	74.15	40.32	27.05	21.88	23.28	41.61	13.51

Figure 81: Explaining GDP Movements at Various Horizons

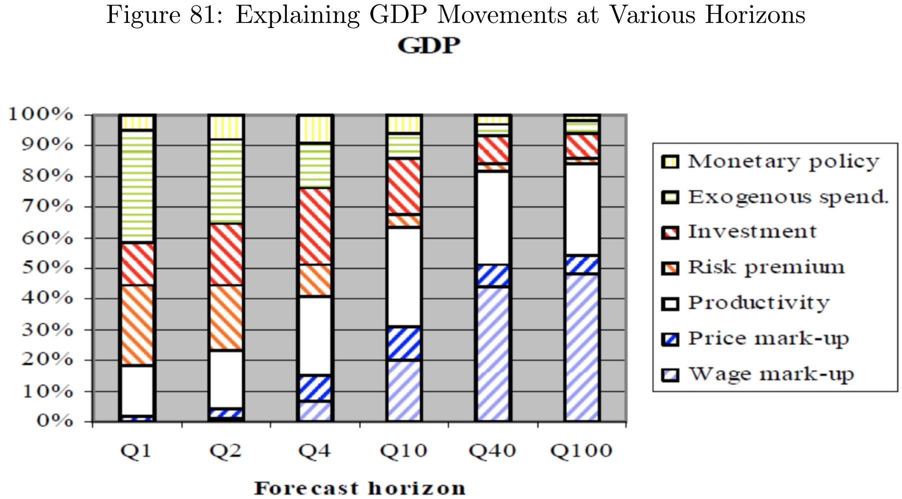


Figure 82: Explaining Inflation Movements at Various Horizons

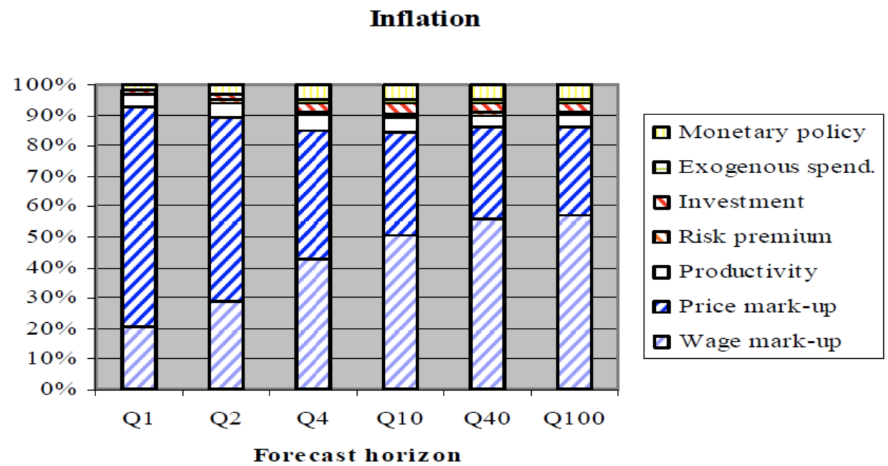


Figure 83: Explaining Fed Funds Movements at Various Horizons

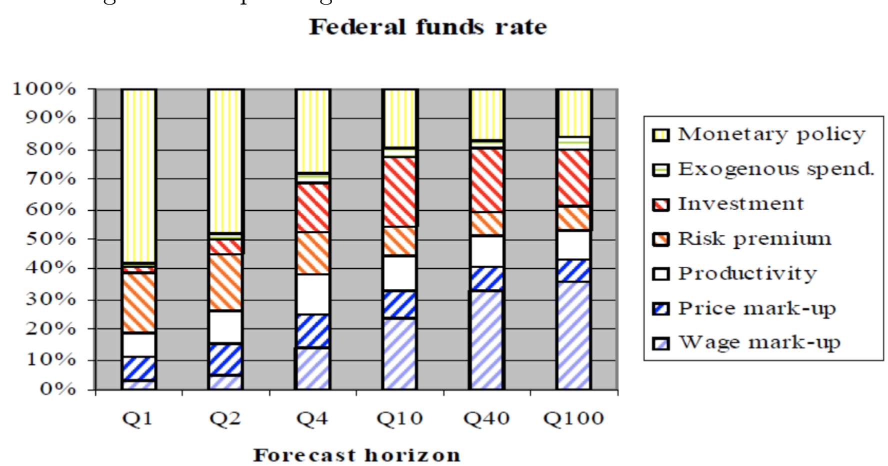
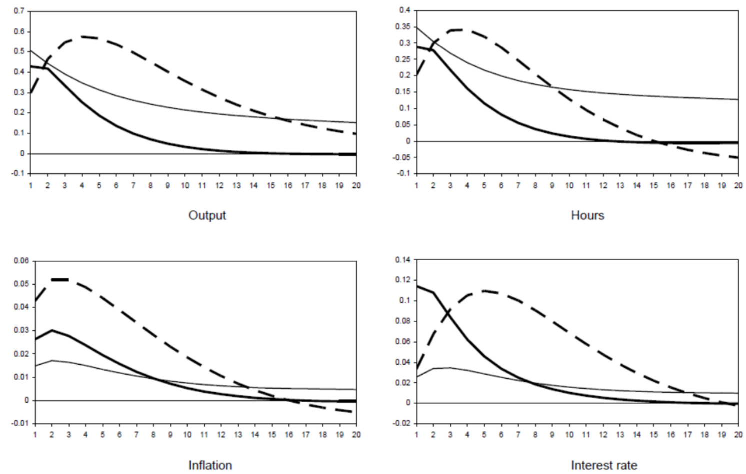


Figure 84: The Impact of Various "Demand" Shocks



Notes: Bold solid line: risk premium shock; thin solid line: exogenous spending shock; dashed line: investment shock.

Figure 85: Impulse Response for a Monetary Policy Shock

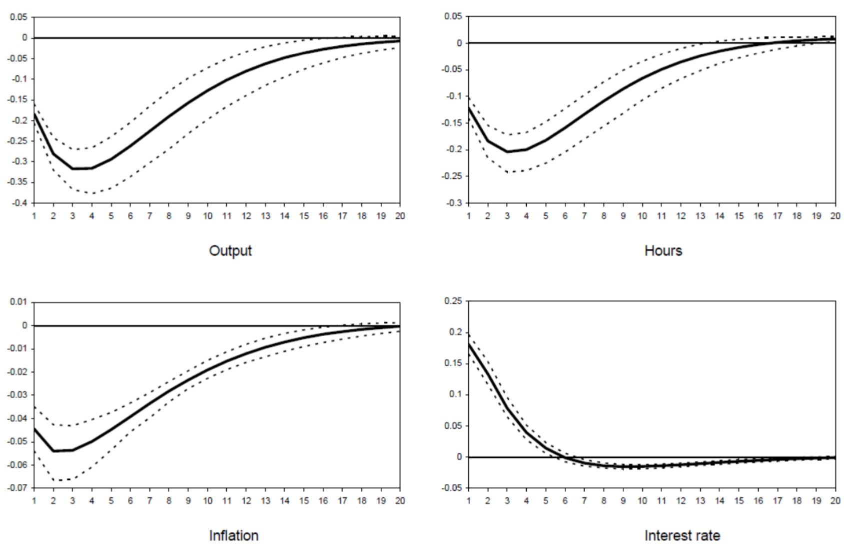


Figure 86: Impulse Response for a Wage Mark-Up Shock

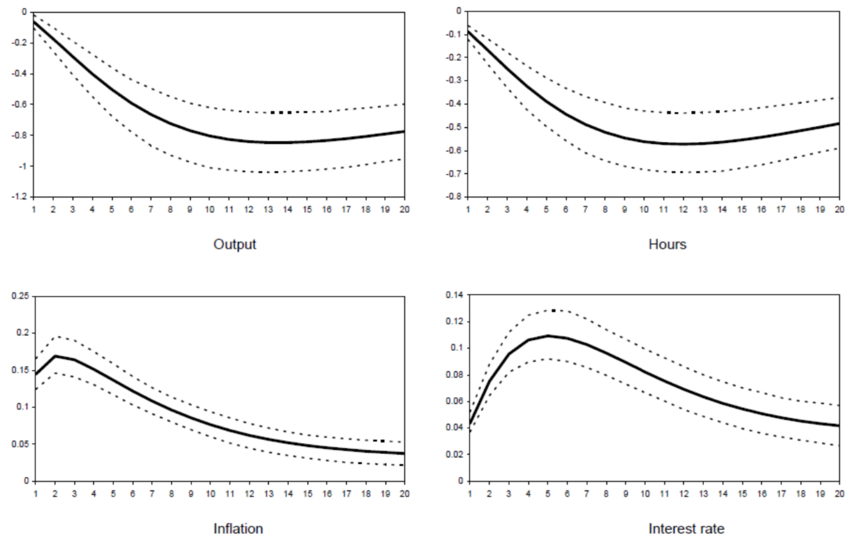
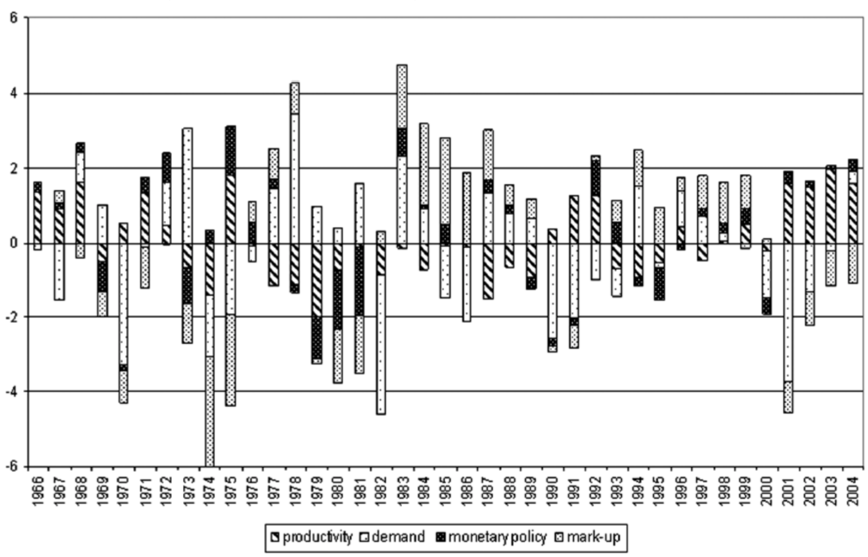


Figure 87: Decomposing the Growth Rate of GDP



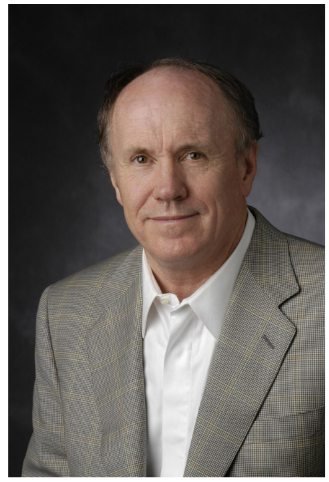
11.17 Comments and key readings

I think a good summary of our coverage of the New Keynesian DSGE model in this chapter is the meme "This isn't even my final form!". I won't go over key readings in this conclusion – there are plenty of references to be found throughout this chapter, and also in the texts Galí (2015), Walsh (2010), Romer (2012), and Woodford (2003). Although, some good readings for Bayesian estimation are: "Methods to Estimate Dynamic Stochastic General Equilibrium Models" Ruge-Murcia (2007), "A Method for Taking Models to the Data" Ireland (2004), and "The Econometrics of DSGE Models" Fernández-Villaverde (2010).

New Keynesian DSGE models – in particular, the medium-scale ones such as the Smets-Wouters model – have a lot of strengths: They fit the data well (see Christiano, Trabandt, et al. (2011)) and allow us to answer many "what if" questions regarding policy and economic shocks.

However, they also have a list of weaknesses:

- A large number of ad-hoc economic mechanisms designed mainly to fit persistence properties of the data rather than because economists have a strong belief in these particular stories;
- A large amount of unexplained shocks which are often highly persistent;
- A minimal treatment of banking and financial markets (still true despite current ongoing work);
- Very limited modelling of policy tools or details of national accounts;
- Plenty of evidence that pure Rational Expectations assumption is flawed; and
- Claims that they are based on stable structural parameters and thus immune to the Lucas Critique are silly, and would most likely upset these two:

Robert E. Lucas Jr. Edward C. Prescott

Abstracting from New Keynesian models, DSGE models in general have come under heavy criticism – rather, the herd mentality and group think surrounding the DSGE models in modern macroeconomics has been heavily criticised. An example would be the paper "The Trouble with Macroeconomics" by Paul Romer[78] – once you've read a few DSGE papers (or sat in a conference where these DSGE models are presented), I would highly recommend reading Romer's paper.

Finally, I would like to share a quote from Lawrence J. Christiano, Martin S. Eichenbaum, and Mathias Trabandt in their essay "On DSGE Models" (2018) – which was a response to critics like Paul Romer:

> The enterprise of dynamic stochastic general equilibrium modelling is an organic process that involves the constant interaction of data and theory. Pre-crisis DSGE models had shortcomings that were highlighted by the financial crisis and its aftermath. Substantial progress has occurred since then. We have emphasised the incorporation of financial frictions and heterogeneity into DSGE models. However, we should also mention that other exciting work is being done in this area, like research on deviations from conventional rational expectations. These deviations include k-level thinking, robust control, social

[78] https://paulromer.net/the-trouble-with-macro/WP-Trouble.pdf. See also: https://paulromer.net/trouble-with-macroeconomics-update/

learning, adaptive learning, and relaxing the assumption of common knowledge.

Frankly, we do not know which of these competing approaches will play a prominent role in the next generation of mainstream DSGE models. Will the future generation of DSGE models predict the time and nature of the next crisis? Frankly, we doubt it. As far as we know, there is no sure, time-tested way of foreseeing the future. The proximate cause for the financial crisis was a failure across the economics profession, policymakers, regulators, and financial market professionals to recognise and to react appropriately to the growing size and leverage of the shadow-banking sector. DSGE models are evolving in response to that failure as well as to the treasure trove of micro data available to economists. We don't know where that process will lead. But we do know that DSGE models will remain central to how macroeconomists think about aggregate phenomena and policy. There is simply no credible alternative to policy analysis in a world of competing economic forces operating on different parts of the economy.

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Part III

Optimal Policy

12 Determinacy and Interest Rate Rules in the New Keynesian Model

12.1 Introduction

In the previous section we developed the New Keynesian DSGE model, and explored its strengths and weaknesses. While it isn't perfect, it's the best tool we have when it comes to understanding macroeconomics fluctuations, and to explain the business cycle data that we observe. More importantly, the New Keynesian model allowed us to integrate a role for policy – most notably monetary policy – into a macroeconomic DSGE model framework. In this chapter, and forthcoming chapters, we address the question of how monetary policy should be conducted, using as reference the canonical New Keynesian model we have just developed.

The objectives of optimal monetary policy are first determined, and then issues pertaining to its implementation are addressed. As will soon be demonstrated, when prices are sticky, it is popular to characterise monetary policy with simple interest rate rules instead of exogenous money supply rules. Such roles focus in on the instrument central banks seem to care about (i.e., interest rates, not measures of the money supply), and seem to fit the data reasonably well, and often have good normative properties.

A complicating factor with interest rate rules is that issues of determinacy arise. In general, interest rate rules must react sufficiently strong enough to endogenous variables (like inflation and/or output) in order to guarantee a determinant Rational Expectations equilibrium. By "determinate" we mean "unique." If a rule does not respond aggressively enough to endogenous variables then there may be indeterminacy, which can give rise to non-fundamental "sunspot" equilibria. If there is an indeterminate equilibrium in these models, then that means that there is no unique non-explosive value of current inflation that satisfies the equilibrium conditions of the model given the current state. In a model with no nominal rigidities this just means there is nominal indeterminacy. But if there are nominal rigidities, then nominal indeterminacy also gives rise to real indeterminacy in the sense that there may be non-fundamental fluctuations in real quantities. From a welfare perspective, real indeterminacy is undesirable, so we would like to understand the restrictions on policy rules giving rise to determinacy.

Oh, one more thing: for this part of the course, we'll change notation. Let lower case variables denote log levels, so $x_t = \ln X_t$, while lower case "hatted" variables will be log deviations from steady state, $\hat{x}_t = \ln X_t - \ln \bar{X}$. Unless otherwise stated, i_t, r_t, and π_t will denote net rates. Also, you'll probably see me interchangeably use π_t and $\hat{\pi}_t$: for a zero inflation steady state $\pi_t = \hat{\pi}_t = \pi_t - \bar{\pi}$, where $\bar{\pi} = 0$. So I'll generally specify $\bar{\pi}$ and π_t if we're dealing with the odd case of non-zero steady state inflation.

12.2 The Taylor Rule and Taylor's original intuition

The progenitor of interest rate rules is widely considered to be John Taylor, after whom the "Taylor Rule" is named. His famous paper on the topic was "Discretion Verses Policy Rules in Practice" (1993). In this paper, Taylor documented a policy rule of the following form:

$$i_t = \bar{i} + \phi_\pi(\pi_t - \bar{\pi}) + \phi_x \hat{x}_t,$$

where π_t is inflation, x_t is the gap between actual and potential output, and i_t is the interest rate controlled by a central bank (i.e., the Federal Funds Rate). We say that a rule like this implies that monetary policy "leans against" inflation and output gaps by raising the interest rate when these increase. Taylor argued that values of $\phi_\pi = 1.5$ and $\phi_x = 0.5$ fit the data well. He also argued that the coefficient on inflation needed to be greater than 1. This came to be known as the "Taylor Principle".

Taylor's logic for this parameter restriction was loosely as follows. Total aggregate demand depends on the real interest rate, and inflation depends on aggregate demand. The real interest rate is approximately $r = i - \pi$ by the Fisher Relation.[79] Whenever inflation increases, if $\phi_\pi > 1$, the nom-

[79] Technically, it should be expected future inflation, not current inflation. But, for convenience, let's assume that it is current inflation.

inal interest rate increases by more than inflation. This means real interest rates increase whenever inflation increases. Higher real interest rates depress aggregate demand, which brings inflation down. In contrast, suppose that $\phi_\pi < 1$. This implies that whenever inflation increases the real interest rate declines. This decline in the real rate fuels more inflation, and so inflation can "spiral" out of control.

This is stabilising logic. Implicitly, it sounds like you need a sufficient reaction to inflation to generate a stable root to keep the system from exploding. Though a similar restriction is obtained in a forward-looking DSGE model, such a restriction is not really about stabilising per se. Rather, we need a sufficient response to endogenous variables in a policy rule to impart a sufficient number of unstable roots into the system. This makes the policy functions unique (recall the discussion we had in Section 7). We rule out explosions by assumption.

12.3 Determinacy in a model with flexible prices

Suppose that we have a very simple model. There is no capital, so all output must be consumed. Prices are flexible, meaning that the classical dichotomy holds and there is no effect of nominal variables on real variables. Money demand is implicitly generated via money in the utility function, additively separable from consumption. The demand side of the economy is summarised by the DISE (assuming $\sigma = 1$),

$$\hat{y}_t = \mathbb{E}_t \hat{y}_{t+1} - (\hat{i}_t - \mathbb{E}_t \pi_{t+1}).$$

Suppose the policy rule just reacts to inflation with a random mean zero shock, u_t:

$$\hat{i}_t = \phi_t \pi_t + u_t,$$

and suppose that u_t follows a stationary AR(1) process,

$$u_t = \rho u_{t-1} + e_t, \quad \rho \in (0, 1).$$

To make life as simple as possible, suppose that real output is constant. This means that $\hat{y}_t = \mathbb{E}_t \hat{y}_{t+1} = 0$. The Euler equation then becomes

$$\hat{i}_t = \mathbb{E}_t \pi_{t+1}.$$

If we combine these expressions we get

$$\mathbb{E}_t \pi_{t+1} = \phi_\pi \pi_t + u_t.$$

This is a forward looking difference equation for which there exists many different solutions as a general matter. To get a solution we use the equivalent of a transversality condition, requiring that

$$\lim_{T\to\infty} \mathbb{E}_t \pi_{t+T} = 0.$$

Recall our Blanchard-Kahn conditions. For there to be a unique non-explosive solution, you need the difference equation to be explosive. Basically, this is a system of one forward looking variable, π_t, and one state variable, u_t. The eigenvalue associated with u_t will be ρ, which is stable (since $0 < \rho < 1$). For saddle path stability, we need an unstable eigenvalue with π_t. This eigenvalue is ϕ_π. If $\phi_\pi < 1$, then there is no unique solution – any value of π_t will have expected inflation go to zero in the limit for any u_t. If $\phi_\pi > 1$ there will exist a unique solution, with that solution given by:

$$\pi_t = -\frac{u_t}{\phi_\pi - \rho}.$$

We went over this in Section 5.2.6, but just as a reminder:

$$\begin{aligned}
\mathbb{E}_t \pi_{t+1} &= \phi_\pi \pi_t + u_t \\
\Leftrightarrow \pi_t &= \phi_\pi^{-1} \mathbb{E}_t \pi_{t+1} - \phi_\pi^{-1} u_t,
\end{aligned}$$

and so the solution takes the form

$$\pi_t = -\frac{1}{\phi_\pi}\left[\sum_{k=0}^{\infty}(\phi_\pi^{-1})^k \mathbb{E}_t u_{t+k}\right].$$

We have

$$\mathbb{E}_t u_{t+k} = \rho^k u_t,$$

so we can write

$$\pi_t = -\frac{1}{\phi_\pi}\left[\sum_{k=0}^{n}\left(\frac{\rho}{\phi_\pi}\right)^k u_t\right],$$

and we require that $|\rho/\phi_\pi| < 1$, so we can write

$$\begin{aligned}\sum_{k=0}^{n}\left(\frac{\rho}{\phi_\pi}\right)^k &= \frac{1}{1-\frac{\rho}{\phi_\pi}}, \\ \implies \pi_t &= -\left[\frac{1}{1-\frac{\rho}{\phi_\pi}}\right]\frac{1}{\phi_\pi}u_t \\ &= -\frac{u_t}{\phi_\pi - \rho}.\end{aligned}$$

12.4 Determinacy in a simple New Keynesian model

Now, consider a standard New Keynesian model. There is the NKPC, the DISE, and an exogenous process for the flexible price level of output (recall that there are multiple ways to write down the equilibrium conditions). The equations of the model are:

$$\begin{aligned}\pi_t &= \beta\mathbb{E}_t\pi_{t+1} + \kappa(\hat{y}_t - \hat{y}_t^f), \\ \hat{y}_t &= \mathbb{E}_t\hat{y}_{t+1} - \hat{i}_t + \mathbb{E}_t\pi_{t+1}, \\ \hat{y}_t^f &= \rho\hat{y}_{t-1}^f + \epsilon_t,\end{aligned}$$

where the slope coefficient of the NKPC is $\kappa = \frac{(1-\phi)(1-\phi\beta)}{\phi}(\sigma+\eta)$, the coefficient of relative risk aversion is $\sigma = 1$, η is the inverse Frisch labour supply elasticity, and the AR coefficient for flex-price output is $\rho \in (0,1)$. Suppose that the nominal interest rate obeys a simple Taylor rule of the form:

$$\hat{i}_t = \phi_\pi \pi_t + \phi_x(\hat{y}_t - \hat{y}_t^f).$$

We want to know the following: what restrictions on ϕ_π and ϕ_x must be made in order to ensure a determinate rational expectations equilibrium? To see this, eliminate i_t and form a three variable system (the canonical New Keynesian model). After substitution, the DISE becomes:

$$\hat{y}_t = \mathbb{E}_t \hat{y}_{t+1} - \phi_\pi \pi_t - \phi_x \hat{y}_t + \phi_x \hat{y}_t^f + \mathbb{E}_t \pi_{t+1}.$$

We can form a vector system:

$$\mathbb{E}_t \begin{bmatrix} \pi_{t+1} \\ \hat{y}_{t+1} \\ \hat{y}_{t+1}^f \end{bmatrix} = \begin{bmatrix} \frac{1}{\beta} & -\frac{\kappa}{\beta} & \frac{\kappa}{\beta} \\ \phi_\pi - \frac{1}{\beta} & 1 + \phi_x + \frac{\kappa}{\beta} & -\frac{\gamma}{\beta} - \phi_x \\ 0 & 0 & \rho \end{bmatrix} \begin{bmatrix} \pi_t \\ \hat{y}_t \\ \hat{y}_t^f \end{bmatrix}.$$

We need to find the eigenvalues of this system. One of the eigenvalues is clearly ρ, which is less than 1 in absolute value and hence stable. To find the other two eigenvalues, we just need to find the eigenvalues of the upper 2×2 block of the coefficient matrix. That is, we need to find the λ which makes:

$$\det \begin{bmatrix} \frac{1}{\beta} - \lambda & -\frac{\kappa}{\beta} \\ \phi_\pi - \frac{1}{\beta} & 1 + \phi_x + \frac{\kappa}{\beta} - \lambda \end{bmatrix} = 0.$$

The determinant of a 2×2 matrix is just the different of the product of the diagonals:

$$\left(\frac{1}{\beta} - \lambda\right)\left(1 + \phi_x + \frac{\kappa}{\beta} - \lambda\right) + \frac{\kappa}{\beta}\left(\phi_\pi - \frac{1}{\beta}\right) = 0.$$

Now, two useful facts about eigenvalues and determinants. First, the product of the eigenvalues is just equal to the determinant of the matrix. Second, the sum of the eigenvalues is equal to the trace of the matrix. The determinant and trace of the upper 2×2 matrix are:

$$\lambda_1 \lambda_2 = \det \begin{bmatrix} \frac{1}{\beta} & -\frac{\kappa}{\beta} \\ \phi_\pi - \frac{1}{\beta} & 1 + \phi_x + \frac{\kappa}{\beta} \end{bmatrix} = \frac{1}{\beta} + \frac{\phi_x}{\beta} + \frac{\kappa \phi_\pi}{\beta},$$
$$\lambda_1 + \lambda_2 = \operatorname{trace} \begin{bmatrix} \frac{1}{\beta} & -\frac{\kappa}{\beta} \\ \phi_\pi - \frac{1}{\beta} & 1 + \phi_x + \frac{\kappa}{\beta} \end{bmatrix} = \frac{1}{\beta} + 1 + \phi_x + \frac{\kappa}{\beta}.$$

Since both the determinant and trace must be positive given standard assumptions on parameter values, we know that both eigenvalues must be positive as well.

For a unique equilibrium, we need both of these eigenvalues to be explosive (we already have one stable root for the single state variable, and we have two jump variables). Since we know from above that both these eigenvalues must be positive, then (ignoring complex roots), the necessary condition for stability is that:

$$(\lambda_1 - 1)(\lambda_2 - 1) > 0.$$

Multiply this out:

$$\begin{aligned} \lambda_1 \lambda_2 - (\lambda_1 + \lambda_2) + 1 &> 0 \\ \lambda_1 \lambda_2 - (\lambda_1 + \lambda_2) &> -1. \end{aligned}$$

Plug in our expressions from above and simplify:

$$\begin{aligned} \frac{1}{\beta} + \frac{\phi_x}{\beta} + \frac{\kappa \phi_\pi}{\beta} - \left(\frac{1}{\beta} + 1 + \phi_x + \frac{\kappa}{\beta} \right) &> -1 \\ \phi_x \left(\frac{1}{\beta} - 1 \right) + \frac{\kappa \phi_\pi}{\beta} - \frac{\kappa}{\beta} &> 0 \\ \phi_x (1 - \beta) + \kappa \phi_\pi - \kappa &> 0. \end{aligned}$$

The last line follows from multiplying both sides by β. Now divide both sides by κ and simplify:

$$\phi_x \frac{1 - \beta}{\kappa} + \phi_\pi > 1.$$

This is the condition that must be satisfied for there to exist a determinate equilibrium. We can see that $\phi_\pi > 1$ is slightly too strong of a restriction – determinacy also depends on the response to the output gap in the policy rule. But if $\beta \approx 1$, then unless κ is very small, the determinacy condition is still roughly $\phi_\pi > 1$.

You can trick up the model along a number of dimensions but something like this basic condition usually emerges. Recall that this condition is needed to pin down a unique, non-explosive equilibrium, which is (quite) different to Taylor's original intuition.

12.4.1 Alternative derivation using state-space representation

Consider the Euler/IS equation and NKPC:[80]

$$\begin{aligned} \hat{x}_t &= \mathbb{E}_t \hat{x}_{t+1} - \sigma^{-1}(\hat{i}_t - \mathbb{E}_t \pi_{t+1} - \hat{r}_t^f), \\ \pi_t &= \beta \mathbb{E}_t \pi_{t+1} + \kappa \hat{x}_t + \epsilon_t, \end{aligned}$$

where ϵ_t is a "cost-push" shock, i_t is the short term nominal interest rate, and $\hat{r}_t^f$ is the Wicksellian natural rate of interest – the rate of interest that is associated with a stable price level. $\hat{r}_t^f$ can be defined as:

$$\hat{r}_t^f = \sigma(\mathbb{E}_t \hat{y}_{t+1}^f - \hat{y}_t^f),$$

where $\hat{y}_t^f$ is period t output when prices are fully flexible, which would then imply that what drives $\hat{r}_t^f$ are technology shocks and preferences – these are both independent of monetary policy.

Furthermore, if we combine the NKPC and the DISE, we can write the NKPC as:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa \mathbb{E}_t \hat{x}_{t+1} - \frac{\kappa}{\sigma}(\hat{i}_t - \mathbb{E}_t \pi_{t+1} - \hat{r}_t^f) + \epsilon_t,$$

then gathering this NKPC and the original IS equation, we can write the two equations in matrix form

[80] As you can see, the ordering of the NKPC and DISE does not matter.

as:

$$\begin{bmatrix} \hat{x}_t \\ \pi_t \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{\sigma} \\ \kappa & \beta + \frac{\kappa}{\sigma} \end{bmatrix} \begin{bmatrix} \mathbb{E}_t \hat{x}_{t+1} \\ \mathbb{E}_t \hat{\pi}_{t+1} \end{bmatrix} + \begin{bmatrix} \frac{1}{\sigma}(\hat{r}_t^f - \hat{i}_t) \\ \frac{\kappa}{\sigma}(\hat{r}_t^f - \hat{i}_t) + \epsilon_t \end{bmatrix}.$$

From Blanchard and Kahn (1980), in order for DSGE models of the form $\mathbf{X}_t = \mathbf{A}\mathbb{E}_t\mathbf{X}_{t+1} + \mathbf{B}\mathbf{V}_t$ to have a stable unique solution, we need all the eigenvalues of $\mathbf{A}$ to be less than one.[81] In our case we have:

$$\mathbf{A} = \begin{bmatrix} 1 & \frac{1}{\sigma} \\ \kappa & \beta + \frac{\kappa}{\sigma} \end{bmatrix}.$$

The eigenvalues satisfy:

$$\begin{aligned} P(\lambda) &= (1-\lambda)\left(\beta + \frac{\kappa}{\sigma} - \lambda\right) - \frac{\kappa}{\sigma} = 0 \\ \Leftrightarrow P(\lambda) &= \lambda^2 - \left(1 + \beta + \frac{\kappa}{\sigma}\right)\lambda + \beta = 0, \end{aligned}$$

which means that $P(\lambda)$ is a convex polynomial – we can show that $P(0) = \beta > 0$, $P(1) = -\frac{\kappa}{\sigma} < 0$, and that $P(\lambda) > 0$ as λ rises above one, suggesting that one eigenvalue is between 0 and 1 (stable), while the other eigenvalue is greater than one (explosive). This then means that with just these two equations, the model we just specified has no unique stable solution.

To circumvent this, we can specify that monetary policy follows a rule designed to produce a unique stable equilibrium. Adding a third equation – a monetary policy rule – to the NKPC and DISE will yield the canonical New Keynesian model.

Consider a simple Taylor-type Rule,

$$\hat{i}_t = \hat{r}_t^f + \phi_\pi \pi_t + \phi_y \hat{x}_t, \tag{525}$$

[81] There seems to be a lot of confusion over the Blanchard-Kahn conditions. Here, the Blanchard-Kahn conditions require that for the system of difference equations,

$$\mathbf{X}_t = \mathbf{A}\mathbb{E}_t\mathbf{X}_{t+1} + \mathbf{B}\mathbf{V}_t,$$

we need the number of eigenvalues of $\mathbf{A}$ **inside** unit circle to be the same as the number of jump variables. In Section 7.2, we said that the Blanchard-Kahn conditions were that for the system of equations,

$$\mathbb{E}_t\mathbf{X}_{t+1} = \mathbf{C}\mathbf{X}_t,$$

we needed the number of eigenvalues of $\mathbf{C}$ **outside** unit circle to be the same as the number of jump variables.

which can be substituted into the DISE to get

$$\hat{x}_t = \mathbb{E}_t \hat{x}_{t+1} + \sigma^{-1}\mathbb{E}_t \pi_{t+1} - \sigma^{-1}\phi_\pi \pi_t - \sigma^{-1}\phi_y \hat{x}_t.$$

This equation can be combined with the NKPC,[82] to yield a system of first-order difference equations of the form

$$\mathbf{X}_t = \mathbf{A}\mathbb{E}_t \mathbf{X}_{t+1} + \mathbf{B}\mathbf{V}_t,$$

$$\mathbf{X}_t = \begin{bmatrix} x_t \\ \pi_t \end{bmatrix}, \ \mathbf{V}_t = \begin{bmatrix} 0 \\ \epsilon_t \end{bmatrix},$$

and:

$$\mathbf{A} = \frac{1}{\sigma + \phi_y + \kappa\phi_\pi} \begin{bmatrix} \sigma & 1 - \beta\phi_\pi \\ \kappa\sigma & \kappa + \beta\left(\sigma + \phi_y\right) \end{bmatrix},$$

$$\mathbf{B} = \frac{1}{\sigma + \phi_y + \kappa\phi_\pi} \begin{bmatrix} \sigma & -\phi_\pi \\ \kappa\sigma & \sigma + \phi_y \end{bmatrix}.$$

These matrices constitute a system of first order stochastic difference equations, and given certain conditions, can be iterated forward and solved:

$$\mathbf{X}_t = \sum_{i=0}^{\infty} \mathbf{A}^i \mathbf{B} \mathbb{E}_t \mathbf{V}_{t+i}.$$

But more importantly, the canonical New Keynesian model will have a unique stable equilibrium if and only if both eigenvalues of **A** are inside unit circle (less than one in absolute value). It can be shown that both eigenvalues of **A** are inside unit circle if and only if:

$$\phi_\pi + \frac{(1-\beta)\phi_y}{\kappa} > 1,$$

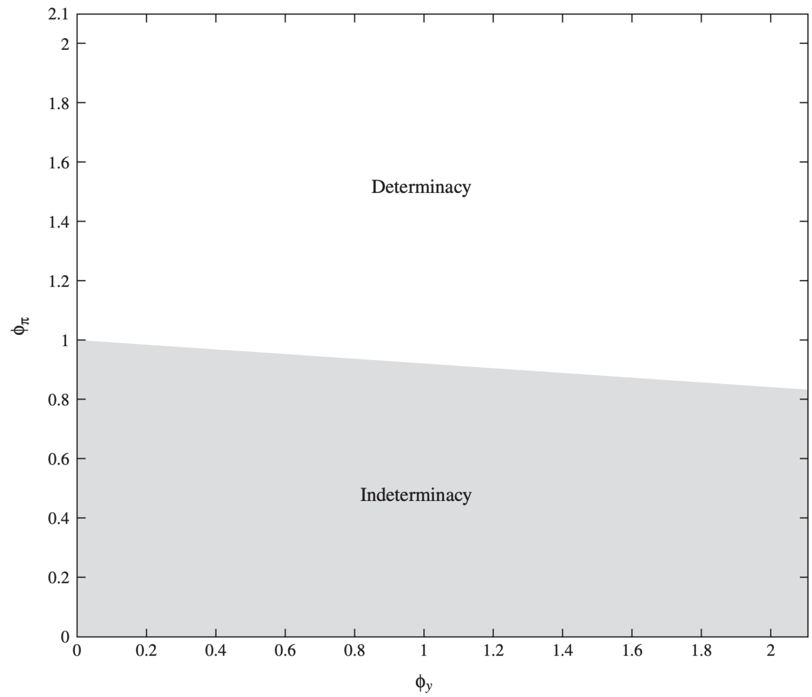
[82] I do a full derivation with a similar model in Section 13.2.

and since the household's discount factor, $\beta \approx 1$, this stability condition boils down to approximately:

$$\phi_\pi > 1.$$

In other words, nominal interest rates must rise by more than inflation, so that the real interest rate increases in response to an increase in inflation. This is called the "Taylor Principle".

Figure 88: Determinacy and Indeterminacy Regions for a Contemporaneous Interest Rate Rule



Source: Galí (2015) for a Taylor-type rule like (525).

12.5 Comments and key readings

This chapter was quite self contained. There really isn't much to add other than to suggest reading the texts from the usual suspects: Galí (2015), McCandless (2008), Walsh (2010), Romer (2012), and Woodford (2003). Actually, of those books, chapter 4 of Galí (2015) gives a good treatment of the notion of the flexible price equilibrium, determinacy, and optimal policy. Seminal papers related to this chapter are Taylor (1993), which introduced the Taylor Rule, and Clarida et al. (2000). The Clarida, Galí, and Gertler paper estimates alternative versions of the Taylor Rule, and examine its (in)stability over the postwar period. They also find that the Taylor Rule was violated in the years before the Volcker era. Orphanides (2003) argues that the bulk of the deviations from the baseline Taylor Rule observed in the pre-Volcker era may have been the result of large biases in real-time measures of the output gap.

One interesting paper to read on the topic of the New Keynesian model not producing a unique stable solution is "Do Higher Interest Rates Raise or Lower Inflation?" by John Cochrane. The New Keynesian model, in general, does not have a unique stable solution. The model turns out to have multiple equilibria and there is nothing to determine which of the equilibria gets chosen. How to deal with this? One way is to accept that there are multiple equilibria and to analyse the impact of interest rate changes on output and inflation across a range of different possible equilibria. The Cochrane paper does this and reaches the conclusion (surprising to some) that higher interest rates lead to higher inflation in the New Keynesian model. This has lead to a debate about the so-called neo-Fisherite predictions of the New Keynesian model. This was quite popular a few years ago, especially in Japan.

But, the conventional New Keynesian literature went with the alternative approach: by specifying a monetary policy rule which abides by the Taylor Principle to close the model.

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13 Optimal Monetary Policy in the New Keynesian Model

13.1 Introduction

How do we think about what is "optimal" for a central bank to do? Clearly, central banks don't like inflation, and they would also like to keep output on a steady path close to potential. But what's the best way to formalise this? For a long time, economists have formulated central banks as behaving in a way that minimises a "loss function", something like:

$$\mathbb{L}_t = \frac{1}{2}\sum_{s=0}^{\infty} \beta^s \mathbb{E}_t(\pi_{t+s}^2 + \vartheta x_{t+s}^2),$$

where, $x_t = y_t - y_t^e$ is the welfare-relevant output gap – yes, we will define this soon – and ϑ indicates the weight put on output stabilisation relative to inflation stabilisation. Economists like quadratic loss functions: When you differentiate things to the power of 2, they give you equations with things to the power of one (i.e., linear relationships). Traditionally, though, the quadratic loss function was purely ad-hoc.

So in this chapter we will try and provide some theoretical reasoning for these loss functions, and we will delve deeper into optimal monetary policy in the canonical New Keynesian model. Also, since initially writing this section, I've decided to follow Galí's treatment of optimal policy, as once you get over some of the notational quirks (and typos), his exposition is quite clear. But in order to align with Gali, we're going to have to rewrite our three-equation New Keynesian model in the format of Galí (2015). However, in order to avoid complete confusion and disaster, I am going to be using the same symbols that we've used thus far for parameters.

13.2 Re-writing the basic model

First things first: Galí defines an optimal monetary policy which sets

$$y_t = y_t^f$$
$$\Leftrightarrow y_t - y_t^f = 0, \quad \forall t.$$

In words: an optimal monetary policy ensures that the output gap (here defined in log levels) is closed. What does this mean for inflation? Well, let's look at our three-equation canonical New Keynesian model: Equations (473)-(476), but using our new notation for this Part[83] and assuming that $\rho_i = 0$, we have

$$\begin{aligned}
\hat{x}_t &= \mathbb{E}_t \hat{x}_{t+1} - \frac{1}{\sigma}\left(\hat{i}_t - \mathbb{E}_t \pi_{t+1} - \hat{r}_t^f\right), \\
\pi_t &= \beta \mathbb{E}_t \pi_{t+1} + \kappa \hat{x}_t, \\
\hat{i}_t &= \phi_\pi \pi_t + \phi_y \hat{x}_t + \epsilon_{i,t}, \\
\hat{r}_t^f &= \rho_a \hat{r}_{t-1}^f + \sigma(\rho_a - 1)\omega \epsilon_{a,t}.
\end{aligned}$$

Remember that we assume that $\bar{\pi} = 0$ and that $\bar{y} = \bar{y}^f$. Thus, we can write out our canonical New Keynesian model as

$$\tilde{y}_t = \mathbb{E}_t \tilde{y}_{t+1} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t \pi_{t+1} - r_t^f\right), \tag{526}$$

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa \tilde{y}_t, \tag{527}$$

$$i_t = \rho + \phi_\pi \pi_t + \phi_y \tilde{y}_t + \phi_y \hat{y}_t^f + \epsilon_{i,t}, \tag{528}$$

$$r_t^f = \rho + \rho_a r_{t-1}^f + \sigma(\rho_a - 1)\omega \epsilon_{a,t},$$

where $\tilde{y}_t = y_t - y_t^f$, and $\bar{i} = \bar{r}^f = \rho$ because we have $\bar{\pi} = 0$. The key point is that we want to get accustomed to Galí's notation of $\tilde{y}_t$ and x_t (as well as their hatted versions too), and take careful not of what is a steady state variable, a log level, and a log deviation from steady state.

Now, because we want to try and establish an optimal policy for monetary policy, let's **<u>forget about the Taylor Rule for now</u>**, and let's consider what happens when we assume that $y_t = y_t^f$, $\forall t$. The NKPC then implies that

$$\pi_t = 0, \quad \forall t,$$

[83]Pay close attention to what is a log deviation from steady state, and what is in log levels.

that is, inflation is kept constant at a zero rate (or, equivalently, the aggregate price level is fully stabilised). Roll this forward and substitute this into the DISE, and we get

$$i_t = r_t^f, \tag{529}$$

that is, for all t, the equilibrium nominal interest rate, i_t, which equals the real rate of interest, ρ, when inflation is zero and the output gap is closed, is equal to the "flexible price real interest rate" or the Wicksellian natural rate of interest. Now, we know what's going to happen if we adopt (529) as a monetary policy – we're going to get indeterminacy, which we showed previously. To see this, substitute (529) into (526), and the new $\tilde{y}_t$ expression into the NKPC, and write the system out:

$$\begin{aligned}
\tilde{y}_t &= \mathbb{E}_t \tilde{y}_{t+1} + \frac{1}{\sigma}\mathbb{E}_t \pi_{t+1}, \\
\pi_t &= \kappa \mathbb{E}_t \tilde{y}_{t+1} + \left(\beta + \frac{\kappa}{\sigma}\right)\mathbb{E}_t \pi_{t+1}, \\
\Leftrightarrow \begin{bmatrix} \tilde{y}_t \\ \pi_t \end{bmatrix} &= \begin{bmatrix} 1 & \frac{1}{\sigma} \\ \kappa & \beta + \frac{\kappa}{\sigma} \end{bmatrix} \mathbb{E}_t \begin{bmatrix} \tilde{y}_{t+1} \\ \pi_{t+1} \end{bmatrix},
\end{aligned}$$

or more compactly,

$$\mathbf{X}_t = \mathbf{A}\mathbb{E}_t \mathbf{X}_{t+1}.$$

The Blanchard-Kahn conditions for this system are that the number of eigenvalues of $\mathbf{A}$ inside unit circle be equal to the number of jump variables in our system. But as we just saw in the previous section, for the two jump variables in our system, we only have one eigenvalue of $\mathbf{A}$ inside unit circle. So while $\tilde{y}_t = \pi_t = 0$ is a solution, it is not unique, and we will have additional multiple equilibria to this system. Simply put: the Blanchard-Kahn conditions have not been met. We're being repetitious because it's worth stressing that the Taylor Rule can really be thought of as a sort of "patch" to the DISE and NKPC to bring about determinacy.

Just to see this again with our slightly new, adjusted notation, consider Equation (525), which we rewrote as in (528). Again, substitute (528) into (526),

$$
\begin{aligned}
\tilde{y}_t &= \mathbb{E}_t\tilde{y}_{t+1} - \frac{1}{\sigma}\left(\rho + \phi_\pi\pi_t + \phi_y\tilde{y}_t + \phi_y\hat{y}_t^f + \epsilon_{i,t} - \mathbb{E}_t\pi_{t+1} - r_t^f\right) \\
&= \frac{\sigma}{\sigma+\phi_y}\mathbb{E}_t\tilde{y}_{t+1} + \frac{1}{\sigma+\phi_y}\mathbb{E}_t\pi_{t+1} - \frac{1}{\sigma+\phi_y}\left(\rho + \phi_\pi\pi_t + \phi_y\hat{y}_t^f + \epsilon_{i,t} - r_t^f\right),
\end{aligned}
$$

and then the new $\tilde{y}_t$ expression into the NKPC to get:

$$
\begin{aligned}
\pi_t &= \beta\mathbb{E}_t\pi_{t+1} + \kappa\left[\frac{\sigma}{\sigma+\phi_y}\mathbb{E}_t\tilde{y}_{t+1} + \frac{1}{\sigma+\phi_y}\mathbb{E}_t\pi_{t+1} - \frac{1}{\sigma+\phi_y}\left(\rho + \phi_\pi\pi_t + \phi_y\hat{y}_t^f + \epsilon_{i,t} - r_t^f\right)\right] \\
\pi_t\left(1 + \frac{\kappa\phi_\pi}{\sigma+\phi_y}\right) &= \left[\beta + \frac{\kappa}{\sigma+\phi_y}\right]\mathbb{E}_t\pi_{t+1} + \kappa\left[\frac{\sigma}{\sigma+\phi_y}\mathbb{E}_t\tilde{y}_{t+1} - \frac{1}{\sigma+\phi_y}\left(\rho + \phi_y\hat{y}_t^f + \epsilon_{i,t} - r_t^f\right)\right] \\
\pi_t\left(\frac{\sigma+\phi_y+\kappa\phi_\pi}{\sigma+\phi_y}\right) &= \left(\frac{\beta(\sigma+\phi_y)+\kappa}{\sigma+\phi_y}\right)\mathbb{E}_t\pi_{t+1} + \frac{\kappa\sigma}{\sigma+\phi_y}\mathbb{E}_t\tilde{y}_{t+1} + \frac{\kappa}{\sigma+\phi_y}\left(\hat{r}_t^f - \phi_y\hat{y}_t^f - \epsilon_{i,t}\right) \\
\pi_t &= \frac{\beta(\sigma+\phi_y)+\kappa}{\sigma+\phi_y+\kappa\phi_\pi}\mathbb{E}_t\pi_{t+1} + \frac{\kappa\sigma}{\sigma+\phi_y+\kappa\phi_\pi}\mathbb{E}_t\tilde{y}_{t+1} + \frac{\kappa}{\sigma+\phi_y+\kappa\phi_\pi}\left(\hat{r}_t^f - \phi_y\hat{y}_t^f - \epsilon_{i,t}\right) \\
\pi_t &= \Omega\left[\beta(\sigma+\phi_y)+\kappa\right]\mathbb{E}_t\pi_{t+1} + \Omega\kappa\sigma\mathbb{E}_t\tilde{y}_{t+1} + \Omega\kappa\left(\hat{r}_t^f - \phi_y\hat{y}_t^f - \epsilon_{i,t}\right),
\end{aligned}
$$

where $\Omega = 1/(\sigma + \phi_y + \kappa\phi_\pi)$. Now, put this expression for π_t back into the DISE (with the Taylor Rule), to get the following mess:

$$
\begin{aligned}
\tilde{y}_t &= \mathbb{E}_t\tilde{y}_{t+1} - \frac{1}{\sigma}\left(\rho + \phi_\pi\pi_t + \phi_y\tilde{y}_t + \phi_y\hat{y}_t^f + \epsilon_{i,t} - \mathbb{E}_t\pi_{t+1} - r_t^f\right) \\
\sigma\tilde{y}_t &= \sigma\mathbb{E}\tilde{y}_{t+1} + \left(\hat{r}_t^f - \phi_\pi\pi_t - \phi_y\tilde{y}_t - \phi_y\hat{y}_t^f - \epsilon_{i,t} + \mathbb{E}_t\pi_{t+1}\right) \\
\sigma\tilde{y}_t + \phi_y\tilde{y}_t &= \sigma\mathbb{E}\tilde{y}_{t+1} + \mathbb{E}_t\pi_{t+1} - \phi_\pi\pi_t + \left(\hat{r}_t^f - \phi_y\hat{y}_t^f - \epsilon_{i,t}\right) \\
\sigma\tilde{y}_t + \phi_y\tilde{y}_t &= \sigma\mathbb{E}\tilde{y}_{t+1} + \mathbb{E}_t\pi_{t+1} - \phi_\pi\Omega\left[\beta(\sigma+\phi_y)+\kappa\right]\mathbb{E}_t\pi_{t+1} - \phi_\pi\Omega\kappa\sigma\mathbb{E}_t\tilde{y}_{t+1} \\
&\quad - \phi_\pi\Omega\kappa\left(\hat{r}_t^f - \phi_y\hat{y}_t^f - \epsilon_{i,t}\right) + \left(\hat{r}_t^f - \phi_y\hat{y}_t^f - \epsilon_{i,t}\right) \\
\sigma\tilde{y}_t + \phi_y\tilde{y}_t &= \sigma\left(1 - \phi_\pi\Omega\kappa\right)\mathbb{E}_t\tilde{y}_{t+1} + \left(1 - \phi_\pi\Omega\left[\beta(\sigma+\phi_y)+\kappa\right]\right)\mathbb{E}_t\pi_{t+1} + \left(1 - \phi_\pi\Omega\kappa\right)\left(\hat{r}_t^f - \phi_y\hat{y}_t^f - \epsilon_{i,t}\right) \\
\tilde{y}_t &= \frac{\sigma\left(\frac{\sigma+\phi_y+\kappa\phi_\pi-\kappa\phi_\pi}{\sigma+\phi_y+\kappa\phi_\pi}\right)}{\sigma+\phi_y}\mathbb{E}_t\tilde{y}_{t+1} + \frac{\frac{\sigma+\phi_y+\kappa\phi_\pi-\phi_\pi\beta\sigma-\phi_\pi\beta\phi_y-\phi_\pi\kappa}{\sigma+\phi_y+\kappa\phi_\pi}}{\sigma+\phi_y}\mathbb{E}_t\pi_{t+1} + \frac{\left(\frac{\sigma+\phi_y+\kappa\phi_\pi-\phi_\pi\kappa}{\sigma+\phi_y+\kappa\phi_\pi}\right)}{\sigma+\phi_y}\left(\hat{r}_t^f - \phi_y\hat{y}_t^f - \epsilon_{i,t}\right) \\
\tilde{y}_t &= \sigma\Omega\mathbb{E}_t\tilde{y}_{t+1} + \Omega(1-\beta\phi_\pi)\mathbb{E}_t\pi_{t+1} + \Omega\left(\hat{r}_t^f - \phi_y\hat{y}_t^f - \epsilon_{i,t}\right),
\end{aligned}
$$

which matches up with the solution in Bullard and Mitra (2002) and Galí's textbook. So now we have the following system of equations:

$$\begin{aligned}
\tilde{y}_t &= \sigma\Omega\mathbb{E}_t\tilde{y}_{t+1} + \Omega(1-\beta\phi_\pi)\mathbb{E}_t\pi_{t+1} + \Omega\left(\hat{r}_t^f - \phi_y\hat{y}_t^f - \epsilon_{i,t}\right), \\
\pi_t &= \Omega\kappa\sigma\mathbb{E}_t\tilde{y}_{t+1} + \Omega\left[\beta(\sigma+\phi_y)+\kappa\right]\mathbb{E}_t\pi_{t+1} + \Omega\kappa\left(\hat{r}_t^f - \phi_y\hat{y}_t^f - \epsilon_{i,t}\right), \\
\Leftrightarrow \begin{bmatrix}\tilde{y}_t \\ \pi_t\end{bmatrix} &= \Omega\begin{bmatrix}\sigma & 1-\beta\phi \\ \kappa\sigma & \beta(\sigma+\phi)+\kappa\end{bmatrix}\mathbb{E}_t\begin{bmatrix}\tilde{y}_{t+1} \\ \pi_{t+1}\end{bmatrix} + \Omega\begin{bmatrix}1 \\ \kappa\end{bmatrix}\left(\hat{r}_t^f - \phi_y\hat{y}_t^f - \epsilon_{i,t}\right),
\end{aligned}$$

or, more compactly as

$$\mathbf{X}_t = \mathbf{A}\mathbb{E}_t\mathbf{X}_{t+1} + \mathbf{B}\mathbf{V}_t.$$

This was basically what we had in Section 12.4.1, except for a few slight differences for the $\mathbf{B}$ and $\mathbf{V}_t$ matrices (because we have assumed different shocks). But all our logic which applied to determinacy requirements applies here to the $\mathbf{A}$ matrix. In fact, the two matrices are identical.

One more thing, we should cover in order to bolt down our new notation. Write the NKPC and DISE as the following:

$$\begin{aligned}
\pi_t &= \beta\mathbb{E}_t\pi_{t+1} + \kappa\tilde{y}_t, \\
\tilde{y}_t &= \mathbb{E}_t\tilde{y}_{t+1} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t\pi_{t+1} - r_t^f\right).
\end{aligned}$$

As we know by now, the Wicksellian natural rate of interest is the rate that would be obtained if prices were fully flexible. We can solve for this by looking at the DISE assuming that $y_t = \mathbb{E}_t y_{t+1} = y_t^f$:

$$\begin{aligned}
0 &= -\sigma\left(y_t^f - \mathbb{E}_t y_{t+1}^f\right) + i_t - \mathbb{E}_t\pi_{t+1} - r_t^f \\
r_t^f &= \rho - \sigma\left(y_t^f - \mathbb{E}_t y_{t+1}^f\right).
\end{aligned}$$

Now, we can use a couple tricks, similar to how we derived (468) when we looked at the New Keynesian model. Note that we have

$$y_t^f = \psi_y + \omega a_t,$$

where $\psi_y = \frac{1}{\sigma+\eta} \log\left(\frac{1}{\psi}\frac{\epsilon-1}{\epsilon}\right)$ and $\omega = \frac{1+\eta}{\sigma+\eta}$, and we can manipulate the fact that a_t follows a AR(1) process:

$$\begin{aligned}
y_{t+1}^f &= \psi_y + \omega a_{t+1} \\
a_{t+1} &= \frac{y_{t+1}^f - \psi_y}{\omega} \\
\implies \rho_a a_t + \epsilon_{a,t} &= \frac{y_{t+1}^f - \psi_y}{\omega} \\
y_{t+1}^f &= \omega \rho_a a_t + \omega \epsilon_{a,t} + \psi_y.
\end{aligned}$$

So we do some substitutions and write

$$\begin{aligned}
r_t^f &= \rho - \sigma\left(\psi_y + \omega a_t - \mathbb{E}_t\left[\omega\rho_a a_t + \omega\epsilon_{a,t} + \psi_y\right]\right) \\
&= \rho - \sigma\left(\omega a_t - \omega\rho_a a_t\right) \\
&= \rho - \sigma(1-\rho_a)\omega a_t.
\end{aligned}$$

We can then summarise the main equations of the model as:

$$\begin{aligned}
\pi_t &= \beta \mathbb{E}_t \pi_{t+1} + \kappa \tilde{y}_t, \\
\tilde{y}_t &= \mathbb{E}_t \tilde{y}_{t+1} - (i_t - \mathbb{E}_t \pi_{t+1} - r_t^f), \\
r_t^f &= \rho - \sigma(1-\rho_a)\omega a_t.
\end{aligned}$$

In the background, there is also the money demand relationship and a Fisher relationship. For now, we can think about the central bank effectively being able to choose i_t, given an implied path for $\mathbb{E}_t\pi_{t+1}$. Given that, as well as r_t^f (which is exogenously given), $\tilde{y}_t$ and π_t will be determined.

Now, onto optimal monetary policy!

13.3 Distortions and welfare in the New Keynesian model

There are two welfare-reducing distortions in the NK model, one of which is essentially "long-run" and the other which is "short-run". The "long-run" distortion is that the flexible price level of output will be lower than what would be obtained in the first best allocation of the economy. This is because in the flexible price version of the model, firms will set prices equal to a markup over marginal cost due to monopolistic competition. Hence there will be too little employment and output. You can think of this as the lost output relative to an RBC model – and hence why studying the RBC model was so important. The "short-run" distortion is due to price stickiness, and leads to non-optimal fluctuations in relative prices.

13.3.1 Distortions due to monopolistic competition

Recall that each intermediate firm perceives the demand for its differentiated product to be imperfectly elastic. Endowing the firm with some market power leads to pricing above marginal cost. Recall that the price for intermediate good j can be written as

$$P_t(j) = \frac{\epsilon}{\epsilon - 1} \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s u'(C_{t+s}) \phi^s mc_{t+s} P_{t+s}^{\epsilon} Y_{t+s}}{\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s u'(C_{t+s}) \phi^s P_{t+s}^{\epsilon-1} Y_t},$$

and remember that none of the variables on the RHS of this equation depend on j. This means that any firm that is able to update their prices will update their price to the same reset price which we denoted by $P_t^{\#}$. We can write $P_t^{\#}$ compactly as

$$P_t^{\#} = \mathcal{M} \frac{X_{1,t}}{X_{2,t}},$$

where $\mathcal{M} = \frac{\epsilon}{\epsilon-1} > 1$ is the [gross] markup charged by the monopolistically competitive firms, and the auxiliary variables $X_{1,t}$ and $X_{2,t}$ are:

$$X_{1,t} = u'(C_t) mc_t P_t^{\epsilon} Y_t + \phi\beta \mathbb{E}_t X_{1,t+1},$$

$$X_{2,t} = u'(C_t) P_t^{\epsilon-1} Y_t + \phi\beta \mathbb{E}_t X_{2,t+1}.$$

Now, to better isolate the distortions caused by monopolistic competition, let's assume that all prices are flexible ($\phi = 0$). When all firms are able to freely adjust their prices, their ideal reset price becomes:

$$P_t^{\#} = \mathcal{M} mc_t P_t$$
$$\Leftrightarrow P_t = \mathcal{M} \varphi_t$$

where φ_t denoted the [nominal] marginal cost for an intermediate firm and we can simply denote the ideal reset price as P_t. Furthermore, recall that in the simple New Keynesian model without capital, φ_t was equal to the nominal wage, W_t, over the marginal product of labour, A_t, so we can write:

$$P_t = \mathcal{M} \frac{W_t}{A_t}.$$

Then, recall from the household's optimality conditions that:

$$-\frac{U_{n,t}}{U_{c,t}} = \frac{W_t}{P_t} = \frac{A_t}{\mathcal{M}} < A_t,$$

recalling that A_t is the marginal product of labour for the intermediate firm. Hence, in equilibrium and in the presence of monopolistic competition, the presence of a markup distortion leads to an inefficiently low level of employment and output.

The inefficiency can be eliminated through an employment subsidy, financed by means of lump-sum taxes. Let τ denote the rate at which the cost of employment is subsidised. Then, under flexible prices:

$$P_t = \mathcal{M} \frac{(1-\tau) W_t}{A_t}, \tag{530}$$

and accordingly:

$$-\frac{U_{n,t}}{U_{c,t}} = \frac{W_t}{P_t} = \frac{A_t}{\mathcal{M}(1-\tau)}.$$

Hence, the optimal allocation can be attained if and only if $\mathcal{M}(1-\tau) = 1$, or, equivalently, by setting:

$$\tau = \frac{1}{\epsilon}. \tag{531}$$

In much of the analysis that follows, we assume that such an optimal subsidy is in place. By construction, the equilibrium under flexible prices is allocatively efficient in that case.

13.3.2 Distortions due to sticky prices

We now discuss the "short-run" distortion in the baseline New Keynesian model. This short-run distortion constitutes two sources of inefficiency: i) the fact that firms do not adjust their prices continuously implies that the economy's average markup will change over time in response to shocks, and will generally differ from the constant frictionless markup, $\mathcal{M}$; and ii) the presence of staggered price setting is a source of inefficiency as the relative price of different goods will vary in a way unwarranted by changes in preferences or technologies, and as a result of the lack of synchronisation in price adjustments.

First, to show how markups change over time, let $\mathcal{M}_t$ denote the period t average markup charged by intermediate firms and use (530) to write:

$$\begin{aligned} P_t &= \mathcal{M}_t \frac{(1-\tau)W_t}{A_t} \\ \Leftrightarrow \mathcal{M}_t &= \frac{P_t A_t}{(1-\tau)W_t} \\ &= \frac{P_t}{(1-\tau)W_t/A_t}, \end{aligned}$$

and then use a "trick" noting that $\mathcal{M} = \frac{1}{1-\tau}$:

$$\mathcal{M}_t = \mathcal{M} \frac{P_t}{W_t/A_t}. \tag{532}$$

I say "trick" but it's really just manipulating the fact that we've assumed that the tax perfectly offsets the long-run distortion due to monopolistic competition. We can then write:

$$-\frac{U_{n,t}}{U_{c,t}} = \frac{W_t}{P_t} = A_t \frac{\mathcal{M}}{\mathcal{M}_t},$$

which violates our typical efficiency condition which demands that the real wage be equal to the marginal product of labour. Clearly, here, this efficiency condition is violated whenever $\mathcal{M} \neq \mathcal{M}_t$. The efficiency of the equilibrium allocation can be restored if and only if the economy's average markup is stabilised at its frictionless level – this is a huge part of why central banks dislike unstable prices!

Next, think about the effects of staggered pricing on the relative price of goods. Normally when we think of price changes due to technological progress (or regression) or preference shocks, we find that to be allocatively efficient – which was an important lesson we learned from RBC theory. However, if prices change simply because some firms were faster/slower than others (in the model this is due to the Calvo fairy), then it's deemed to be simply inefficient. What this means is that $P_t(i) \neq P_t(j)$ for any pair of goods (i, j) whose prices do not happen to have been adjusted in the same period. Such a relative price distortion will lead, in turn, to different quantities of the different goods being produced and consumed, that is, $C_t(i) \neq C_t(j)$, and, as a result, different quantities of labour employed by different firms, that is, $N_t(i) \neq N_t(j)$. Recall from our intermediate goods producer problem, we can use results from Blanchard and Kiyotaki (1987) to write:

$$C_t(j) \equiv \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} C_t \implies C_t = \left(\int_0^1 C_t(j)^{\frac{\epsilon-1}{\epsilon}} dj\right)^{\frac{\epsilon}{\epsilon-1}},$$
$$N_t(j) \equiv \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon} N_t \implies N_t = \left(\int_0^1 N_t(j)^{\frac{\epsilon-1}{\epsilon}} dj\right)^{\frac{\epsilon}{\epsilon-1}},$$

and where:

$$P_t = \left(\int_0^1 P_t(j)^{1-\epsilon} dj\right)^{\frac{1}{1-\epsilon}}.$$

These imply the following optimality conditions:

$$C_t(j) = C_t, \ \forall j \in [0,1],$$
$$N_t(j) = N_t, \ \forall j \in [0,1].$$

But if $C_t(i) \neq C_t(j)$ then these efficient optimality conditions are not satisfied. Attaining the efficient allocation thus requires that the quantities produced and consumed of all goods be equalised, which in turn requires that their prices and marginal costs be equalised. We will consider the policy that attains those objectives next.

13.3.3 "The Divine Coincidence" and the efficient natural allocation

Up until now,[84] we've assessed monetary policy rules (Taylor-type rules) in the New Keynesian model for the case in which the equilibrium allocation under flexible prices – the natural allocation – is efficient. That is, there is some kind of Pigouvian tax which offsets the distortion caused by intermediate firms being monopolistically competitive (and not perfectly competitive). In other words, the "long-run" distortion from monopolistic competition has been taken care of via some kind of subsidy for labour equal to the inverse price markup. This means we can interpret y_t^f as the optimal equilibrium. We assume that the central bank is concerned with the "short-run" distortion (due to sticky prices) and, other things being equal, the central bank would like to eliminate output gaps. To keep the analysis simple, let's also assume that we do not inherit any price distortions from the previous period – i.e., assume that $P_{t-1}(j) = P_{t-1},\ \forall j \in [0, 1]$.

Under these assumptions, the efficient allocation can be attained by a policy that stabilises firms' marginal cost at a level consistent with their desired markup, $\mathcal{M}$, at unchanged prices. If that policy is expected to remain in place indefinitely, no firm ever has an incentive to adjust its price, because it is currently charging its optimal markup and expects to keep doing so in the future. As a result, $P_t^{\#} = P_{t-1}$, and, hence, $P_t = P_{t-1}$, for $t = 0, 1, 2, ...$ In other words, the aggregate price level is fully stabilised and no relative price distortions emerge. In addition, $\mathcal{M}_t = \mathcal{M}$, $\forall t$, and output and employment match their counterparts in the flexible price equilibrium allocation (which, in turn, corresponds to the efficient allocation given the assumed subsidy).

Thus, and it's worth repeating, the optimal policy requires that:

$$y_t = y_t^f,$$

[84] When I originally wrote these notes, I wanted to discuss the case of the efficient natural allocation and "The Divine Coincidence" described by Blanchard and Galí (2007) after going through the optimal discretionary and commitment policies. But doing so caused confusion, as students would go through the optimal policies looking at cost push shocks without looking at the case of productivity shocks and the implications they have on optimal policy. So in this section we will go over the efficient natural allocation – it may be worth coming back to this section again after reading the optimal discretionary and commitment policies. Understandably, all of this optimal policy discussion is at first a bit confusing.

or, equivalently:

$$\tilde{y}_t = 0, \quad \forall t,$$

that is, the output gap should be closed at all times. In that case, the NKPC implies

$$\pi_t = 0, \quad \forall t.$$

That is, inflation is kept constant at a zero rate (or, equivalently, the aggregate price level is fully stabilised). The DISE then implies

$$i_t = r_t^f, \quad \forall t,$$

i.e., the equilibrium nominal interest rate (which equals the real rate, given zero inflation) must be equal to the Wicksellian natural rate of interest.

Two features of the optimal policy are worth emphasising. First, stabilising output is not desirable in and of itself. Instead, output should vary one-for-one with the flex price level of output, that is, $y_t = y_t^f$, $\forall t$. There is no reason, in principle, why the flex price level of output should be constant or take the form of a smooth trend, because all kinds of real shocks are a potential source of variation in its level – this is the main lesson from RBC research. In that context, policies that stress output stability (possibly around a smooth trend) may generate potentially large deviations of output from its flex price level – or, rather, natural – level and, thus, be suboptimal.[85]

Second, price stability emerges as a feature of the optimal policy even though, a priori, the policymaker does not attach any weight to such an objective. But under the assumptions made, price stability implies an efficient level of output, and vice versa. The previous finding, often referred to as "The Divine Coincidence" (Blanchard and Galí 2007) in the literature, implies that a central bank doesn't need to know or worry about what the efficient level of output is at each point in time, for the latter can be attained automatically as a byproduct of a successful price stabilisation policy.

A bit of derivation should solidify our understanding. In a first best Pareto-efficient allocation with perfect competition and no market distortions (think RBC model), we have the following equilibrium

[85] Think about what would happen if a central bank made a rule like $i_t = r_t^f$.

conditions:

$$mp_{n,t} = y_t^e - n_t^e,$$
$$mrs_{c,n;t} = \sigma y_t^e + \eta n_t^e,$$

where both the marginal product of labour and marginal rate of substitution are equal to the real wage, $\ln Y_t^e$ denotes the first best allocation output, and we have market clearing so that $C_t^e = Y_t^e$.[86] Setting these conditions equal to one another we have

$$\begin{aligned} y_t^e - n_t^e &= \sigma y_t^e + \eta n_t^e \\ n_t^e(1+\eta) &= y_t^e(1-\sigma) \\ n_t^e &= \frac{(1-\sigma)}{1+\eta} y_t^e. \end{aligned}$$

Using the production function, $y_t^e = a_t + n_t^e$, we have

$$\begin{aligned} y_t^e - a_t &= \frac{(1-\sigma)}{1+\eta} y_t^e \\ y_t^e \left[1 - \frac{1-\sigma}{1+\eta}\right] &= a_t \\ y_t^e &= \frac{1+\eta}{\eta+\sigma} a_t. \end{aligned} \tag{533}$$

Then, consider the flexible price equilibrium (the second best allocation relative to perfect compet-

[86] Just to be clear, recall that we have the following optimality conditions in the New Keynesian model:

$$MP_{N,t} = (1-\alpha)\frac{Y_t}{N_t},$$
$$\psi N_t^\eta = C_t^{-\sigma}\frac{W_t}{P_t},$$

where, for convenience, we sometimes assume that $\alpha = 0$ in our derivations. If we take logs for these two equations, we get

$$\begin{aligned} \ln MP_{N,t} &= \ln Y_t - \ln N_t \\ \ln\left(\frac{W_t}{P_t}\right) &= \ln\psi + \eta \ln N_t + \sigma \ln C_t, \end{aligned}$$

and we can also assume $\ln\psi = 0$. You could include it if you want, but it doesn't really make a difference to our results. Finally, we assume that $C_t = Y_t$, and that's how we end up with the expression for $mp_{n,t}$ and $mrs_{c,n;t}$.

ition), where we know that intermediate firms' optimal price is given by

$$P_t = \mathcal{M}\frac{W_t}{A_t},$$

this then implies

$$\begin{aligned} 1 &= \frac{\mathcal{M}W_t}{A_t P_t} \\ &= mc_t \mathcal{M}. \end{aligned}$$

Using the definition of marginal cost of firms, we have[87]

$$\begin{aligned} 1 &= mc_t \mathcal{M} \\ 1 &= \frac{w_t}{MP_{N,t}}\mathcal{M} \\ MP_{N,t} &= w_t \mathcal{M} \\ \frac{Y_t^f}{N_t^f} &= w_t \mathcal{M} \\ \implies y_t^f - n_t^f &= w_t + \ln \mathcal{M}, \end{aligned}$$

hence,

$$w_t = y_t^f - n_t^f - \ln \mathcal{M}.$$

Combining this with the FOC for households,

$$\begin{aligned} w_t &= mrs_{c,n;t} \\ &= \sigma y_t^f + \eta n_t^f, \end{aligned}$$

[87] Apologies for the poor notation: I use mc_t and w_t to mean the real marginal cost and real wage, respectively, and I also reuse these variables to denote their log level counterparts too. This is something that most, if not all, macro textbooks don't make clear.

yields

$$y_t^f - n_t^f - \ln \mathcal{M} = \sigma y_t^f + \eta n_t^f,$$

where y_t^f and n_t^f denote the flex price allocations. With a bit of rearranging, we have

$$n_t^f = \frac{y_t^f(1-\sigma) - \ln \mathcal{M}}{1+\eta},$$

and then using the production function, $y_t^f = a_t + n_t^f$, we get

$$\begin{aligned} y_t^f - a_t &= \frac{y_t^f(1-\sigma) - \ln \mathcal{M}}{1+\eta} \\ y_t^f \left[1 - \frac{1-\sigma}{1+\eta}\right] &= \frac{(1+\eta)a_t - \ln \mathcal{M}}{1+\eta} \\ y_t^f &= \frac{(1+\eta)a_t - \ln \mathcal{M}}{\eta+\sigma}. \end{aligned} \tag{534}$$

Then simply take the difference between y_t^e and y_t^f:

$$\begin{aligned} y_t^e - y_t^f &= \frac{1+\eta}{\eta+\sigma}a_t - \frac{(1+\eta)a_t - \ln \mathcal{M}}{\eta+\sigma} \\ &= \frac{\ln \mathcal{M}}{\eta+\sigma} \equiv \Upsilon, \end{aligned} \tag{535}$$

where we say that the difference between the first best and second best equilibrium allocations are a constant wedge, Υ.[88] As mentioned in the previous section, the introduction of Pigouvian lump-sum taxes is assumed to completely offset this "long run" distortion arising from monopolistic competition. By construction, the equilibrium under flexible prices is efficient in that case. i.e., $y_t^f = y_t^e$.

Now, we look at distortions related to sticky prices. Introduce price rigidities (a la Calvo):

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \lambda(mc_t + \ln \mathcal{M}_t),$$

[88] If you actually take log deviations from steady state, you would find that there are no differences in the transition dynamics between the "RBC-like" equilibrium and the flex price equilibrium, i.e., $\hat{y}_t^e - \hat{y}_t^f = 0$.

where $\lambda = \frac{(1-\phi)(1-\beta\phi)}{\phi}$ and the derivation for λ is exactly as we had in the standard model previously. Now – and yes this does require a lot of back and forth and recalling – we have that

$$w_t = mrs_{c,n;t} = \sigma y_t + \eta n_t,$$

and the marginal cost,

$$\begin{aligned} mc_t &= w_t - mp_{n,t} \\ &= w_t - (y_t - n_t), \end{aligned}$$

and combining these two yields

$$\begin{aligned} mc_t + (y_t - n_t) &= \sigma y_t + \eta n_t \\ mc_t &= y_t(\sigma - 1) + (y_t - a_t)(1 + \eta) \\ &= y_t(\sigma + \eta) - a_t(1 + \eta), \end{aligned} \tag{536}$$

using the fact that $y_t \equiv a_t + n_t$. Okay, now, we need an expression for $\ln \mathcal{M}_t$. We know that

$$y_t^f - n_t^f - \ln \mathcal{M}_t = \sigma y_t^f + \eta n_t^f,$$

and

$$y_t^f = a_t + n_t^f,$$

which come from the definitions of the second best allocation and production function, respectively. Combining them to get an expression for $\ln \mathcal{M}_t$ gives

$$\begin{aligned} \ln \mathcal{M}_t &= y_t^f - n_t^f - \sigma y_t^f - \eta n_t^f \\ &= y_t^f (1 - \sigma) - (1 + \eta)(y_t^f - a_t) \\ &= a_t(1 + \eta) - y_t^f (\sigma + \eta). \end{aligned} \tag{537}$$

So then we can put the expressions for marginal cost and the log markup together to get

$$
\begin{aligned}
mc_t + \ln \mathcal{M} &= y_t(\sigma + \eta) - a_t(1 + \eta) + a_t(1 + \eta) - y_t^f(\sigma + \eta) \\
&= (\sigma + \eta)(y_t - y_t^f) \\
&= (\sigma + \eta)(y_t - y_t^f),
\end{aligned}
$$

Now, putting this back into our expression for inflation π_t will give us the familiar NKPC,

$$
\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa(y_t - y_t^f),
$$

where $\kappa = \lambda(\sigma + \eta)$. Thus, stabilising inflation so that $\pi_t = \beta \mathbb{E}_t \pi_{t+1}$ is equivalent to stabilising the output gap, $y_t - y_t^f$, and vice versa. But remember that $y_t^e - y_t^f = \Upsilon$ is constant, so stabilising $y_t - y_t^f$ is the same as stabilising $y_t^e - y_t^f$. In other words, stabilising inflation, π_t, is the same as stabilising the welfare-relevant distance of output from the first-best allocation. This is the divine coincidence.

As stated at the start of this subsection, it's worth noting that most of the work that we've done with the New Keynesian model up until now was working with the assumptions of the divine coincidence. That, as well as the fact that we implicitly assumed that the flex price equilibrium (which in turn was the natural rate of output) was the efficient equilibrium due to the provision of employment subsidies. In other words, we've been working with the assumption that under flexible prices ($\phi = 0$), the fiscal authority addressed the "long-run" distortion which arose from monopolistic competition. So the optimal interest rate rules we looked at (and the conditions for determinacy and so on) have all been under the assumption of an efficient flex price equilibrium, $y_t^f = y_t^e$. Thus, the optimal interest rate rules implied that stabilising inflation is the same as stabilising the output gap – i.e., the central bank did not face an inflation-output tradeoff.

In the next section, we will change the assumption of an efficient flexible price equilibrium, introducing a tradeoff between inflation and output for the central bank.

13.4 Optimal policy for an efficient steady state: Discretion vs commitment

When nominal rigidities coexist with real imperfections, the flexible price equilibrium allocation is generally inefficient – when distortions from monopolistically competitive firms are left in the economy with the combination of sticky prices. In this section we make a few amendments which will give us some interesting case studies. First, as we just alluded to, we assume that the flexible price equilibrium is no longer efficient (so $y_t^f \neq y_t^e$), but we assume that **the steady state is still efficient**. So, unlike our previous analysis, in the short-run there are deviations between the efficient level of output and the flexible price level of output. More precisely, the gap between the two can be modelled to follow some stationary process with a zero mean.

We will need a framework for the central bank to follow to optimise inflation and output objectives, and so we assume that the welfare of the central bank (or the broader economy) is a present discounted value of a quadratic loss function in inflation and the output gap:

$$\mathbb{L}_t = \frac{1}{2}\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left(\pi_{t+s}^2 + \vartheta x_{t+s}^2\right), \tag{538}$$

where $x_t \equiv y_t - y_t^e$ is the welfare-relevant output gap, with y_t^e denoting the (log) efficient level of output. This loss function can actually be derived from taking a quadratic approximation to household welfare, while using the linearised equilibrium conditions (see Galí or Woodford's textbooks for a formal derivation). You may wonder why the central bank cares about inflation over and above the output gap (which, via the logic above, the central bank would like to eliminate). If you go back to the CES aggregator over intermediate goods, you will note that it is concave, meaning that households (or the final goods firm, if you like) would like to smooth demand over intermediate inputs. In a flexible price world, all intermediate producers would choose the same price (e.g., they all desire a relative price of 1). If aggregate inflation is different from zero, with price stickiness, relative prices at the intermediate firm level get distorted (i.e., there is price dispersion). This leads to a non-smooth allocation of intermediates, which results in a welfare loss. Another way to think about this is to recall some basic microeconomics: if individuals have well-behaved utility curves, then they are prepared to pay a resource cost in order to attain certainty. You can think of price dispersion as being analogous to uncertainty – if price dispersion exists,[89] you're not sure if the price for a particular immediate good is above or below its close substitute. So the quadratic approximation to household welfare attempts to capture this uncertainty component.

Using the identity $\tilde{y}_t \equiv (y_t - y_t^e) + (y_t^e - y_t^f)$ to substitute for the output gap, $\tilde{y}_t$, in the NKPC, a structural equation relating inflation and the welfare-relevant output gap can be written as:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + u_t, \tag{539}$$

where $u_t = \kappa(y_t^e - y_t^f)$. Hence, the central bank will seek to minimise the loss function (538) subject to the sequence of constraints given by (539). Critically, time variations in the gap between the efficient and flexible price levels of output (the "natural level") – reflected in fluctuations in u_t – generate a tradeoff for the central bank, because they make it impossible to simultaneously attain zero inflation and an efficient level of activity. This is a key difference from the canonical New Keynesian models that we previously analysed, where $y_t^f = y_t^e \implies u_t = 0, \forall t$.

We can also consider the disturbance term u_t in (539) as a "cost push shock", which follows the exogenous AR(1) exogenous process:

$$u_t = \rho_u u_t + \epsilon_{u,t},$$

where $\rho_u \in [0, 1)$, and $\{\epsilon_{u,t}\}$ is a white-noise process with constant variance, σ_u^2.

13.4.1 Technical aside: Second-order approximation of an objective function

It's actually worth going through this derivation. We will be using Galí's book as reference to derive a second-order approximation to the utility of the representative household when the economy remains in a neighbourhood of an efficient steady state. We will not be covering the generalisation of a distorted steady state here.

A second-order approximation of utility is derived around a given steady state allocation. Frequent

[89] Remember that price dispersion only exists when we have Calvo pricing, or pricing mechanisms similar to Calvo. With Rotemberg pricing, all firms charge the same price, and thus there is no dispersion. That's why, up to a first-order approximation, Calvo and Rotemberg pricing are identical. Second-order approximations capture the variance/uncertainty component, and so the costs to welfare are different for the Calvo and Rotemberg models.

use is made of the following second-order approximation of relative deviations in terms of log deviations:

$$\frac{X_t - \bar{X}}{\bar{X}} \approx \hat{x}_t + \frac{1}{2}\hat{x}_t^2,$$

where $\hat{x}_t \equiv \log(X_t/\bar{X})$ is the log deviation from steady state for a generic variable X_t. All along it is assumed that utility is separable in consumption and hours (so $U_{cn} = 0$). In order to lighten the notation, define:

$$\begin{aligned} U_t &\equiv U(C_t, N_t; Z_t), \\ \bar{U} &\equiv U(\bar{C}, \bar{N}; \bar{Z}), \end{aligned}$$

where Z_t is a preference shock.

The second order-Taylor expansion of U_t around a steady state $(\bar{C}, \bar{N})$ yields:

$$\begin{aligned} U_t - \bar{U} \approx{} & \bar{U}_c\bar{C}\left(\frac{C_t - \bar{C}}{\bar{C}}\right) + \bar{U}_n\bar{N}\left(\frac{N_t - \bar{N}}{\bar{N}}\right) + \frac{1}{2}\bar{U}_{cc}\bar{C}^2\left(\frac{C_t - \bar{C}}{\bar{C}}\right)^2 \\ & + \frac{1}{2}\bar{U}_{nn}\bar{N}^2\left(\frac{N_t - \bar{N}}{\bar{N}}\right)^2 + \bar{U}_c\bar{C}\left(\frac{C_t - \bar{C}}{\bar{C}}\right)\left(\frac{Z_t - \bar{Z}}{\bar{Z}}\right) \\ & + \bar{U}_n\bar{N}\left(\frac{N_t - \bar{N}}{\bar{N}}\right)\left(\frac{N_t - \bar{N}}{\bar{N}}\right) + TIP, \end{aligned}$$

where TIP denotes terms independent of policy. In terms of log deviations, and ignoring terms independent of policy, we have:

$$U_t - \bar{U} \approx \bar{U}_c\bar{C}\left(\hat{y}_t(1+z_t) + \frac{1-\sigma}{2}\hat{y}_t^2\right) + \bar{U}_n\bar{N}\left(\hat{n}_t(1+z_t) + \frac{1+\varphi}{2}\hat{n}_t^2\right) + TIP,$$

where $\sigma = -\frac{\bar{U}_{cc}}{\bar{U}_c}\bar{C}$ and $\varphi = \frac{\bar{U}_{nn}}{\bar{U}_n}\bar{N}$, and where we use the market clearing condition $c_t = y_t$.

The next step consists of rewriting $\hat{n}_t$ in terms of output. Using the fact that:

$$N_t = \left(\frac{Y_t}{A_t}\right)^{\frac{1}{1-\alpha}} \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\frac{\epsilon}{1-\alpha}} dj,$$

we can write:

$$(1-\alpha)\hat{n}_t = \hat{y}_t - a_t + d_t,$$

where:

$$d_t = (1-\alpha)\ln \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\frac{\epsilon}{1-\alpha}} dj.$$

The following lemma shows that d_t is proportional to the cross-sectional variance of relative prices.

Lemma 1 (Gali, 2015): In a neighbourhood of a symmetric steady state, and up to a second-order approximation:

$$d_t = \frac{\epsilon}{2\Theta}\operatorname{Var}_j(\ln P_t(j)),$$

where $\Theta = \frac{1-\alpha}{1-\alpha+\alpha\epsilon}$.[90]

Proof: See the Appendix 3.4 of Galí (2015).

Now, the period t utility can be rewritten (ignoring terms of third or higher order) as:

$$\begin{aligned}
U_t - \bar{U} = \bar{U}_c\bar{C}&\left(\hat{y}_t(1+z_t) + \frac{1-\sigma}{2}\hat{y}^2\right) \\
&+ \frac{\bar{U}_n\bar{N}}{1-\alpha}\left(\hat{y}_t(1+z_t) + \frac{\epsilon}{2\Theta}\operatorname{Var}_j(\ln P_t(j)) + \frac{1+\varphi}{2(1-\alpha)}(\hat{y}_t - a_t)^2\right) + TIP.
\end{aligned}$$

Efficiency of the steady state implies $-\frac{\bar{U}_n}{\bar{U}_c} = \overline{MP}_n$. Thus, and using the fact that $\overline{MP}_n = (1-\alpha)\bar{Y}/\bar{N}$ and $\bar{Y} = \bar{C}$,

$$\begin{aligned}
\frac{U_t - \bar{U}}{\bar{U}_c\bar{C}} &\approx -\frac{1}{2}\left[\frac{\epsilon}{\Theta}\operatorname{Var}_j(\ln P_t(j)) - (1-\sigma)\hat{y}_t^2 + \left(\frac{1+\varphi}{1-\alpha}\right)(\hat{y}_t - a_t)^2\right] + TIP \\
&= -\frac{1}{2}\left[\frac{\epsilon}{\Theta}\operatorname{Var}_j(\ln P_t(j)) - \left(\sigma - \frac{\varphi+\alpha}{1-\alpha}\right)\hat{y}_t^2 + 2\left(\frac{1+\varphi}{1-\alpha}\right)\hat{y}_t a_t\right] + TIP \\
&= -\frac{1}{2}\left[\frac{\epsilon}{\Theta}\operatorname{Var}_j(\ln P_t(j)) - \left(\sigma - \frac{\varphi+\alpha}{1-\alpha}\right)(\hat{y}_t^2 - 2\hat{y}_t\hat{y}_t^f)\right] + TIP \\
&= -\frac{1}{2}\left[\frac{\epsilon}{\Theta}\operatorname{Var}_j(\ln P_t(j)) - \left(\sigma - \frac{\varphi+\alpha}{1-\alpha}\right)\tilde{x}_t^2\right] + TIP,
\end{aligned}$$

[90]It would be worth noting that in our own derivation of the New Keynesian model we assumed that $\alpha = 0$.

$$Y_t(j) = A_t N_t(j)^{1-\alpha}.$$

In other words, we assumed that the production technology for the intermediate goods producer was simply linear in aggregate technology and labour:

where $\hat{y}_t^f = y_t^f - \bar{y}^f$, and where the fact was used that $y_t^f = \frac{1+\varphi}{\sigma(1-\alpha)+\varphi+\alpha} a_t$ and $\tilde{y}_t = \hat{y}_t - \hat{y}_t^f$.

Accordingly, a second-order approximation to the consumer's welfare losses can be written and expressed as a fraction of steady state consumption (and up to additive terms independent of policy) as:

$$
\begin{aligned}
\mathbb{L}_t &= -\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left(\frac{U_{t+s} - \bar{U}}{\bar{U}_c \bar{C}} \right) \\
&= \frac{1}{2} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left(\frac{\epsilon}{\Theta} \operatorname{Var}_j(\ln P_{t+s}(j)) + \left(\sigma + \frac{\varphi + \alpha}{1 - \alpha} \right) \tilde{y}_{t+s}^2 \right).
\end{aligned}
$$

The final step consists of rewriting the terms involving the price dispersion variable as a function of inflation. In order to do so, make use of the following lemma:

Lemma 2 (Gali, 2015):

$$
\sum_{s=0}^{\infty} \beta^s \operatorname{Var}_j(\ln P_{t+s}(j)) = \frac{\phi}{(1 - \beta\phi)(1 - \phi)} \sum_{s=0}^{\infty} \beta^s \pi_{t+s}^2.
$$

Proof: See Woodford (2003, Chapter 6).

Using the fact that $\lambda = \frac{(1-\phi)(1-\beta\phi)}{\phi} \Theta$, the previous lemma can be combined with the expression above to obtain:

$$
\begin{aligned}
\mathbb{L}_t &= \frac{1}{2} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left[\left(\frac{\epsilon}{\lambda} \right) \pi_{t+s}^2 + \left(\sigma + \frac{\varphi + \alpha}{1 - \alpha} \right) \tilde{y}_{t+s}^2 \right] \\
&= \frac{1}{2} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left[\pi_{t+s}^2 + \frac{\lambda}{\epsilon} \left(\sigma + \frac{\varphi + \alpha}{1 - \alpha} \right) \tilde{y}_{t+s}^2 \right] \\
&= \frac{1}{2} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left[\pi_{t+s}^2 + \frac{\kappa}{\epsilon} \tilde{y}_{t+s}^2 \right] \\
\mathbb{L}_t &= \frac{1}{2} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left[\pi_{t+s}^2 + \vartheta \tilde{y}_{t+s}^2 \right].
\end{aligned}
$$

Recall that κ is the slope of the NKPC and ϵ is the price elasticity of demand. The $1/2$ on the outside is just a scaling term that doesn't affect the optimum but simplifies things a bit (when we take differentials). As noted above, we can think about the central bank as choosing inflation and the output gap, given its choice of i_t, which then determines r_t given a path of $\mathbb{E}_t \pi_{t+1}$. This must be done subject to the constraint of the NKPC, however.

The rationale for the two terms is as follows. i) $\tilde{y}_t^2$ term: risk averse households prefer smooth consumption paths. Keeping output close to its natural rate achieves this. ii) π_t^2 term: households don't just care about the level of consumption but also its allocation. With inflation, sticky prices implies different prices for the symmetric goods and thus different consumption levels. Optimality requires equal consumption of all items in the bundle – rationale for the welfare effect of inflation, independent of its effect on output (though perhaps you can think of other – better – explanations for a negative effect of inflation on welfare).

We now consider two cases of central bank optimal policy. The first is called "discretion," where the central bank solves the one period problem each period. In the other, called "commitment," the central bank solves the entire problem at the beginning of time and commits to its policy.

13.4.2 Optimal discretionary policy

We first start with the discretion case. The problem can be written as:

$$\min_{\pi_t, x_t} \frac{1}{2} \left(\pi_t^2 + \vartheta x_t^2 \right),$$

subject to:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + u_t, \tag{540}$$

where $\beta \mathbb{E} \pi_{t+1}$ and u_t are taken as given by the central bank. Because u_t is exogenous and $\beta \mathbb{E}_t \pi_{t+1}$ is a function of expectations about future output gaps (as well as future u_t's), they cannot be influenced by the central bank in period t. To be precise, the term $\beta \mathbb{E}_t \pi_{t+1}$ can be treated as given by the central bank because there are no endogenous state variables (e.g. past inflation) affecting current inflation. Otherwise, the central bank would have to take into account the influence that its current actions, through their impact on those state variables, would have on future inflation.

Set the problem up as the Lagrangian,

$$\mathcal{L} = -\frac{1}{2}(\pi_t^2 + \vartheta x_t^2) + \lambda\left(\pi_t - \beta \mathbb{E}_t \pi_{t+1} - \kappa x_t\right),$$

and the FOCs are

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \pi_t} &= -\pi_t + \lambda = 0, \\ \frac{\partial \mathcal{L}}{\partial x_t} &= -\vartheta x_t - \lambda \kappa = 0.\end{aligned}$$

Combining FOCs, so as to eliminate the Lagrange multipliers, we get

$$x_t = -\frac{\kappa}{\vartheta}\pi_t. \tag{541}$$

Loosely speaking, this optimality condition can be interpreted as a "lean against the wind" policy. If the output gap is positive, the central bank will want to pursue a policy in which it lowers inflation (and vice-versa), up to the point where condition (541) is satisfied. Thus, one can view that condition as a relation between target variables that the discretionary central bank will seek to maintain at all times, and it is in that sense that it may be labeled a "targeting rule."

Using (541) to substitute for x_t in (540), yields the following difference equation for inflation:

$$\pi_t = \frac{\vartheta \beta}{\vartheta + \kappa^2}\mathbb{E}_t \pi_{t+1} + \frac{\vartheta}{\vartheta + \kappa^2} u_t.$$

Iterating the previous equation forward,[91] an expression is obtained for equilibrium inflation under the optimal discretionary policy:

$$\pi_t = \frac{\vartheta}{\kappa^2 + \vartheta(1 - \beta \rho_u)} u_t. \tag{542}$$

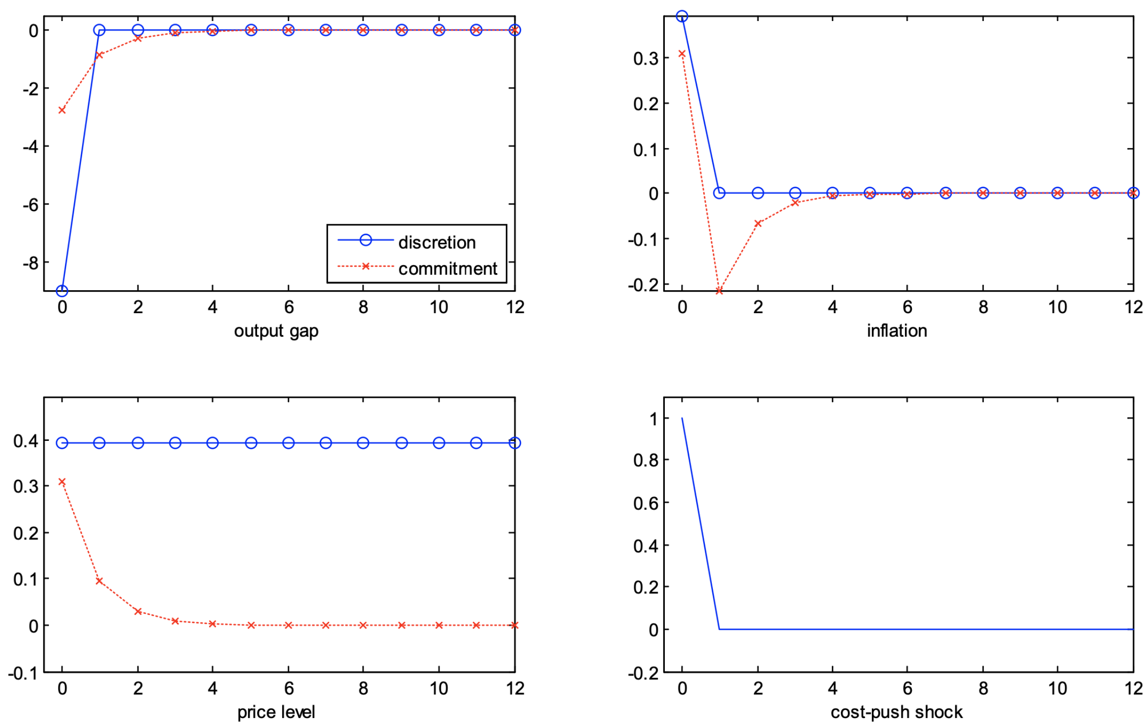
[91] Use the techniques in Section 5.2.6.

Combining (542) and (541) obtains an analogous expression for the output gap:

$$x_t = -\frac{\kappa}{\kappa^2 + \vartheta(1 - \beta\rho_u)} u_t. \tag{543}$$

Thus, under the optimal discretionary policy, the central bank lets the output gap and inflation fluctuate in proportion to the current value of the cost-push shock. This is illustrated graphically by the circled lines in Figures 89 and 90, which represent the responses under the optimal discretionary policy of the welfare-relevant output gap, inflation, and the price level to a one percent increase in u_t.

Figure 89: Optimal Responses to a Transitory Cost Push Shock ($\rho_u = 0$)



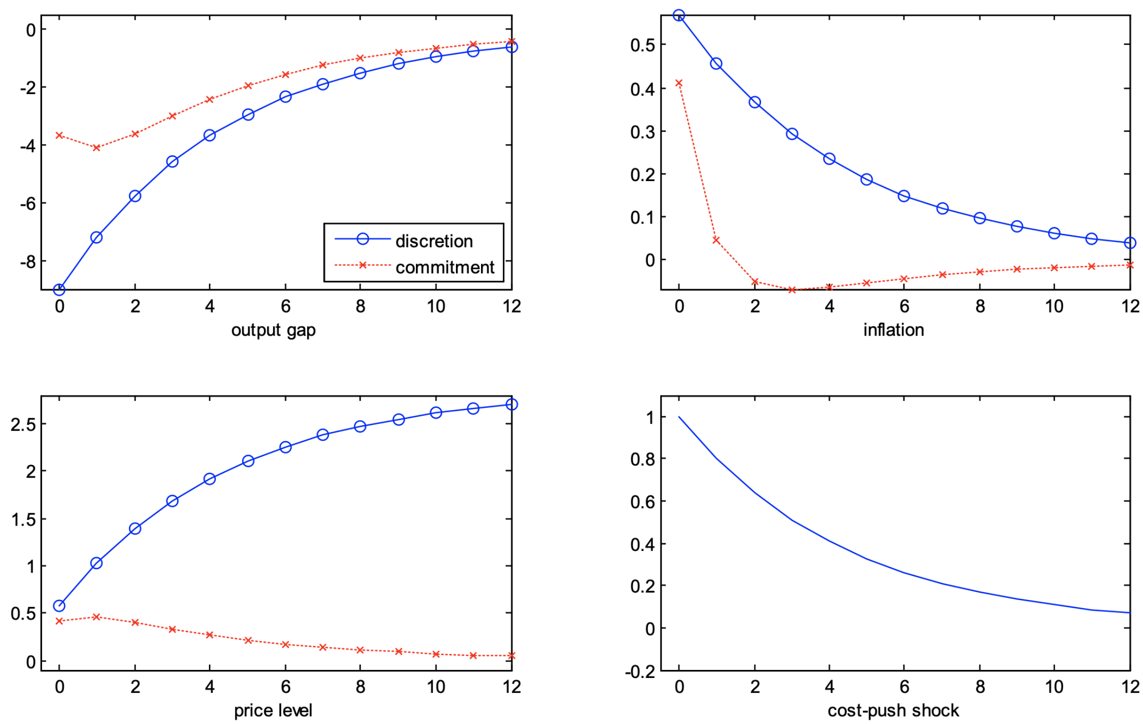
Source: Galí (2015)

The path of the cost push shock, u_t, after a one percent increase, is displayed in the bottom right plot of Figures 89 and 90. In both cases, the central bank finds it optimal to partly accommodate the inflationary pressures resulting from the cost push shock, and thus lets inflation rise. Note, however, that the increase in inflation is smaller than the increase that would be obtained if the output gap remained unchanged. In the latter case it is easy to check that inflation would be given by

$$\pi_t = \frac{1}{1 - \beta\rho_u} u_t,$$

thus implying a larger response of inflation (in absolute value) at all horizons in response to the cost push shock. Instead, under the optimal discretionary policy, the impact on inflation is dampened by the negative response of the output gap. Finally, it is seen that the implied response of inflation leads naturally to a permanent change in the price level, whose size is increased in the persistence of the shock.

Figure 90: Optimal Responses to a Persistent Shock ($\rho_u = 0.8$)



Source: Galí (2015)

The analysis above implicitly assumes that the central bank can choose its desired level of inflation and the output gap at each point in time. Of course, in practice, a central bank cannot directly set either variable. One possible approach to implementing that policy is to adopt an interest rate rule which guarantees that the desired outcome is attained. Before deriving the form that such a rule may take, it is convenient to determine the equilibrium interest rate under the optimal discretionary policy as a function of the exogenous driving forces. Begin with the DISE:

$$x_t = \mathbb{E}_t x_{t+1} - \sigma^{-1}(i_t - \mathbb{E}_t \pi_{t+1} - r_t^f), \tag{544}$$

where the natural interest rate, r_t^e, is the real interest that is consistent with the efficient level of output:

$$\begin{aligned} r_t^f &\equiv \rho + \sigma \mathbb{E}_t \Delta y_{t+1}^f \\ &= \rho - \sigma(1-\rho_a)\omega a_t. \end{aligned}$$

where we made use of the fact that $\mathbb{E}_t y_{t+1}^f = \rho_a y_t^f$ and we used the AR(1) process, $y_t^f = \rho_a y_{t-1}^f + \omega \epsilon_{a,t}$.

Thus, combining (542), (543), and (544) yields:

$$i_t = r_t^f + \Psi_i u_t, \tag{545}$$

where $\Psi_i = \frac{\vartheta \rho_u + \sigma\kappa(1-\rho_u)}{\kappa^2 + \vartheta(1-\beta\rho_u)} > 0$.

Applying the arguments we made previously, it is easy to see that (545) cannot be viewed as a desirable interest rate rule, for it does not guarantee a unique equilibrium and, hence, the attainment of the desired outcome. In particular, if "rule" (545) is used to eliminate the nominal rate in (544), the resulting equilibrium dynamics are represented by the system:

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathbb{E}_t x_{t+1} \\ \mathbb{E}_t \pi_{t+1} \end{bmatrix} + \mathbf{B} u_t,$$

where:

$$\mathbf{A} = \begin{bmatrix} 1 & \frac{1}{\sigma} \\ \kappa & \beta + \frac{\kappa}{\sigma} \end{bmatrix}, \ \mathbf{B} = \begin{bmatrix} -\frac{\Psi_i}{\sigma} \\ 1 - \frac{\kappa \Psi_i}{\sigma} \end{bmatrix}.$$

Here, matrix $\mathbf{A}$ always has one eigenvalue outside the unit circle. Remember that for models of the form $\mathbf{X}_t = \mathbf{A}\mathbb{E}_t\mathbf{X}_{t+1} + \mathbf{B}\mathbf{V}_t$ to have a unique stable solution, we need all the eigenvalues of $\mathbf{A}$ to be less than one. Here, the eigenvalues for $\mathbf{A}$ satisfy:

$$
\begin{aligned}
P(\lambda) &= (1-\lambda)\left(\beta + \frac{\kappa}{\sigma} - \lambda\right) - \frac{\kappa}{\sigma} = 0 \\
&= \lambda^2 - \left(1 + \beta + \frac{\kappa}{\sigma}\right)\lambda + \beta = 0, \\
\implies P(0) &= \beta > 0, \\
\implies P(1) &= -\frac{\kappa}{\sigma} < 0, \\
\implies P(1) &> 0 | \lambda > 1.
\end{aligned}
$$

thus implying that the system has a multiplicity of solutions, only one of which corresponds to the desired outcome given by (542) and (543).

In the context of the present model, one can always derive a rule that guarantees equilibrium uniqueness (independently of parameter values), by appending to the expression for the equilibrium nominal rate under the optimal discretionary policy (545), a term proportional to the deviation between inflation and the equilibrium value of the latter under that policy, with a coefficient of proportionality greater than one (in order to satisfy the Taylor Principle). Formally:

$$
\begin{aligned}
i_t &= r_t^e + \Psi_i u_t + \phi_\pi \left(\pi_t - \frac{\vartheta}{\kappa^2 + \vartheta(1-\beta\rho_u)} u_t\right) \\
&= r_t^e + \Theta_i u_t + \phi_\pi \pi_t, \qquad (546)
\end{aligned}
$$

where $\Theta_i = \frac{\sigma\kappa(1-\rho_u) - \vartheta(\phi_\pi - \rho_u)}{\kappa^2 + \vartheta(1-\beta\rho_u)}$ and for an arbitrary inflation coefficient satisfying $\phi_\pi > 1$.

In practice, a rule like (546) is not easy to implement. It requires knowledge of the model's parameters, and real time observation of variations in the cost push shock and the efficient interest rate. This has led to some macroeconomists to emphasise "targeting rules", such as (541), as practical guides for monetary policy, as opposed to "instrument rules", such as (546). Under a targeting rule, the central bank would adjust its instrument until a certain optimal relation between target variables is satisfied. In the aforementioned example, however, following such a targeting rule requires that the efficient level of output, y_t^e, be observed in real time in order to determine the output gap, thus limiting its practical appeal.

13.4.3 Optimal policy under commitment

Next, we consider the problem under commitment. Here, the objective of the central bank is not just the current objective, but the present discounted value of the flow objective functions. In other words, the problem for the central bank is

$$\min_{\{\pi_t, x_t\}} \frac{1}{2}\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left(\pi_{t+s}^2 + \vartheta x_{t+s}^2\right),$$

subject to the series of constraints,

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + u_t. \tag{547}$$

The Lagrangian is

$$\mathcal{L} = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left[-\frac{1}{2}\left(\pi_t^2 + \vartheta x_t^2\right) + \lambda_t \left(\pi_t - \kappa x_t - \beta \mathbb{E}_t \pi_{t+1} - u_t\right)\right].$$

Before proceeding, it's worth mentioning that when we deal with lagged state variables (e.g. capital) when constructing the Lagrangian, we append an extra $t+1$ constraint onto the Lagrangian and then differentiate. Here, we have a $t+1$ variable, so we now append a $t-1$ constraint onto the Lagrangian to get

$$\mathcal{L} = -\frac{1}{2}\left(\pi_t^2 + \vartheta x_t^2\right) + \lambda_t \left[\pi_t - \kappa x_t - \beta \mathbb{E}_t \pi_{t+1} - u_t\right] + \lambda_{t-1}\left[\pi_{t-1} - \kappa x_{t-1} - \mathbb{E}_{t-1}\pi_t - u_{t-1}\right],$$

which gives the following FOCs:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \pi_t} &= -\pi_t + \lambda_t - \lambda_{t-1} = 0, \\
\frac{\partial \mathcal{L}}{\partial x_t} &= -\vartheta x_t - \lambda_t \kappa = 0,
\end{aligned}$$

where we can use law of iterated expectations to drop the $\mathbb{E}_{t-1}$ operator, and where $\lambda_{t-1} = 0$ because the inflation equation corresponding to period $t-1$ is not an effective constraint for the central bank choosing its optimal plan in period t. We can combine the FOCs to get the following:

$$x_t = -\frac{\kappa}{\vartheta}\pi_t. \tag{548}$$

Now, we need to use a trick to proceed. Since the optimal x_t only depends on π_t, we can also write

$$\begin{aligned}
\Delta x_{t+1} = \mathbb{E}_t x_{t+1} - x_t &= -\frac{\kappa}{\vartheta}\mathbb{E}_t \pi_{t+1} \\
\implies \mathbb{E}_t x_{t+1} &= x_t - \frac{\kappa}{\vartheta}\mathbb{E}_t \pi_{t+1}.
\end{aligned} \tag{549}$$

Now, we can begin at period t and substitute forward using these optimality conditions to get

$$\begin{aligned}
x_t &= -\frac{\kappa}{\vartheta}\pi_t \\
\mathbb{E}_t x_{t+1} &= x_t - \frac{\kappa}{\vartheta}\mathbb{E}_t \pi_{t+1} = -\frac{\kappa}{\vartheta}\pi_t - \frac{\kappa}{\vartheta}\mathbb{E}_t \pi_{t+1} \\
\mathbb{E}_t x_{t+2} &= \mathbb{E}_t x_{t+1} - \frac{\kappa}{\vartheta}\mathbb{E}_t \pi_{t+2} = -\frac{\kappa}{\vartheta}\pi_t - \frac{\kappa}{\vartheta}\mathbb{E}_t \pi_{t+1} - \frac{\kappa}{\vartheta}\mathbb{E}_t \pi_{t+2} \\
&\vdots \\
\mathbb{E}_t x_{t+k} &= -\frac{\kappa}{\vartheta}\sum_{k=0}^{\infty}\pi_{t+k}.
\end{aligned}$$

Now, what is the sum of the inflation rates between period t and $t+k$? It is the [log] price level minus the price level in the period before t. Thus, the FOC becomes

$$\begin{aligned}\mathbb{E}_t x_{t+1} &= -\frac{\kappa}{\vartheta}\mathbb{E}_t p_{t+1} \\ &= -\frac{\kappa}{\vartheta}\mathbb{E}_t\left[p_{t+1} - p_{-1}\right],\end{aligned}$$

where P_{-1} is an "implicit target" given by the price level one period before the central bank chooses its optimal plan. Since this must hold in expectation, and there are no disturbances that show up here either, it must also hold ex-post. This means we can get rid of the expectations operator:

$$x_t = -\frac{\kappa}{\vartheta}\hat{p}_t. \tag{550}$$

This looks familiar to the FOC under discretion (541), but it features the price level as opposed to price inflation. As such, this kind of rule is called a price level targeting rule. As $k \to \infty, x_{t+k} \to 0$, which means that:

$$\lim_{k\to\infty} \mathbb{E}_t p_{t+k} = p_t.$$

This means that the policy under commitment implies that the price level always returns to trend.

It is worth pointing out the difference between (550) and the corresponding targeting rule for the discretion case (541). The optimal discretionary policy requires that the central bank keeps output below (above) its efficient level as long as inflation is positive (negative). By way of contrast, under the optimal policy with commitment, the central bank sets the sign and size of the output gap in proportion to the deviations of the price level from its implicit target. As is discussed next, this has important consequences for the economy's equilibrium response to a cost push shock.

By combining optimality condition (550) with the NKPC (540) (after rewriting it in terms of the price level), the stochastic difference equation satisfied by p_t under the optimal policy is derived:

$$\hat{p}_t = \gamma\hat{p}_{t-1} + \gamma\beta\mathbb{E}_t\hat{p}_{t+1} + \gamma u_t, \tag{551}$$

where $\gamma = \frac{\vartheta}{\vartheta(1+\beta)+\kappa^2}$. The stationary solution[92] to this difference equation is given by:

$$\hat{p}_t = \delta \hat{p}_{t-1} + \frac{\delta}{1-\delta\beta\rho_u} u_t, \tag{552}$$

where $\delta = \frac{1-\sqrt{1-4\beta\gamma^2}}{2\gamma\beta} \in (0,1)$. Then (550) is used to derive the equilibrium process for the output gap:

$$x_t = \delta x_{t-1} - \frac{\kappa\delta}{\vartheta(1-\delta\beta\rho_u)} u_t, \tag{553}$$

with the response at the time of the shock being given by:

$$x_t = -\frac{\kappa\delta}{\vartheta(1-\delta\beta\rho_u)} u_t.$$

Now, look at Figures 89 and 90 again. A look at the case of a transitory cost push shock illustrates the difference most clearly. In the case of discretionary policy, both the output gap and inflation return to their zero initial value once the shock has vanished. By contrast, as implied by the policy functions under commitment (552) and (553), under the optimal policy, the deviations in the output gap and inflation from target persist well beyond the life of the shock. Given that a zero inflation zero output gap outcome is feasible once the shock has vanished, why does the central bank find it optimal to maintain a persistently negative output gap and inflation?

The reason is simple: By committing to such a response, the central bank manages to improve the output gap/inflation tradeoff in the period when the shock occurs. In the case illustrated in Figure 89, it lowers the initial impact of the cost push shock on inflation (relative to the discretion case), while incurring smaller output gap losses in the same period. This is possible because of the forward looking nature of inflation, which can be highlighted by iterating the NKPC (540) forward to yield:

$$\pi_t = \kappa x_t + \kappa \sum_{k=1}^{\infty} \beta^k \mathbb{E}_t x_{t+k} + \frac{1}{1-\beta\rho_u} u_t.$$

[92] To derive the stationary solution, first guess that it takes the form:

$$p_t = \delta p_{t-1} + b u_t,$$

for some pair (δ, b) with $|\delta| < 1$. The guess can be verified and coefficients (δ, b) determined by plugging the guess into (551) and using the method of undetermined coefficients.

Hence, it is seen that the central bank can offset the inflationary impact of a cost push shock by lowering the current output gap, x_t, but also by committing to lower future output gaps (or, equivalently, future reductions in the price level). If credible, such "promises" will bring about a downward adjustment in the sequence of expectations $\mathbb{E}_t x_{t+k}$ for $k = 1, 2, 3....$ As a result, and in response to a positive realisation of the cost push shock, u_t, the central bank may achieve any given level of current inflation, π_t, with a smaller decline in the current output gap, x_t. That is the sense in which the output gap/inflation tradeoff is improved by the possibility of commitment. Given the convexity of the loss function in inflation and output gap deviations, the dampening of those deviations in the period of the shock brings about an improvement in overall welfare relative to the case of discretion, because the implied benefits are not offset by the (relatively small) losses generated by the deviations in subsequent periods (and which are absent in the discretion case).

Figure 90 displays analogous IRFs under the assumption that $\rho_u = 0.8$. Note that in this case the economy reverts back to the initial position only asymptotically, even under discretion. Yet, some of the key qualitative features emphasised above above are all still present. In particular, the optimal policy with commitment manages once again to attain both lower inflation and a smaller output gap (in absolute value) at the time of the shock, relative to the discretionary policy. Note also that under commitment, the price level reverts back to its original level, albeit at a slower rate than in the case of a transitory shock. As a result, inflation displays some positive short-run autocorrelation, illustrating the fact that the strong negative short-run autocorrelation observed in the case of a purely transitory shock is not a necessary implication of the policy with commitment.

Either way, a feature of discretionary policy is the attempt to stabilise the output gap in the medium term more than the case with commitment, without internalising the benefits in terms of short term stability that result from allowing larger deviations of the output gap at future horizons. This characteristic, which is most clearly illustrated by the example of a purely transitory cost push shock, is referred to as the stabilisation bias associated with discretion.

As in the case of discretion, we might be interested in deriving an interest rate rule. Such a rule is derived for the special case of serially uncorrelated cost push shocks ($\rho_u = 0$). In that case, combining

(544), (552), and (553) yields:

$$\begin{aligned} i_t &= r_t^e - (1-\delta)\left(1 - \frac{\sigma\kappa}{\vartheta}\right) p_t \\ &= r_t^e - (1-\delta)\left(1 - \frac{\sigma\kappa}{\vartheta}\right) \sum_{k=0}^{t} \delta^{k+1} u_{t-k}. \end{aligned}$$

Thus, one possible rule that would bring about the desired allocation as the unique equilibrium is given by:

$$i_t = r_t^e - \left(\phi_p + (1-\delta)\left(1 - \frac{\sigma\kappa}{\vartheta}\right)\right) \sum_{k=0}^{t} \delta^{k+1} u_{t-k} + \phi_p \hat{p}_t,$$

for any $\phi_p > 0$. Note that the central bank stands ready to respond to any deviations of the price level from the path prescribed by (552), though this will not be necessary in equilibrium.

13.5 Optimal policy with an inefficient steady state: Discretion vs commitment

So far in this chapter looking at optimal policy, we've assumed that the steady state equilibrium is efficient. So while the short-run allocation lead to a gap between the flex price equilibrium and the efficient equilibrium (i.e., $y_t^f \neq y_t^e$), in the steady state we still had an efficient allocation ($\bar{y}^f = \bar{y}^e$). Now, we make one more assumption: that the flex price/natural steady state equilibrium is not efficient (so now we have $\bar{y}^f \neq \bar{y}^e$). The gap between the steady state levels of output is denoted by a parameter Φ, representing the wedge between the marginal product of labour and the marginal rate substitution between consumption and hours, both evaluated at the steady state. Formally, Φ is defined by

$$-\frac{\bar{U}_n}{\bar{U}_c} = (1-\Phi)\overline{MP}_n,$$

where $\Phi > 0$. An example or motivation for Φ could be the presence of firms' market power in the goods market being unaccounted for. This would create a distortion in the steady state, where,

$$\Phi \equiv 1 - \frac{1}{\mathcal{M}} > 0,$$

where $\mathcal{M}$ is the steady state gross markup.

Now, our quadratic welfare function can be described as:

$$\mathbb{W}_t = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left[\pi_{t+s}^2 + \vartheta \hat{x}_t^2 - \Lambda \hat{x}_t \right], \tag{554}$$

where $\Lambda = \Phi\lambda/\epsilon > 0$, $\hat{x}_t \equiv x_t - \bar{x}$, and where $\vartheta = \frac{(1-\phi)(1-\beta\phi)}{\phi}\Theta$ and $\Theta = \frac{1-\alpha}{1-\alpha+\alpha\epsilon}$. Note that any marginal increase in the output gap relative to its steady state value has a positive first-order effect on welfare (thus decreasing welfare losses), because output is assumed to be below its efficient level at that steady state, that is, $\bar{x} < 0$. Similarly, the constraint can be written based on the NKPC:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa \hat{x}_t + u_t, \tag{555}$$

where now $u_t \equiv \kappa(\hat{y}_t^e - \hat{y}_t^f)$.

13.5.1 Optimal discretionary policy

Similar to the case under the efficient steady state, the central bank faces the following problem:

$$\min_{\pi_t, x_t} \frac{1}{2}(\pi_t^2 + \vartheta \hat{x}_t^2) + \Lambda \hat{x}_t,$$

subject to the constraint:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa \hat{x}_t + u_t.$$

The optimality condition from the FOCs is:

$$\hat{x}_t = \frac{\Lambda}{\vartheta} - \frac{\kappa}{\vartheta}\pi_t. \tag{556}$$

Comparing (556) with (541), we see that (556) implies a more expansionary policy at any given level of inflation. This is a consequence of the desire by the central bank to partly correct for the inefficient flexible price, low average level of activity.

Plugging (556) into the NKPC constraint gives the following policy function for inflation:

$$\pi_t = \frac{\Lambda\kappa}{\kappa^2 + \vartheta(1-\beta)} + \frac{\vartheta}{\kappa^2 + \vartheta(1-\beta\rho_u)} u_t, \tag{557}$$

and then combining this with (556) gives the policy function for the equilibrium output gap:

$$\hat{x}_t = \frac{\Lambda(1-\beta)}{\kappa^2 + \vartheta(1-\beta)} - \frac{\kappa}{\kappa^2 + \vartheta(1-\beta\rho_u)} u_t. \tag{558}$$

Now, comparing (557) and (558) with (542) and (543), we see that the presence of a distorted steady state does not affect the dynamics of the output gap and inflation to cost push shocks under the optimal discretionary policy. It has, however, an effect on the average levels of inflation and the output gap around about which the economy fluctuates.

In particular, when the natural/flex price level of output and employment are inefficiently low ($\Lambda > 0$), we have that:

$$\begin{aligned} \bar{\pi} &= \frac{\Lambda\kappa}{\kappa^2 + \vartheta(1-\beta)}, \\ \bar{x} &= \frac{\Lambda(1-\beta)}{\kappa^2 + \vartheta(1-\beta)}. \end{aligned}$$

The incentive to push output above its natural steady state increases with the degree of inefficiency of the natural flex price steady state, which explains the fact that the average inflation is increasing in Φ (and hence in Λ), giving rise to the classical inflation bias phenomenon.

13.5.2 Optimal policy under commitment

Now, under commitment with an inefficient steady state, the central bank faces the following problem:

$$\min_{\pi_t, x_t} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left[\frac{1}{2}(\pi_{t+s}^2 + \vartheta \hat{x}_{t+s}^2) - \Lambda \hat{x}_{t+s} \right],$$

subject to

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t + u_t.$$

The Lagrangian for the problem is:

$$\mathcal{L} = -\frac{1}{2}(\pi_t^2 + \vartheta \hat{x}_t^2) + \Lambda \hat{x}_t + \xi_t \left(\pi_t - \kappa \hat{x}_t - \beta \mathbb{E}_t \pi_{t+1} - u_t\right) + \xi_{t-1}(\pi_{t-1} - \kappa \hat{x}_{t-1} - \pi_t - u_{t-1}),$$

where ξ_t are the Lagrangian multipliers. The FOCs are

$$\frac{\partial \mathcal{L}}{\partial \pi_t} = -\pi_t + \xi_t - \xi_{t-1} = 0,$$
$$\frac{\partial \mathcal{L}}{\partial \hat{x}_t} = -\vartheta \hat{x}_t + \Lambda - \xi_t \kappa = 0.$$

After a bit of rearranging, we get rid of the Lagrangian multipliers (making use of the fact that $\xi_{t-1} = 0$) and combine the FOCs:

$$\vartheta \Delta \hat{x}_t = \Lambda - \kappa \pi_t,$$
$$\vartheta \hat{x}_t = \Lambda - \kappa \hat{p}_t, \ \forall t, \tag{559}$$

where $\hat{p}_t = p_t - p_{-1}$. Combining (559) with the NKPC yields the following difference equation for the [log] price level:

$$\hat{p}_t = \gamma \hat{p}_{t-1} + \gamma \beta \mathbb{E}_t \hat{p}_{t+1} + \frac{\gamma \kappa \Lambda}{\vartheta} + \gamma u_t, \tag{560}$$

where $\gamma = \frac{\vartheta}{\vartheta(1+\beta)+\kappa^2} \in (0,1)$.

The stationary solution to (560) describes the evolution of the equilibrium price level under the optimal policy with commitment, and takes the form:

$$\hat{p}_t = \delta \hat{p}_{t-1} + \frac{\delta}{1 - \delta \beta \rho_u} u_t + \frac{\delta}{1 - \delta \beta} \left(\frac{\kappa \Lambda}{\vartheta}\right),$$

where $\delta = \frac{1 - \sqrt{1 - 4\beta\gamma^2}}{2\gamma\beta} \in (0,1)$. Solving this equation backward yields

$$\hat{p}_t = \left(\frac{1 - \delta^{t+1}}{1 - \delta}\right) \left(\frac{\delta}{1 - \delta\beta}\right) \left(\frac{\kappa \Lambda}{\vartheta}\right) + \frac{\delta}{1 - \delta \beta \rho_u} \sum_{k=0}^{t} \delta^k u_{t-k}. \tag{561}$$

Combining (559) and (561), the corresponding path for output can be derived:

$$\hat{x}_t = \delta \hat{x}_{t-1} - \frac{\kappa\delta}{\vartheta(1-\delta\beta\rho_u)} u_t + \frac{\Lambda}{\vartheta}\left[1 - \delta\left(1 - \frac{\kappa^2}{\vartheta(1-\delta\beta)}\right)\right], \tag{562}$$

with the response at $t = 0$ being given by:

$$\hat{x}_0 = -\frac{\kappa\delta}{\vartheta(1-\delta\beta\rho_u)} u_0 + \frac{\Lambda}{\vartheta}\left[1 - \frac{\kappa^2\delta}{\vartheta(1-\delta\beta)}\right]. \tag{563}$$

Equivalently, solving (562) backward and combining it with (563) yields:

$$x_t = \frac{\Lambda(1-\delta)\delta^k}{\vartheta} - \frac{\kappa\delta}{\vartheta(1-\delta\beta\rho_u)} \sum_{k=0}^{t} \delta u_{t-k}. \tag{564}$$

Looking (561) and (564) reveals the main implications of the presence of a distorted steady state on the path of the price level and output under the optimal policy with commitment: It leads to some persistent, deterministic transitional dynamics. Thus, in the absence of cost push shocks, the central bank chooses an output gap and inflation persistently above their steady state levels, and converging to the latter only asymptotically. In the presence of shocks, that deterministic component is added to the stochastic one, which is otherwise identical to the case of an efficient steady state. That modification thus generates a persistent inflationary bias, resulting from the benefits from a higher output, even at the cost of higher inflation.

In the long-run, however, the marginal benefits of increasing output, Λ, equals the cost of [cumulative] inflation, $\lim_{T\to\infty} \kappa\hat{p}_T$, so it is optimal to keep output at its natural level, $\lim_{T\to\infty} \bar{x}_T = \bar{x}$.

There is one very important difference between commitment and discretion in the case of an inefficient steady state: Under discretion, we have shown that the central bank as perpetual inflation bias as a result of perpetual increase of output above its natural flex price level. In the case of commitment, however, the price level converges asymptotically to a constant, given by:

$$\lim_{T\to\infty} p_T = p_{-1} + \frac{\delta}{(1-\delta\beta)(1-\delta)}\left(\frac{\kappa\Lambda}{\vartheta}\right).$$

Hence, under the optimal plan, the economy eventually converges to an equilibrium characterised by zero average inflation and a zero average output gap (relative to the flex price natural output). In that sense, it is asymptotically equivalent to the outcome of an economy with an efficient steady state. As the central bank desires zero long-run inflation, and the public is aware of this, it allows the central bank to raise output above its natural level (with consequent welfare improvements) with more subdued effect on inflation (since the public anticipates a gradual return of output to its natural level) – it essentially allows smoothing. Thus, the central bank's ability to commit avoids (at least asymptotically) the inflation bias that characterises the outcome of the discretionary policy. As we can see from Figures 89 and 90, the response to a cost push shock under the optimal policy with commitment is not affected by the presence of a distorted steady state – likewise with the discretionary policy case.

13.6 Comments and key readings

There was a lot to process in this chapter – and mostly due to difficult concepts. Let's go over them again. The main idea of this chapter was to start with the basic New Keynesian model but without specifying an interest rate rule – similar to where we were when we derived the New Keynesian model with just money. Combined with what we learned in the previous chapter, we found that absent of any cost push shocks, and assuming that the flexible price equilibrium is equal to the efficient allocation, the central bank is able to stabilise inflation and close the output gap by adhering to an interest rate rule which satisfies the requirements of determinacy. This was called the "The Divine Coincidence".

We derived then moved to a situation where we loosened the assumption that the flex price equilibrium was equal to the efficient allocation – something we implicitly assumed in the previous chapters. We motivated this by saying this could be due to distortions associated with monopolistic competition being left in the economy. With the flex price equilibrium not necessarily being efficient, and because we no longer assumed a specific form of the Taylor Rule, we needed a new framework to derive optimal monetary policy. We used a quadratic loss function framework proposed by Woodford (2003), where the central bank minimised losses subject to the NKPC acting as a constraint. The central bank could also adopt two broad strategies when it came to formulating optimal policy: discretion and commitment.

Discretion basically saw the central bank trying to re-optimise each period, with no regard for future commitments. Commitment, as the name suggests, amended the central bank's objective function to include all future discounted household welfare/losses due to inflation and output gaps. The basic idea was that commitment allowed a smoothing effect, letting the central bank reap the rewards of higher output (and slightly higher inflation) over the short to medium term before closing both the inflation and output gap. Furthermore, what emerged from the commitment case was that the central bank would use "price level targeting" when it acted optimally.

Finally, our last case study was one where even in the steady state, the flex price equilibrium allocation was inefficient. The key takeaway from this chapter was that by assuming a quadratic welfare/loss framework, we could pin down the optimal policy of a monetary authority. However, as we saw, while this may be great in theory, it is very difficult to practically implement. The parameters of the optimal policy – whether in discretion or commitment – would be difficult to estimate and calibrate.

The key readings for this chapter are really chapters 4 and 5 of Galí (2015). Almost all of the notes here are based on Galí's texts, but with slight adjustments made to notation and expanding derivations. There are also the seminal papers "The Science of Monetary Policy: A New Keynesian Perspective" Clarida et al. (1999) and "Real Wage Rigidities and the New Keynesian Model" Blanchard and Galí (2007) which are worth a look.

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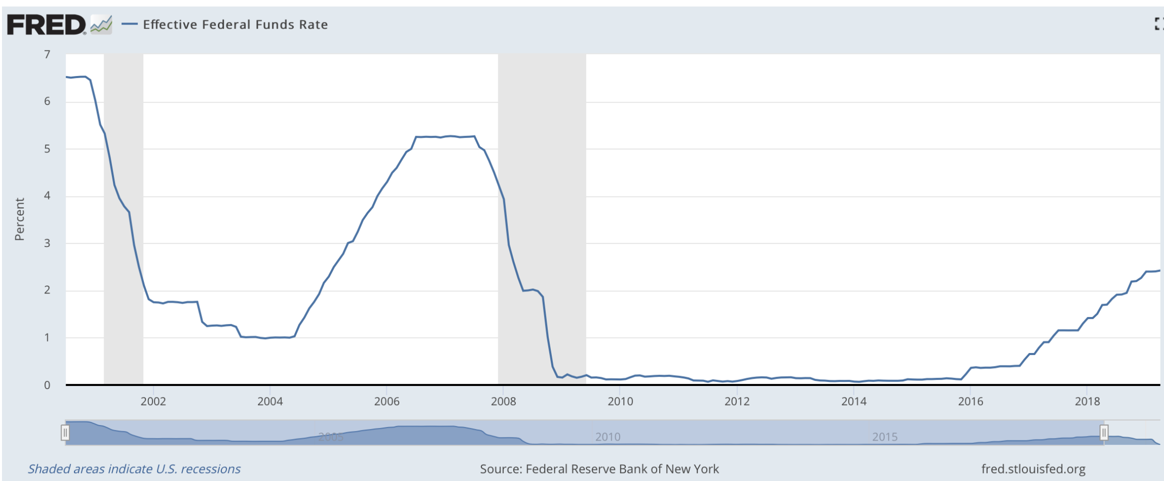
14 The Zero Lower Bound

14.1 Introduction

In the standard New Keynesian model, monetary policy is often described by an interest rate rule (e.g. a Taylor Rule) that moves the interest rate in response to deviations of inflation and some measure of economic activity from target. Nominal interest rates are bound from below by 0. How does the behaviour of the New Keynesian model change when interest rates hit zero and cannot freely adjust in response to changing economic conditions?



As we proceed, we will start to adopt more notation and terminology found in the literature, and as we've gone through developing the New Keynesian model quite thoroughly up until this point, we will skip a lot details on the model. I will also note that in discussing optimal policy with the ZLB, I will first adopt a fairly quantitative approach. Those that prefer less dialogue and a more equation-based treatment should find themselves at home. However, upon writing this section, I found that it would be worth going through a more descriptive approach than initially planned. Thus I have included Galí's treatment of optimal policy and the ZLB.

14.2 Optimal policy in a simple two-period model

We first delve into the ZLB by considering a two-period setup, which captures some of the key findings in classical papers on optimal policy and the ZLB: Eggertson and Woodford (2003), Jung et al. (2005), Adam and Billi (2006), and Nakov (2008).

14.2.1 Model overview:

The economy starts at $t = 1$ and ends in $t = 2$, and the private sector's equilibrium conditions are given by the DISE and the NKPC. At $t = 1$:

$$y_1 = y_2 - \sigma(i_1 - \pi_2 - r_1^f),$$
$$\pi_1 = \kappa y_1 + \beta\pi_2,$$
$$i_1 \geq 0.$$

At $t = 2$:

$$y_2 = -\sigma(i_2 - r_2^f),$$
$$\pi_2 = \kappa y_2,$$
$$i_2 \geq 0.$$

Where here y_t denotes output, π_t is inflation, i_t is the short term interest rate, and r_t^f is driven by an exogenous process with $r_1^f < 0$ and $r_2^f = r^* > 0$.

Society's welfare is given by the standard quadratic objective function:

$$\mathbb{W} = u(\pi, y) = -\frac{1}{2}(\pi^2 + \vartheta y^2).$$

The central bank either: Optimises every period, taking as given future allocations (optimal discretionary policy or "Markov-Perfect Policy"); optimises once at $t = 1$ (optimal commitment policy or "Ramsey Policy"); and, follows an interest-rate feedback rule.

14.2.2 Optimal discretionary policy (Markov-Perfect Policy)

At $t = 1$, the central bank's problem is:

$$V_1 = \max_{\pi_1, y_1, i_1} \; u(\pi_1, y_1) + \beta V_2,$$

subject to the DISE, NKPC, and ZLB constraint, and taking V_2, y_2, and π_2 as given.

At $t = 2$, its problem is:

$$V_2 = \max_{\pi_2, y_2, i_2} \; u(\pi_2, y_2),$$

subject to the DISE, NKPC, and ZLB constraint.

A Markov-Perfect Equilibrium (MPE) is defined as a vector $\{y_1, \pi_1, i_1, y_2, \pi_2, i_2\}$ that solves these two problems.

We can begin to solve this problem by first constructing a Lagrangian[93] for the period 2 problem as:

$$\begin{aligned}\mathcal{L}_{MP,2} = &-\frac{1}{2}\left[\pi_2^2 + \vartheta y_2^2\right] \\ &+ \xi_{EE,2}(y_2 + \sigma i_2 - \sigma r_2^f) \\ &+ \xi_{PC,2}(\pi_2 - \kappa y_2) \\ &+ \xi_{ELB,2} i_2.\end{aligned}$$

At $t = 2$, the FOCs for the Lagrangian are:

$$\frac{\partial \mathcal{L}_{MP,2}}{\partial y_2} = -\vartheta y_2 + \xi_{EE,2} - \kappa \xi_{PC,2} = 0, \tag{565}$$

$$\frac{\partial \mathcal{L}_{MP,2}}{\partial \pi_2} = -\pi_2 + \xi_{PC,2} = 0, \tag{566}$$

$$\frac{\partial \mathcal{L}_{MP,2}}{\partial i_2} = \sigma \xi_{EE,2} + \xi_{ELB,2} = 0, \tag{567}$$

[93] I'm adopting the notation from Nakata here for the Lagrangian multipliers. "EE" is Euler Equation, "PC" is Phillips Curve, and "ELB" is Extended Lower Bound, a term that macroeconomists began to use as some policy rates went below zero.

and where $\xi_{ELB,2} \geq 0$. The first two FOCs can be combined to yield an expression without $\xi_{EE,2}$ and $\xi_{PC,2}$:

$$\xi_{ELB,2} = \vartheta y_2 + \kappa \pi_2. \tag{568}$$

We now have two potential solutions for (568):

$$\begin{aligned} &\text{Case 1: } i_2 > 0 \Leftrightarrow \xi_{ELB,2} = 0, \text{ or,} \\ &\text{Case 2: } i_2 = 0 \Leftrightarrow \xi_{ELB,2} \geq 0. \end{aligned}$$

Let's assume that we have the first case where $i_2 > 0$ and the ZLB constraint is non-binding ($\xi_{ELB,2} = 0$). Our period 2 DISE and NKPC are:

$$\begin{aligned} y_2 &= -\sigma[i_2 - r_2^f], \\ \pi_2 &= \kappa y_2, \end{aligned}$$

and (568) implies:

$$0 = \vartheta y_2 + \kappa \pi_2.$$

Since ϑ, σ, and κ cannot be equal to 0, it must be the case that $y_2 = \pi_2 = 0$, which must then imply that $i_2 = r_2^f$.

Now, consider the second case for (568), where the ZLB constraint is binding since $i_2 = 0$. From (568) we have that:

$$0 \geq \vartheta y_2 + \kappa \pi_2,$$

and since $i_2 = 0$, from our DISE we have that $y_2 = \sigma r_2^f$, which must then imply that $y_2 > 0$ and $\pi_2 > 0$.

Now, we look at the problem at time $t = 1$. The Lagrangian is:

$$
\begin{aligned}
\mathcal{L}_{MP,1} = &-\frac{1}{2}\left[\pi_1^2 + \vartheta y_1^2\right] \\
&+ \xi_{EE,1}(y_1 - y_2 + \sigma i_1 - \sigma\pi_2 - \sigma r_1^f) \\
&+ \xi_{PC,1}(\pi_1 - \kappa y_1 - \beta\pi_2) \\
&+ \xi_{ELB,1} i_1,
\end{aligned}
$$

and the FOCs are:

$$\frac{\partial \mathcal{L}_{MP,1}}{\partial y_1} = -\vartheta y_1 + \xi_{EE,1} - \kappa\xi_{PC,1} = 0, \tag{569}$$

$$\frac{\partial \mathcal{L}_{MP,1}}{\partial \pi_1} = -\pi_1 + \xi_{PC,1} = 0, \tag{570}$$

$$\frac{\partial \mathcal{L}_{MP,1}}{\partial i_1} = \sigma\xi_{EE,1} + \xi_{ELB,1} = 0, \tag{571}$$

and where $\xi_{ELB,1} \geq 0$. As before, the first two FOCs can be combined to yield an expression without $\xi_{EE,1}$ and $\xi_{PC,1}$:

$$\xi_{ELB,1} = \vartheta y_1 + \kappa\pi_1. \tag{572}$$

and we have two potential solutions for (572):

$$
\begin{aligned}
&\text{Case 1: } i_1 > 0 \Leftrightarrow \xi_{ELB,1} = 0, \text{ or,} \\
&\text{Case 2: } i_1 = 0 \Leftrightarrow \xi_{ELB,1} \geq 0.
\end{aligned}
$$

For the first case ($i_1 > 0$) we would have the following for the DISE and NKPC:

$$
\begin{aligned}
y_1 &= -\sigma(i_1 - r_1^f), \\
\pi_1 &= \kappa y_1,
\end{aligned}
$$

as this time we are in period 1 and the central bank is operating with discretion. Additionally, from our FOCs we have:

$$0 = \vartheta y_1 + \kappa \pi_1.$$

With some rearranging we can say that:

$$y_1 = \pi_1 = 0,\ i_i = r_1^f.$$

For the second case we have:

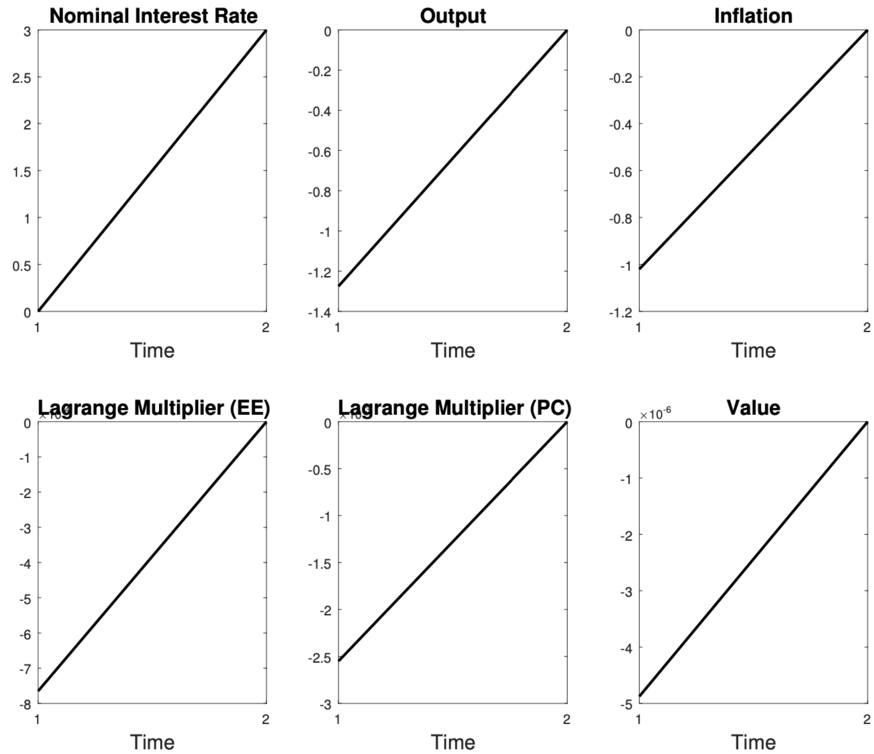
$$\begin{aligned} y_1 &= -\sigma(i_1 - r_1^f), \\ \pi_1 &= \kappa y_1, \\ 0 &\geq \vartheta y_1 + \kappa \pi_1, \\ i_1 &= 0, \end{aligned}$$

which implies that:

$$y_1 < 0, \pi_1 < 0.$$

We can characterise the optimal discretionary policy with the following figure:

Figure 91: Optimal Discretionary Policy



Key features are that the economy is at its steady state at $t = 2$, as the output gap is closed and inflation is stabilised, and also note that the $V_{MP,1} < 0$ and $V_{MP,2} = 0$.

14.2.3 Optimal commitment policy (Ramsey Policy)

Now in period 1, the central bank chooses a sequence $\{y_t, \pi_t, i_t\}_{t=1}^{2}$ to maximise time-one welfare:

$$V_{RAM,1} = \max_{\pi_1, y_1, i_1, \pi_2, y_2, i_2} u(\pi_1, y_1) + \beta u(\pi_2, y_2),$$

subject to the DISE, NKPC, and ZLB constraints for $t = 1$ and $t = 2$. Thus, our Lagrangian is:

$$\begin{aligned}
\mathcal{L}_{RAM} = & -\frac{1}{2}[\pi_1^2 + \vartheta y_1^2] - \beta\frac{1}{2}[\pi_2^2 + \lambda y_2^2] \\
& + \xi_{EE,1}(y_1 - y_2 + \sigma i_1 - \sigma\pi_2 - \sigma r_1^f] \\
& + \xi_{PC,1}(\pi_1 - \kappa y_1 - \beta\pi_2) \\
& + \xi_{ELB,1} i_1 \\
& + \xi_{EE,2}(y_2 + \sigma i_2 - \sigma r_2^f) \\
& + \xi_{PC,2}(\pi_2 - \kappa y_2) \\
& + \xi_{ELB,2} i_2,
\end{aligned}$$

with the following FOCs:

$$\frac{\partial \mathcal{L}_{RAM}}{\partial y_1} = -\vartheta y_1 + \xi_{EE,1} - \kappa\xi_{PC,1} = 0, \tag{573}$$
$$\frac{\partial \mathcal{L}_{RAM}}{\partial \pi_1} = -\pi_1 + \xi_{PC,1} = 0, \tag{574}$$
$$\frac{\partial \mathcal{L}_{RAM}}{\partial i_1} = \sigma\xi_{EE,1} + \xi_{ELB,1} = 0, \tag{575}$$

and $\xi_{ELB,1} \geq 0$,

$$\frac{\partial \mathcal{L}_{RAM}}{\partial y_2} = -\vartheta y_2 - \frac{1}{\beta}\xi_{EE,1} + \xi_{EE,2} - \kappa\xi_{PC,2} = 0, \tag{576}$$
$$\frac{\partial \mathcal{L}_{RAM}}{\partial \pi_2} = -\pi_2 - \frac{1}{\beta}\xi_{EE,1} - \xi_{PC,1} + \xi_{PC,2} = 0, \tag{577}$$
$$\frac{\partial \mathcal{L}_{RAM}}{\partial i_2} = \sigma\xi_{EE,2} + \xi_{ELB,2} = 0, \tag{578}$$

and $\xi_{ELB,2} \geq 0$.

We have 4 potential cases to deal with:

$$\text{Case 1: } i_1 = 0, i_2 > 0; \xi_{ELB,1} \geq 0, \xi_{ELB,2} = 0,$$

$$\text{Case 2: } i_1 = 0, i_2 = 0; \xi_{ELB,1} \geq 0, \xi_{ELB,2} \geq 0,$$

$$\text{Case 3: } i_1 > 0, i_2 > 0; \xi_{ELB,1} = 0, \xi_{ELB,2} = 0,$$

$$\text{Case 4: } i_1 > 0, i_2 = 0; \xi_{ELB,1} = 0, \xi_{ELB,2} \geq 0.$$

Plotting the optimal commitment policy, we see:

Figure 92: Optimal Commitment Policy

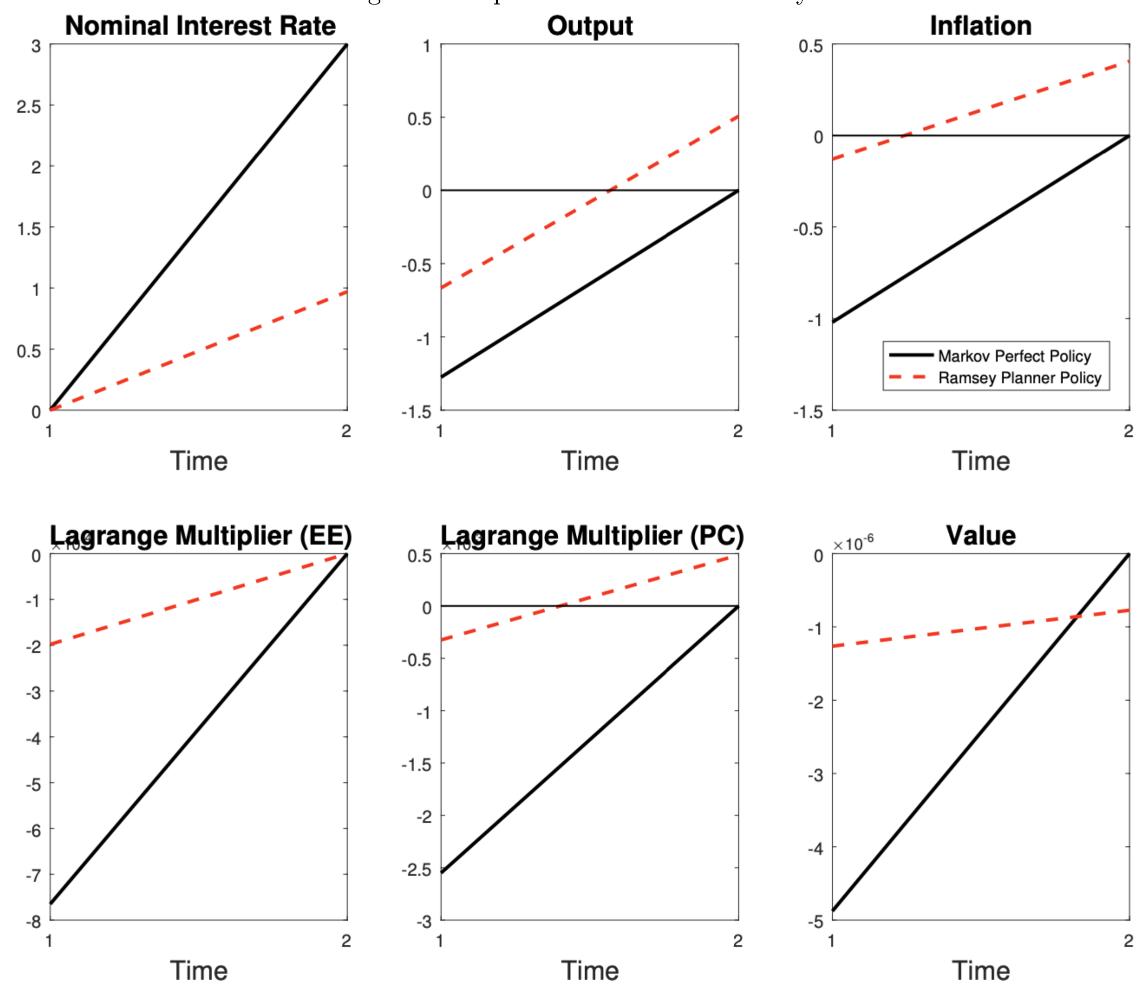
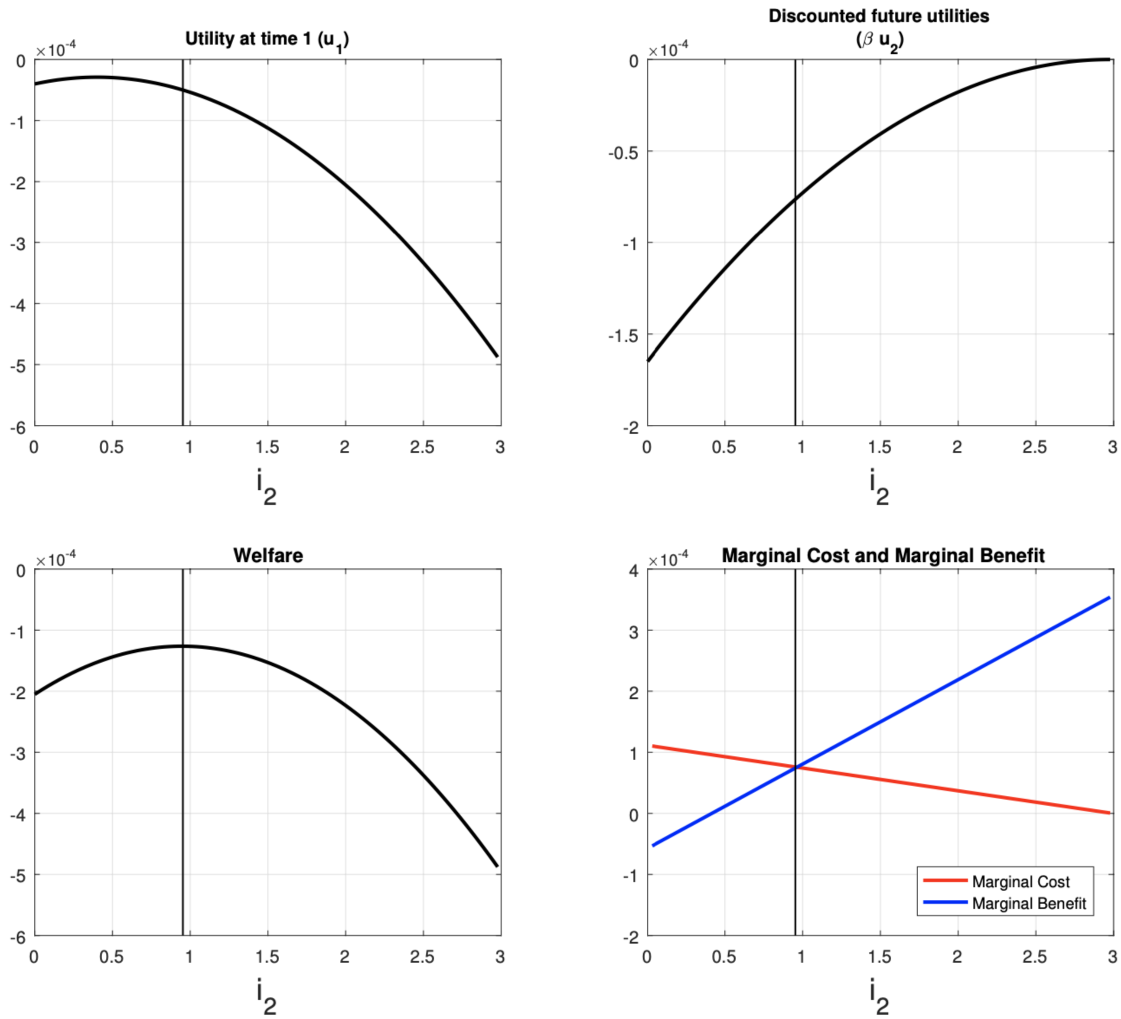


Figure 93: Optimal Commitment Policy

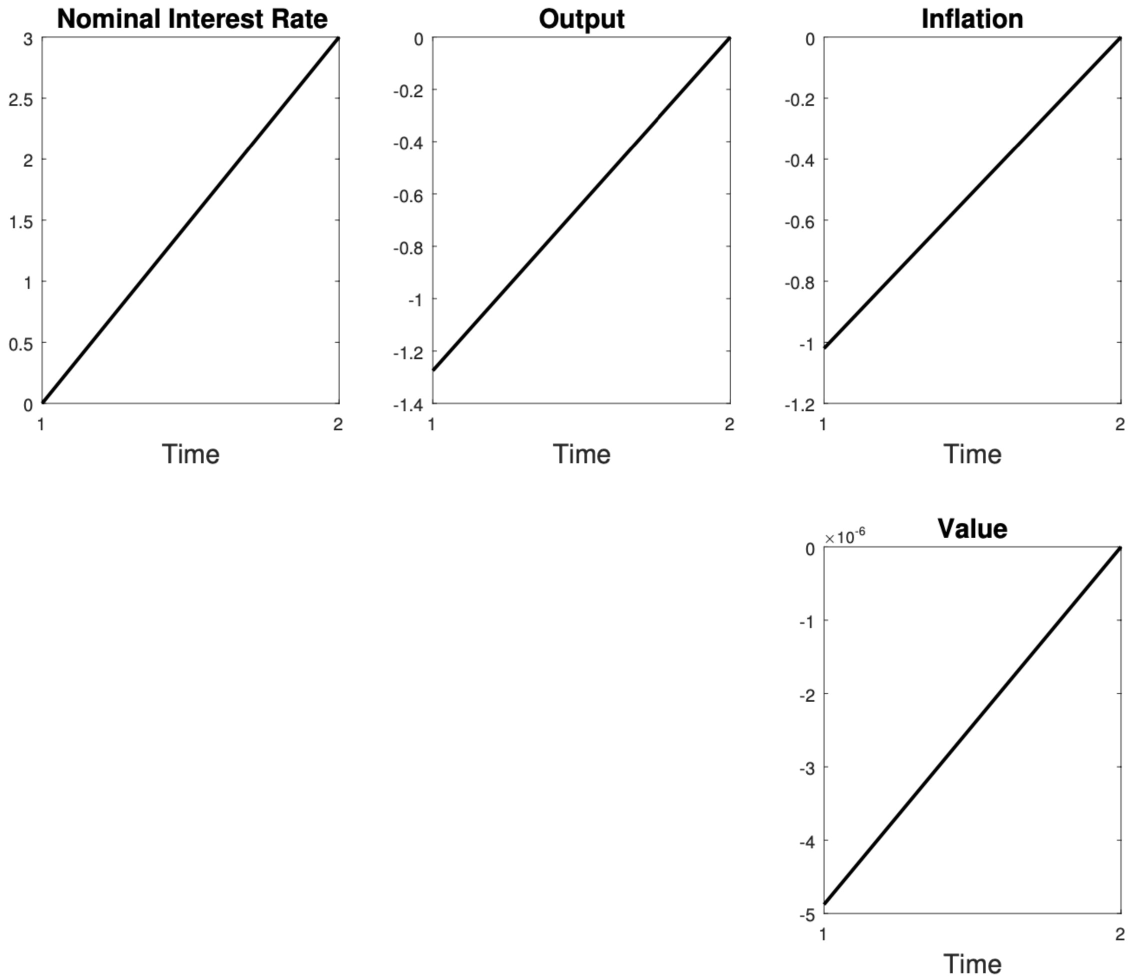


Bare in mind that we are still dealing with a very simple setup here, but some key features emerge when we look at optimal policy under commitment: The policy rate is kept "low-for-long"; inflation and output "overshoot" their target levels at $t = 2$; the declines in output and inflation are small, and $V_{RAM,1} > V_{MP,1}$; and, $V_{RAM,2} < 0 = V_{MP,2}$, which implies that there is some time-inconsistency.

14.2.4 Equilibrium under interest-rate feedback rules

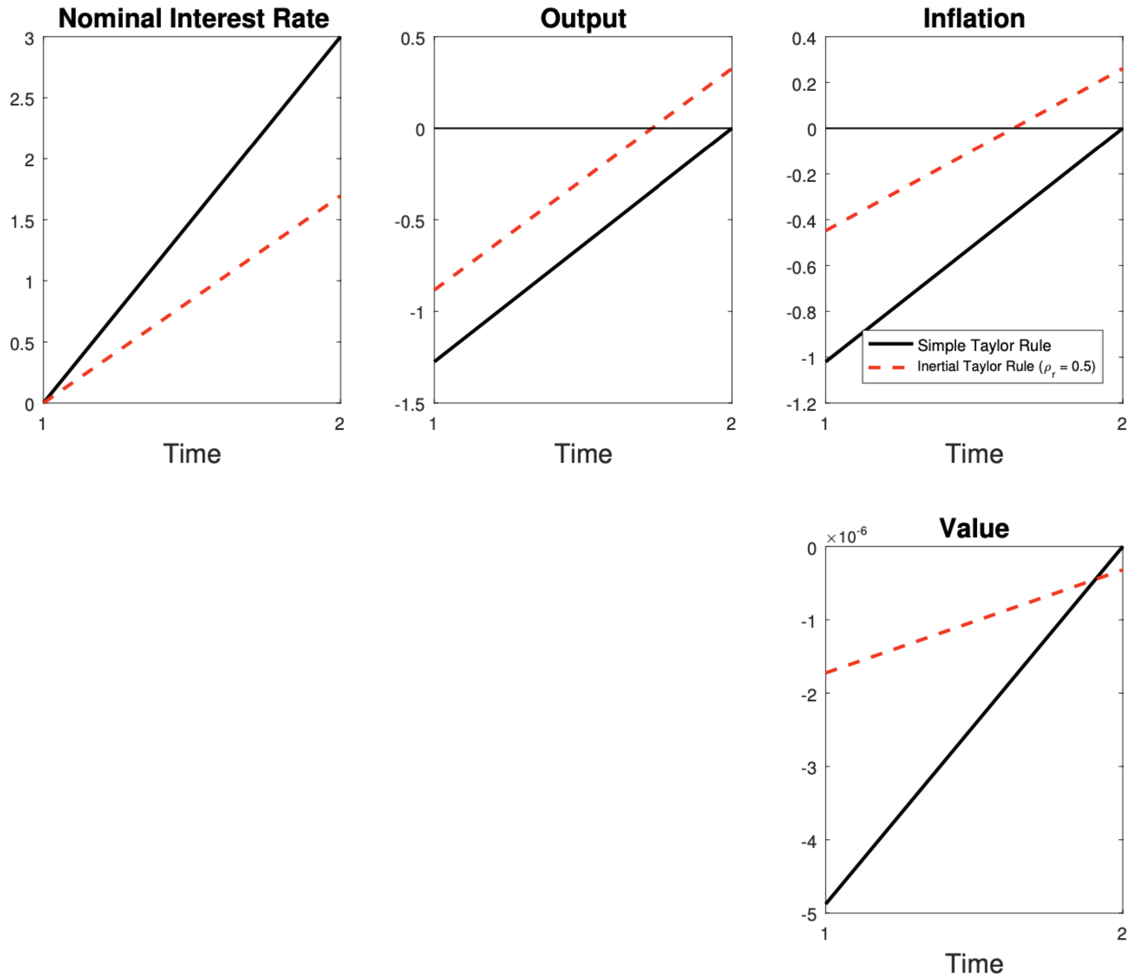
Before moving onto our familiar infinite-horizon setup, it's worth looking at some interest rate rules that are quite popular in the literature. We won't go through them too much – instead we will plot out the equilibrium paths and the rules in the figures below.

Figure 94: Taylor Rule



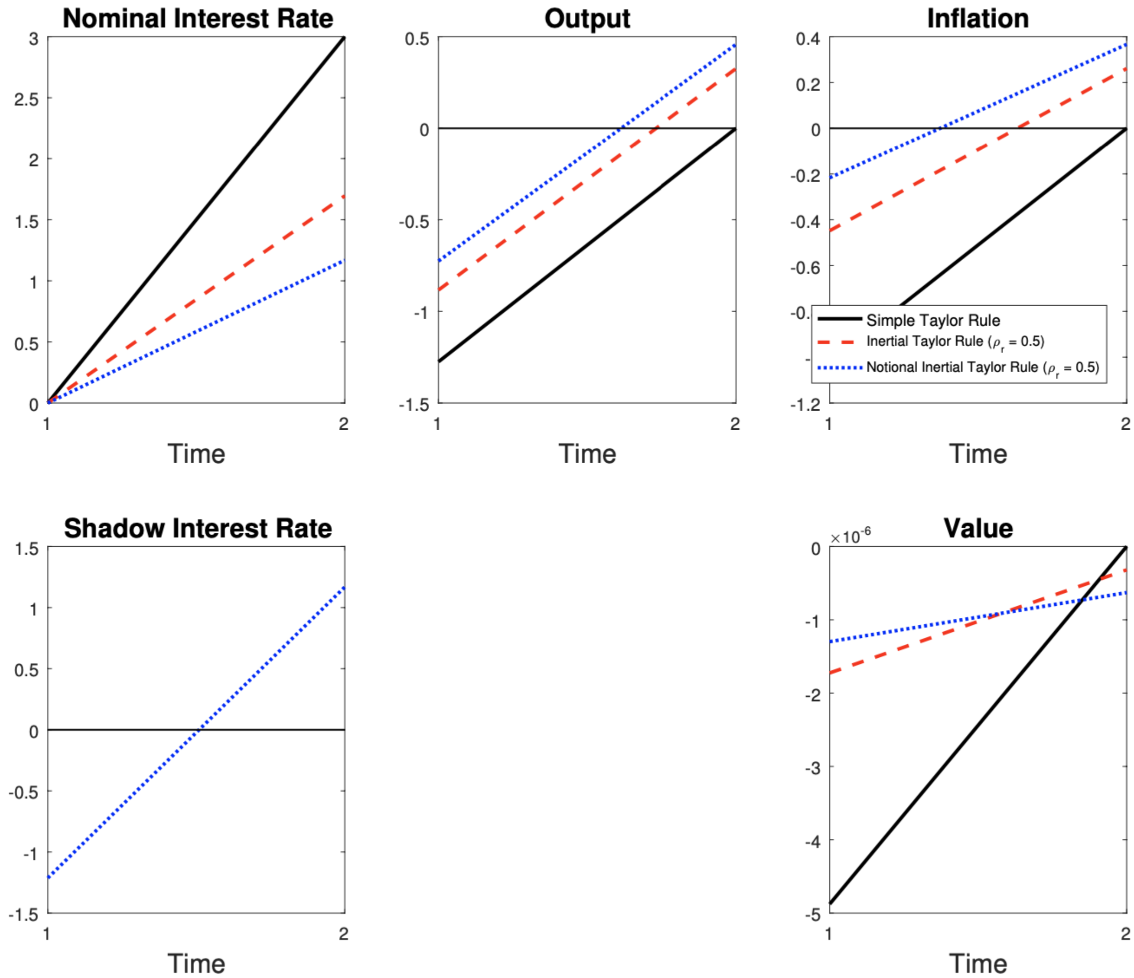
$$
\begin{aligned}
i_t &= \max[0, i_t^*] \\
i_t^* &= \bar{r}^f + \phi_\pi \pi_t
\end{aligned}
$$

Figure 95: Inertial Taylor Rule with Lagged Actual Rate



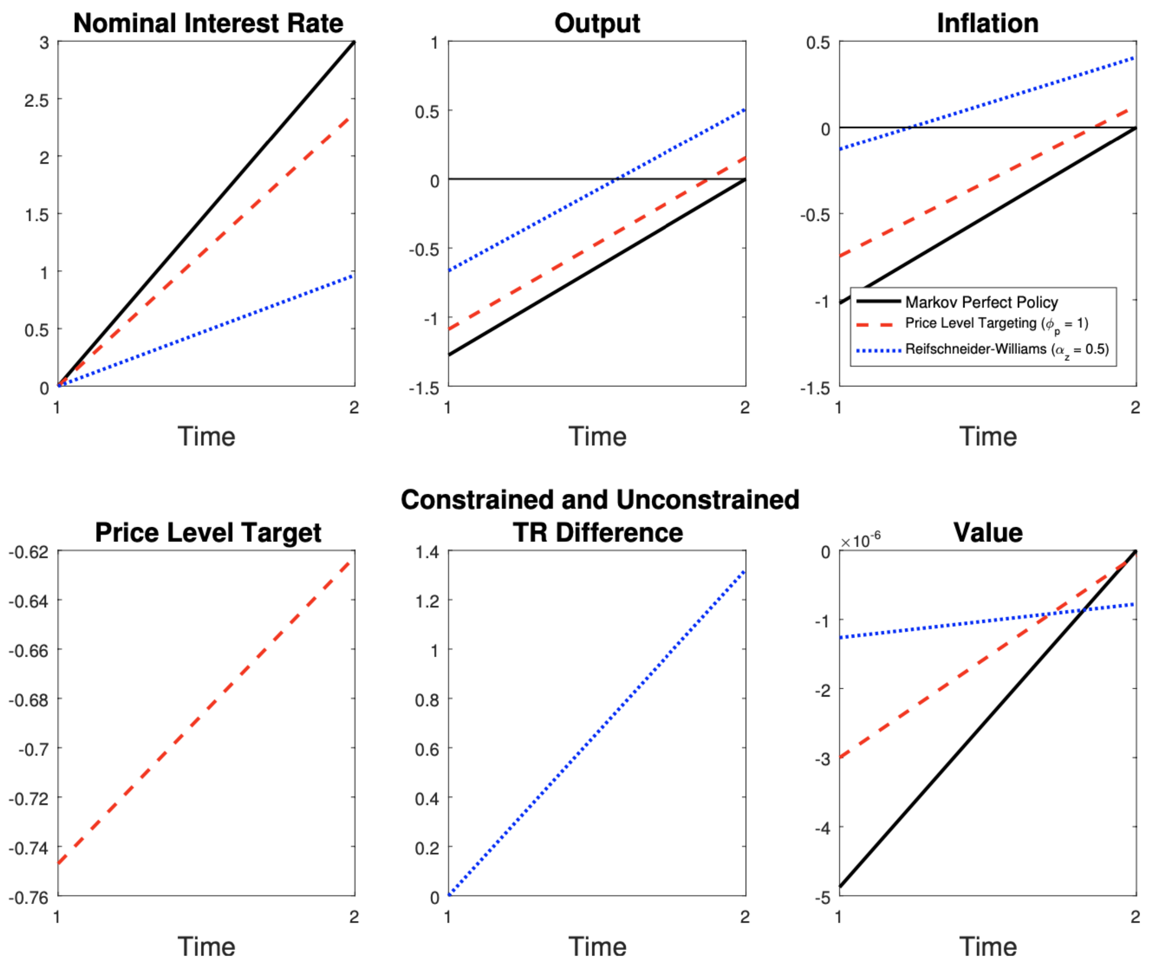
$$
\begin{aligned}
i_t &= \max[0, i_t^*] \\
i_t^* &= (1-\rho)\bar{r}^f + +\rho i_{t-1} + (1-\rho)\phi_\pi \pi_t
\end{aligned}
$$

Figure 96: Inertial Taylor Rule with Lagged Shadow Rate



$$i_t = \max[0, i_t^*]$$
$$i_t^* = (1-\rho)\bar{r}^f + \rho i_{t-1}^* + (1-\rho)\phi_\pi \pi_t$$

Figure 97: Price-Level Targeting (PLT) and Reifscshneider-Williams Rule (RW)



$$\text{PLT: } i_t = \max[0, i_t^*]$$
$$i_t^* = \bar{r}^f + \phi_p(p_t - \bar{p}) + (1 - \phi_p)\phi_\pi \pi_t$$

$$\text{RW Rule: } i_t = \max[0, i_t^* - \alpha Z_t]$$
$$i_t^* = \bar{r}^f + \phi_\pi \pi_t$$
$$Z_t = Z_{t-1} + (i_{t-1} - i_{t-1}^*)$$
$$Z_0 = 0$$

14.3 Optimal policy in an infinite-horizon model

We now move to a setup more frequently used in the literature, and stick to the notation used by Eggertson and Woodford (2003) and Jung et al. (2005). Again, the private sector/non-policy block equilibrium conditions are based on the DISE and NKPC:

$$
\begin{aligned}
y_t &= \mathbb{E}_t y_{t+1} - \sigma(i_t - \mathbb{E}_t \pi_{t+1} - r_t^f), \\
\pi_t &= \beta \mathbb{E}_t \pi_{t+1} + \kappa y_t,
\end{aligned}
$$

where we also have $i_t \geq 0$, so our nominal interest is still bound from below by 0. In this setup, r_t^f takes the value of either r_H or r_L, where:

$$
\begin{aligned}
r_H &= r^* > 0, \\
r_L &< 0,
\end{aligned}
$$

whereby r_H is the high (normal) state and r_L is the low (crisis) state. The transition probabilities are given by:

$$
\begin{aligned}
\Pr(r_{t+1}^f = r_L | r_t^f = r_H) &= p_H \text{ [crisis frequency]}, \\
\Pr(r_{t+1}^f = r_L | r_t^f = r_L) &= p_L \text{ [crisis persistence]}.
\end{aligned}
$$

So, p_H denotes the transition probably that we go from a normal state to a crisis state, and p_L denotes the probability that we stay in a crisis state.

14.3.1 Optimal policy under discretion (Markov-Perfect Policy)

At each t, the discretionary central bank chooses $\{y_t, \pi_t, i_t\}$ to maximise:

$$
V_t(r_t^f) = \max \ u(\pi_t, y_t) + \beta \mathbb{E}_t V_{t+1}(r_{t+1}^f),
$$

subject to the DISE, NKPC, and ZLB constraints, and taking $V_{t+1}(\cdot)$, $y_{t+1}(\cdot)$, and $\pi_{t+1}(\cdot)$ as given.

The MPP equilibrium is defined as a set of time-invariant value and policy functions $\{V(\cdot), y(\cdot), \pi(\cdot), i(\cdot)\}$ that solves the central bank's problem. There are four potential equilibria:

$$\begin{aligned}
\text{Type I: } & i_H > 0, i_L = 0, \\
\text{Type II: } & i_H = 0, i_L = 0, \\
\text{Type III: } & i_H = 0, i_L > 0, \\
\text{Type IV: } & i_H > 0, i_L > 0.
\end{aligned}$$

We now go through each of the four types of equilibria.

First, consider Type I. The DISE, NKPC, and FOCs imply:

$$\begin{aligned}
y_H &= [(1-p_H)y_H + p_H y_L] - \sigma\left[i_H - (1-p_H)\pi_H - p_H\pi_L - r^*\right], \\
\pi_H &= \beta\left[(1-p_H)\pi_H + p_H\pi_L\right] + \kappa y_H, \\
0 &= \vartheta y_H + \kappa\pi_H, \\
y_L &= [(1-p_L)y_H + p_L y_L] - \sigma\left[i_L - (1-p_L)\pi_H - p_L\pi_L - r_L\right], \\
\pi_L &= \beta\left[(1-p_L)\pi_H + p_L\pi_L\right] + \kappa y_L, \\
i_L &= 0,
\end{aligned}$$

and it satisfies the following two inequality constraints:

$$\begin{aligned}
i_H &> 0, \\
\vartheta y_L + \kappa\pi_L &< 0.
\end{aligned}$$

Now, there's Type II ($i_H = 0, i_L = 0$):

$$
\begin{aligned}
y_H &= [(1-p_H)y_H + p_H y_L] - \sigma\left[i_H - (1-p_H)\pi_H - p_H\pi_L - r^*\right], \\
\pi_H &= \beta\left[(1-p_H)\pi_H + p_H\pi_L\right] + \kappa y_H, \\
i_H &= 0, \\
y_L &= [(1-p_L)y_H + p_L y_L] - \sigma\left[i_H - (1-p_H)\pi_H - p_H\pi_L - r_L\right], \\
\pi_L &= \beta\left[(1-p_L)\pi_H + p_L\pi_L\right] + \kappa y_L, \\
i_L &= 0,
\end{aligned}
$$

and it satisfies the following two inequality constraints:

$$
\begin{aligned}
\vartheta y_H + \kappa\pi_H &\leq 0, \\
\vartheta y_L + \kappa\pi_L &\leq 0.
\end{aligned}
$$

Type III ($i_H = 0, i_L > 0$):

$$
\begin{aligned}
y_H &= [(1-p_H)y_H + p_H y_L] - \sigma\left[i_H - (1-p_H)\pi_H - p_H\pi_L - r^*\right], \\
\pi_H &= \beta\left[(1-p_H)\pi_H + p_H\pi_L\right] + \kappa y_H, \\
i_H &= 0, \\
y_L &= [(1-p_L)y_H + p_L y_L] - \sigma\left[i_H - (1-p_H)\pi_H - p_H\pi_L - r_L\right], \\
\pi_L &= \beta\left[(1-p_L)\pi_H + p_L\pi_L\right] + \kappa y_L, \\
0 &= \vartheta y_L + \kappa\pi_L,
\end{aligned}
$$

and it satisfies the following two inequality constraints:

$$
\begin{aligned}
\vartheta y_H + \kappa\pi_H &\leq 0, \\
i_L &> 0.
\end{aligned}
$$

...and Type IV ($i_H > 0, i_L > 0$):

$$
\begin{aligned}
y_H &= [(1-p_H)y_H + p_H y_L] - \sigma\left[i_H - (1-p_H)\pi_H - p_H\pi_L - r^*\right], \\
\pi_H &= \beta\left[(1-p_H)\pi_H + p_H\pi_L\right] + \kappa y_H, \\
0 &= \vartheta y_H + \kappa\pi_H, \\
y_L &= [(1-p_L)y_H + p_L y_L] - \sigma\left[i_H - (1-p_H)\pi_H - p_H\pi_L - r_L\right], \\
\pi_L &= \beta\left[(1-p_L)\pi_H + p_L\pi_L\right] + \kappa y_L, \\
0 &= \vartheta y_L + \kappa\pi_L,
\end{aligned}
$$

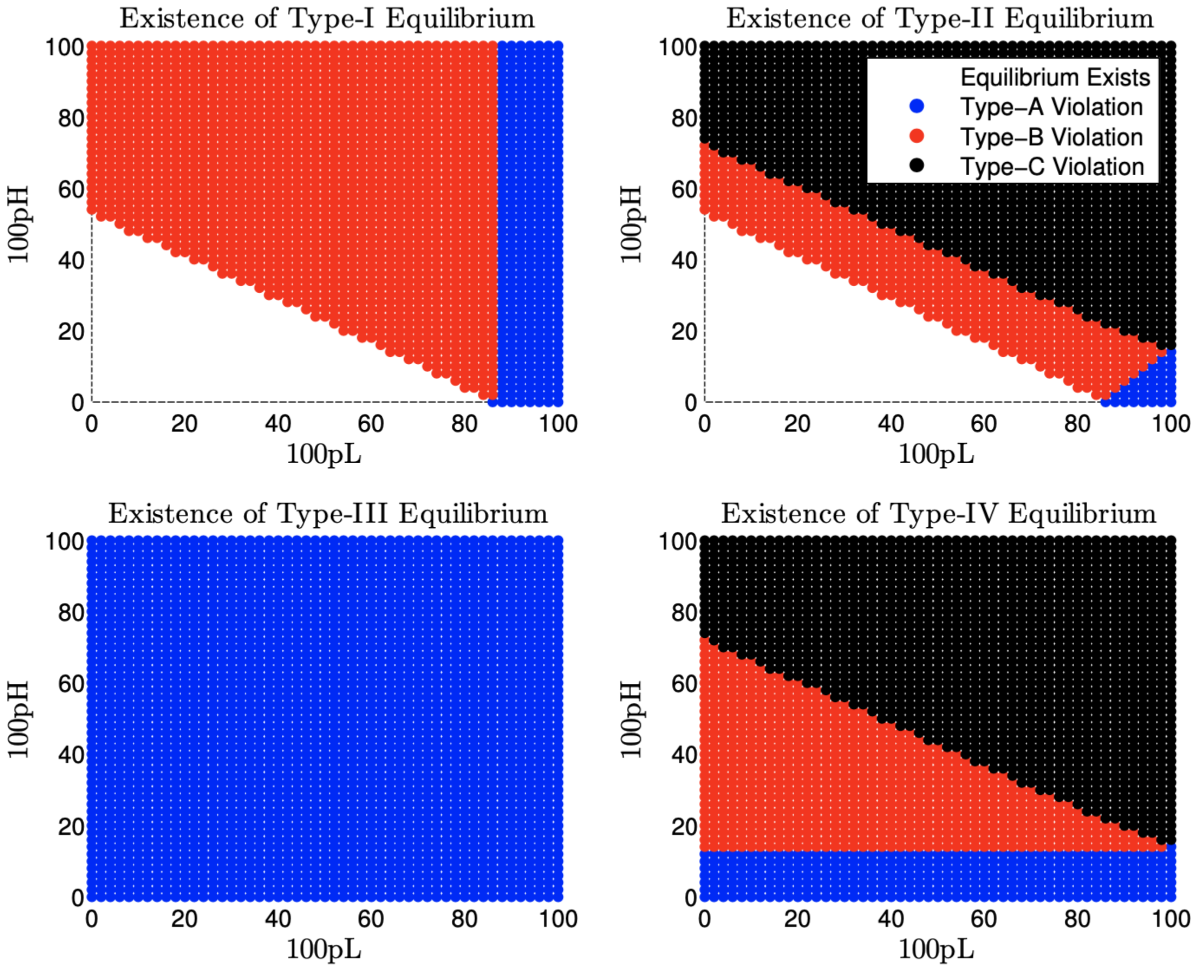
and it satisfies the following two inequality constraints:

$$
\begin{aligned}
i_H &> 0, \\
i_L &> 0.
\end{aligned}
$$

Nakata Nakata (2018) and Nakata and Schmidt (2019a) show that:

- Type I: Exists if p_H and p_L are sufficiently low;
- Type II: Exists if p_H and p_L are sufficiently low;
- Type III: Does not exist; and
- Type IV: Does not exist.

Figure 98: Equilibrium Existence



So, we focus on the Type I MPP equilibrium in which $i_H > 0$ and $i_L = 0$. We also assume that the normal state is an absorbing state (i.e., $p_H = 0$), as is common in the literature. In the normal state

we have:

$$
\begin{aligned}
y_H &= y_H - \sigma[i_H - \pi_H - r^*], \\
\pi_H &= \beta \pi_H + \kappa y_H, \\
0 &= \vartheta y_H + \kappa \pi_H, \\
i_H &> 0,
\end{aligned}
$$

which implies:

$$y_H = \pi_H = 0, i_H = r_H^f.$$

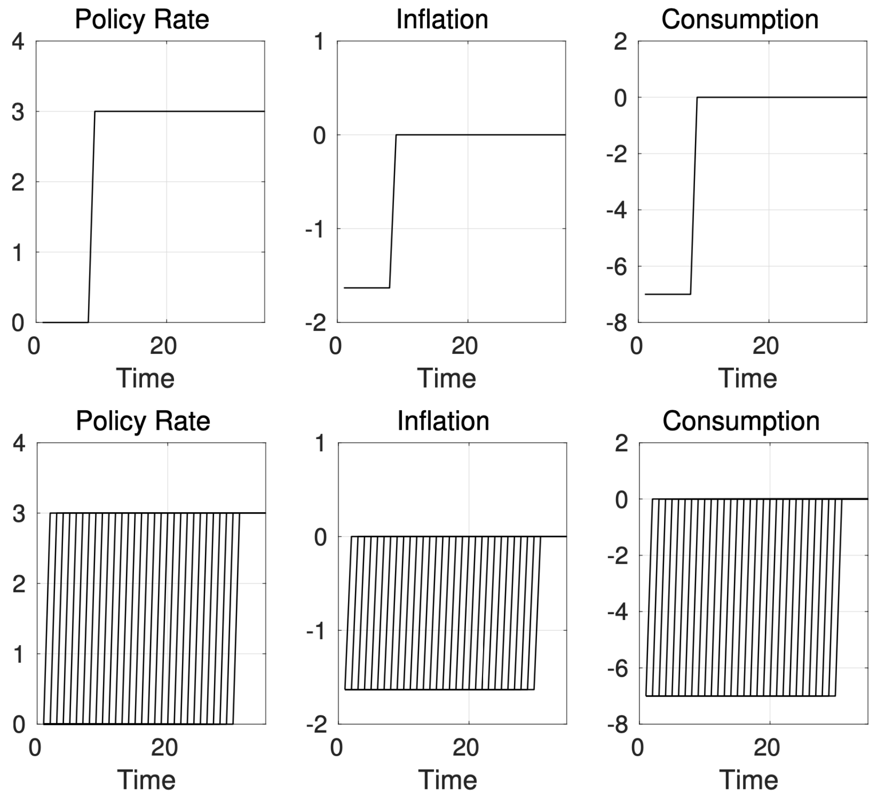
In the crisis state:

$$
\begin{aligned}
y_L &= p_L y_L - \sigma[i_L - p_L \pi_L - r_L], \\
\pi_L &= \beta p_L \pi_L + \kappa y_L, \\
i_L &= 0, \\
\vartheta y_L + \kappa \pi_L &< 0,
\end{aligned}
$$

which implies:

$$y_L < 0, \pi_L < 0, i_L = 0.$$

Figure 99: Type I Discretionary Policy



14.3.2 Optimal policy under commitment (Ramsey Policy)

At the beginning of $t = 1$, the central bank chooses the state-contingent sequence of $\{y_t, \pi_t, i_t\}$ in order to maximise the expected discounted sum of future utility flows at time one:

$$\max \sum_{t=0}^{\infty} \sum_{s^t \in \mathbb{S}^t} \beta^{t-1} u(y_t(s^t), \pi_t(s^t)),$$

subject to the DISE, NKPC, and ZLB constraints for all $t \geq 1$ and for all $s^t \in \mathbb{S}^t$, and the Ramsey equilibrium is defined as the $\arg\max$ of this optimisation problem. Note, on notation: $s^t = \{s_k\}_{k=1}^t$, where s_t is generic notation for exogenous shocks (here, $s_t = r_t^f$).

The partial derivatives with respect to the time-one variables are:

$$\frac{\partial \mathcal{L}}{\partial y_1} = -\vartheta y_1 + \xi_{EE,1} - \kappa \xi_{PC,1} = 0, \tag{579}$$

$$\frac{\partial \mathcal{L}}{\partial \pi_1} = -\pi_1 + \xi_{PC,1} = 0, \tag{580}$$

$$\frac{\partial \mathcal{L}}{\partial i_1} = \sigma \xi_{EE,1} - \xi_{ELB,1} = 0, \tag{581}$$

and $\xi_{ELB,1} \geq 0$. Partial derivatives with respect to the time t variables are:

$$\frac{\partial \mathcal{L}}{\partial y_t} = -\vartheta y_t - \frac{1}{\beta}\xi_{EE,t-1} + \xi_{EE,t} - \kappa \xi_{PC,t} = 0, \tag{582}$$

$$\frac{\partial \mathcal{L}}{\partial \pi_t} = -\pi_t - \frac{1}{\beta}\xi_{EE,t-1} + \xi_{PC,t-1} + \xi_{PC,t} = 0, \tag{583}$$

$$\frac{\partial \mathcal{L}}{\partial i_t} = \sigma \xi_{EE,t} + \xi_{ELB,t} = 0, \tag{584}$$

and $\xi_{ELB,t} \geq 0$.

In linear models (be it deterministic or stochastic), one can use standard LQ methods to solve (e.g. via Gensys (by Chris Sims) or Dynare). In nonlinear models, solution methods are outlined by Marcet and Marimon (2019) and Eggertson and Woodford (2003), for when the shock a two-state Markov and there is an absorbing state.

For $\mathbb{S}_t = (\xi_{PC,t-1}, \xi_{EE,t-1}, r_t^f)$ we have:

$$0 = y(\mathbb{S}_t) - \mathbb{E}_t y(\mathbb{S}_{t+1}) - \mathbb{E}_t \pi(\mathbb{S}_{t+1}) + i(\mathbb{S}_t) + r_t^f, \tag{585}$$

$$0 = \pi(\mathbb{S}_t) - \kappa y(\mathbb{S}_t) - \beta \mathbb{E}_t \pi(\mathbb{S}_{t+1}), \tag{586}$$

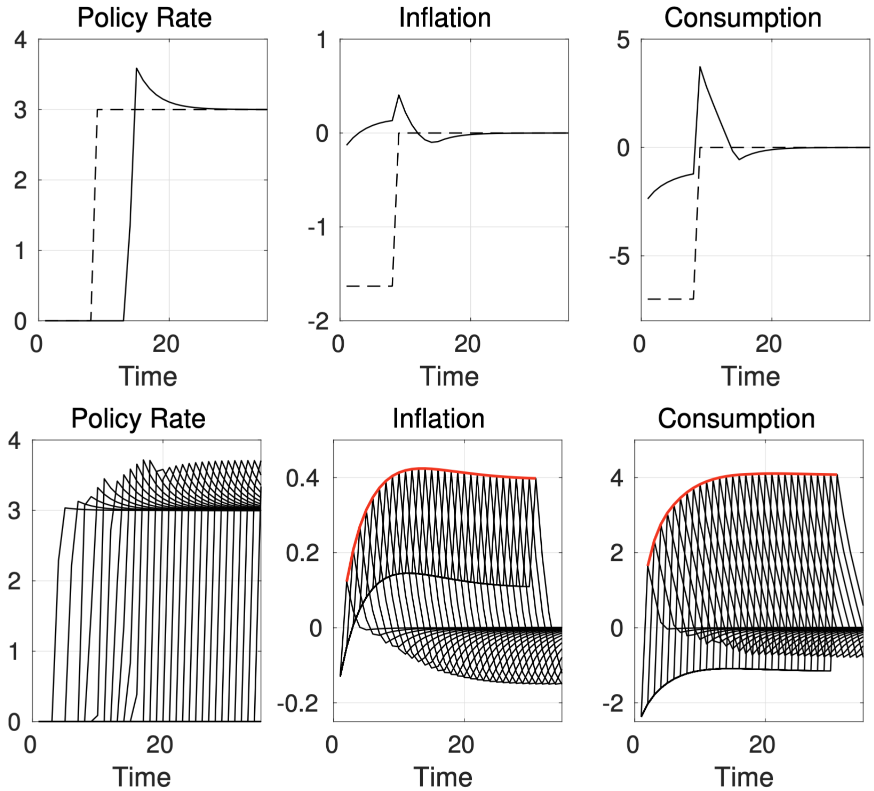
$$0 = -\vartheta y(\mathbb{S}_t) - \frac{1}{\beta}\xi_{EE,t-1} + \xi_{EE}(\mathbb{S}_t) - \kappa \xi_{PC}(\mathbb{S}_t), \tag{587}$$

$$0 = -\pi(\mathbb{S}_t) - \frac{1}{\beta}\xi_{EE,t-1} - \xi_{PC,t-1} + \xi_{PC}(\mathbb{S}_t), \tag{588}$$

$$0 = \sigma \xi_{EE}(\mathbb{S}_t) + \xi_{ELB}(\mathbb{S}_t), \tag{589}$$

and $\xi_{ELB}(\mathbb{S}_t) \geq 0$.

Figure 100: Optimal Commitment Policy



Key features of optimal policy under commitment: The policy rate is kept “low for long”; inflation and output gap “overshoot” their target levels after the crisis is gone; there is time inconsistency; and, state contingency – the additional period to keep the policy rate at the ZLB after the crisis shock is gone – depends on the realised duration of the crisis, and the magnitude of the overshoot depends on the realised duration of the crisis.

But how does this theory stack up against the reality? With the exception of the Bank of Japan (BOJ), central banks have not adopted “low for long” policies. See articles by Bernanke such as “Temporary Price-Level Targeting” (2018) or “Monetary Policy in a New Era” (2019), and Yellen’s 2018 speech “Comments on monetary policy at the effective lower bound”, where she states:

> I believe the FOMC should seriously consider pursuing a lower-for-longer or makeup strategy for setting short rates when the zero lower bound bind and should articulate its intension

to do so before the next zero lower bound episode.

So why have central banks been squeamish to adopt "lower-for-longer" monetary policies during ZLB crises? There have so far been two arguments against the policy: i) Time inconsistency (e.g. "Credibility of Optimal Forward Guidance at the Interest Rate Lower Bound" Nakata (2015)), and ii) Destabilising inflation expectations (see for example Donald Kohn's 2009 speech).

Read the following quotes in regards to concerns over time inconsistency:

> *The optimal forward guidance policy is not time-consistent. According to the theory, for this policy to have the desired effects, the central bank must commit to two things: keeping the short-term policy rate lower than it otherwise would in the future, and allowing inflation to rise higher than it otherwise would. However, when the time comes for the central bank to fulfil this commitment, it may not want to do so. It might find it hard to resist the temptation to raise rates earlier than promised to avoid the rise in inflation.*
>
> – John Williams, San Francisco Fed (2012)

> *Today, to achieve a better path for the economy over time, a central bank may need to commit credibly to maintaining highly accommodative policy even after the economy and, potentially, inflation picks up. Market participants may doubt the willingness of an inflation-targeting central bank to respect this commitment if inflation goes temporarily above target. These doubts reduce the effective stimulus of the commitment and delay the recovery.*
>
> – Mark Carney, Bank of Canada/Bank of England (2012)

> *The "Woodford period" approach to forward guidance [i.e., optimal commitment policy] relies on a credible announcement made today that future monetary policy will deviate from normal. The central bank does not actually behave differently today. One might argue that such an announcement is unlikely to be believed. Why should future monetary policy deviate from normal once the economy is growing and inflation is rising? But if the announcement is not credible, then the private sector will not react with more consumption and investment today. That is, any effects would be minimal.*

– James Bullard, St Louis Fed (2013)

Then there are the concerns over destabilising inflation expectations. In the model, the assumption of rational expectations means that the private sector's long-run expectations over inflation are well anchored. The fear is that private agents may not correctly understand the temporary nature of the overshooting of inflation from its longer run target and long-run inflation expectations can be de-anchored from the central bank's target.

> T*o be sure, we have not followed the theoretical prescription of promising to keep rates low enough for long enough to create a period of above-normal inflation. The arguments in favour of such a policy hinge on a clear understanding on the part of the public that the central bank will tolerate increased inflation only temporarily–say, for a few years once the economy has recovered–before returning to the original inflation target in the long term. In standard theoretical model environments, long-run inflation expectations are perfectly anchored.*
>
> *In reality, however, the anchoring of inflation expectations has been a hard-won achievement of monetary policy over the past few decades, and we should not take this stability for granted. Models are by their nature only a stylised representation of reality, and a policy of achieving "temporarily" higher inflation over the medium term would run the risk of altering inflation expectations beyond the horizon that is desirable. Were that to happen, the costs of bringing expectations back to their current anchored state might be quite high.*
>
> – Donald Kohn, Federal Reserve Board (2009)

14.4 Galí's treatment of optimal policy and the ZLB

So far in investigating the ZLB, we've used compact notation which appeared in Jung et al. (2005), Nakata, and Eggertson and Woodford (2003). In this chapter, to make sure things are clear, we will go through the treatment of optimal policy under a ZLB on the nominal interest rate by Galí (2015).

To simplify the analysis, we make the following assumptions:

- The ZLB constraint is given by $i_t \geq 0$;

- No cost push shocks;
- The steady state is efficient, so $\bar{Y}_t^f = \bar{Y}_t^e$. Combined with the assumption of no cost push shocks, the model fulfils "The Divine Coincidence", so the central bank faces no tradeoff between stabilising inflation and closing the output gap (i.e. the optical policy satisfies $\pi_t = x_t = 0$); and
- The non-policy block of the model is represented by the DISE and NKPC:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t,$$
$$x_t = \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}(i_t - \mathbb{E}_t \pi_{t+1} - r_t^f),$$

  and where the Wicksellian natural rate of interest is assumed to be exogenous and remain constant at the steady state level $\rho > 0$. In other words, $r_t^f = \rho$ until the ZLB becomes binding.

It is important to note first that ZLB constraint restricts the set of feasible equilibrium paths. In particular, the constraint prevents, while binding, the attainment of the optimal allocation, characterised by zero inflation and a zero output gap at all times. This is the case, even though the NKPC is in principle consistent with such an outcome. The reason is that, as discussed earlier, supporting the efficient outcome as an equilibrium requires that $i_t = r_t^f$ for all t, which violates the NKPC whenever $r_t^f < 0$, as in the example considered here. The optimal policy will thus necessarily involve a second best outcome.

As in the previous chapter, we will look at at cases where the central bank is operating under discretion and commitment. In both cases, it is assumed that up to period $t = 0$, the economy's equilibrium involved $\pi_t = x_t = 0$ and $i_t = \rho$ for all $t < 0$. The unexpected shift to the natural rate, which causes the ZLB constraint to bind, occurs at period $t = 0$.

14.4.1 Optimal discretionary policy in the presence of ZLB constraint

Monetary policy does not commit to future actions, and the problem that the central banker faces is:

$$\min_{x_t, \pi_t} \frac{1}{2}(\pi_t^2 + \vartheta x_t^2),$$

subject to:

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t, \tag{590}$$
$$x_t = \mathbb{E}_t x_{t+1} - \frac{1}{\sigma}(i_t - \mathbb{E}_t \pi_{t+1} - r_t^f),$$
$$i_t \geq 0.$$

But, we can combine the constraints by first substituting in the ZLB constraint into the DISE constraint:

$$x_t \leq \mathbb{E}_t x_{t+1} + \frac{1}{\sigma}(\mathbb{E}_t \pi_{t+1} + r_t^f). \tag{591}$$

Thus, the Lagrangian[94] is:

$$\mathcal{L} = -\frac{1}{2}(\pi_t^2 + \vartheta x_t^2) + \xi_{1,t}(\pi_t - \beta \mathbb{E}_t \pi_{t+1} - \kappa x_t) + \xi_{2,t}\left(\mathbb{E}_t x_{t+1} + \frac{1}{\sigma}(\mathbb{E}_t \pi_{t+1} + r_t^f) - x_t\right),$$

with the following FOCs:

$$\frac{\partial \mathcal{L}}{\partial \pi_t} = -\pi_t + \xi_{1,t} = 0, \tag{592}$$
$$\frac{\partial \mathcal{L}}{\partial x_t} = -\vartheta x_t - \kappa \xi_{1,t} - \xi_{2,t} = 0, \tag{593}$$

and where we have the slackness conditions:

$$\xi_{2,t} \geq 0, i_t \geq 0, \xi_{2,t} i_t = 0,$$

so that if $i_t \geq 0$, the DISE is not binding (i.e. $\xi_{2,t} = 0$).

Combining the FOCs, we can eliminate the the Lagrangian multiplier on the NKPC constraint and

[94] Be careful with the signs here. In Galí's textbook, he sets up the problem as a maximisation problem, and then brings the variables to the LHS of the $\leq$ sign. Here, we've set things up as a minimisation problem – which is conceptually "more correct" as we're trying to minimise the society welfare loss function – and we bring things to the RHS of the $\leq$ sign. Either method will yield the same outcome, so long as you're consistent.

get:

$$x_t = -\frac{\kappa}{\vartheta}\pi_t - \frac{\xi_{2,t}}{\vartheta}. \tag{594}$$

As stated, from period $t_Z + 1$ onward (when the ZLB is not binding), we have:

$$i_t = \rho > 0,$$

which implies the usual equilibrium condition (i.e. $i_t > 0, \xi_{2,t} = 0$ and the DISE constraint is not binding):

$$x_t = -\frac{\kappa}{\vartheta}\pi_t,$$

which is consistent with the no-dilemma, first-best outcome: $\pi_t = x_t = 0$.[95]

However, for $t = 1, 2, ..., t_Z$, the ZLB constraint is binding implying $i_t = 0, \xi_{2,t} > 0$. The equilibrium for inflation and the output gap (when the ZLB binds) can be determined recursively backward using the system:

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \frac{1}{\sigma} \\ \kappa & \beta + \frac{\kappa}{\sigma} \end{bmatrix}}_{\mathbf{A}} \begin{bmatrix} x_{t+1} \\ \pi_{t+1} \end{bmatrix} - \underbrace{\begin{bmatrix} \frac{1}{\sigma} \\ \frac{\kappa}{\sigma} \end{bmatrix}}_{\mathbf{B}} \boldsymbol{\epsilon}, \tag{595}$$

with terminal conditions $x_{t_Z+1} = \pi_{t_Z+1} = 0$.

It can also be checked that along the equilibrium path, $x_t < 0$ and $\pi_t < 0$ for $t = 0, 1, 2, ..., t_Z$, thus guaranteeing that $\xi_{2,t} > 0$ given by the optimal condition (594) (in equilibrium, the NKPC holds). This representation is analogous to what we had before previously.

[95] In order to guarantee that the desired outcome is not only consistent with equilibrium, but also the only possible equilibrium outcome, the central bank could adopt an interest rate rule of the form:

$$i_t = \rho + \phi_\pi \pi_t, \ \phi_\pi > 1.$$

Figure 101: Discretion vs Commitment in the Presence of a ZLB

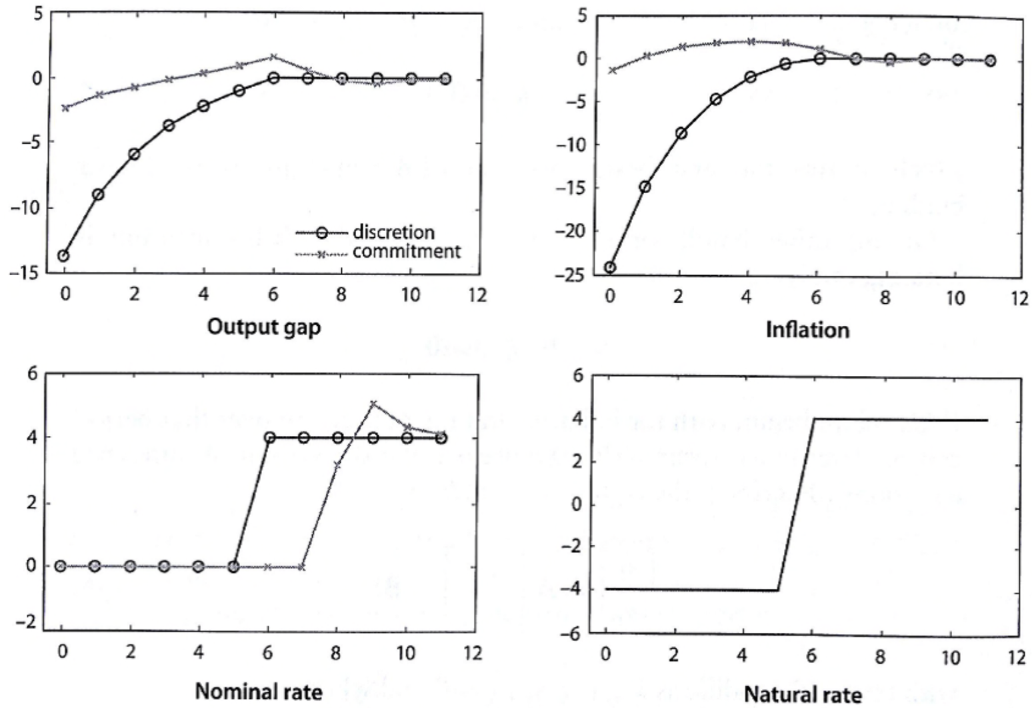


Figure 101 is a simulation of the model for discretion and commitment (which we will cover next). It is assumed that the unexpected drop in the natural rate, from 1 percent to -1 percent (4 to -4 percent in annualised rates) lasts 6 quarters (from $t = 0$ to $t_Z = 5$). The remaining parameters are set at their baseline values. Note that both the output gap and inflation experience a large decline on impact and remain below their optimal values until the negative shock vanishes. The presence of the ZLB is the ultimate source of the welfare losses resulting from the adverse demand shock. Those losses cannot be fully avoided, but are considerably reduced when the central bank ca commit credibly to a future policy plan.

14.4.2 Optimal policy under commitment in the presence of a ZLB constraint

Recall that the central bank now makes binding promises about future behaviour. Starting in period $t = 0$, in every period the central bank solves:

$$\min_{\{x_t,\pi_t\}} \frac{1}{2}\mathbb{E}_t \sum_{t=0}^{\infty} \beta^t(\pi_t^2 + \vartheta x_t^2),$$

subject to:

$$\begin{aligned}
\pi_t &= \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t, \\
x_t &\leq \mathbb{E}_t x_{t+1} + \frac{1}{\sigma}(\mathbb{E}_t \pi_{t+1} + r_t^f),
\end{aligned}$$

where $r_t^f = -\epsilon$ for $t = 0, 1, 2, ..., t_Z$ and $r_t^f = \rho$ for $t = t_Z + 1, t_Z + 2, ...$ The Lagrangian for this problem is:

$$\mathcal{L} = \mathbb{E}_t \sum_{t=0}^{\infty} \beta^t \left\{ -\frac{1}{2}(\pi_t^2 + \vartheta x_t^2) + \xi_{1,t}\left[\pi_t - \beta \mathbb{E}_t \pi_{t+1} - \kappa x_t\right] + \xi_{2,t}\left[\mathbb{E}_t x_{t+1} + \frac{1}{\sigma}(\mathbb{E}_t \pi_{t+1} + r_t^f) - x_t\right]\right\},$$

with the following FOCs:

$$\frac{\partial \mathcal{L}}{\partial \pi_t} = -\pi_t + \xi_{1,t} - \xi_{1,t-1} + \frac{1}{\beta\sigma}\xi_{2,t-1} = 0, \tag{596}$$

$$\frac{\partial \mathcal{L}}{\partial x_t} = -\vartheta x_t - \kappa \xi_{1,t} - \xi_{2,t} + \frac{1}{\beta}\xi_{2,t-1} = 0, \tag{597}$$

with the slackness conditions:

$$\xi_{2,t} \geq 0, i_t \geq 0, \xi_{2,t} i_t = 0,$$

and initial conditions:

$$\xi_{1,-1} = \xi_{2,-1} = 0.$$

The solution is conjectured to be the following. From period 0 to $t_C \geq t_Z$ the nominal rate remains at 0. It becomes positive in period $t_C + 1$ and remains positive from then onward.

The equilibrium dynamics for $t = t_C + 2, t_C + 3, ...$ are described by the difference equations:

$$0 = -\pi_t + \xi_{1,t} - \xi_{1,t-1}, \tag{598}$$

$$0 = -\vartheta x_t - \kappa \xi_{1,t}, \tag{599}$$

$$\pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa x_t, \tag{600}$$

together with an initial condition for ξ_{1,t_C+1} (which we will derive below). Note that (598) and (599) can be combined to get:

$$x_t = -\frac{\kappa}{\vartheta}(\ln p_t - p^*), \tag{601}$$

for $t = t_C + 2, t_C + 3, ...$, and where $p^* = \ln p_{t_C+1} + \xi_{1,t_C+1}$. Combining this with the NKPC yields the following second-order difference equation:

$$\hat{p}_t = \gamma \hat{p}_{t-1} + \gamma \beta \mathbb{E}_t \hat{p}_{t+1},$$

where $\hat{p}_t = p_t - p^*$ and $\gamma = \frac{\vartheta}{\vartheta(1+\beta)+\kappa^2}$. The unique stationary solution to the previous difference equation is:

$$\hat{p}_t = \delta \hat{p}_{t-1}, \tag{602}$$

for $t = t_C+2, t_C+3, ...$, with initial condition $p_{t_C+1} = -\xi_{1,t_C+1} < 0$, and where $\delta = \frac{1-\sqrt{1-4\beta\gamma^2}}{2\gamma\beta} \in (0,1)$. By combining (601) and (602) the path of the output gap for $t = t_C + 2, t_C + 3, ...$ can be determined. Note (602) implies:

$$\hat{p}_{t_C+2+k} = -\delta^{k+1} \xi_{1,t_C+1},$$

and

$$x_{t_C+2+k} = \frac{\kappa \delta^{k+1}}{\vartheta} \xi_{1,t_C+1} > 0, \tag{603}$$

for $k = 0, 1, 2, ...$, as well as:

$$\pi_{t_C+2+k} = (1-\delta)\delta^k \xi_{1,t_C+1} > 0. \tag{604}$$

Thus, under the optimal policy with commitment, inflation and the output gap converge to zero asymptotically.

Consider the next the equilibrium conditions in period $t_C + 1$, the first period in which the ZLB is not binding. They are given by:

$$0 = -\pi_{t_C+1} + \xi_{1,t_C+1} - \xi_{1,t_C} + \frac{1}{\beta\sigma}\xi_{2,t_C}, \tag{605}$$

$$0 = -\vartheta x_{t_C+1} - \kappa\xi_{1,t_C+1} + \frac{1}{\beta}\xi_{2,t_C}, \tag{606}$$

$$\pi_{t_C+1} = \beta(1-\delta)\xi_{1,t_{C+1}} + \kappa x_{t_C+1}. \tag{607}$$

We can use the third condition to substitute out ξ_{1,t_C+1} from the first two conditions to get the following linear relation:

$$\begin{bmatrix} x_{t_C+1} \\ \pi_{t_C+1} \end{bmatrix} = \underbrace{\begin{bmatrix} -\kappa & 1+\beta(1-\delta) \\ \beta(1-\delta)+\frac{\kappa^2}{\vartheta} & -\frac{\kappa}{\vartheta} \end{bmatrix}^{-1} \begin{bmatrix} \beta(1-\delta) & \frac{1-\delta}{\sigma} \\ 0 & \frac{1-\delta}{\vartheta} \end{bmatrix}}_{\mathbf{M}} \begin{bmatrix} \xi_{1,t_C} \\ \xi_{2,t_C} \end{bmatrix}. \tag{608}$$

Finally, consider the equilibrium trajectory between periods 0 and t_C. During this phase, the ZLB is binding, with $i_t = 0$ and the equilibrium trajectory is given by:

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \mathbf{A} \begin{bmatrix} x_{t+1} \\ \pi_{t+1} \end{bmatrix} - \mathbf{B}\boldsymbol{\epsilon},\ t = 0, 1, ..., t_Z, \tag{609}$$

and:

$$\begin{bmatrix} x_t \\ \pi_t \end{bmatrix} = \mathbf{A} \begin{bmatrix} x_{t+1} \\ \pi_{t+1} \end{bmatrix} + \mathbf{B}\boldsymbol{\rho},\ t = t_{Z+1}, ..., t_C, \tag{610}$$

where $\mathbf{A}$ and $\mathbf{B}$ are defined in (595). In addition, we can write the equilibrium system for the Lagrangian multipliers as:

$$\begin{bmatrix} \xi_{1,t} \\ \xi_{2,t} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \frac{1}{\beta\sigma} \\ \kappa & \frac{1}{\beta}(1+\frac{\kappa}{\sigma}) \end{bmatrix}}_{\mathbf{H}} \begin{bmatrix} \xi_{1,t-1} \\ \xi_{2,t-1} \end{bmatrix} - \underbrace{\begin{bmatrix} 0 & 1 \\ \vartheta & \kappa \end{bmatrix}}_{\mathbf{J}} \begin{bmatrix} x_t \\ \pi_t \end{bmatrix}. \tag{611}$$

Given an initial guess for t_C, the equilibrium path under the optimal policy with commitment can be determined as follows. Equations (608), (609), (610), and (611) make up a system of $4(t_C + 2)$ equations with an equal number of unknowns, namely, $(x_t, \pi_t, \xi_{1,t}, \xi_{2,t})$ for $t = 0, 1, ..., t_C + 1$. The value for ξ_{1,t_C+1} is associated with that solution can then be combined with (603) and (604) in order to determine (x_t, π_t) for $t = t_C + 2, ...$

Given the path of inflation and the output gap is determined, one can solve for the interest rate implied by the DISE:

$$i_t = r_t^f + \mathbb{E}_t\pi_{t+1} + \sigma(\mathbb{E}_t x_{t+1} - x_t),$$

and check that indeed $i_t = 0$ for $t = 0, 1, ..., t_C$ and $i_t > 0$ for $t = t_C + 1, t_C + 2, ...$ If those conditions are not verified, the procedure is repeated for a different value for t_C.

The lines with crosses in Figure 101 display the equilibrium paths for the output gap, inflation, and the nominal interest rate under the optimal policy with commitment, when the economy experiences the same adverse demand shock analysed for the discretionary case. The results are analogous to what we had before, where society's welfare was higher than the case with discretion as the central bank adopts a "lower for longer" strategy. Here, the nominal interest rate remains at 0 for two additional periods once the natural rate is back at its normal level, and below the natural rate for a third period. The anticipation of such a promise by the central bank reduces the initial impact of the excessively tight policy implied by the binding ZLB, leading to much smaller deviations of the output gap and inflation from target between $t = 0$ and $t = t_Z$, which more than offset, from a welfare point of view, the subsequent deviations.

This analysis can be viewed as providing the theoretical underpinning to the so-called "forward guidance" strategy adopted by the Fed, ECB, and BOJ during the aftermath of the GFC.

14.5 An approach to overcome time-inconsistency

There are three common approaches to overcoming the central bank's time-inconsistency problem in macroeconomics, which we saw when evaluating the Markov-Perfect and Ramsey policies:

- Policy delegation;

- Introducing an endogenous state variable; and
- Reputation.

This material is beyond what is necessary in a first-year macroeconomics course, so for now we will just cover policy delegation using a simple two-period model for illustration. The basic idea of policy delegation is that society states a welfare function, $\mathbb{W}$, which is essentially a mandate for the central bank to maximise.

In models without the ZLB, examples of policy delegation are:

- The conservative central banker by Rogoff (1985):

$$\mathbb{W} = -\frac{1}{2}\pi_t^2.$$

- Interest rate smoothing by Woodford (2003):

$$\mathbb{W} = -\frac{1}{2}\left[\pi_t^2 + \vartheta y_t^2 + (i_t - i_{t-1})^2\right],$$

  where i_{t-1} is an endogenous state variable.

- Speed-limit policy by Walsh (2003):

$$\mathbb{W} = -\frac{1}{2}\left[\pi_t^2 + \alpha(y_t - y_{t-1})^2\right],$$

  which converts y_{t-1} into an endogenous state variable.

The first is in a model with inflation bias, while the second two are in models with stabilisation bias.

In models with the ZLB, examples of policy delegation are:

- "Conservatism and Liquidity Traps" Nakata and Schmidt (2019*a*), which is based on the idea of Rogoff's conservative central banker. The central bank improves welfare by mitigating deflationary biases.

- "Gradualism and Liquidity Traps" by Nakata and Schmidt (2019*b*), which is based on Woodford's interest rate smoothing. The central bank improves welfare without creating a "low for long" policy.
- "Speed-Limit Policy and Liquidity Traps" by Nakata, Schmidt, and Yoo (2018), which is based on Walsh's speed limit policy. The central bank can worsen welfare by generating "lower for shorter" policy.

We shall take a look at the second paper by Nakata and Schmidt (2019*a*).

14.5.1 Gradualism in a two-period model

First, we introduce some additional notation:

$$\hat{i}_t = i_t - r^*,$$
$$\hat{r}_1^f = r_1^f - r^*,$$

so the hatted interest rate expressions denote the gap between the interest rate and its steady state natural level. Thus, similar to our setup previously, we have in period $t = 1$:

$$y_1 = y_2 - \sigma(\hat{i}_1 - \pi_2 - \hat{r}_1^f),$$
$$\pi_1 = \kappa y_1 + \beta \pi_2,$$
$$i_t \geq 0 \Leftrightarrow \hat{i}_1 \geq -r^*,$$

and in period $t = 2$ we have:

$$y_2 = -\sigma \hat{i}_2,$$
$$\pi_2 = \kappa y_2,$$
$$i_2 \geq 0 \Leftrightarrow \hat{i}_2 \geq -r^*,$$

where $r^* > 0$. We consider three cases:

$$\text{Case 1 (no shock): } r_1^f = r^* \Leftrightarrow \hat{r}_1^f = 0,$$
$$\text{Case 2 (small shock): } r_1^f \in (0, r^*) \Leftrightarrow \hat{r}_1^f \in (-r^*, 0),$$
$$\text{Case 3 (large shock): } r_1^f < 0 \Leftrightarrow \hat{r}_1^f < -r^*.$$

The welfare function which the central is mandated to maximise is:

$$\mathbb{W} = u^{CB}(\pi_t, y_t, \hat{i}_t, \hat{i}_{t-1}) = -\frac{1}{2}\left[(1-\alpha)(\pi_t^2 + \vartheta y_t^2) + \alpha(\hat{i}_t - \hat{i}_{t-1})^2\right],$$

and so we have the following function at $t = 1$:

$$V_1^{CB} = \max_{\pi_1, y_1, \hat{i}_1} \; u^{CB}(\pi_1, y_1, \hat{i}_1, \hat{i}_0) + \beta V_2^{CB}(\hat{i}_1), \tag{612}$$

subject to the DISE, NKPC, and ZLB constraints, taking $V_2^{CB}(\cdot), y_2(\cdot)$, and $\pi_2(\cdot)$ as given. At $t = 2$:

$$V_2^{CB} = \max_{\pi_2, y_2, \hat{i}_2} \; u^{CB}(\pi_2, y_2, \hat{i}_2, \hat{i}_1), \tag{613}$$

subject to the DISE, NKPC, and ZLB constraints. A Markov-Perfect Equilibrium is defined as a set of value and policy functions, $\{V_1^{CB}(\cdot), y_1(\cdot), \pi_1(\cdot), \hat{i}_1(\cdot), V_2^{CB}(\cdot), y_2(\cdot), \pi_2(\cdot), \hat{i}_2(\cdot)\}$ that solves these two problems. The FOCs at $t = 2$ are:

$$\frac{\partial \mathcal{L}_2}{\partial y_2} = -(1-\alpha)\vartheta y_2 + \xi_{EE,2} - \kappa \xi_{PC,2} = 0, \tag{614}$$
$$\frac{\partial \mathcal{L}_2}{\partial \pi_2} = -(1-\alpha)\pi_2 + \xi_{PC,2} = 0, \tag{615}$$
$$\frac{\partial \mathcal{L}_2}{\partial \hat{i}_2} = -\alpha(\hat{i}_2 - \hat{i}_1) + \sigma \xi_{EE,2} + \xi_{ELB,2} = 0. \tag{616}$$

With a bit of rearranging to eliminate the Lagrangian multipliers, and substituting our $t = 2$ parameter values, we get:

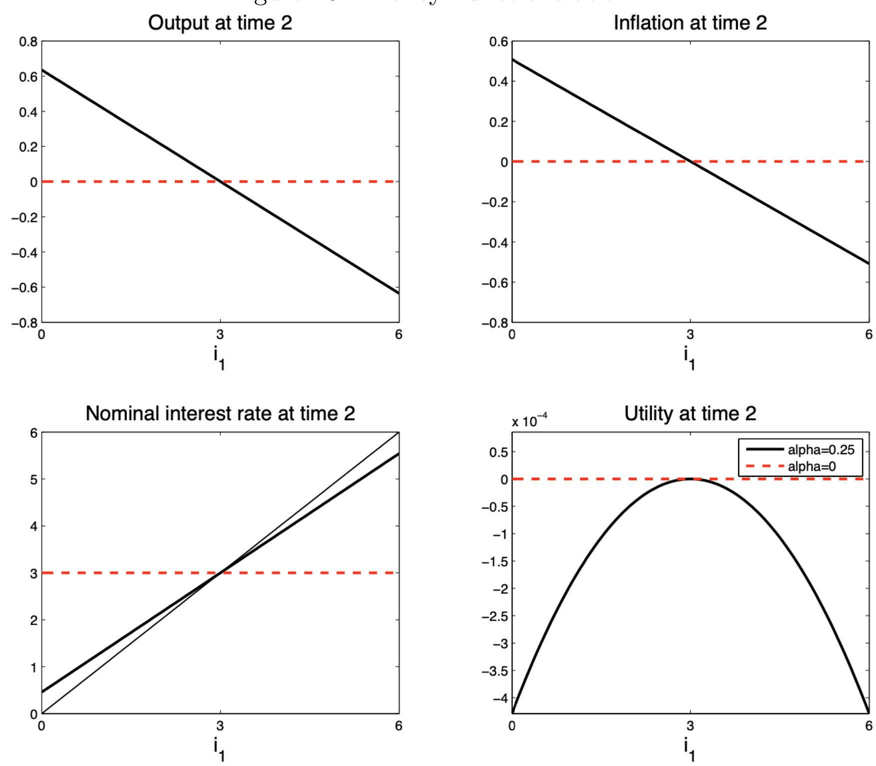
$$\hat{i}_2 = \frac{\alpha}{\alpha+\gamma}\hat{i}_1, \tag{617}$$

$$y_2 = -\frac{\sigma\alpha}{\alpha+\gamma}\hat{i}_1, \tag{618}$$

$$\pi_2 = -\frac{\kappa\sigma\alpha}{\alpha+\gamma}\hat{i}_1, \tag{619}$$

where $\gamma = \sigma^2(\vartheta + \kappa^2)(1-\alpha)$.

Figure 102: Policy Functions at $t = 2$



The central bank's problem at $t = 1$ is:

$$V_1^{CB} = \max_{\pi_1, y_1, \hat{i}_1} u^{CB}(\pi_1, y_1, \hat{i}_1, \hat{i}_0) + \beta V_2^{CB}(\hat{i}_1),$$

subject to the DISE, NKPC, and ZLB constraints, and taking $V_2^{CB}(\cdot)$, $y_2(\cdot)$, and $\pi_2(\cdot)$ as given:

$$
\begin{aligned}
y_1 &= y_2(\hat{i}_1) - \sigma(\hat{i}_1 - \pi_2(\hat{i}_1) - \hat{r}_1^f), \\
\pi_1 &= \kappa y_1 + \beta \pi_2(\hat{i}_1).
\end{aligned}
$$

Recall that:

$$
u^{CB}(\pi_1, y_1, \hat{i}_1, \hat{i}_0) = -\frac{1}{2}\left[(1-\alpha)(\pi_1^2 + \vartheta y_1^2) + \alpha(\hat{i}_1 - \hat{i}_0)^2\right],
$$

and the FOCs at $t = 1$ are:

$$
\begin{aligned}
\frac{\partial \mathcal{L}_1}{\partial y_1} &= -(1-\alpha)\vartheta y_1 + \xi_{EE,1} - \kappa \xi_{PC,1} = 0, && (620) \\
\frac{\partial \mathcal{L}_1}{\partial \pi_1} &= -(1-\alpha)\pi_1 + \xi_{PC,1} = 0, && (621) \\
\frac{\partial \mathcal{L}_1}{\partial \hat{i}_1} &= -\alpha(\hat{i}_1 - \hat{i}_0) + \beta \frac{\partial V_2^{CB}(\hat{i}_1)}{\partial \hat{i}_1} - \xi_{EE,1} \frac{\partial y_2(\hat{i}_1)}{\partial \hat{i}_1} \\
&\quad + \sigma \xi_{EE,1} - \sigma \xi_{EE,1} \frac{\partial \pi_2(\hat{i}_1)}{\partial \hat{i}_1} - \xi_{PC,1} \beta \frac{\partial \pi_2(\hat{i}_1)}{\partial \hat{i}_1} \\
&\quad + \xi_{ELB,1} = 0 && (622)
\end{aligned}
$$

Figure 103: Policy Functions at $t = 1$: Case 1 (no shock; $\hat{r}_1^f = 0$)

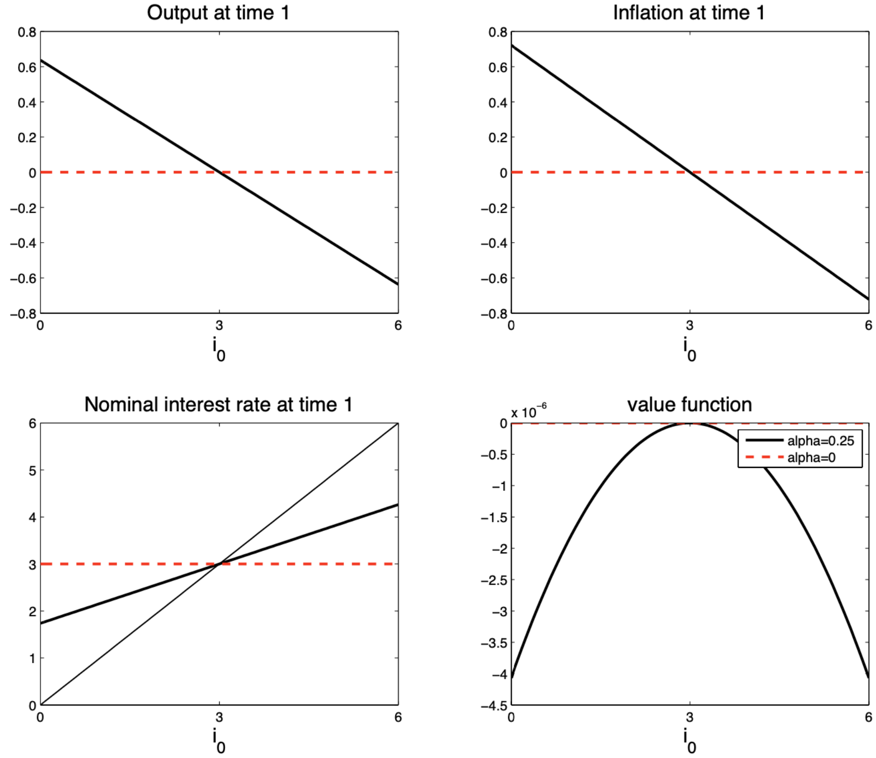


Figure 104: IRFs: Case 1 (no shock) with $i_0 = r^*$

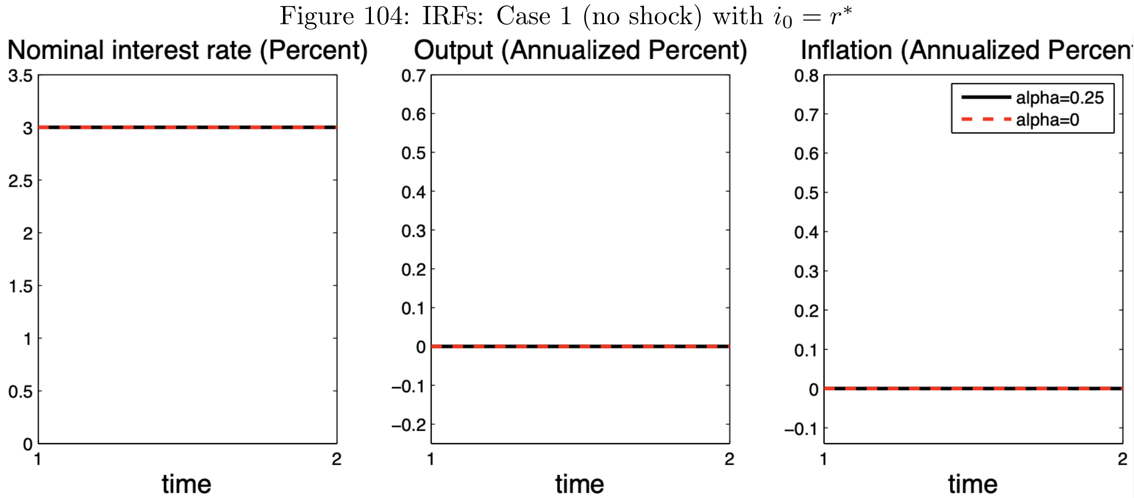


Figure 105: IRFs: Case 1 (no shock) with $i_0 = 0$

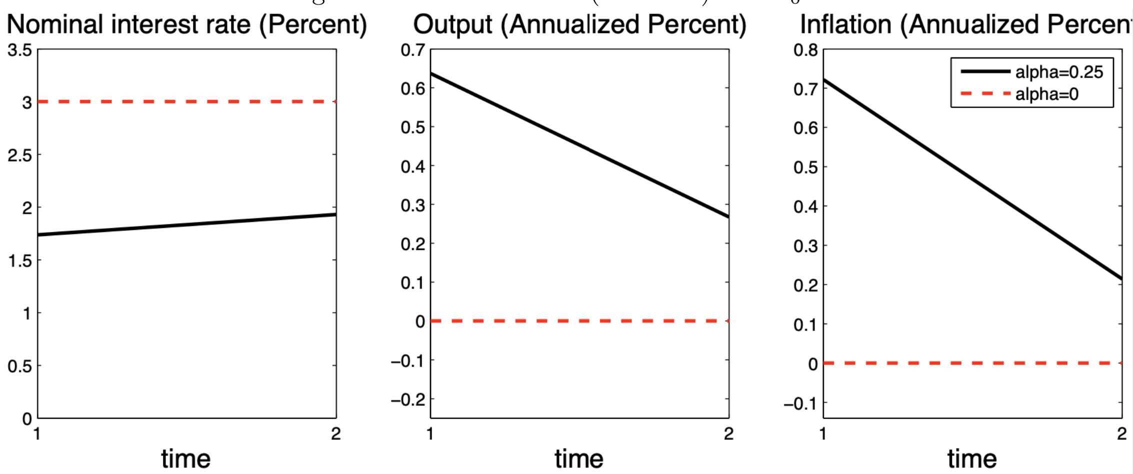


Figure 106: Policy Functions at $t = 1$: Case 2 ($-r^* < \hat{r}_1^f < 0$)

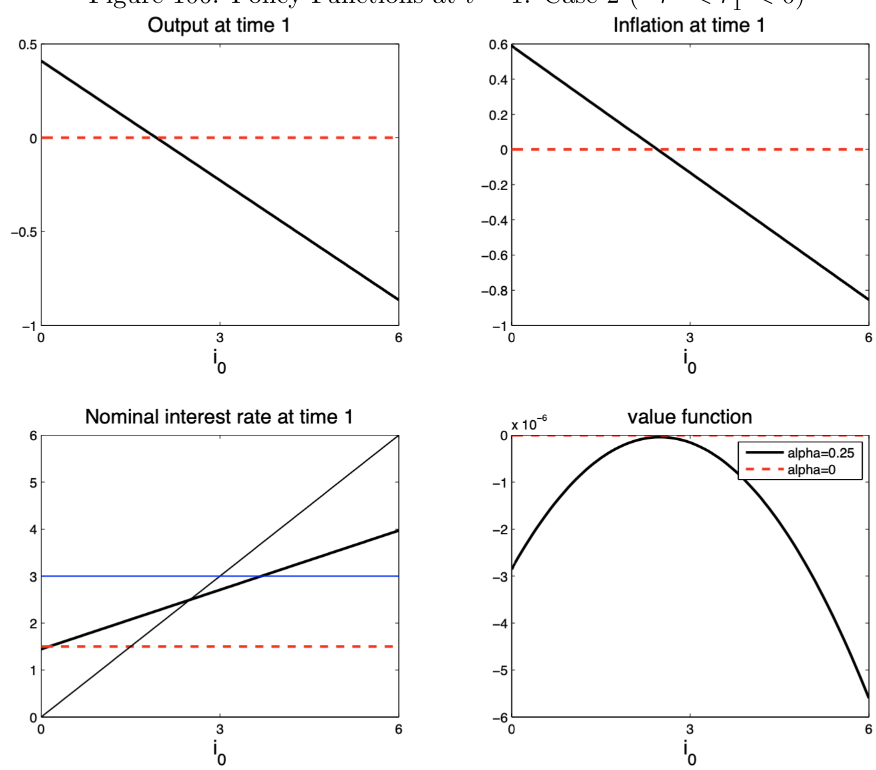


Figure 107: IRFs: Case 2 (small shock) with $i_0 = r^*$

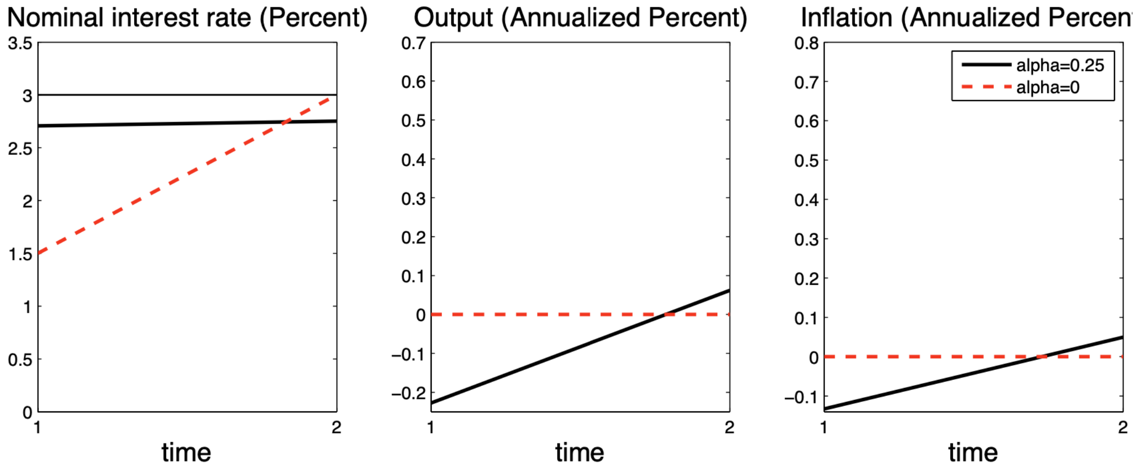


Figure 108: Policy Functions at $t = 1$: Case 3 ($\hat{r}_1^f < -r^*$)

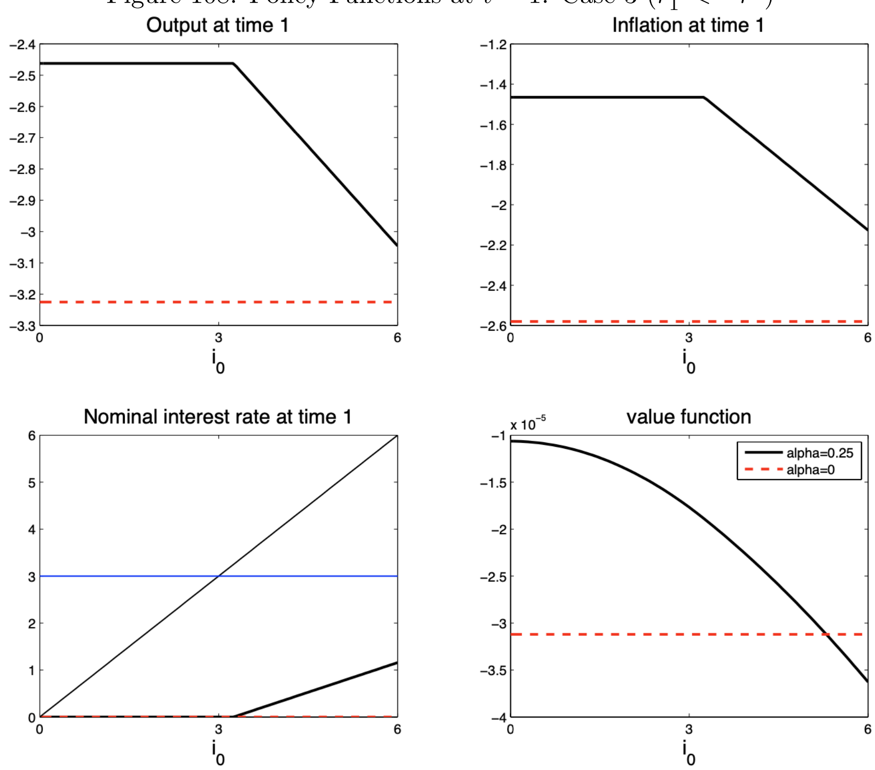
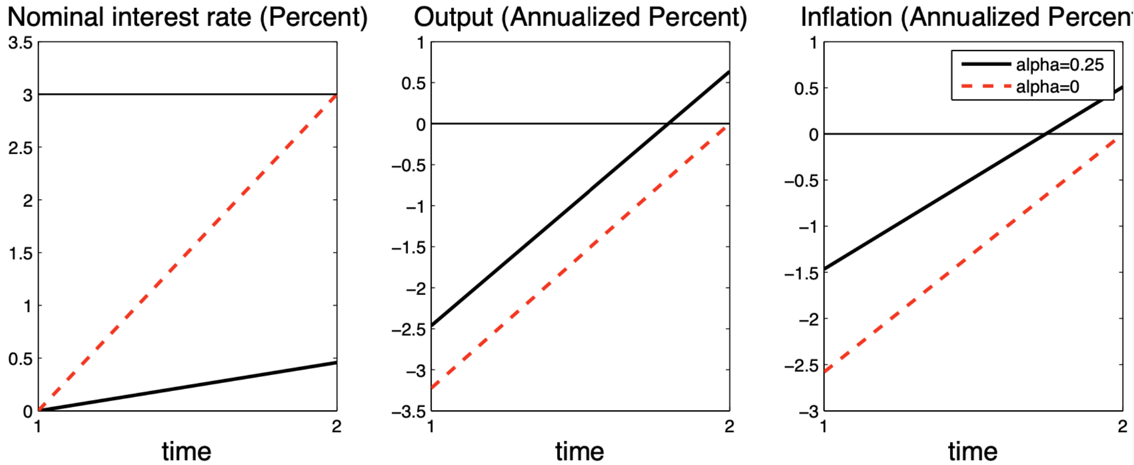


Figure 109: IRFs: Case 3 (large shock) with $i_0 = r^*$



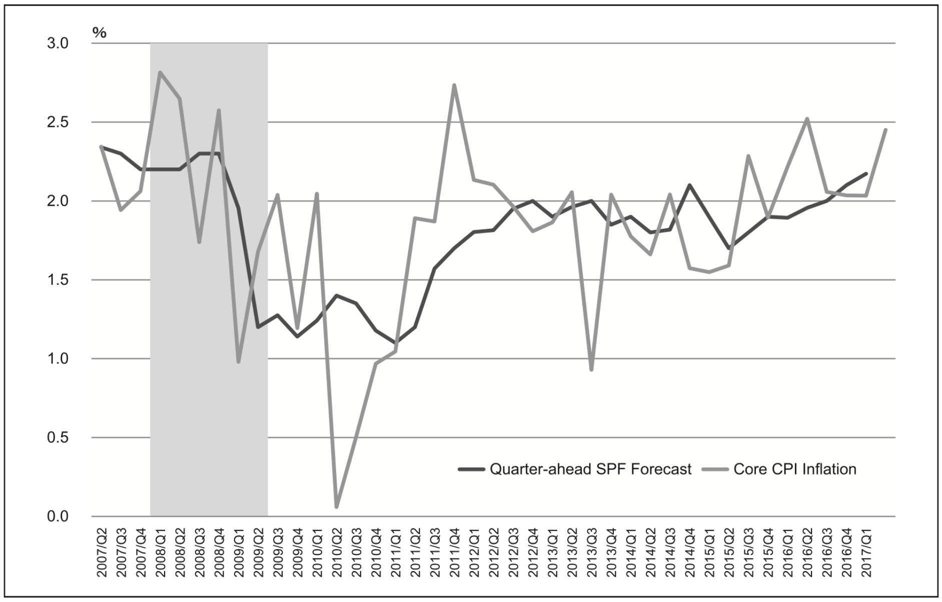
14.6 The forward guidance puzzle

As seen, the standard New Keynesian model predicts a large effect of "forward guidance" on the economy. The key effect of forward guidance is to generate large inflation by retaining – or promising to retain – a low inflation rate for longer. Despite the prediction of the effects of forward guidance in these theoretical models, we have not seen this manifest so much in reality in the post-GFC era. The key questions we will try to answer in this section are: i) Why does the does the standard New Keynesian model fail to describe reality? ii) What might be needed to reconcile theory with data?

14.6.1 Empirics on forecasts and expectations

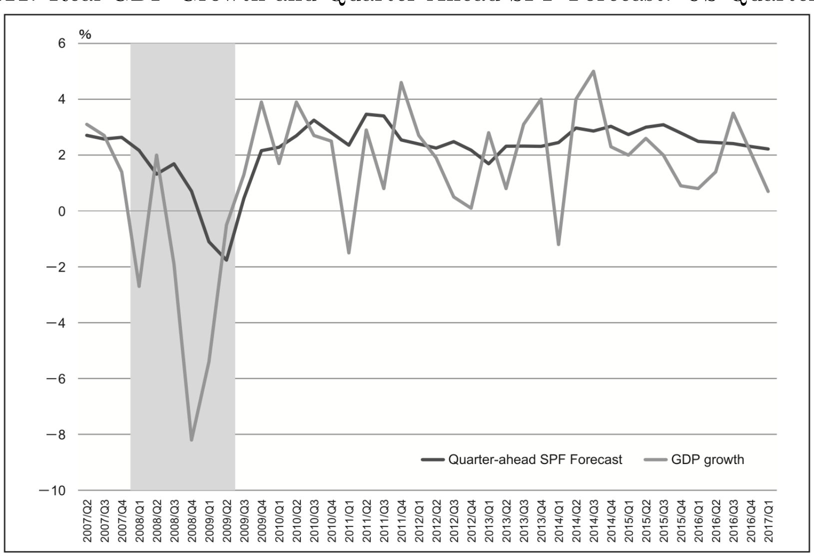
Figure 110 shows that the forecast of inflation tends to follow movements in inflation – implying some kind of adaptive mechanics or persistence for inflation expectations. Of note: i) Forecast errors are serially correlated; ii) Actual inflation tends to lead movements in expected inflation; and iii) Expectations fall below 2 percent for some years after the GFC, but they return to percent thereafter. The first two points tend to imply an adaptive mechanism for inflation, while the third implies that the 2 percent inflation target likely anchors inflation expectations. Meanwhile, Figure 111 shows properties to Figure 110, only this time for output.

Figure 110: Core CPI Inflation and Quarter-Ahead SPF Forecast: US Quarterly Data



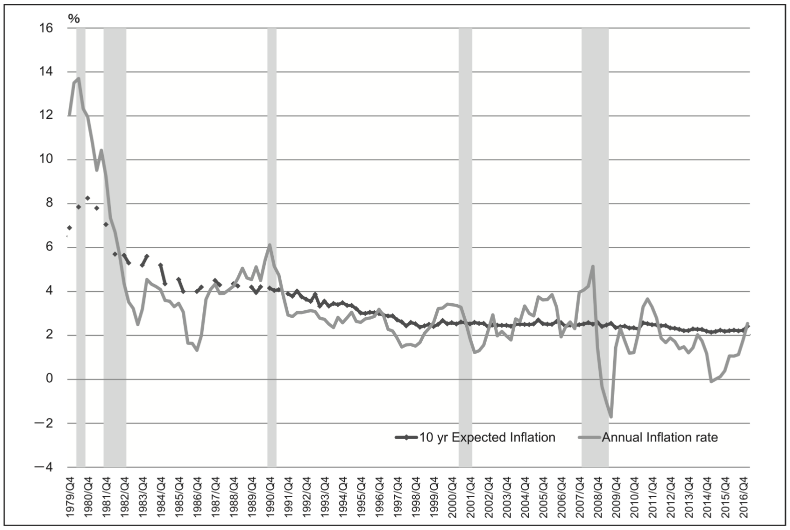
Survey of Professional Forecasters (SPF) of core CPI vs the realised value. Source: Federal Reserve Bank of Philadelphia; US Bureau of Labor Statistics

Figure 111: Real GDP Growth and Quarter-Ahead SPF Forecast: US Quarterly Data



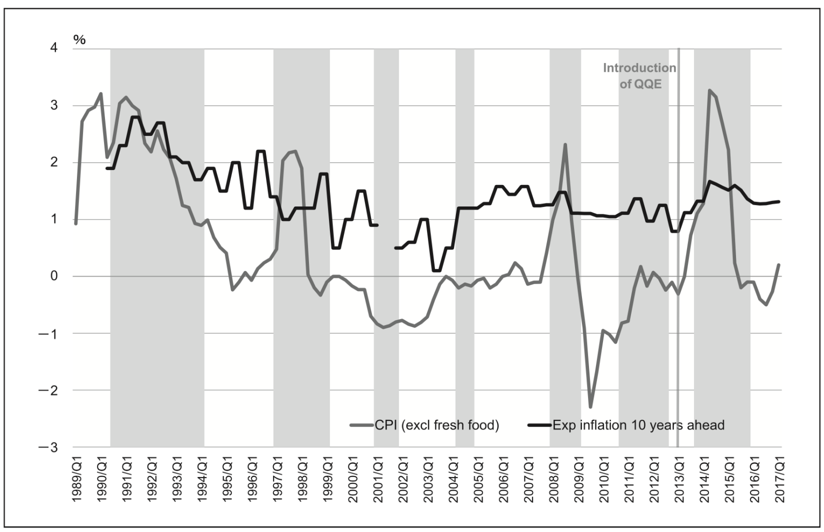
Survey of Professional Forecasters (SPF) of real output vs the realised value. Source: The Federal Reserve Bank of Philadelphia; US Bureau of Labour Statistics

Figure 112: Annual Headline CPI and Expected Rate for Next 10 Years (SPF+Blue Chip): US Data



Median SPF of headline CPI inflation ten years ahead vs the realised value. Source: The Federal Reserve Bank of Philadelphia; Blue Chip Economic Indicators; US Bureau of Labour Statistics

Figure 113: Inflation and Expected Rate for Next 10 Years: Japanese Data



Survey expectations of CPI inflation sex to ten years vs actual inflation. Source: Consensus Economics Inc.; Ministry of Internal Affairs and Communication

Figures 112 and 113show that expectations are strongly and equally adaptive in the US and Japan, and that the inflation target anchors inflation in the US, but not in Japan.

14.6.2 Attenuating the forward guidance puzzle in the New Keynesian model

Consider the canonical New Keynesian model and the effect of an anticipated policy shock at $t = k$ on allocations at $t = 1$ (and assuming that the interest rate is constant before $t = k$).

Suppose $k = 2$, then at $t = 2$ we have:

$$\begin{aligned} y_2 &= y_3 - \sigma(i_2 - \pi_3 - \bar{r}^f), \\ \pi_2 &= \kappa y_2 + \beta\pi_3, \\ i_2 &= \bar{r}^f - \epsilon, \end{aligned}$$

with $y_3 = 0$ and $\pi_3 = 0$, we then get:

$$y_2 = \sigma\epsilon, \tag{623}$$

$$\pi_2 = \kappa\sigma\epsilon. \tag{624}$$

Then at $t = 1$ we have:

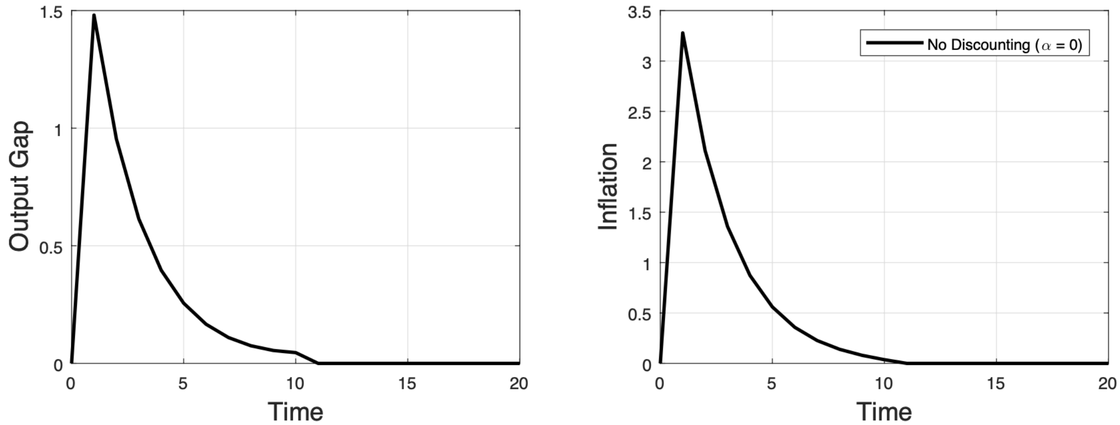
$$\begin{aligned} y_1 &= y_2 - \sigma(i_1 - \pi_2 - \bar{r}^f), \\ \pi_1 &= \kappa y_1 + \beta\pi_2, \\ i_1 &= \bar{r}^f, \end{aligned}$$

and using our expressions for y_2 and π_2 we can write:

$$y_1 = \sigma\epsilon(1 + \sigma\kappa) > y_2, \tag{625}$$

$$\pi_1 = \kappa\sigma\epsilon(1 + \sigma\kappa + \beta) > \pi_2. \tag{626}$$

Figure 114: Anticipated Monetary Policy Shock for $k = 10$



There have been many ways to resolve the puzzle:

- Sticky-information models of pricey (Kiley (2016) and Carlstrom et al. (2015));
- Perpetual youth model of Blanchard and Yaari (Del Negro et al. (2015));
- Incomplete markets with idiosyncratic income risk (McKay et al. (2016) and Kaplan et al. (2018));
- Bounded rationality (Gabaix 2016);
- Lack of common knowledge (Angeletos and Lian 2018); and
- Imperfect credibility (Haberis et al. 2017).

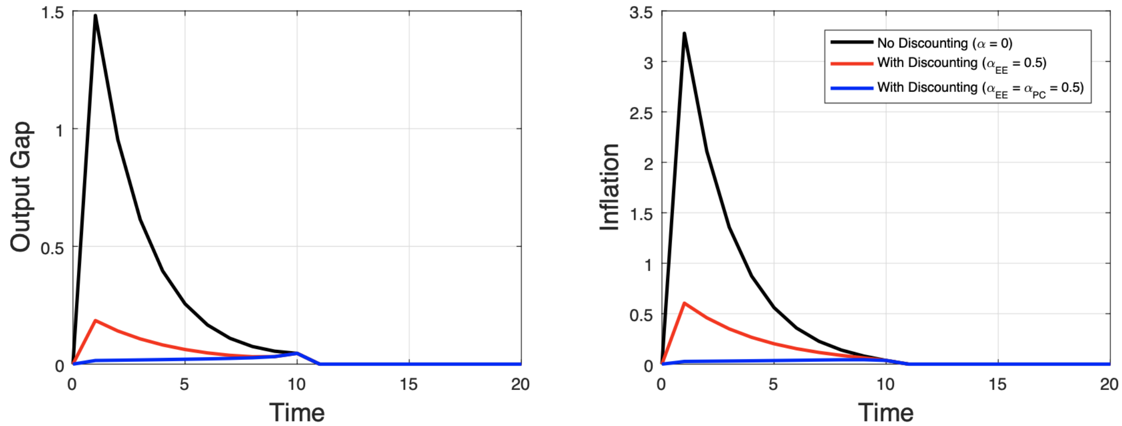
All of these proposal make the choices of the private sector (the DISE and NKPC) today less dependent on future economic outcomes.

A reduced-form way to capture the attenuated forward guidance puzzle is to use discounting in both the DISE and NKPC:

$$y_t = (1-\alpha_1)\mathbb{E}_t y_{t+1} - \sigma(i_t - \mathbb{E}_t \pi_{t+1} - r_t^f),$$
$$\pi_t = \beta(1-\alpha_2)\mathbb{E}_t \pi_{t+1} + \kappa y_t,$$

which was used in papers by McKay et al. (2016), Gabaix (2016), and Angeletos and Lian (2018).

Figure 115: Anticipated Monetary Policy Shock $k = 10$



So what are the implications of attenuating the forward guidance puzzle for “low-for-long” policy? “Attenuating the Forward Guidance Puzzle: Implications for Optimal Monetary Policy” by Nakata, Ogaki, et al. (2019) provides some answers.

Consider a three-period model where at $t = 1$ and $t = 2$ we have:

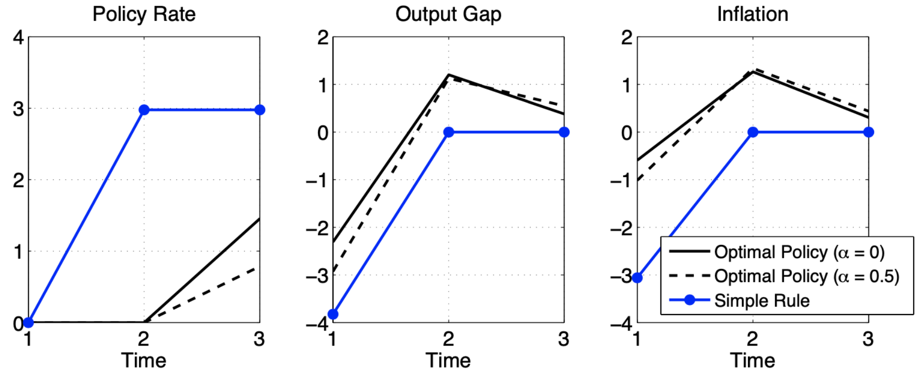
$$y_t = (1 - \alpha_1)\mathbb{E}_t y_{t+1} - \sigma(i_t - \mathbb{E}_t \pi_{t+1} - r_t^f),$$
$$\pi_t = \beta(1 - \alpha_2)\mathbb{E}_t \pi_{t+1} + \kappa y_t,$$

and $t = 3$:

$$y_3 = -\sigma(i_3 - r_3^f),$$
$$\pi_3 = \kappa y_3,$$

where $r_1^f < 0, r_2^f = r_3^f = \bar{r}^f > 0$.

Figure 116: Optimal Commitment Policy in a Three-Period Model with and without Discounting



Note: Units are annualised percent rate, percent deviation, and annualised percentage points for the policy rate, output gap, and inflation, respectively

Figure 117: Tradeoff from Adjusting the Future Policy Rate

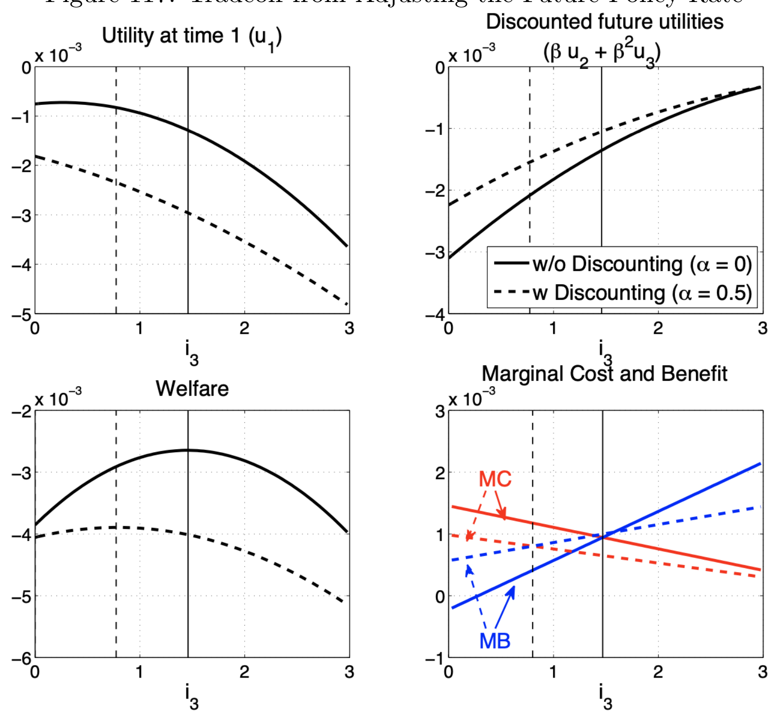
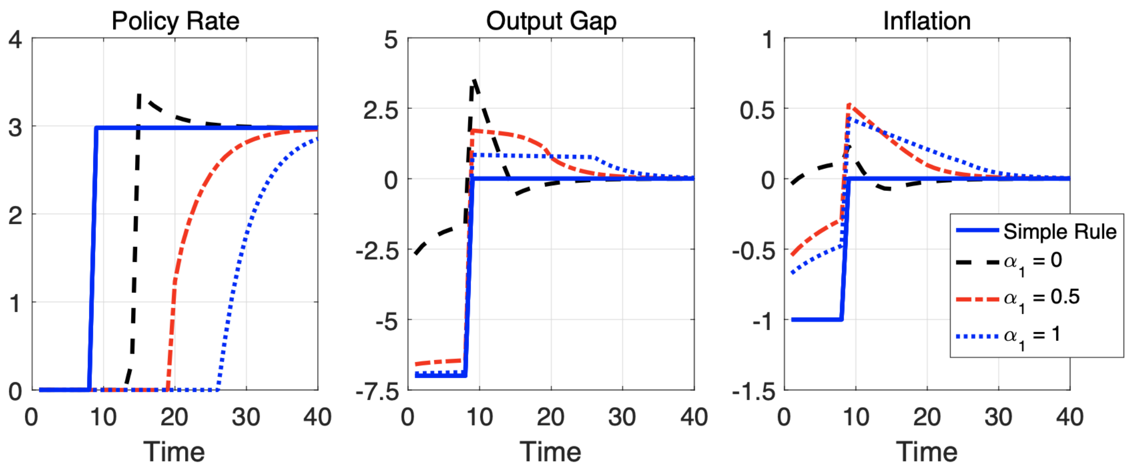
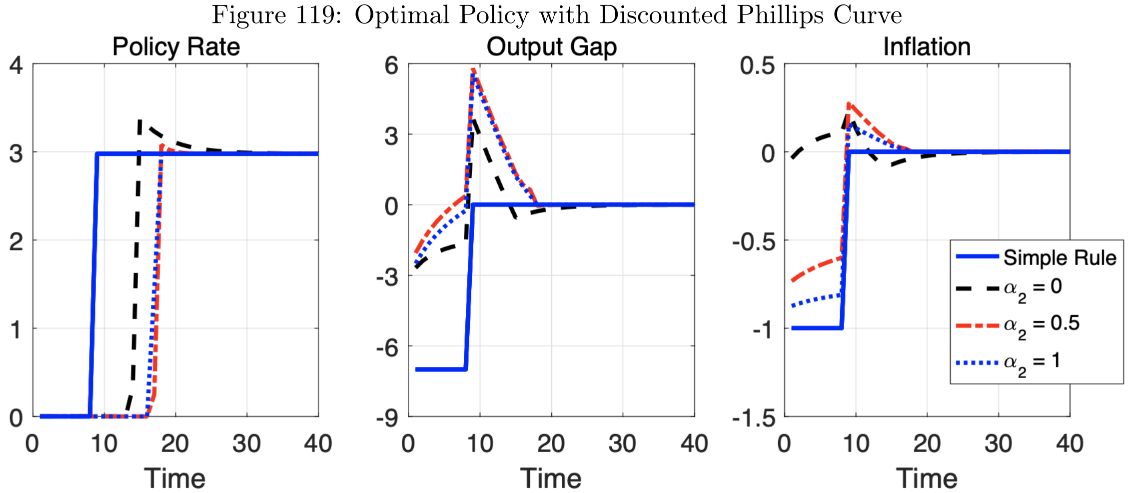


Figure 118: Optimal Policy with Discounted Euler Equation



Note: Here $\alpha_2 = 0$, so the NKPC is standard

Figure 119: Optimal Policy with Discounted Phillips Curve



Note: Here $\alpha_1 = 0$, so the DISE is standard

Figure 120: Expected ZLB Duration

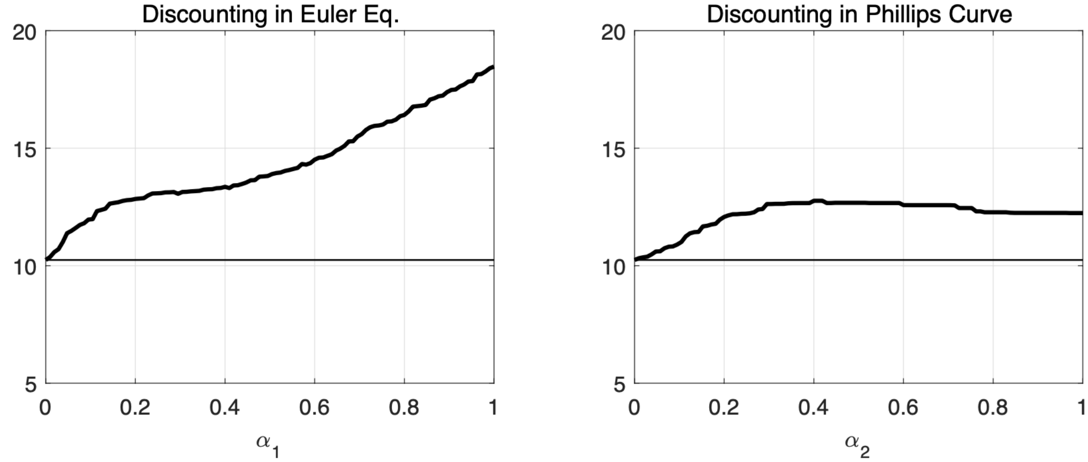
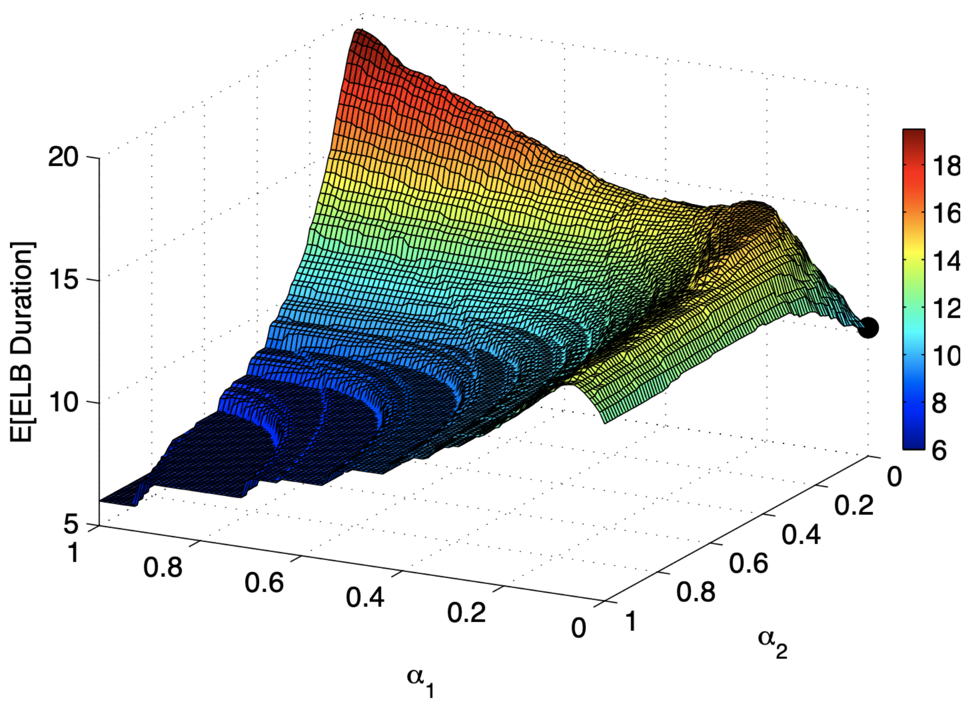


Figure 121: Expected ZLB Duration



What all these plots are saying are that expectations in the standard New Keynesian model are very powerful (due to Rational Expectations). If a policy announcement is made such that the policy change occurs 10 periods in the future, the effects today are very strong. If Rational Expectations were the correct mechanism to describe the DISE and NKPC, then one could expect an economy to move out of a deflation/recession by the central bank simply making the announcement that it would hold rates lower for longer than necessary – the central bank could, in a way, talk its way out of sluggish growth. Of course, this isn't what we observed in the data.

The idea then is to adjust the Rational Expectations assumption in our model by introducing discounting. By doing this, we can see that policy announcements, while stimulatory, are more subtle due to the discounting adjustments we make in the DISE and NKPC.

14.7 Comments and key readings

That completes a brief overview of the ZLB/ELB. There were a few topics that we didn't cover: government spending multipliers, ELB risk, and deflationary equilibrium, for instance. For those interested, I recommend you to check out the publications of Taisuke Nakata, who specialises on optimal monetary policy and the ZLB. In fact, most of the notes in this chapter are based on his notes.

I won't bother reiterating the key readings for literature on the ZLB – there have been plenty of references made throughout this section, and I think we've covered the key points quite well. It's worth finally noting that the ZLB still remains a very active area of research, particularly when exploring optimal policy under the ZLB for a heterogenous agent New Keynesian model.

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15 Labour Market Rigidities in the New Keynesian Model

15.1 Introduction

All of our analysis of the New Keynesian model (and RBC model) thus far has assumed that the labour market is perfectly competitive. We know that this is most likely not an accurate representation of labour market dynamics. As such, in this chapter we will augment our New Keynesian model to include rigidities in the labour market, in particular, wage rigidities. Wage rigidities are introduced in an analogous way to price rigidities: via Calvo pricing which facilitates aggregation. As with price setting, to get wage setting we need to introduce some kind of monopoly power in wage setting. To do this, we assume that households supply differentiated labour. This imperfect substitutability between types of labour gives them some market power, and allows us to think about the consequences of wage stickiness.

It's also worth pointing that I'll be basing the notes for this section on the amazing set of notes by Eric Sims and Galí (2015). My only contribution here is to catch typos, and clear up any notational confusion or inconsistencies – particularly when it comes to log-linearisation.

15.2 Production

Production in this model economy is virtually identical to what we had before. There is a continuum of intermediate firms which are monopolistically competitive and produce intermediate goods which are slightly differentiated. There is a representative perfectly competitive final goods firms which purchases intermediate goods and produces a final good for consumption.

15.2.1 The final goods sector

The final output good is a CES aggregate, using the Dixit-Stiglitz aggregator, of a continuum of intermediates:

$$Y_t = \left(\int_0^1 Y_t(j)^{\frac{\epsilon_p - 1}{\epsilon_p}} dj \right)^{\frac{\epsilon_p}{\epsilon_p - 1}}, \tag{627}$$

where $\epsilon_p > 1$. Note that we use a p subscript as we will need to use similar notation to denote the price elasticity for differentiated labour. Profit maximisation by the final goods firm yields a downward-sloping demand curve for each intermediate:

$$Y_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} Y_t. \tag{628}$$

In words, this states that the relative demand for the j-th intermediate good is a function of its relative price, with ϵ_p the price elasticity of demand. The price index (derived from the definition of nominal output as the sum of prices times quantities of intermediates) can be seen to be:

$$P_t = \left(\int_0^1 P_t(j)^{1-\epsilon_p} dj\right)^{\frac{1}{1-\epsilon_p}} \tag{629}$$

15.2.2 Intermediate producers

A typical intermediate producer produces output according to a constant returns to scale technology in labour, with a common productivity shock process, A_t:

$$Y_t(j) = A_t N_t(j). \tag{630}$$

Note that in order to keep the model fairly tractable – and to keep our sanity – we assume no capital. Intermediate producers must pay a common wage. They are not freely able to adjust prices so as to maximise profits each period, but they will always act to minimise cost. The cost minimisation problem is to minimise total cost subject to the constraint of producing enough to meet demand:

$$\min_{N_t(j)} \; W_t N_t(j),$$

subject to:

$$A_t N_t(j) \geq \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} Y_t.$$

The Lagrangian for the intermediate goods producer problem is:

$$\mathcal{L} = -W_t N_t(j) + \varphi_t(j)\left(A_t N_t(j) - \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} Y_t\right).$$

The FOC is:

$$\frac{\partial \mathcal{L}}{\partial N_t(j)} - W_t + \varphi_t(j) A_t = 0,$$

which obviously implies that:

$$\begin{aligned}\varphi_t(j) &= \frac{W_t}{A_t}\\ \Leftrightarrow \varphi_t &= \frac{W_t}{A_t},\end{aligned} \tag{631}$$

where we can remove the j reference as all intermediate goods firms have the same marginal cost, φ_t.

Real flow profit for intermediate producer j is:

$$D_t(j) = \frac{P_t(j)}{P_t} Y_t(j) - \frac{W_t}{P_t} N_t(j).$$

But from (631) we know that $W_t = \varphi_t A_t$, and so plugging this into the expression for profits we get:

$$D_t(j) = \frac{P_t(j)}{P_t} Y_t(j) - mc_t Y_t(j), \tag{632}$$

where $mc_t = \frac{\varphi_t}{P_t}$ is the real marginal cost.

Intermediate firms are not freely able to adjust price each period. In particular, each period there is a fixed probability of $1 - \phi_p$ that a firm can adjust its price.[96] This means that the probability that a firm will be stuck with a price one period is ϕ_p, for two periods it is ϕ_p^2, and so on. Consider the pricing problem of a firm given the opportunity to adjust its price in a given period. Since there is a chance that the firm will get stuck with its price for multiple periods, the pricing problem becomes dynamic. Firms will discount profits s periods into the future by $\tilde{M}_{t+s}\phi_p^s$, where $\tilde{M}_{t+s} = \beta^s \frac{u'(C_{t+s})}{u'(C_t)}$ is

[96] i.e., that it gets a visit from the Calvo fairy and is allowed to change its price.

the stochastic discount factor. The dynamic problem for an intermediate firm can be written as:

$$\max_{P_t(j)} \mathbb{E}_t \sum_{s=0}^{\infty} \phi_p^s \tilde{M}_{t+s} \left[\frac{P_t(j)}{P_{t+s}} \left(\frac{P_t(j)}{P_{t+s}} \right)^{-\epsilon_p} Y_{t+s} - mc_{t+s} \left(\frac{P_t(j)}{P_{t+s}} \right)^{-\epsilon_p} Y_{t+s} \right],$$

where we impose that output will equal demand. Multiplying out, we get:

$$\max_{P_t(j)} \mathbb{E}_t \sum_{s=0}^{\infty} \phi_p^s \hat{M}_{t+s} \left[P_t(j)^{1-\epsilon_p} P_{t+s}^{\epsilon_p - 1} Y_{t+s} - mc_{t+s} P_t(j)^{-\epsilon_p} P_{t+s}^{\epsilon_p} Y_{t+s} \right],$$

and the FOC is:

$$\begin{aligned} 0 = (1-\epsilon_p) P_t(j)^{-\epsilon_p} \mathbb{E}_t \sum_{s=0}^{\infty} \phi_p^s \beta^s u'(C_{t+s}) P_{t+s}^{\epsilon_p - 1} Y_{t+s} \\ + \epsilon_p P_t(j)^{-\epsilon_p - 1} \mathbb{E}_t \sum_{s=0}^{\infty} \phi_p^s \beta^s u'(C_{t+s}) mc_{t+s} P_{t+s}^{\epsilon_p} Y_{t+s}. \end{aligned}$$

Simplifying this expression, we get the price for intermediate good j:

$$P_t(j) = \mathcal{M}_p \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \phi_p^s \beta^s u'(C_{t+s}) mc_{t+s} P_{t+s}^{\epsilon_p} Y_{t+s}}{\mathbb{E}_t \sum_{s=0}^{\infty} \phi_p^s \beta^s u'(C_{t+s}) P_{t+s}^{\epsilon_p - 1} Y_{t+s}},$$

where $\mathcal{M}_p = \frac{\epsilon_p}{\epsilon_p - 1}$ is the markup charged by the intermediate firms. Note that nothing on the RHS of this equation is specific to firm j, which drives the motivation that all firms that get a visit from the Calvo fairy will update prices to the same reset price, which we can denote with $P_t^{\#}$. So, we can rewrite this expression compactly as:

$$P_t^{\#} = \mathcal{M}_p \frac{X_{1,t}}{X_{2,t}}, \tag{633}$$

where

$$X_{1,t} = u'(C_t) mc_t P_t^{\epsilon_p} Y_t + \phi_p \beta \mathbb{E}_t X_{1,t+1}, \tag{634}$$

$$X_{2,t} = u'(C_t) P_t^{\epsilon_p - 1} Y_t + \phi_p \beta \mathbb{E}_t X_{2,t+1}. \tag{635}$$

Note here that if $\phi_p = 0$, then the RHS would reduce to $mc_t P_t = \varphi_t$. In this case, the optimal price would be a fixed markup, $\mathcal{M}_p$, over nominal marginal cost, φ_t. This will come in handy when we calculate the flexible price equilibrium.

15.3 Households

We now amend households slightly to get sticky wages. As stated earlier, we assume that households supply slightly differentiated labour which gives them some pricing power in setting their own wage. In a similar way to the final goods firm, we introduce the concept of a labour bundler (like a union) which combines different types of labour into a composite labour contract that it then leases to firms at wage rate W_t. We first consider the problem of the competitive labour bundler, and then the problem of the household.

15.3.1 Labour bundler

Total labour input is equal to:

$$N_t = \left(\int_0^1 N_t(l)^{\frac{\epsilon_w - 1}{\epsilon_w}} dl \right)^{\frac{\epsilon_w}{\epsilon_w - 1}}, \tag{636}$$

where here $\epsilon_w > 1$, and l indexes the differentiated labour inputs which populate a unit interval. The profit maximisation problem of the competitive labour bundler is:

$$\max_{N_t(l)} W_t \left(\int_0^1 N_t(l)^{\frac{\epsilon_w - 1}{\epsilon_w}} dl \right)^{\frac{\epsilon_w}{\epsilon_w - 1}} - \int_0^1 W_t(l) N_t(l) dl.$$

The FOC for the choice of labour of variety l is:

$$W_t \frac{\epsilon_w}{\epsilon_w - 1} \left(\int_0^1 N_t(l)^{\frac{\epsilon_w - 1}{\epsilon_w}} dl \right) \frac{\epsilon_w - 1}{\epsilon_w} N_t(l)^{\frac{\epsilon_w - 1}{\epsilon_w} - 1} = W_t(l),$$

and this can be simplified to:

$$
\begin{aligned}
\frac{W_t(l)}{W_t} &= N_t(l)^{-\frac{1}{\epsilon_w}} \left(\int_0^1 N_t(l)^{\frac{\epsilon_w - 1}{\epsilon_w}} dl \right)^{\frac{\epsilon_w}{\epsilon_w - 1}} \\
\left(\frac{W_t(l)}{W_t} \right)^{-\epsilon_w} &= N_t(l) \left(\int_0^1 N_t(l)^{\frac{\epsilon_w - 1}{\epsilon_w}} dl \right)^{-\frac{\epsilon_w}{\epsilon_w - 1}} \\
N_t(l) &= \left(\frac{W_t(l)}{W_t} \right)^{-\epsilon_w} N_t
\end{aligned}
\tag{637}
$$

In a way exactly analogous to intermediate goods, the relative demand for labour of type l is a function of its relative wage with elasticity ϵ_w. Likewise, using the results from Blanchard and Kiyotaki (1987), we can derive an aggregate wage index in a similar way to above, by defining:

$$
\begin{aligned}
W_t N_t &= \int_0^1 W_t(l) N_t(l) dl \\
&= \int_0^1 W_t(l) \left(\frac{W_t(l)}{W_t} \right)^{-\epsilon_w} N_t dl \\
W_t &= \int_0^1 W_t(l)^{1-\epsilon_w} W_t^{\epsilon_w} dl \\
W_t^{1-\epsilon_w} &= \int_0^1 W_t(l)^{1-\epsilon_w} dl \\
W_t &= \left(\int_0^1 W_t(l)^{1-\epsilon_w} dl \right)^{\frac{1}{1-\epsilon_w}} .
\end{aligned}
\tag{638}
$$

15.3.2 Differentiated labour

Households are heterogenous are indexed by $l \in (0, 1)$, supplying differentiated labour input to the labour bundler. We're going to assume that preferences are additively separable in consumption and labour, which turns out to be important (so that we don't get any weird cross elasticities). If wages are subject to frictions like the Calvo (1983) pricing friction, households will charge different wages, meaning they will work different hours, meaning they will have different incomes and therefore different consumption and bond-holding decisions. Erceg et al. (2000) show that if there exists state contingent claims that insure households against idiosyncratic wage risk, and if preferences are separable in consumption and leisure, households will be identical in their choice of consumption and bold-holdings, and will only differ in the wage they charge and labour they supply. As such, in the notation below, we will suppress dependence on l for consumption and bonds, but leave it for wages and labour input. We also abstract from money altogether, nothing that we could include real balances as a separable argument in the utility function without any effects on the rest of them model.

The household problem is:

$$\max_{C_t, N_t(l), W_t(l), B_t} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left[\frac{C_{t+s}^{1-\sigma}}{1-\sigma} - \psi \frac{N_t(l)^{1+\eta}}{1+\eta} \right],$$

subject to:

$$\begin{aligned} P_t C_t + B_t &\leq W_t(l) N_t(l) + D_t + R_{t-1} B_{t-1}, \\ N_t(l) &= \left(\frac{W_t(l)}{W_t} \right)^{-\epsilon_w} N_t. \end{aligned}$$

P_t is the nominal price of goods, D_t are nominal profits distributed from firms, and B_t is the nominal stock of bonds which pay a return in period t, paying the nominal interest rate known in period $t-1$.

Imposing that labour supply equal labour demand, which allows us to switch notation from choosing $N_t(l)$ to instead choosing $W_t(l)$, we can write the Lagrangian for the household's problem as:

$$\mathcal{L} = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left\{ \begin{array}{c} \frac{C_{t+s}^{1-\sigma}}{1-\sigma} - \psi \frac{\left[\left(\frac{W_{t+s}(l)}{W_{t+s}} \right)^{-\epsilon_w} N_{t+s} \right]^{1+\eta}}{1+\eta} \\ + \lambda_{t+s} \left[W_{t+s}(l) \left(\frac{W_{t+s}(l)}{W_{t+s}} \right)^{-\epsilon_w} N_{t+s} + D_{t+s} + R_{t+s-1} B_{t+s-1} - P_{t+s} C_{t+s} - B_{t+s} \right] \end{array} \right\}.$$

The FOCs with respect to C_t and B_t are:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial C_t} &= C_t^{-\sigma} - P_t \lambda_t = 0, \\ \frac{\partial \mathcal{L}}{\partial B_t} &= -\lambda_t + \beta \mathbb{E}_t \lambda_{t+1} R_t = 0. \end{aligned}$$

Combining these, we get:

$$C_t^{-\sigma} = \beta \mathbb{E}_t C_{t+1}^{-\sigma} R_t \frac{P_t}{P_{t+1}}, \tag{639}$$

which is the standard Euler equation for bonds.

Now, let's think about wage setting. In writing the Lagrangian, we have eliminated $N_t(j)$ as a choice variable, instead writing the problem as choosing $W_t(l)$. As with prices, assume that households are not freely able to choose their wage each period. In particular, each period they face the probability of getting a visit from the Calvo fairy of $1 - \phi_w$ and are able to adjust their wage. With probability ϕ_w they are stuck with a wage for one period, ϕ_w^2 for two periods, and so on.

Before proceeding, let's rewrite the problem in terms of choosing the real wage instead of the nominal wage. The reason we may want to do this is that, depending on the monetary policy rule, inflation could be non-stationary, which would make nominal wages non-stationary, but real wages stationary. Define the real wage a household charges as:

$$w_t(l) = \frac{W_t(l)}{P_t},$$

and similarly for the aggregate real wage:

$$w_t = \frac{W_t}{P_t}.$$

Since both of these real wages are divided by the same price level, the relative demand for labour of variety l can be written either in terms of the ratio of nominal wages or the ratio of real wages, as these are equivalent.

Now, let's consider the problem of a household who can update its nominal wage in period t. The probability that the nominal wage will still be operative in period $t+s$ is ϕ_w^s. The real wage a household charges in period $t+s$ if it is stuck with the nominal wage it chose in period t is:

$$w_{t+s}(l) = \frac{W_t(l)}{P_{t+s}},$$

which can be written in terms of the period t real wage as:

$$w_{t+s}(l) = \frac{W_t(l)}{P_t} \frac{P_t}{P_{t+s}}.$$

We can create something out of nothing. Define $\Pi_{t,t+s} = \frac{P_{t+s}}{P_t}$ as the gross inflation between t and $t+s$. This is just equal to the product of period-over-period gross inflation. Define $\pi_t = \frac{P_t}{P_{t-1}} - 1$ as the period-over-period net inflation, so we have:

$$\begin{aligned}\Pi_{t,t+s} &= \prod_{m=1}^{s}(1-\pi_{t+m}) \\ &= \frac{P_{t+1}}{P_t} \times \frac{P_{t+2}}{P_{t+1}} \times ... \times \frac{P_{t+s}}{P_{t+s-1}} \\ &= \frac{P_{t+s}}{P_t}.\end{aligned}$$

This means that the real wage a household with a stuck nominal wage will charge in period $t+s$ can be written as:

$$w_{t+s}(l) = w_t(l)\Pi_{t,t+s}^{-1},$$

where $w_t(l)$ is the real wage chosen in period t.

Now, when choosing $w_t(l)$, households will discount the future not just by β^s but by ϕ_w^s as well, since the latter is the probability that a household will be stuck with that wage in period $t+s$. reproducing just the parts of the Lagrangian that related to the choice of labour, we have:

$$\tilde{\mathcal{L}} = \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \left\{ -\psi \frac{\left[\frac{w_t(l)\Pi_{t,t+s}^{-1}}{w_{t+s}}\right]^{-\epsilon_w(1+\eta)} N_{t+s}^{1+\eta}}{1+\eta} + \lambda_{t+s}P_{t+s}\left[w_t(l)\Pi_{t,t+s}^{-1}\left(\frac{w_t(l)\Pi_{t,t+s}^{-1}}{w_{t+s}}\right)^{-\epsilon_w} N_{t+s}\right]\right\}.$$

Note that the multiplier, λ_{t+s}, gets multiplied by P_{t+s} because we're writing the real wage in real terms here (so we're de-facto multiplying and dividing by P_{t+s}). By multiplying out, this can be rewritten as:

$$\tilde{\mathcal{L}} = \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \left\{ -\psi \frac{w_t(l)^{-\epsilon_w(1+\eta)} w_{t+s}^{\epsilon_w(1+\eta)} \Pi_{t,t+s}^{\epsilon_w(1+\eta)} N_{t+s}^{1+\eta}}{1+\eta} + \lambda_{t+s}P_{t+s}\left[w_t(l)^{1-\epsilon_w} w_{t+s}^{\epsilon_w} \Pi_{t,t+s}^{\epsilon_w - 1} N_{t+s}\right]\right\}.$$

The FOC is:

$$\frac{\partial \tilde{\mathcal{L}}}{\partial w_t(l)} = \epsilon_w w_t(l)^{-\epsilon_w(1+\eta)-1} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \psi w_{t+s}^{\epsilon_w(1+\eta)} \Pi_{t,t+s}^{\epsilon_w(1+\eta)} N_{t+s}^{1+\eta}$$
$$+ (1-\epsilon_w) w_t(l)^{-\epsilon_w} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \lambda_{t+s} P_{t+s} w_{t+s}^{\epsilon_w} \Pi_{t,t+s}^{\epsilon_w - 1} N_{t+s} = 0,$$

and with a bit simplifying this becomes:

$$\epsilon_w w_t(l)^{-\epsilon_w(1+\eta)-1} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \psi w_{t+s}^{\epsilon_w(1+\eta)} \Pi_{t,t+s}^{\epsilon_w(1+\eta)} N_{t+s}^{1+\eta}$$
$$= (\epsilon_w - 1) w_t(l)^{-\epsilon_w} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \lambda_{t+s} P_{t+s} w_{t+s}^{\epsilon_w} \Pi_{t,t+s}^{\epsilon_w - 1} N_{t+s},$$

or:

$$(w_t^{\#})^{1+\epsilon_w \eta} = \mathcal{M}_w \frac{\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \psi w_{t+s}^{\epsilon_w(1+\eta)} \Pi_{t,t+s}^{\epsilon_w(1+\eta)} N_{t+s}^{1+\eta}}{\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\phi_w)^s \lambda_{t+s} P_{t+s} w_{t+s}^{\epsilon_w} \Pi_{t,t+s}^{\epsilon_w - 1} N_{t+s}}, \tag{640}$$

where $\mathcal{M}_w = \frac{\epsilon_w}{\epsilon_w - 1}$. Above, we have gotten rid of the dependence on the l index because nothing on the RHS is dependent on l, meaning that all updating households will update to the same wage, which we call $w_t^{\#}$, the reset wage. This an be written more compactly as:

$$(w_t^{\#})^{1+\epsilon_w \eta} = \mathcal{M}_w \frac{H_{1,t}}{H_{2,t}}, \tag{641}$$

where:

$$H_{1,t} = \psi w_t^{\epsilon_w(1+\eta)} N_t^{1+\eta} + \beta\phi_w \mathbb{E}_t \Pi_{t+1}^{\epsilon_w(1+\eta)} H_{1,t+1}, \tag{642}$$

$$H_{2,t} = C_t^{-\sigma} w_t^{\epsilon_w} N_t + \beta\phi_w \mathbb{E}_t \Pi_{t+1}^{\epsilon_w - 1} H_{2,t+1}. \tag{643}$$

These lines follow because $\Pi_{t,t} = 1$, and $\Pi_{t,t+1} = (1+\pi_{t+1}) = \Pi_{t+1}$, so the $\Pi_{t,t+s}$ is effectively like an additional part of the discount factor, and $\lambda_t P_t = C_t^{-\sigma}$.

Now, consider the case where wages flexible, i.e. when $\phi_w = 0$:

$$
\begin{aligned}
(w_t^{\#})^{1+\epsilon_w \eta} &= \mathcal{M}_w \frac{\psi w_t^{\epsilon_w (1+\eta)} N_t^{1+\eta}}{C_t^{-\sigma} w_t^{\epsilon_w} N_t} \\
&= \mathcal{M}_w \frac{\psi w_t^{\epsilon_w \eta} N^{\eta}}{C_t^{-\sigma}},
\end{aligned}
$$

and if $\phi_w = 0$, then all households update, so the reset wage is equal to the actual real wage ($w_t^{\#} = w_t$):

$$
\implies w_t = \mathcal{M}_w \frac{\psi N^{\eta}}{C_t^{-\sigma}}.
$$

Since $\epsilon_w > 1$, we have $\mathcal{M}_w > 1$. What this says is that the wage is a markup over the marginal rate of substitution between labour and consumption ($\psi N_t^{\eta}/C_t^{-\sigma}$ is the MRS). If $\epsilon_w \to \infty$, this would be exactly the FOC that we had in the flexible wage case.

15.4 Equilibrium and aggregation

Let's first assume that the central bank sets interest rates according to a Taylor Rule. In the Taylor Rule, the central bank only targets inflation, but it would be straightforward to also target the output gap (or output growth). As long as households get utility from real balances in an additively separable way, this will determine the price level and we can ignore money:

$$
i_t = (1 - \rho_i)\bar{i} + \rho_i i_{t-1} + \phi_\pi (\pi_t - \bar{\pi}) + \epsilon_{i,t} \tag{644}
$$

Productivity follows an AR(1) process in logs:

$$
\ln A_t = \rho_a \ln A_{t-1} + \epsilon_{a,t}. \tag{645}
$$

In equilibrium, bond-holding is always zero: $B_t = 0$. Using this, the household budget constraint can be written in real terms:

$$
C_t = \frac{W_t(l)}{P_t} N_t(l) + \frac{D_t}{P_t}, \tag{646}
$$

and integrating over l:

$$C_t = \int_0^1 \frac{W_t(l)}{P_t} N_t(l) dl + \frac{D_t}{P_t}. \tag{647}$$

Real dividends received the household are just the sum of real profits from intermediate goods firms:

$$\frac{D_t}{P_t} = \int_0^1 \left(\frac{P_t(j)}{P_t} Y_t(j) - \frac{W_t}{P_t} N_t(j) \right) dj.$$

This can be written as:

$$\frac{D_t}{P_t} = \int_0^1 \frac{P_t(j)}{P_t} Y_t(j) dj - w_t \int_0^1 N_t(j) dj,$$

where we used the definition that $w_t \equiv W_t / P_t$. Now, market clearing requires that the sum of labour used by firms equals the total labour supplied by the labour bundler, so $\int_0^1 N_t(j) dj = N_t$. Hence:

$$\frac{D_t}{P_t} = \int_0^1 \frac{P_t(j)}{P_t} Y_t(j) dj - w_t N_t.$$

Plug this into the integrated household budget constraint (647):

$$C_t = \int_0^1 \frac{W_t(l)}{P_t} N_t(l) dl + \int_0^1 \frac{P_t(j)}{P_t} Y_t(j) dj - w_t N_t,$$

and then plug in the demand for labour of type l (637):

$$C_t = \int_0^1 \frac{W_t(l)}{P_t} \left(\frac{W_t(l)}{W_t} \right)^{-\epsilon_w} N_t dl + \int_0^1 \frac{P_t(j)}{P_t} Y_t(j) dj - w_t N_t,$$

simplify:

$$C_t = N_t P_t^{-1} W_t^{\epsilon_w} \int_0^1 W_t(l)^{1-\epsilon_w} dl + \int_0^1 \frac{P_t(j)}{P_t} Y_t(j) dj - w_t N_t.$$

Now, using the aggregate (nominal) wage index, we know:

$$\int_0^1 W_t(l)^{1-\epsilon_w} dl = W_t^{1-\epsilon_w},$$

and so we can substitute this in:

$$\begin{aligned} C_t &= N_t P_t^{-1} W_t^{\epsilon_w} W_t^{1-\epsilon_w} + \int_0^1 \frac{P_t(j)}{P_t} Y_t(j) dj - w_t N_t \\ &= N_t \frac{W_t}{P_t} + + \int_0^1 \frac{P_t(j)}{P_t} Y_t(j) dj - w_t N_t. \end{aligned}$$

Since $w_t \equiv W_t / P_t$, we have:

$$C_t = \int_0^1 \frac{P_t(j)}{P_t} Y_t(j) dj, \tag{648}$$

which says that aggregate consumption must equal the sum of real quantities of intermediates. Now, plug in the demand curve for intermediate variety j (628):

$$C_t = \int_0^1 \frac{P_t(j)}{P_t} \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon_p} Y_t dj,$$

and clean up a bit:

$$C_t = \frac{Y_t}{P_t} \int_0^1 P_t(j)^{1-\epsilon_p} dj.$$

Now, from the definition of the aggregate price level and using Blanchard and Kiyotaki (1987) again, we have:

$$\int_0^1 P_t(j)^{1-\epsilon_p} dj = P_t^{1-\epsilon_p}.$$

This means the terms involving P_t cancel, so we're left with:[97]

$$C_t = Y_t. \tag{649}$$

Now, what is Y_t? From the demand for intermediate j (628), we know:

$$Y_t(j) = \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon_p} Y_t,$$

[97] Adding either government spending or capital to this model will obviously throw a wrench into this result.

and using the production function for each intermediate (630), we can write (628) as:

$$A_t N_t(j) = \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} Y_t,$$

and then integrate over j:

$$\begin{aligned} \int_0^1 A_t N_t(j) dj &= \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} Y_t dj \\ \Leftrightarrow A_t \int_0^1 N_t(j) dj &= Y_t \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} dj. \end{aligned}$$

Then, define a new variable, v_t^p, as:

$$v_t^p = \int_0^1 \left(\frac{P_t(j)}{P_t}\right)^{-\epsilon_p} dj, \tag{650}$$

which is a measure of price dispersion. If there were no pricing frictions, all firms would charge the same price, and $v_t^p = 1$. If prices are different, one can show that this expression is bound from below by unity. Using the definition of aggregate labour input, we can therefore write:

$$Y_t = \frac{A_t N_t}{v_t^p}. \tag{651}$$

This is the aggregate production function. Since $v_t^p \geq 1$, price dispersion results in an output loss – you produce less output than you would given A_t and aggregate labour input if prices are disperse.

Since we've written the FOCs for labour in terms of the real wage, let's rewrite the aggregate nominal wage index (638) in terms of real wages. Divide both sides by $P_t^{1-\epsilon_w}$:

$$\begin{aligned} \left(\frac{W_t}{P_t}\right)^{1-\epsilon_w} &= \int_0^1 \left(\frac{W_t(l)}{P_t}\right)^{1-\epsilon_w} dl \\ \Leftrightarrow w_t^{1-\epsilon_w} &= \int_0^1 w_t(l)^{1-\epsilon_w} dl. \end{aligned} \tag{652}$$

The full set of equilibrium conditions can then be characterised by:

$$C_t^{-\sigma} = \beta \mathbb{E}_t C_{t+1}^{-\sigma} R_t \frac{P_t}{P_{t+1}}, \tag{653}$$

$$(w_t^{\#})^{1+\epsilon_w \eta} = \mathcal{M}_w \frac{H_{1,t}}{H_{2,t}}, \tag{654}$$

$$H_{1,t} = \psi w_t^{\epsilon_w(1+\eta)} N_t^{1+\eta} + \beta \phi_w \mathbb{E}_t \Pi_{t+1}^{\epsilon_w(1+\eta)} H_{1,t+1}, \tag{655}$$

$$H_{2,t} = C_t^{-\sigma} w_t^{\epsilon_w} N_t + \beta \phi_w \mathbb{E}_t \Pi_{t+1}^{\epsilon_w - 1} H_{2,t+1}, \tag{656}$$

$$mc_t = \frac{w_t}{A_t}, \tag{657}$$

$$C_t = Y_t, \tag{658}$$

$$Y_t = \frac{A_t N_t}{v_t^p}, \tag{659}$$

$$v_t^p = \int_0^1 \left(\frac{P_t(j)}{P_t} \right)^{-\epsilon_p} dj, \tag{660}$$

$$w_t^{1-\epsilon_w} = \int_0^1 w_t(l)^{1-\epsilon_w} dl, \tag{661}$$

$$P_t^{1-\epsilon_p} = \int_0^1 P_t(j)^{1-\epsilon_p} dj, \tag{662}$$

$$P_t^{\#} = \mathcal{M}_p \frac{X_{1,t}}{X_{2,t}}, \tag{663}$$

$$X_{1,t} = C_t^{-\sigma} mc_t P_t^{\epsilon_p} Y_t + \phi_p \beta \mathbb{E}_t X_{1,t+1}, \tag{664}$$

$$X_{2,t} = C_t^{-\sigma} P_t^{\epsilon_p - 1} Y_t + \phi_p \beta \mathbb{E}_t X_{2,t+1}, \tag{665}$$

$$i_t = (1-\rho_i)\bar{i} + \rho_i i_{t-1} + \phi_\pi (\pi_t - \bar{\pi}) + \epsilon_{i,t}, \tag{666}$$

$$\ln A_t = \rho_a \ln A_{t-1} + \epsilon_{a,t}, \tag{667}$$

$$\pi_t = \frac{P_t}{P_{t-1}} - 1. \tag{668}$$

This is a system of 16 equations in 16 aggregate variables:

$$\{C_t, i_t, P_t, w_t^{\#}, H_{1,t}, H_{2,t}, w_t, N_t, \pi_t, mc_t, A_t, Y_t, v_t^p, P_t^{\#}, X_{1,t}, X_{2,t}\}.$$

15.5 Re-writing equilibrium conditions

There are two issues with the way in which we've written out these conditions. First, we still have heterogeneity – we still have j and l indexes showing up. Second, we have the price level showing up, which, as we mentioned before, may not be stationary. So we will rewrite these equilibrium conditions only in terms of inflation, and independent of any heterogeneity.

The Euler equation is easy to deal with and can be rewritten as:

$$C_t^{-\sigma} = \beta \mathbb{E}_t C_{t+1}^{-\sigma} R_t \Pi_{t+1}^{-1}. \tag{669}$$

Now, consider the expression for the price level and the real wage. The expression for the price level is:

$$P_t^{1-\epsilon_p} = \int_0^1 P_t(j)^{1-\epsilon_p} dj,$$

where a fraction $(1-\phi_p)$ of these firms will update their price to the same reset price, $P_t^{\#}$. The other fraction ϕ_p will charge the price they charged in the previous period. This means we can break up the integral on the RHS as:

$$\begin{aligned} P_t^{1-\epsilon_p} &= \int_0^{1-\phi_p} (P_t^{\#})^{1-\epsilon_p} dj + \int_{1-\phi_p}^1 P_{t-1}(j)^{1-\epsilon_p} dj \\ &= (1-\phi_p)(P_t^{\#})^{1-\epsilon_p} + \int_{1-\phi_p}^1 P_{t-1}(j)^{1-\epsilon_p} dj. \end{aligned}$$

We can then use a trick to take advantage of Calvo pricing. Because the firms who get to update are randomly chosen, and because there are a large number (continuum) of firms, the integral of individual prices over some subset of the unit interval will simply be proportional to the integral over the entire unit interval, where the proportion is equal to the subset of the unity interval over which the integral is taken. This means:

$$\begin{aligned} \int_{1-\phi_p}^1 P_{t-1}(j)^{1-\epsilon_p} dj &= \phi_p \int_0^1 P_{t-1}(j)^{1-\epsilon_p} dj \\ &= \phi_p P_{t-1}^{1-\epsilon_p}. \end{aligned}$$

This means that the aggregate price level (raised to $1-\epsilon_p$) is a convex combination of the reset price and lagged price level (raised to the same power). So:

$$P_t^{1-\epsilon_p} = (1-\phi_p)(P_t^{\#})^{1-\epsilon_p} + \phi_p P_{t-1}^{1-\epsilon_p},$$

and just like that we've gotten rid of the heterogeneity. The Calvo assumption allows us to integrate out of the heterogeneity and not worry about keeping track of what each firm is doing from the perspective of looking at the behaviour of aggregates. Now, we still have the issue here of things being written in terms of the price level and not inflation. To get things in terms of inflation, divide both sides by $P_{t-1}^{1-\epsilon_p}$, and define $\pi_t^{\#} = \frac{P_t^{\#}}{P_{t-1}} - 1$ as reset price inflation:

$$(1+\pi_t)^{1-\epsilon_p} = (1-\phi_p)(1+\pi_t^{\#})^{1-\epsilon_p} + \phi_p. \tag{670}$$

We can do exactly the same thing for wages. The aggregate real wage index is:

$$w_t^{1-\epsilon_w} = \int_0^1 w_t(l)^{1-\epsilon_w} dl.$$

Since $1-\phi_w$ of households will update to the same reset wage, and ϕ_w will be stuck with last period's nominal wage, this is:

$$\begin{aligned} w_t^{1-\epsilon_w} &= \int_0^{1-\phi_w} (w_t^{\#})^{1-\epsilon_w} dl + \int_{1-\phi_w}^1 \left(\frac{W_{t-1}}{P_t}\right)^{1-\epsilon_w} dl \\ &= (1-\phi_w)(w_t^{\#})^{1-\epsilon_w} + \int_{1-\phi_w}^1 \left(\frac{W_{t-1}}{P_t}\right)^{1-\epsilon_w} dl, \end{aligned}$$

and note that we have written this in terms of nominal wages in terms of the non-updated wages. We can rewrite in terms of real wages as:

$$w_t^{1-\epsilon_w} = (1-\phi_w)(w_t^{\#})^{1-\epsilon_w} + \int_{1-\phi_w}^1 \left(\frac{W_{t-1}}{P_{t-1}}\right)^{1-\epsilon_w} \left(\frac{P_{t-1}}{P_t}\right)^{1-\epsilon_w} dl.$$

Written in terms of inflation, and moving things out of the integral, we get:

$$w_t^{1-\epsilon_w} = (1-\phi_w)(w_t^{\#})^{1-\epsilon_w} + (1+\pi_t)^{\epsilon_w - 1} \int_{1-\phi_w}^{1} w_{t-1}(l)^{1-\epsilon_w} dl.$$

Again, the Calvo assumption allows us to get rid of the integral on the RHS which will just be proportional to last period's aggregate real wage. So we're left with:

$$w_t^{1-\epsilon_w} = (1-\phi_w)(w_t^{\#})^{1-\epsilon_w} + \phi_w (1+\pi_t)^{\epsilon_w - 1} w_{t-1}^{1-\epsilon_w}. \tag{671}$$

We can also use the Calvo assumption to break up the price dispersion term, by again noting that $(1-\phi_p)$ of firms will update to the same price, and ϕ_p firms will be stuck with last period's price. Hence:

$$v_t^p = \int_0^{1-\phi_p} \left(\frac{P_t^{\#}}{P_t} \right)^{-\epsilon_p} dj + \int_{1-\phi_p}^{1} \left(\frac{P_{t-1}(j)}{P_t} \right)^{-\epsilon_p} dj.$$

This can be written in terms of inflation by multiplying and dividing by powers of P_{t-1} where necessary:

$$v_t^p = \int_0^{1-\phi_p} \left(\frac{P_t^{\#}}{P_t} \right)^{-\epsilon_p} \left(\frac{P_{t-1}}{P_t} \right)^{-\epsilon_p} dj + \int_{1-\phi_p}^{1} \left(\frac{P_{t-1}(j)}{P_{t-1}} \right)^{-\epsilon_p} \left(\frac{P_{t-1}}{P_t} \right)^{-\epsilon_p} dj.$$

Take stuff out of the integral:

$$v_t^p = (1-\phi_p)(1+\pi_t^{\#})^{-\epsilon_p}(1+\pi_t)^{\epsilon_p} + (1+\pi_t)^{\epsilon_p} \int_{1-\phi_p}^{1} \left(\frac{P_{t-1}(j)}{P_{t-1}} \right)^{-\epsilon_p} dj.$$

By the same Calvo logic, the term inside the integral is just going to be proportional to v_{t-1}^p . This means we can write the price dispersion term as:

$$v_t^p = (1-\phi_p)(1+\pi_t^{\#})^{-\epsilon_p}(1+\pi_t)^{\epsilon_p} + (1+\pi_t)^{\epsilon_p} \phi_p v_{t-1}^p. \tag{672}$$

In other words, we just have to keep track of v_t^p, not the individual prices.

Now, we need to adjust the reset price expression. First, define two new auxiliary variables as:

$$x_{1,t} = \frac{X_{1,t}}{P_t^{\epsilon_p}},$$
$$x_{2,t} = \frac{X_{2,t}}{P_t^{\epsilon_p - 1}}.$$

Divide both sides of the reset price expressions by the appropriate power of P_t, and we get:

$$x_{1,t} = C_t^{-\sigma} mc_t Y_t + \phi_p \beta \mathbb{E}_t \frac{X_{1,t+1}}{P_t^{\epsilon_p}},$$
$$x_{2,t} = C_t^{-\sigma} Y_t + \phi_p \beta \mathbb{E}_t \frac{X_{2,t+1}}{P_t^{\epsilon_p - 1}}.$$

Multiplying and dividing the $t+1$ terms by the appropriate power of P_{t+1}, we have:

$$x_{1,t} = C_t^{-\sigma} mc_t Y_t + \phi_p \beta \mathbb{E}_t \frac{X_{1,t+1}}{P_{t+1}^{\epsilon_p}} \left(\frac{P_{t+1}}{P_t} \right)^{\epsilon_p},$$
$$x_{2,t} = C_t^{-\sigma} Y_t + \phi_p \beta \mathbb{E}_t \frac{X_{2,t+1}}{P_{t+1}^{\epsilon_p - 1}} \left(\frac{P_{t+1}}{P_t} \right)^{\epsilon_p - 1},$$

and, in terms of inflation as:

$$x_{1,t} = C_t^{-\sigma} mc_t Y_t + \phi_p \beta \mathbb{E}_t \left(1 + \pi_{t+1}\right)^{\epsilon_p} x_{1,t+1}, \tag{673}$$

$$x_{2,t} = C_t^{-\sigma} Y_t + \phi_p \beta \mathbb{E}_t \left(1 + \pi_{t+1}\right)^{\epsilon_p - 1} x_{2,t+1}. \tag{674}$$

Now, in terms of the reset price expression, since we divided $X_{1,t}$ by $P_t^{\epsilon_p}$, and $X_{2,t}$ by $P_t^{\epsilon_p - 1}$, we de-facto multiply the ratio of $\frac{X_{1,t}}{X_{2,t}}$ by P_t^{-1}. Hence, to keep equality, we need to multiply the RHS by P_t. Hence, the reset price expression can now be written as:

$$P_t^{\#} = \mathcal{M}_p P_t \frac{x_{1,t}}{x_{2,t}},$$

and then divide both sides by P_{t-1} to have everything in terms of inflation rates:

$$\Pi_t^{\#} = \mathcal{M}_p \Pi_t \frac{x_{1,t}}{x_{2,t}}. \tag{675}$$

The full set of equilibrium conditions can now be expressed as:

$$C_t^{-\sigma} = \beta \mathbb{E}_t C_{t+1}^{-\sigma} \frac{R_t}{\Pi_{t+1}}, \tag{676}$$

$$(w_t^{\#})^{1+\epsilon_w \eta} = \mathcal{M}_w \frac{H_{1,t}}{H_{2,t}}, \tag{677}$$

$$H_{1,t} = \psi w_t^{\epsilon_w(1+\eta)} N_t^{1+\eta} + \beta \phi_w \mathbb{E}_t \Pi_{t+1}^{\epsilon_w(1+\eta)} H_{1,t+1}, \tag{678}$$

$$H_{2,t} = C_t^{-\sigma} w_t^{\epsilon_w} N_t + \beta \phi_w \mathbb{E}_t \Pi_{t+1}^{\epsilon_w - 1} H_{2,t+1}, \tag{679}$$

$$mc_t = \frac{w_t}{A_t}, \tag{680}$$

$$C_t = Y_t, \tag{681}$$

$$Y_t = \frac{A_t N_t}{v_t^p}, \tag{682}$$

$$v_t^p = (1-\phi_p)(\Pi_t^{\#})^{-\epsilon_p} \Pi_t^{\epsilon_p} + \Pi_t^{\epsilon_p} \phi_p v_{t-1}^p, \tag{683}$$

$$w_t^{1-\epsilon_w} = (1-\phi_w)(w_t^{\#})^{1-\epsilon_w} + \phi_w \Pi_t^{\epsilon_w - 1} w_{t-1}^{1-\epsilon_w}, \tag{684}$$

$$\Pi_t^{1-\epsilon_p} = (1-\phi_p)(\Pi_t^{\#})^{1-\epsilon_p} + \phi_p, \tag{685}$$

$$\Pi_t^{\#} = \mathcal{M}_p \Pi_t \frac{x_{1,t}}{x_{2,t}}, \tag{686}$$

$$x_{1,t} = C_t^{-\sigma} mc_t Y_t + \phi_p \beta \mathbb{E}_t \Pi_{t+1}^{\epsilon_p} x_{1,t+1}, \tag{687}$$

$$x_{2,t} = C_t^{-\sigma} Y_t + \phi_p \beta \mathbb{E}_t \Pi_{t+1}^{\epsilon_p - 1} x_{2,t+1}, \tag{688}$$

$$i_t = (1-\rho_i)\bar{i} + \rho_i i_{t-1} + \phi_\pi(\pi_t - \bar{\pi}) + \epsilon_{i,t}, \tag{689}$$

$$\ln A_t = \rho_a \ln A_{t-1} + \epsilon_{a,t}. \tag{690}$$

This is now 15 equations in fifteen variables, where we have eliminated P_t as a variable, replaced $P_t^{\#}$ with $\pi_t^{\#}$, and replaced $X_{1,t}$ and $X_{2,t}$ with $x_{1,t}$ and $x_{2,t}$:

$\{C_t, i_t, w_t^{\#}, H_{1,t}, H_{2,t}, w_t, N_t, \pi_t, mc_t, A_t, Y_t, v_t^p, \pi_t^{\#}, x_{1,t}, x_{2,t}\}$.

15.6 The steady state

In the non-stochastic steady state, $\bar{A} = 1$. Steady state inflation will be equal to target. We can solve for steady state reset price inflation as:

$$\bar{\Pi}^{\#} = \left(\frac{\bar{\Pi}^{1-\epsilon_p} - \phi_p}{1 - \phi_p} \right)^{\frac{1}{1-\epsilon_p}}, \tag{691}$$

where if $\bar{\pi} = 0$, then $\bar{\pi}^{\#} = 0$ too (in gross terms: if $\bar{\Pi} = 1$, then $\bar{\Pi}^{\#} = 1$). Steady state price dispersion is:

$$\bar{v}^p = \frac{(1 - \phi_p) \left(\frac{\bar{\Pi}}{\bar{\Pi}^{\#}} \right)^{\epsilon_p}}{1 - \bar{\Pi}^{\epsilon_p} \phi_p}, \tag{692}$$

where again we see that if $\bar{\pi} = \bar{\pi}^{\#} = 0$, then $\bar{v}^p = 1$.

The steady state nominal interest rate is:

$$1 + \bar{i} = \frac{1}{\beta}(1 + \bar{\pi}). \tag{693}$$

The steady state auxiliary pricing variables are:

$$\bar{x}_1 = \frac{\bar{Y}^{1-\sigma} \bar{mc}}{1 - \phi_p \beta (1 + \bar{\pi})^{\epsilon_p}}, \tag{694}$$

$$\bar{x}_2 = \frac{\bar{Y}^{1-\sigma}}{1 - \phi_p (1 + \bar{\pi})^{\epsilon_p - 1}}, \tag{695}$$

which means that the ratio is:

$$\frac{\bar{x}_1}{\bar{x}_2} = \bar{mc} \frac{1 - \phi_p \beta (1 + \bar{\pi})^{\epsilon_p - 1}}{1 - \phi_p \beta (1 + \bar{\pi})^{\epsilon_p}}.$$

Hence, we can solve for the steady state marginal cost as:

$$\bar{mc} = \mathcal{M}_p^{-1} \left(\frac{1 - \phi_p \beta (1 + \bar{\pi})^{\epsilon_p - 1}}{1 - \phi_p \beta (1 + \bar{\pi})^{\epsilon_p}} \right) \frac{1 + \bar{\pi}^{\#}}{1 + \bar{\pi}} \tag{696}$$

Again, we can see that if $\bar{\pi} = \bar{\pi}^{\#} = 0$, then $\bar{mc} = \bar{\mathcal{M}}_p^{-1} = \frac{\epsilon_p - 1}{\epsilon_p}$, which is the desired flexible price markup.

Let's solve for the optimal reset wage in terms of the steady state real wage:

$$\bar{w}^{\#} = \left[\frac{\left(1 - \phi_w (1 + \bar{\pi})^{\epsilon_w - 1}\right)}{1 - \phi_w} \right]^{\frac{1}{1-\epsilon_w}} \bar{w}. \tag{697}$$

This says that the reset wage is proportional to the steady state wage. Note that, if $\phi_w = 0$ (wages fully flexible), we would have $\bar{w}^{\#} = \bar{w}$. Let's now solve for the steady states of the auxiliary variables related to wage setting. We have:

$$H_1 = \frac{\psi \bar{w}^{\epsilon_w (1+\eta)} \bar{N}^{1+\eta}}{1 - \phi_w \beta (1 + \bar{\pi})^{\epsilon_w (1+\eta)}}, \tag{698}$$

$$H_2 = \frac{\bar{Y}^{-\sigma} \bar{w}^{\epsilon_w} \bar{N}}{1 - \phi_w \beta (1 + \bar{\pi})^{\epsilon_w - 1}}. \tag{699}$$

The ratio is just:

$$\frac{H_1}{H_2} = \psi \bar{w}^{\epsilon_w} \bar{Y}^{\sigma} \bar{N}^{\eta} \frac{1 - \phi_w \beta \bar{\Pi}^{\epsilon_w - 1}}{1 - \phi_w \beta \bar{\Pi}^{\epsilon_w (1+\eta)}}.$$

Now, consider the case of flexible wages, so $\phi_w = 0$. This would mean $\bar{w}^{\#} = \bar{w}$. Combining these, we would have:

$$\mathcal{M}_w^{-1} \bar{Y}^{-\sigma} \bar{w} = \psi \bar{N}^{\eta}.$$

This is fairly intuitive: If $\epsilon_w \to \infty$, this would be the same static FOC that we've had before – the marginal disutility of labour must equal the marginal utility of consumption ($\bar{Y}^{-\sigma}$ here, since $\bar{C} = \bar{Y}$) times the real wage. If you define the MRS between labour and consumption as $\psi \bar{N}^{\eta} \bar{Y}^{\sigma}$, then you could re-write this as:

$$\bar{w} = \mathcal{M}_w MRS.$$

In other words, households set the wage as a markup over the MRS, in an analogous way to how firms set price as a markup over marginal cost.

Now, go back to our earlier expression from the FOC for labour (641). Eliminating the reset wage by using (697), and using the steady state ratio H_1/H_2, we have:

$$\left[\frac{\left(1-\phi_w(1+\bar{\pi})^{\epsilon_w-1}\right)}{1-\phi_w}\right]^{\frac{1+\epsilon_w\eta}{1-\epsilon_w}} \bar{w}^{1+\epsilon_w\eta} = \mathcal{M}_w\psi\bar{w}^{\epsilon_w\eta}\bar{Y}^{\sigma}\bar{N}^{\eta}\frac{1-\phi_w\beta\bar{\Pi}^{\epsilon_w-1}}{1-\phi_w\beta\bar{\Pi}^{\epsilon_w(1+\eta)}},$$

and simplifying:

$$\bar{N}^{\eta} = \mathcal{M}_w^{-1}\psi^{-1}\bar{Y}^{-\sigma}\bar{w}\left[\frac{\left(1-\phi_w(1+\bar{\pi})^{\epsilon_w-1}\right)}{1-\phi_w}\right]^{\frac{1+\epsilon_w\eta}{1-\epsilon_w}}\frac{1-\phi_w\beta\bar{\Pi}^{\epsilon_w(1+\eta)}}{1-\phi_w\beta\bar{\Pi}^{\epsilon_w-1}}.$$

Now, what does this tell us? Well, we know that $\bar{w} = \bar{mc}$, and we know that $\bar{N} = \bar{Y}\bar{v}^p$. Plugging these in, we get:

$$\bar{N}^{\eta} = \mathcal{M}_w^{-1}\psi^{-1}\bar{N}^{-\sigma}(\bar{v}^p)^{\sigma}\bar{mc}\left[\frac{\left(1-\phi_w(1+\bar{\pi})^{\epsilon_w-1}\right)}{1-\phi_w}\right]^{\frac{1+\epsilon_w\eta}{1-\epsilon_w}}\frac{1-\phi_w\beta\bar{\Pi}^{\epsilon_w(1+\eta)}}{1-\phi_w\beta\bar{\Pi}^{\epsilon_w-1}},$$

and now we can solve for $\bar{N}$:

$$\bar{N} = \left\{\mathcal{M}_w^{-1}\psi^{-1}(\bar{v}^p)^{\sigma}\bar{mc}\left[\frac{\left(1-\phi_w(1+\bar{\pi})^{\epsilon_w-1}\right)}{1-\phi_w}\right]^{\frac{1+\epsilon_w\eta}{1-\epsilon_w}}\frac{1-\phi_w\beta\bar{\Pi}^{\epsilon_w(1+\eta)}}{1-\phi_w\beta\bar{\Pi}^{\epsilon_w-1}}\right\}^{\frac{1}{\sigma+\eta}}. \tag{700}$$

Once we know $\bar{N}$, we know $\bar{Y}$ too.

15.7 Numerical analysis of the model

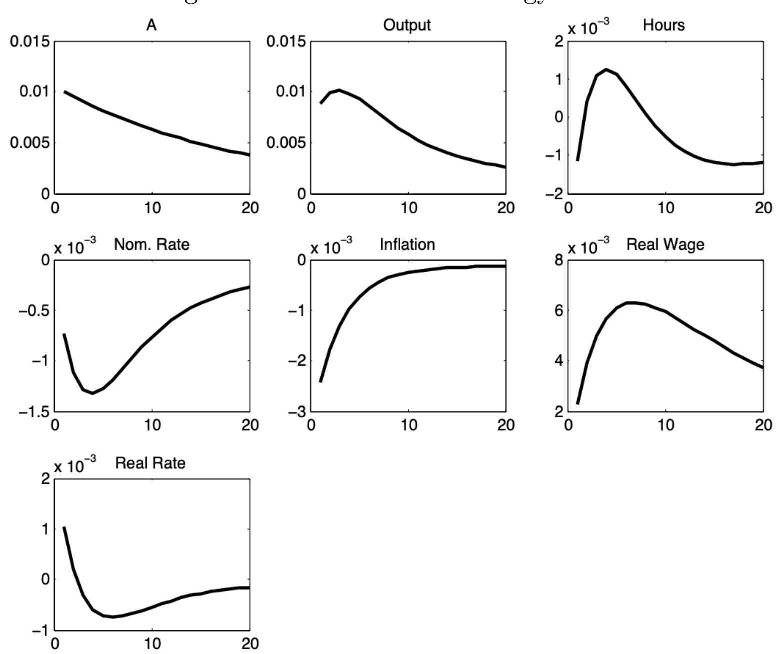
We solve the model numerically using a first order approximation in Dynare, using the following parameter values.

Table 6: Parameter Values

ϵ_p	10	Price elasticity for intermediate good
ϵ_w	10	Price elasticity for labour
ϕ_p	0.75	Calvo parameter for intermediate firm
ϕ_w	0.75	Calvo parameter for household
β	0.99	Household discount factor
σ	1	Risk aversion coefficient
ψ	1	Labour disutility parameter
η	1	Labour supply elasticity
ρ_a	0.95	Technology persistence
ρ_i	0.8	Interest rate persistence
ϕ_π	1.5	Inflation weight
$\text{Var}(\epsilon_{a,t})$	0.01	Variance of technology shock
$\text{Var}(\epsilon_{i,t})$	0.0025	Variance of interest rate shock

Below are the IRFs to a productivity shock. Output rises, by an amount fairly close to its "flexible price" level, hours decline, the nominal interest rate declines, the real interest rate rises, and the real wage rises.

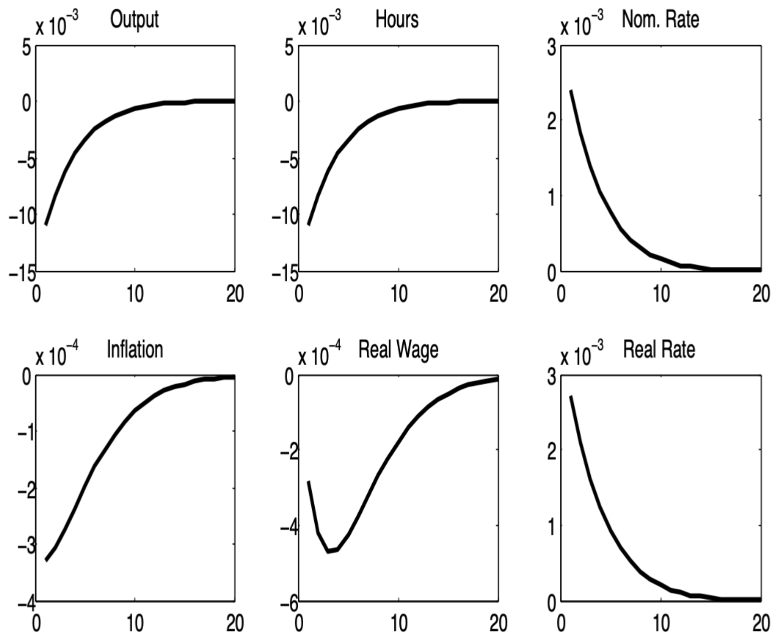
Figure 122: IRFs to a Technology Shock



Source: Sims (2017)

The responses to a monetary policy shock are shown below. Both the real and nominal interest rates rise. Output, hours, inflation, and the real wage all decline.

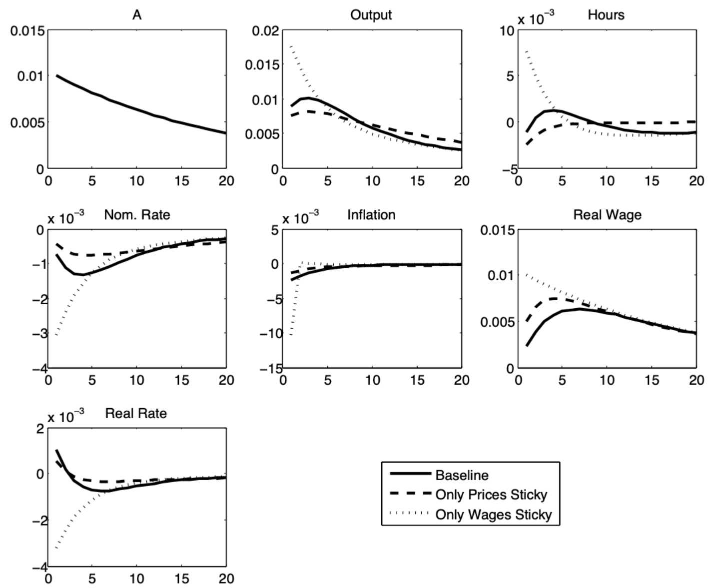
Figure 123: IRFs to an Interest Rate Shock



Source: Sims (2017)

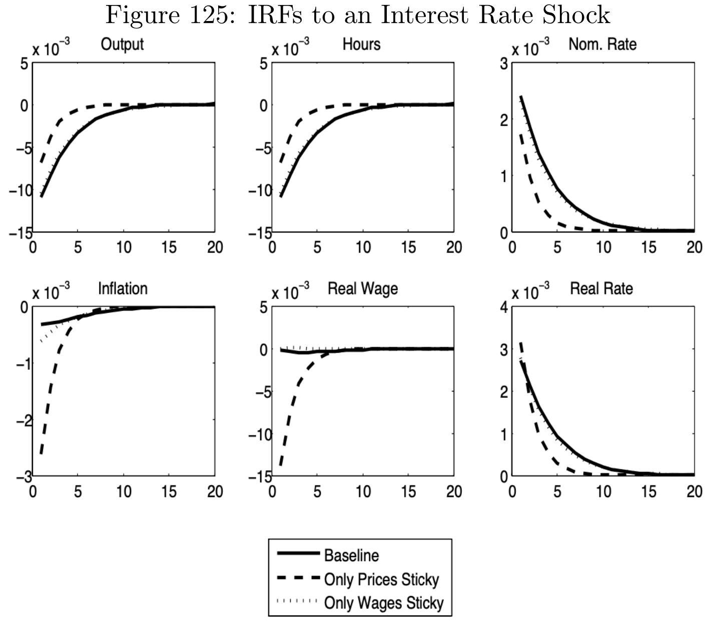
What is the relative importance of price and wage stickiness in accounting for this pattern of impulse responses? We observe that wage rigidity [in isolation] actually amplifies the responses of real variables to a productivity shock, whereas price rigidity [in isolation] dampens those responses. The responses with both price and wage rigidity are somewhere in between. In contrast, the reactions to the monetary policy shock are pretty similar with either wage or price rigidity.

Figure 124: IRFs to a Technology Shock



Source: Sims (2017)

Figure 125: IRFs to an Interest Rate Shock

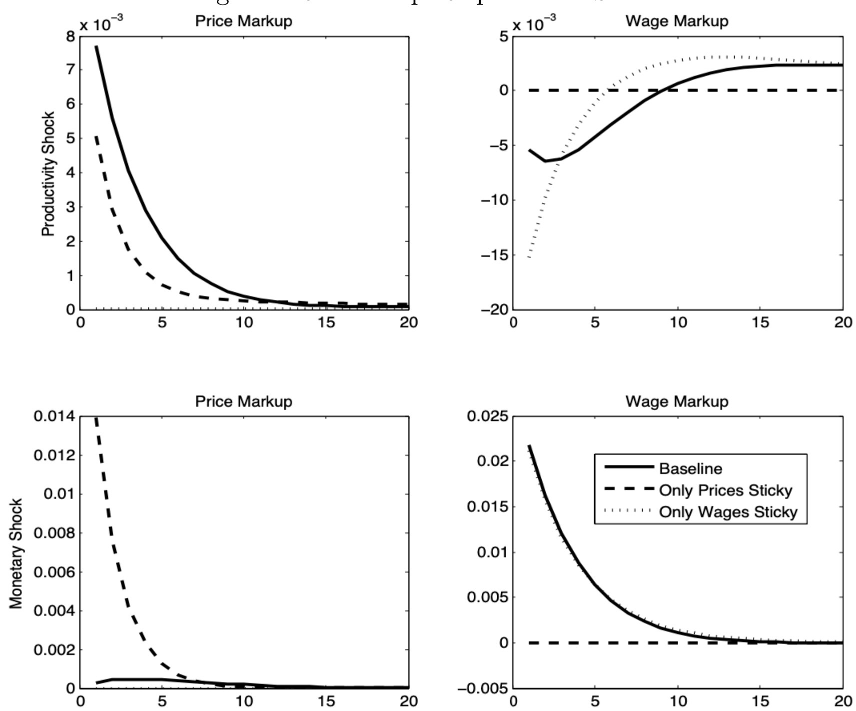


Source: Sims (2017)

We can understand the pattern of IRFs by noting that there are two monopoly distortions in the model. One relates to price setting (price would be a fixed markup over marginal cost if prices were flexible) and one relates to wage setting (the real wage would be a fixed markup over the MRS). We can think of price and wage rigidity causing these markups to vary endogenously in the short-run in response to shocks. The price markup is just the negative of real marginal cost, and the wage markup is the ratio of the real wage to the MRS. Output will respond by less than it would if prices were flexible if either of these markups rise in response to a shock; if either markup falls, this is relatively expansionary.

In the figure below, we plot the price and wage price markups to both a productivity and monetary policy shock under three regimes: one where both prices and wages are sticky, one where only prices are sticky, and one where only wages are sticky. In response to either shock, when only wages are sticky, the price markup is fixed. In contrast, when only prices are sticky, the wage markup is fixed.

Figure 126: Markup Responses to Shocks



Source: Sims (2017)

We can see that both the price and wage markups go up in response to the monetary policy shock, but they go in opposite directions after a productivity shock. A productivity shock puts upward pressure on real wages and downward pressure on prices. Downward pressure on prices means that some firms will end up with prices that are higher than they would like, hence, if prices are sticky, the price markup rises, which effectively means the economy is more distorted. This is why output rises by less than it would if prices were flexible in a stick price model in response to a productivity shock.

The opposite pattern occurs with wages. Real wages need to rise after a positive productivity shock; because some households can't adjust their nominal wages, they end up with wage markups that are too low. Hence, the aggregate wage markup falls, which means that the economy is relatively undistorted along that dimension, which is relatively expansionary. This is why, when only wages are sticky, output rises by more than it would under flexible prices and wages, because the wage markup gets "squeezed". The differential behaviour of the price and wage markups in response to the productivity shock accounts for why the output responses to the shock look so different when one of the stickiness parameters is "turned off".

In response to the monetary policy shock, both the wage and price markups move in the same direction. The contractionary monetary policy shock puts downward pressure on prices, so some firms end up with prices that are too high relative to what they would optimally like – the price markup rises. If prices and wages were both flexible, there would be no effect on real wages of a monetary policy shock. The downward pressure on prices therefore means that there is downward pressure on wages. Since some households can't adjust their wages downward, they end up with wages that are too high, and the economy wide wage markup rises. The increases in both the price and wage markups are contractionary in the case of a monetary policy shock. Since the markups behave in the same way, we observe that there is a much smaller difference in the responses to a monetary policy shock when either prices or wages are sticky, relative to the case of a productivity shock.

15.8 Log-linearisation

Now, let's log-linearise the equilibrium conditions. We are going to do this about a zero inflation steady state, which will make life much easier.

Start with the Euler equation, going ahead and imposing the accounting identity that $C_t = Y_t$. We have:

$$-\sigma \ln Y_t = \ln \beta - \sigma \mathbb{E}_t \ln Y_{t+1} + i_t - \mathbb{E}_t \pi_{t+1}$$
$$-\sigma \hat{Y}_t = -\sigma \mathbb{E}_t \hat{Y}_{t+1} + \hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1},$$

where we use our standard log-linearisation: $\hat{Y}_t = \frac{Y_t - \bar{Y}}{\bar{Y}}$, $\hat{i}_t = i_t - \bar{i}$, and $\hat{\pi}_t = \pi_t - \bar{\pi}$. In other words, the variables already in rate firm (interest rates and inflation) are expressed as absolute deviations, and variables not already in rate form as percent (log) deviations. We can rewrite this as:

$$\hat{Y}_t = \mathbb{E}_t \hat{Y}_{t+1} - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1}), \tag{701}$$

which is the DISE – aka "New Keynesian IS Curve".

Real marginal cost is already log-linear:

$$\widehat{mc}_t = \hat{w}_t - \hat{A}_t. \tag{702}$$

The production function is:

$$\hat{Y}_t = \hat{A}_t + \hat{N}_t + \hat{v}_t^p.$$

But what is $\hat{v}_t^p$? Let's take logs and go from there:

$$\ln v_t^p = \ln \left((1-\phi_p)(1+\pi_t^{\#})^{-\epsilon_p}(1+\pi_t)^{\epsilon_p} + (1+\pi_t)^{\epsilon_p} \phi_p v_{t-1}^p \right).$$

Based on our previous discussion, we know that $\bar{v}^p = 1$ when $\bar{\pi} = 0$. So totally differentiating gives:

$$\hat{v}_t^p = \frac{1}{\bar{v}^p} \left\{ \begin{array}{c} -\epsilon_p(1-\phi_p)(1+\bar{\pi}^{\#})^{-\epsilon_p - 1}(1+\bar{\pi})^{\epsilon_p}(\pi_t^{\#} - \bar{\pi}) \\ +\epsilon_p(1-\phi_p)(1+\bar{\pi}^{\#})^{-\epsilon_p}(1+\bar{\pi})^{\epsilon_p - 1}(\pi_t - \bar{\pi}) + \epsilon_p(1+\bar{\pi})^{\epsilon_p - 1}\phi_p \bar{v}^p(\pi_t - \bar{\pi}) \\ +(1+\bar{\pi})^{\epsilon_p}\phi_p(v_{t-1}^p - \bar{v}^p) \end{array} \right\},$$

and we can clean this up:

$$\begin{aligned}\hat{v}_t^p &= -\epsilon_p(1-\phi_p)\hat{\pi}_t^{\#} + \epsilon_p(1-\phi_p)\hat{\pi}_t + \epsilon_p\phi_p\hat{\pi}_t + \phi_p\hat{v}_{t-1}^p \\ &= -\epsilon_p(1-\phi_p)\hat{\pi}_t^{\#} + \epsilon_p\hat{\pi}_t + \phi\hat{v}_{t-1}^p.\end{aligned}$$

Now, log-linearise the equation for the evolution of inflation:

$$\begin{aligned}(1-\epsilon_p)\pi_t &= \ln\left((1-\phi_p)(1+\pi_t^{\#})^{1-\epsilon_p} + \phi_p\right) \\ (1-\epsilon_p)(\pi_t - \bar{\pi}) &= (1+\bar{\pi})^{\epsilon_p - 1}\left((1-\epsilon_p)(1-\phi_p)(1+\pi^{\#})^{-\epsilon_p}(\pi_t^{\#} - \bar{\pi}^{\#})\right).\end{aligned}$$

This trick always catches me off guard; the reason why $(1+\bar{\pi})^{\epsilon_p-1}$ shows up on the RHS of the above expression is because the term inside the brackets is equal to $(1+\bar{\pi})^{1-\epsilon_p}$ evaluated at the steady state, and when taking the derivative of the log this term gets inverted evaluated at that point. Using facts about the zero inflation steady state, we have:

$$(1-\epsilon_p)\hat{\pi}_t = (1-\epsilon_p)(1-\phi_p)\hat{\pi}_t^{\#},$$

or:

$$\hat{\pi}_t = (1-\phi_p)\hat{\pi}_t^{\#}. \tag{703}$$

In other words, actual inflation is just proportional to reset price inflation, where the constant is equal to the fraction of firms that are updating their prices. This is pretty intuitive. Now, use this in the expression for price dispersion:

$$\hat{v}_t^p = \epsilon_p\left(\hat{\pi}_t - (1-\phi_p)\hat{\pi}_t^{\#}\right) + \phi_p\hat{v}_{t-1}^p.$$

But from the above, the first term drops out, so we are left with:

$$\hat{v}_t^p = \phi_p\hat{v}_{t-1}^p. \tag{704}$$

If we are approximating about the zero inflation steady state in which $\bar{v}^p = 1$, then we're starting from a position in which $\hat{v}^p_{t-1} = 0$, so this means that $\hat{v}^p_t = 0$ at all times. In other words, about a zero inflation steady state, price dispersion is a second order phenomenon, and we can just ignore it in a first order approximation about a zero inflation steady state.

Given this, the log-linearised production function is just:

$$\hat{Y}_t = \hat{A}_t + \hat{N}_t. \tag{705}$$

Now, let's log-linearise the reset price expression. This is multiplicative, so is already in log-linear form. We have:

$$\hat{\pi}^{\#}_t = \hat{\pi}_t + \hat{x}_{1,t} - \hat{x}_{2,t}. \tag{706}$$

Now, we need to log-linearise the auxiliary variables. Imposing the identity that $Y_t = C_t$ we have:

$$\ln x_{1,t} = \ln \left(Y_t^{1-\sigma} mc_t + \phi_p \beta \mathbb{E}_t (1+\pi_{t+1})^{\epsilon_p} x_{1,t+1} \right).$$

Totally differentiating:

$$\frac{x_{1,t} - \bar{x}_1}{\bar{x}_1} = \frac{1}{\bar{x}_1} \left\{ \begin{array}{c} (1-\sigma)\bar{Y}^{-\sigma} \bar{mc}_t (Y_t - \bar{Y}) + \bar{Y}^{1-\sigma}(mc_t - \bar{mc}) \\ +\epsilon_p \phi_p \beta (1+\bar{\pi})^{\epsilon_p - 1} \bar{x}_1 (\mathbb{E}_t \pi_{t+1} - \bar{\pi}) + \phi_p \beta (1+\bar{\pi})^{\epsilon_p} (\mathbb{E}_t x_{1,t+1} - \bar{x}_1) \end{array} \right\},$$

and distributing the $\frac{1}{\bar{x}}$ and multiplying and dividing where necessary to get into percent deviation terms, and making use of the continued assumption of linearisation about a zero inflation steady state, we have:

$$\hat{x}_{1,t} = \frac{(1-\sigma)\bar{Y}^{1-\sigma}\bar{mc}}{\bar{x}_1}\hat{Y}_t + \frac{\bar{Y}^{1-\sigma}\bar{mc}}{\bar{x}_1}\widehat{mc}_t + \epsilon_p \phi_p \beta \mathbb{E}_t \hat{\pi}_{t+1} + \phi_p \beta \mathbb{E}_t \hat{x}_{1,t+1}.$$

Now, with zero steady state inflation, we know that $\bar{x}_1 = \frac{\bar{Y}^{1-\sigma}\bar{mc}}{1-\phi_p\beta}$. This simplifies the first two terms:

$$\hat{x}_{1,t} = (1-\sigma)(1-\phi_p\beta)\hat{Y}_t + (1-\phi_p\beta)\widehat{mc}_t + \epsilon_p \phi_p \beta \mathbb{E}_t \hat{\pi}_{t+1} + \phi_p \beta \mathbb{E}_t \hat{x}_{1,t+1}. \tag{707}$$

Now we do $x_{2,t}$:

$$\ln x_{2,t} = \ln\left(Y_t^{1-\sigma} + \phi_p \beta \mathbb{E}_t (1+\pi_{t+1})^{\epsilon_p - 1} x_{2,t+1}\right).$$

Totally differentiate:

$$\frac{x_{2,t} - \bar{x}_2}{\bar{x}_2} = \frac{1}{\bar{x}_2}\left\{\begin{array}{c}(1-\sigma)\bar{Y}^{-\sigma}(Y_t - \bar{Y}) \\ +(\epsilon_p - 1)\phi_p\beta(1+\bar{\pi})^{\epsilon_p - 2}\bar{x}_2(\mathbb{E}_t \pi_{t+1} - \bar{\pi}) + \phi_p\beta(1+\bar{\pi})^{\epsilon_p-1}(\mathbb{E}_t x_{2,t+1} - \bar{x}_2)\end{array}\right\}.$$

Distribute the $\frac{1}{\bar{x}_2}$, multiply and divide by appropriate terms, and make use of the fact that $\bar{\pi} = 0$, and we have:

$$\hat{x}_{2,t} = \frac{(1-\sigma)\bar{Y}^{1-\sigma}}{\bar{x}_2}\hat{Y}_t + (\epsilon_p - 1)\phi_p\beta\mathbb{E}_t\hat{\pi}_{t+1} + \phi_p\beta\mathbb{E}_t\hat{x}_{2,t+1}.$$

Since $\bar{x}_2 = \frac{\bar{Y}^{1-\sigma}}{1-\phi_p\beta}$, this can be written as:

$$\hat{x}_{2,t} = (1-\sigma)(1-\phi_p\beta)\hat{Y}_t + (\epsilon_p - 1)\phi_p\beta\mathbb{E}_t\hat{\pi}_{t+1} + \phi_p\beta\mathbb{E}_t\hat{x}_{2,t+1}. \tag{708}$$

Now, subtracting $\hat{x}_{2,t}$ from $\hat{x}_{1,t}$, we have:

$$\hat{x}_{1,t} - \hat{x}_{2,t} = (1-\phi_p\beta)\widehat{mc}_t + \phi_p\beta\mathbb{E}_t\hat{\pi}_{t+1} + \phi_p\beta\mathbb{E}_t(\hat{x}_{1,t+1} - \hat{x}_{2,t+1}).$$

From above, we also know that:

$$\hat{x}_{1,t} - \hat{x}_{2,t} = \hat{\pi}_t^{\#} - \hat{\pi}_t,$$

but:

$$\hat{\pi}_t^{\#} = \frac{1}{1-\phi_p}\hat{\pi}_t,$$

so we must also have:

$$\hat{x}_{1,t} - \hat{x}_{2,t} = \frac{\phi_p}{1-\phi_p}\hat{\pi}_t.$$

Make this substitution above to get:

$$\frac{\phi_p}{1-\phi_p}\hat{\pi}_t = (1-\phi_p\beta)\widehat{mc}_t + \phi_p\beta\mathbb{E}_t\hat{\pi}_{t+1} + \phi_p\beta\mathbb{E}_t\left(\frac{\phi_p}{1-\phi_p}\hat{\pi}_{t+1}\right),$$

and then multiply through:

$$\hat{\pi}_t = \frac{(1-\phi_p)(1-\phi_p\beta)}{\phi_p}\widehat{mc}_t + (1-\phi_p)\beta\mathbb{E}_t\hat{\pi}_{t+1} + \phi_p\beta\mathbb{E}_t\hat{\pi}_{t+1},$$

or:

$$\hat{\pi}_t = \frac{(1-\phi_p)(1-\phi_p\beta)}{\phi_p}\widehat{mc}_t + \beta\mathbb{E}_t\hat{\pi}_{t+1}. \tag{709}$$

This is the standard NKPC. It's basic structure is unaltered by the presence of wage rigidity.

Now, let's log-linearise the wage-setting equations. Begin by taking logs of the aggregate real wage series:

$$(1-\epsilon_w)\ln w_t = \ln\left((1-\phi_w)(w_t^{\#})^{1-\epsilon_w} + \phi_w(1+\pi_t)^{\epsilon_w-1}w_{t-1}^{1-\epsilon_w}\right),$$

and then totally differentiate:

$$(1-\epsilon_w)\frac{w_t-\bar{w}}{\bar{w}} = \frac{1}{\bar{w}^{1-\epsilon_w}}\left\{\begin{array}{c}(1-\epsilon_w)(1-\phi_w)(\bar{w}^{\#})^{-\epsilon_w}(w_t^{\#}-\bar{w}^{\#}) \\ +(\epsilon_w-1)\phi_w(1+\bar{\pi})^{\epsilon_w-2}\bar{w}^{1-\epsilon_w}(\pi_t-\bar{\pi}) + (1-\epsilon_w)\phi_w(1+\bar{\pi})^{\epsilon_w-1}\bar{w}^{-\epsilon_w}(w_{t-1}-\bar{w})\end{array}\right\}.$$

Since we are linearising about a zero inflation steady state we know that $\bar{w}^{\#} = \bar{w}$. Making use of this, we have:

$$(1-\epsilon_w)\hat{w}_t = (1-\epsilon_w)(1-\phi_w)\hat{w}_t^{\#} - (1-\epsilon_w)\phi_w\hat{\pi}_t + (1-\epsilon_w)\phi_w\hat{w}_{t-1}.$$

Simplifying we get:

$$\hat{w}_t = (1-\phi_w)\hat{w}_t^{\#} + \phi_w\hat{w}_{t-1} - \phi_w\hat{\pi}_t. \tag{710}$$

This is pretty intuitive. It says that the current real wage is a convex combination of the reset real wage and last period's real wage, minus an adjustment for inflation. The reason for the adjustment for inflation is because nominal wages are fixed.

Now, let's log-linearise the reset wage equation. Since it is multiplicative, it is already log-linear:

$$(1+\epsilon_w\eta)\hat{w}_t^{\#} = \hat{H}_{1,t} - \hat{H}_{2,t}. \tag{711}$$

Now, we need to log-linearise the auxiliary wage-setting variables. Start with $H_{1,t}$:

$$\ln H_{1,t} = \ln\left(\psi w_t^{\epsilon_w(1+\eta)} N_t^{1+\eta} + \beta\phi_w \mathbb{E}_t(1+\pi_{t+1})^{\epsilon_w(1+\eta)} H_{1,t+1}\right),$$

and then totally differentiate:

$$\hat{H}_{1,t} = \frac{1}{\bar{H}_1}\left\{\begin{array}{c} \epsilon_w(1+\eta)\psi\bar{w}^{\epsilon_w(1+\eta)-1}\bar{N}^{1+\eta}(w_t-\bar{w}) + (1+\eta)\psi\bar{w}^{\epsilon_w(1+\eta)}\bar{N}^{\eta}(N_t-\bar{N}) \\ +\epsilon_w(1+\eta)\beta\phi_w(1+\bar{\pi})^{\epsilon_w(1+\eta)-1}\bar{H}_1(\mathbb{E}_t\pi_{t+1}-\bar{\pi}) + \beta\phi_w(1+\bar{\pi})^{\epsilon_w(1+\eta)}(\mathbb{E}_t H_{1,t+1}-\bar{H}) \end{array}\right\}.$$

We know that $\bar{H}_1 = \frac{\psi\bar{w}^{\epsilon_w(1+\eta)}\bar{N}^{1+\eta}}{1-\phi_w\beta}$, and we distribute the $\bar{H}_1$ and multiply and divide by the appropriate quantities we get:

$$\hat{H}_{1,t} = (1-\phi_w\beta)\epsilon_w(1+\eta)\hat{w}_t + (1-\phi_w\beta)(1+\eta)\hat{N}_t + \epsilon_w(1+\eta)\phi_w\beta\mathbb{E}_t\hat{\pi}_{t+1} + \phi_w\beta\mathbb{E}_t\hat{H}_{1,t+1}. \quad (712)$$

Now do $H_{2,t}$:

$$\ln H_{2,t} = \ln\left(Y_t^{-\sigma} w_t^{\epsilon_w} N_t + \beta\phi_w\mathbb{E}_t(1+\pi_{t+1})^{\epsilon_w-1}H_{2,t+1}\right).$$

Totally differentiate:

$$\hat{H}_{2,t} = \frac{1}{\bar{H}_2}\left\{\begin{array}{c} -\sigma\bar{Y}^{-\sigma-1}\bar{w}^{\epsilon_w}\bar{N}(Y_t-\bar{Y}) + \epsilon_w\bar{Y}^{-\sigma}\bar{w}^{\epsilon_w-1}\bar{N}(w_t-\bar{w}) + \bar{Y}^{-\sigma}\bar{w}^{\epsilon_w}(N_t-\bar{N}) \\ (\epsilon_w-1)\beta\phi_w(1+\bar{\pi})^{\epsilon_w-2}\bar{H}_2(\mathbb{E}_t\pi_{t+1}-\bar{\pi}) + \beta\phi_w(1+\bar{\pi})^{\epsilon_w-1}(\mathbb{E}_t H_{2,t+1}-\bar{H}_2) \end{array}\right\}.$$

We know that $\bar{H}_2 = \frac{\bar{Y}^{-\sigma}\bar{w}^{\epsilon_w}\bar{N}}{1-\phi_w\beta}$, so we can simplify to get an expression about the zero inflation steady state:

$$\hat{H}_{2,t} = -(1-\phi_w\beta)\sigma\hat{Y}_t + (1-\phi_w\beta)\epsilon_w\hat{w}_t + (1-\phi_w\beta)\hat{N}_t + (\epsilon_w-1)\phi_w\beta\mathbb{E}_t\hat{\pi}_{t+1} + \phi_w\beta\mathbb{E}_t\hat{H}_{2,t+1}. \quad (713)$$

Now subtract $\hat{H}_{2,t}$ from $\hat{H}_{1,t}$:

$$\hat{H}_{1,t}-\hat{H}_{2,t} = (1-\phi_w\beta)\epsilon_w\eta\hat{w}_t + (1-\phi_w\beta)\eta\hat{N}_t + (1-\phi_w\beta)\sigma\hat{Y}_t + \phi_w\beta(1+\epsilon_w\eta)\mathbb{E}_t\hat{\pi}_{t+1} + \phi_w\beta(\mathbb{E}_t\hat{H}_{1,t+1} - \mathbb{E}_t\hat{H}_{2,t+1}).$$

The MRS between labour and consumption, as introduced above, is:

$$MRS_t = \psi N_t^{\eta} Y_t^{\sigma},$$

and in log-linear terms it is:

$$\widehat{MRS}_t = \eta \hat{N}_t + \sigma \hat{Y}_t.$$

This means we can write $\hat{H}_{1,t} - \hat{H}_{2,t}$ as:

$$\hat{H}_{1,t} - \hat{H}_{2,t} = (1-\phi_w\beta)\epsilon_w\eta\hat{w}_t + (1-\phi_w\beta)\widehat{MRS}_t + \phi_w\beta(1+\epsilon_w\eta)\mathbb{E}_t\hat{\pi}_{t+1} + \phi_w\beta(\mathbb{E}_t\hat{H}_{1,t+1} - \mathbb{E}_t\hat{H}_{2,t+1}).$$

Now, let's define $\hat{\mu}_t = \widehat{MRS}_t - \hat{w}_t$ as the gap between the MRS and the real wage. Playing around we get:

$$\hat{H}_{1,t} - \hat{H}_{2,t} = (1-\phi_w\beta)\epsilon_w\eta\hat{w}_t + (1-\phi_w\beta)\hat{\mu}_t + (1-\phi_w\beta)\hat{w}_t + \phi_w\beta(1+\epsilon_w\eta)\mathbb{E}_t\hat{\pi}_{t+1} + \phi_w\beta(\mathbb{E}_t\hat{H}_{1,t+1} - \mathbb{E}_t\hat{H}_{2,t+1}),$$

or:

$$\hat{H}_{1,t} - \hat{H}_{2,t} = (1-\phi_w\beta)(1+\epsilon_w\eta)\hat{w}_t + (1-\phi_w\beta)\hat{\mu}_t + \phi_w\beta(1+\epsilon_w\eta)\mathbb{E}_t\hat{\pi}_{t+1} + \phi_w\beta(\mathbb{E}_t\hat{H}_{1,t+1} - \mathbb{E}_t\hat{H}_{2,t+1}). \tag{714}$$

Now, from above we know that we can write the reset wage as:

$$\hat{w}_t^{\#} = \frac{1}{1-\phi_w}\hat{w}_t - \frac{\phi_w}{1-\phi_w}\hat{w}_{t-1} + \frac{\phi_w}{1-\phi_w}\hat{\pi}_t,$$

and we also know that:

$$\hat{H}_{1,t} - \hat{H}_{2,t} = (1+\epsilon_w\eta)\hat{w}_t^{\#}.$$

Combining these expressions, we have:

$$\hat{H}_{1,t} - \hat{H}_{2,t} = \frac{1+\epsilon_w\eta}{1-\phi_w}\hat{w}_t - \frac{(1-\epsilon_w\eta)\phi_w}{1-\phi_w}\hat{w}_{t-1} + \frac{(1+\epsilon_w\eta)\phi_w}{1-\phi_w}\hat{\pi}_t.$$

It is helpful to rewrite this in terms of the nominal wage $\hat{W}_t = \hat{w}_t + \hat{P}_t$. Doing so, we have:

$$\hat{H}_{1,t} - \hat{H}_{2,t} = \frac{1+\epsilon_w \eta}{1-\phi_w}(\hat{W}_t - \hat{P}_t) - \frac{(1+\epsilon_w \eta)\phi_w}{1-\phi_w}(\hat{W}_{t-1} - \hat{P}_{t-1}) + \frac{(1+\epsilon_w \eta)\phi_w}{1-\phi_w}\hat{\pi}_t.$$

Now, define $\hat{\pi}_t^w = \hat{W}_t - \hat{W}_{t-1}$ as nominal wage inflation. We can further simplify:

$$\begin{aligned}
\hat{H}_{1,t} - \hat{H}_{2,t} &= \frac{1+\epsilon_w \eta}{1-\phi_w}\left(\hat{W}_t - \hat{P}_t - \phi_w \hat{W}_{t-1} + \phi_w \hat{P}_{t-1} + \phi_w \hat{\pi}_t\right) \\
&= \frac{1+\epsilon_w \eta}{1-\phi_w}\left(\hat{W}_t - \hat{W}_{t-1} + (1-\phi_w)\hat{W}_{t-1} - \phi_w(\hat{P}_t - \hat{P}_{t-1}) - (1-\phi_w)\hat{P}_t + \phi_w \hat{\pi}_t\right) \\
&= \frac{1+\epsilon_w \eta}{1-\phi_w}\hat{\pi}_t^w + (1+\epsilon_w \eta)\hat{W}_{t-1} - (1+\epsilon_w \eta)\hat{P}_t.
\end{aligned}$$

In terms of the real wage again, this can be rewritten as:

$$\hat{H}_{1,t} - \hat{H}_{2,t} = \frac{1+\epsilon_w \eta}{1-\phi_w}\hat{\pi}_t^w + (1+\epsilon_w \eta)\hat{w}_{t-1} - (1+\epsilon_w \eta)\hat{\pi}_t.$$

Now, combine this with our earlier expression for the difference between the auxiliary variables (714). We have:

$$\begin{aligned}
\frac{1}{1-\phi_w}\hat{\pi}_t^w + \hat{w}_{t-1} - \hat{\pi}_t &= (1-\phi_w \beta)\hat{w}_t + \frac{1-\phi_w \beta}{1+\epsilon_w \eta}\hat{\mu}_t + \phi_w \beta \mathbb{E}_t \hat{\pi}_{t+1} + \frac{\phi_w \beta}{1+\epsilon_w \eta}(\mathbb{E}_t \hat{H}_{1,t+1} - \mathbb{E}_t \hat{H}_{2,t+1}) \\
&= (1-\phi_w \beta)\hat{w}_t + \frac{1-\phi_w \beta}{1+\epsilon_w \eta}\hat{\mu}_t + \phi_w \beta \mathbb{E}_t \hat{\pi}_{t+1} \\
&\quad + \frac{\phi_w \beta}{1+\epsilon_w \eta}\left[\frac{1+\epsilon_w \eta}{1-\phi_w}\mathbb{E}_t \hat{\pi}_{t+1}^w + (1+\epsilon_w \eta)\hat{w}_t - (1+\epsilon_w \eta)\mathbb{E}_t \hat{\pi}_{t+1}\right].
\end{aligned}$$

Simplifying:

$$\frac{1}{1-\phi_w}\hat{\pi}_t^w + \hat{w}_{t-1} - \hat{\pi}_t = (1-\phi_w \beta)\hat{w}_t + \frac{1-\phi_w \beta}{1+\epsilon_w \eta}\hat{\mu}_t + \phi_w \beta \mathbb{E}_t \hat{\pi}_{t+1} + \frac{\phi_w \beta}{1-\phi_w}\mathbb{E}_t \hat{\pi}_{t+1}^w + \phi_w \beta \hat{w}_t - \phi_w \beta \mathbb{E}_t \hat{\pi}_{t+1},$$

simplifying further:

$$\begin{aligned}\frac{1}{1-\phi_w}\hat{\pi}_t^w + \hat{w}_{t-1} - \hat{\pi}_t &= \hat{w}_t + \frac{1-\phi_w\beta}{1+\epsilon_w\eta}\hat{\mu}_t + \frac{\phi_w\beta}{1-\phi_w}\mathbb{E}_t\hat{\pi}_{t+1}^w \\ \frac{1}{1-\phi_w}\hat{\pi}^w &= \hat{w}_t - \hat{w}_{t-1} + \hat{\pi}_t + \frac{1-\phi_w\beta}{1+\epsilon_w\eta}\hat{\mu}_t + \frac{\phi_w\beta}{1-\phi_w}\mathbb{E}_t\hat{\pi}_{t+1}^w,\end{aligned}$$

and then note that $\hat{w}_t - \hat{w}_{t-1} + \hat{\pi}_t = \hat{\pi}_t^w$, so we have:

$$\left(\frac{1}{1-\phi_w} - 1\right)\hat{\pi}_t^w = \frac{1-\phi_w\beta}{1+\epsilon_w\eta}\hat{\mu}_t + \frac{\phi_w\beta}{1-\phi_w}\mathbb{E}_t\hat{\pi}_{t+1}^w,$$

or:

$$\frac{\phi_w}{1-\phi_w}\hat{\pi}_t^w = \frac{1-\phi_w\beta}{1+\epsilon_w\eta}\hat{\mu}_t + \frac{\phi_w\beta}{1-\phi_w}\mathbb{E}_t\hat{\pi}_{t+1}^w.$$

Finally, we get:

$$\hat{\pi}_t^w = \frac{(1-\phi_w)(1-\phi_w\beta)}{\phi_w(1+\epsilon_w\eta)}\hat{\mu}_t + \beta\mathbb{E}_t\hat{\pi}_t^w. \tag{715}$$

This is the wage Phillips Curve (WPC). It looks almost the same as the price Phillips Curve/NKPC, but there is an extra term, $(1+\epsilon_w\eta)$, in the denominator. Since $\epsilon_w\eta > 0$, this means that the WPC is always "flatter" than the NKPC for equal values of the Calvo parameters, ϕ_p and ϕ_w. Also, differently than for prices, the elasticity parameter ϵ_w shows up in the WPC expression.

The full set of log-linearised FOCs can be written as:

$$\hat{Y}_t = \mathbb{E}_t \hat{Y}_{t+1} - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1}), \tag{716}$$

$$\widehat{mc}_t = \hat{w}_t - \hat{A}_t, \tag{717}$$

$$\hat{Y}_t = \hat{A}_t + \hat{N}_t, \tag{718}$$

$$\hat{\pi}_t = \frac{(1-\phi_p)(1-\phi_p \beta)}{\phi_p} \widehat{mc}_t + \beta \mathbb{E}_t \hat{\pi}_{t+1}, \tag{719}$$

$$\hat{\pi}_t^w = \frac{(1-\phi_w)(1-\phi_w \beta)}{\phi_w (1+\epsilon_w \eta)} \hat{\mu}_t + \beta \mathbb{E}_t \hat{\pi}_{t+1}^w, \tag{720}$$

$$\hat{\mu}_t = \widehat{MRS}_t - \hat{w}_t, \tag{721}$$

$$\widehat{MRS}_t = \eta \hat{N}_t + \sigma \hat{Y}_t, \tag{722}$$

$$\hat{\pi}_t^w = \hat{w}_t - \hat{w}_{t-1} + \hat{\pi}_t, \tag{723}$$

$$\hat{i}_t = \rho_i \hat{i}_{t-1} + (1-\rho_i)\phi_\pi \hat{\pi}_t + \epsilon_{i,t}, \tag{724}$$

$$\hat{A}_t = \rho_a \hat{A}_{t-1} + \epsilon_{a,t}. \tag{725}$$

This is 10 equations in 10 unknowns: $\{\hat{Y}_t, \hat{N}_t, \widehat{mc}_t, \hat{i}_t, \widehat{MRS}_t, \hat{\mu}_t, \hat{w}_t, \hat{A}_t, \hat{\pi}_t, \hat{\pi}_t^w\}$.

15.9 Gap notation

As in the simpler New Keynesian model, there are some redundant variables here that could be eliminated, and we might like to write the NKPC for prices and inflation in terms of "gaps".

As we did earlier, let's define the variables with a superscript f as the flexible price/wage variables: the values of endogenous variables which would be obtained in the absence of both price and wage stickiness. If both prices and wages were flexible, we would have $\hat{\mu}_t = \widehat{mc}_t = 0$. This would imply:

$$\hat{w}_t^f = \hat{A}_t,$$
$$\hat{w}_t^f = \eta \hat{N}_t^f + \sigma \hat{Y}_t^f,$$
$$\hat{Y}_t^f = \hat{A}_t + \hat{N}_t^f.$$

Plugging the first and third of the above expressions into the second, we get:

$$\hat{A}_t = \eta \left(\hat{Y}_t^f - \hat{A}_t \right) + \sigma \hat{Y}_t^f.$$

Simplifying:

$$\hat{Y}_t^f = \frac{1+\eta}{\sigma+\eta} \hat{A}_t. \tag{726}$$

Unsurprisingly, this is the same log-linearised expression for the natural rate of output as we had before.

Let's play around with the definition of μ_t a bit:

$$\begin{aligned} \hat{\mu}_t &= \eta \hat{N}_t + \sigma \hat{Y}_t - \hat{w}_t, \\ &= \eta \left(\hat{Y}_t^f - \hat{A}_t \right) + \sigma \hat{Y}_t - \hat{w}_t, \\ &= (\sigma + \eta) \hat{Y}_t - \eta \hat{A}_t - \hat{w}_t. \end{aligned}$$

Now, add and subtract $\hat{A}_t$ from the RHS:

$$\hat{\mu}_t = (\sigma + \eta) \hat{Y}_t - (1+\eta) \hat{A}_t + \hat{A}_t - \hat{w}_t.$$

Simplifying:

$$\hat{\mu}_t = (\sigma + \eta) \left(\hat{Y}_t - \frac{1+\eta}{\sigma+\eta} \hat{A}_t \right) - (\hat{w}_t - \hat{A}_t).$$

Now, define the real wage gap as $\hat{X}_t^w = \hat{w}_t - \hat{A}_t$, since we know that the flexible price real wage would just be $\hat{w}_t^f = \hat{A}_t$. The output gap, $\hat{X}_t = \hat{Y}_t - \hat{Y}_t^f$, is the same as before. This means we can write this expression as:

$$\hat{\mu}_t = (\sigma + \eta) \hat{X}_t - \hat{X}_t^w. \tag{727}$$

We can then plug this into the WPC to get:

$$\hat{\pi}_t^w = \frac{(1-\phi_w)(1-\phi_w \beta)}{\phi_w (1-\epsilon_w \eta)} (\sigma + \eta) \hat{X}_t - \frac{(1-\phi_w)(1-\phi_w \beta)}{\phi_w (1+\epsilon_w \eta)} \hat{X}_t^w + \beta \mathbb{E}_t \hat{\pi}_{t+1},$$

or (and to borrow notation from Galí):

$$\hat{\pi}_t^w = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \varkappa \hat{X}_t - \lambda_w \hat{X}_t^w, \tag{728}$$

where $\varkappa = \lambda_w(\sigma + \eta)$ and $\lambda_w = \frac{(1-\phi_w)(1-\phi_w\beta)}{\phi_w(1+\epsilon_w\eta)}$.

The NKPC can be written in terms of the real wage gap, since real marginal cost is the same thing as the real wage gap:

$$\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \frac{(1-\phi_p)(1-\phi_p\beta)}{\phi_p} \hat{X}_t^w,$$

or:

$$\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \lambda_p \hat{X}_t^w, \tag{729}$$

where $\lambda_p = \frac{(1-\phi_p)(1-\phi_p\beta)}{\phi_p}$.

The DISE can be written in terms of the output gap:

$$\hat{X}_t = \mathbb{E}_t \hat{X}_{t+1} - \frac{1}{\sigma}(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1} - \hat{r}_t^f), \tag{730}$$

where:

$$\hat{r}_t^f = \sigma \frac{1+\eta}{\sigma+\eta}(\rho_a - 1)\hat{A}_t. \tag{731}$$

We can rewrite the wage inflation evolution equation in terms of the real wage gap as well:

$$\begin{aligned} \hat{\pi}_t &= \hat{w}_t - \hat{A}_t + \hat{A}_t - \hat{w}_{t-1} + \hat{A}_{t-1} - \hat{A}_{t-1} + \hat{\pi}_t \\ \Leftrightarrow \hat{\pi}_t &= \hat{X}_t^w - \hat{X}_{t-1}^w + \hat{A}_t - \hat{A}_{t-1} + \hat{\pi}_t. \end{aligned} \tag{732}$$

The full system of equilibrium conditions are:

$$\hat{X}_t = \mathbb{E}_t \hat{X}_{t+1} - \frac{1}{\sigma}\left(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1} - \hat{r}_t^f\right), \tag{733}$$

$$\hat{\pi}_t = \beta \mathbb{E}_t \hat{\pi}_{t+1} + \lambda_p \hat{X}_t^w, \tag{734}$$

$$\hat{\pi}_t^w = \beta \mathbb{E}_t \hat{\pi}_{t+1}^w + \varkappa \hat{X}_t + \lambda_w \hat{X}_t^w, \tag{735}$$

$$\hat{r}_t^f = \sigma \frac{1+\eta}{\sigma+\eta}(\rho_a - 1)\hat{A}_t, \tag{736}$$

$$\hat{\pi}_t^w = \hat{X}_t^w - \hat{X}_{t-1}^w + \hat{A}_t + \hat{A}_{t-1} + \hat{\pi}_t, \tag{737}$$

$$\hat{i}_t = \rho_i \hat{i}_{t-1} + (1-\rho_i)\phi_\pi \hat{\pi}_t + \epsilon_{i,t}, \tag{738}$$

$$\hat{A}_t = \rho_a \hat{A}_{t-1} + \epsilon_{a,t}. \tag{739}$$

There is no way to write the NKPC solely in terms of the output gap when wages are sticky. In the model with just price stickiness, we were able to write the marginal cost in terms of the output gap by eliminating the real wage using the static FOC for labour supply, so we could write marginal cost just in terms of $\hat{Y}_t$ and $\hat{A}_t$. Here, that isn't straightforward since the FOC for labour supply is substantially more complicated.

15.10 Optimal monetary policy

As in the model with just price stickiness, it is possible to derive a welfare loss function from taking a second order approximation to the household's value function while using a first order approximation to the equilibrium conditions. The loss function now depends on the squared values of the output gap, price inflation, and wage inflation:

$$\mathbb{L}_t = \frac{1}{2}\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left\{ \frac{\kappa}{\epsilon_p} \hat{X}_{t+s}^2 + \hat{\pi}_{t+s}^2 + \frac{\kappa}{\lambda_w}\frac{\epsilon_w}{\epsilon_p}(\hat{\pi}_{t+s}^w)^2 \right\}, \tag{740}$$

where $\kappa = (\sigma+\eta)\lambda_p$. Hence, the relative weight on the output gap is the same as in the simpler model. λ_w is just the coefficient on the real wage gap in the WPC. The relative weight on wage inflation depends on: i) The relative coefficients κ and λ_w, and ii) the relative elasticities of goods and labour demand, ϵ_p and ϵ_w.

Why is wage inflation an argument in the loss function? It shows up for an analogous reason to why price inflation shows up.

To think about aggregate welfare, we need to come up with a social welfare function since there isn't a representative agent in this model. The easiest aggregate welfare function is the utilitarian once in which we sum up welfare of individual households. Individual welfare is:

$$V_t(l) = \frac{C_t^{1-\sigma} - 1}{1-\sigma} - \psi \frac{N_t(l)^{1+\eta}}{1+\eta} + \beta \mathbb{E}_t V_{t+1}(l).$$

Define aggregate welfare as:

$$\begin{aligned} \mathcal{W}_t &= \int_0^1 V_t(l) dl \\ &= \frac{C_t^{1-\sigma} - 1}{1-\sigma} - \frac{\psi}{1+\eta} \int_0^1 N_t(l)^{1+\eta} dl + \beta \mathbb{E}_t \mathcal{W}_{t+1}. \end{aligned}$$

Now, note that the demand for labour of variety l:

$$N_t(l) = \left(\frac{W_t(l)}{W_t} \right)^{-\epsilon_w} N_t,$$

and plug this into the expression for welfare above:

$$\mathcal{W}_t = \frac{C_t^{1-\sigma} - 1}{1-\sigma} - \frac{\psi}{1+\eta} \int_0^1 \left(\frac{W_t(l)}{W_t} \right)^{-\epsilon_w (1+\eta)} N_t^{1+\eta} dl + \beta \mathbb{E}_t \mathcal{W}_{t+1}.$$

Then, define:

$$v_t^w = \int_0^1 \left(\frac{W_t(l)}{W_t} \right)^{-\epsilon_w (1+\eta)},$$

which is a measure of wage dispersion, and it is bound from below by 1. This can be written recursively if we want as we did previously for prices, using the assumptions of the Calvo mechanism. Aggregate welfare can be written as:

$$\mathcal{W}_t = \frac{C_t^{1-\sigma} - 1}{1-\sigma} - \frac{\psi}{1+\eta} v_t^w N_t^{1+\eta} + \beta \mathbb{E}_t \mathcal{W}_{t+1}. \tag{741}$$

Hence, the reason why wage inflation matters is that wage dispersion effectively drives a wedge between labour supplied and labour used in production. If $v_t^w > 1$, there is some labour "lost" in the process, in a way analogous to how price dispersion result in some "lost" output.

Going back to the approximated loss function, the intuition for the relative weight on wage inflation is fairly intuitive. Price or wage inflation are costly to the extent to which prices or wages are sticky: If aggregate prices or wages move around, and prices are sticky, this induces price or wage dispersion. The bigger ϵ_p is, the more costly price dispersion is (the lower is the weight on wage inflation); the bigger ϵ_w is, the more costly wage dispersion is (the bigger is the weight on wage inflation). The stickier are prices, the smaller κ is, and hence the smaller is the relative weight on wage inflation (the bigger is the relative weight on price inflation). Conversely, the stickier are wages, the smaller λ_w is, and the bigger the relative weight on inflation.

It is instructive to think about what the relative weight on wage inflation ought to look like by considering some numeric values. Suppose that $\phi_p = \phi_w = 0.75$, and $\epsilon_p = \epsilon_w = 10$, with $\sigma = \eta = 1$, and $\beta = 0.99$. We get $\kappa = 0.1717$, but $\lambda_w = 0.0078$. This means that the relative weight on wage inflation is 22 – i.e., wage inflation is 22 times more important than price inflation. What really drives this is that the WPC is much "flatter" than the NKPC because of the presence of ϵ_w in the denominator of the slope coefficient.

Before doing anything numerical, it is useful to stop and think for a minute. In the basic New Keynesian model, it was possible to completely stabilise both inflation and the output gap, and therefore achieve a welfare loss.[98] This was because stabilising prices led to a stable output gap, and vice-versa. Here, it is in general not possible to simultaneously stabilise price inflation, the output gap, and wage inflation. This is easy to see. For the output gap to be zero, the real wage must equal its natural rate (which is in turn equal to $\hat{A}_t$). But for the real wage to be equal its natural rate, either

[98] See our discussion on "The Divine Coincidence".

wages or prices must adjust to the extent to which $\hat{A}_t$ moves around. Hence, you can't simultaneously get the real wage to fluctuate (which it must if there are real shocks) without either prices or wages moving around. In other words, the presence of wage stickiness makes a central bank face a non-trivial tradeoff without having to resort to including a cost-push shock in the model.

Below we present welfare losses from a quantitative version of the model (using the parameters described above), along with $\rho_a = 0.95$ and $\text{Var}(\epsilon_{a,t}) = 0.01$, for different types of monetary policy:

Table 7: Welfare Losses

Policy	$\mathbb{L}$
Taylor Rule	-0.0020
Price Inflation Targeting	-1.0021
Wage Inflation Targeting	-0.0010
Gap Targeting	-0.0010

Source: Sims (2017). Note: The welfare loss function is only calculated for a single period.

For the Taylor Rule specification, we use: $\hat{i}_t = \rho_i \hat{i}_{t-1} + (1 - \rho_i)(\phi_\pi \hat{\pi}_t + \phi_y \hat{X}_t)$. This is pretty interesting in that we see that price inflation targeting does very poorly. The reason why is fairly transparent. If you target zero price inflation, then the real wage gap must be equal to zero from NKPC. But a zero real wage gap means that real wages must move around one-for-one with $\hat{A}_t$. Given that prices can't move, this means we have to have a lot of wage inflation, and the relative weight on wage inflation is very high. Hence, price inflation targeting does poorly. Wage inflation targeting does very well, which makes sense given the high weight on wage inflation. Interestingly, output gap targeting does well too. Stabilising the gap results in little wage inflation (and comparatively much price less price inflation), but given the relative weights this ends up doing well from a welfare perspective.

As in the simpler model with just sticky prices, we can derive formal FOCs to characterise the optimal policy, but this beyond what is required for the course. Those interested can consult the Galí textbook.

15.11 Comments and key readings

Early examples of non-optimising rational expectations models with nominal wage rigidities can be found in the work of Fischer (1977) and Taylor (1980). Cho and Cooley (1995) and Benassy (1995) were among the first papers that embedded the assumption of sticky nominal wages in a DSGE model, and examined its implications for the properties of a number of variables in the presence of both real and monetary shocks.

Erceg et al. (2000) developed the New Keynesian model with both staggered price and wage contracts a la Calvo that has become the framework of reference in the literature, and on which much of this chapter builds. The focus of their paper was, like this chapter, on the derivation of the implications for monetary policy. A similar focus, including a discussion of the special case in which targeting a weighted average of wage and price inflation is optimal, can be found in Woodford (2003, chap. 6) and Giannoni and Woodford (2004). Other work has focused instead on the impact of staggered wage setting on the persistence of the effects of monetary policy shocks. See, for example, Huang and Liu (2002) and, especially, Woodford (2003, chap. 3) for a detailed discussion of the role of wage stickiness in that regard.

Staggered wage setting is also a common feature of medium-scale models like those of Kim (2000)[99], Smets and Wouters (2003, 2007), and Christiano et al. (2005). Those models also allow for some degree of wage indexation to prices. An analysis of the optimal implementable rules in a medium-scale model can be found in Schmitt-Grohé and Uribe (2010).

This ends our look into the canonical New Keynesian model and analysis of optimal policy. While it may seem obvious now, these DSGE models came under heavy criticism and fire during and after the GFC; much to the dismay and surprise of many macroeconomists. The models had very little to say about the role of financial markets and financial crises in recessions. But, the macroeconomists were not down and out. A new branch of macroeconomic literature flourished, macro-finance, which we will look at in the next chapter.

[99] I had the pleasure of meeting Jinill Kim a few times in Tokyo. One of the nicest and pleasant people you could ever come across.

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Part IV

Macro-Finance

16 Interest Rates and Asset Prices

16.1 Introduction

The course so far has concentrated in how macroeconomic variables, such as consumption, investment, output, and inflation evolve over time. Our efforts have focussed on characterising these dynamics in these variables and trying to understand the underlying forces which produce these fluctuations. Aside from references to interest rates, we have made little mention of financial variables. Earlier chapters introduced money into our standard model but the focus on this chapter is on financial variables: stock prices, bond prices, interest rates, the term structure of interest rates, and so on. In particular, we will extend our analysis to see what implications a neoclassical model has for these real financial variables. The use of neoclassical models to analyse financial variables has generally been seen as less controversial than the same program aimed at economic variables. Many economists believe that goods and credit markets contain fundamental imperfections that make neoclassical analysis irrelevant. However, at least until recently, it is widely perceived that financial markets come closest to approximating the "ideal" market structure of neoclassical macroeconomics. However, as we shall see, the performance of the neoclassical model in explaining financial variables is little better than in explaining economic variables.

The crucial link in economic models of asset pricing is between consumption and rates of return. Individuals can invest in many assets, almost all of which have uncertain returns. Extending our analysis to account for multiple assets and risk raises some new issues concerning both household behaviour and asset markets. The plan for this chapter is that we will re-look at the consumer's optimisation problem, assess the term structure of multiyear securities, introduce Mehra and Prescott's famous Equity Premium Puzzle, describe Lucas' asset pricing model (which Mehra and Prescott's analysis was based upon), and suggest explanations to the Equity Premium Puzzle.

16.2 Conditions for individual optimisation

Consider an individual reducing consumption in period t by an infinitesimal amount and using the resulting saving to buy an asset, i, that produces a potentially uncertain stream of payoffs, $D^i_{t+1}, D^i_{t+2}, \ldots$. If the individual is optimising, the marginal utility he or she forgoes from the reduced consumption in period t must be equal to the expected sum of the discounted marginal utilities from of the future consumption provided by the asset's payoffs. If we let P^i_t denote the price of the asset, this condition is:

$$u'(C_t)P^i_t = \mathbb{E}_t\left[\sum_{k=1}^{\infty}\frac{1}{(1+\rho)^k}u'(C_{t+k})D^i_{t+k}\right], \ \forall i, \tag{742}$$

where ρ is the discount rate. To see the implications of this equation, suppose the individual holds the asset for only one period, and define the return on the asset, r^i_{t+1}, by:

$$r^i_{t+1} = \frac{D^i_{t+1}}{P^i_t} - 1.$$

Note that here the payoff to the asset, D^i_{t+1}, includes not only any dividend payouts in period $t+1$, but also any proceeds from selling the asset. Then (742) becomes:

$$u'(C_t) = \frac{1}{1+\rho}\mathbb{E}_t\left[(1+r^i_{t+1})u'(C_{t+1})\right], \ \forall i, \tag{743}$$

which is nothing but the familiar consumption Euler equation that we've been working with this whole time. Since the expectation of the product of two variables equals the product of their expectations plus their covariance, we can rewrite this expression as:

$$u'(C_t) = \frac{1}{1+\rho}\left\{\mathbb{E}_t\left[1+r^i_{t+1}\right]\mathbb{E}_t\left[u'(C_{t+1})\right] + \mathrm{Cov}_t\left(1+r^i_{t+1}, u'(C_{t+1})\right)\right\} \ \forall i, \tag{744}$$

where $\mathrm{Cov}_t(\cdot)$ is covariance conditional on information available at time t.

If we assume that utility is quadratic:

$$u(C) = C - \frac{aC^2}{2},$$

then the marginal utility of consumption is $1 - aC$. Using this to substitute for the covariance term in (744), we obtain:

$$u'(C_t) = \frac{1}{1+\rho} \left\{ \mathbb{E}_t \left[1 + r^i_{t+1} \right] \mathbb{E}_t \left[u'(C_{t+1}) \right] - a \operatorname{Cov}_t \left(1 + r^i_{t+1}, C_{t+1} \right) \right\}. \tag{745}$$

This equation implies that in deciding whether to hold more of an asset, the individual is not concerned with how risky the asset is: the variance of the asset's return does not appear anywhere in (745), Intuitively, a marginal increase in holdings of an asset that is risky, but whose risk is not correlated with the overall risk the individual faces, does not increase the variance of the individual's consumption. Thus, in evaluating that marginal decision, the individual considers only the asset's expected return.

Equation (745) implies that the aspect of riskiness that matters to the decision of whether to hold more of an asset is the relation between the asset's payoff and consumption. Suppose, for example, that the individual is given an opportunity to buy a new asset whose expected return equals the rate of return on a risk-free asset that the individual is already able to buy. If the payoff to the new asset is typically high when the marginal utility of consumption is high (that is, when consumption is low), buying one unit of the asset raises expected utility by more than buying one of the risk-free asset. Thus (since the individual was previously indifferent about buying more of the risk-free asset), the individual can raise her expected utility buy buying the new asset. As the individual invests more in the asset, her consumption comes to depends more on the asset's payoff, and so the covariance between consumption and the asset's return becomes less negative. In the example we are considering, since the asset's expected return equals the risk-free rate, the individual invests in the asset until the covariance of its return with consumption reaches zero.

This discussion implies that hedging risks is crucial to optimal portfolio choices. A steelworker whose future labour income depends on the health of the US steel industry should avoid – or better yet, sell short – assets whose returns are positively correlated with the fortunes of the steel industry, such as shares in US steel companies. Instead, the worker should invest in assets whose returns move inversely with the health of the US steel industry, such as foreign steel companies or US aluminium companies.[100]

16.3 The consumption CAPM

This discussion takes assets' expected returns as given. But individuals' demands for assets determine these expected returns. If, for example, an asset's payoff is highly correlated with consumption, its price must be driven down to the point where its expected return is high for individuals to hold it.

To see the implications of this observation for asset prices, suppose that all individuals are the same, and return to the general FOC (742). Solving this expression for P_t^i yields:

$$P_t^i = \mathbb{E}_t \left[\sum_{k=1}^{\infty} \frac{1}{(1+\rho)^k} \frac{u'(C_{t+k})}{u'(C_t)} D_{t+k}^i \right]. \tag{746}$$

This should be very familiar. The term $[1/(1+\rho)^k]u'(C_{t+k})/u'(C_t)$ shows how the consumer values future payoffs, and therefore how much she is willing to pay for various assets. It is none other than the pricing kernel or stochastic discount factor, which we previously defined as:

$$M_{t,t+1} = \beta \frac{\mathbb{E}_t \lambda_{t+1}}{\lambda_t}, \tag{747}$$

where β is the discount factor, and λ_t is the marginal utility of consumption. We can find the implications of our analysis for expected returns by solving (745) for $\mathbb{E}_t[1 + r_{t+1}^i]$:

$$\mathbb{E}_t[1 + r_{t+1}^i] = \frac{1}{\mathbb{E}_t[u'(C_{t+1})} \left[(1+\rho)u'(C_t) + a \operatorname{Cov}_t(1 + r_{t+1}^i, C_{t+1}) \right]. \tag{748}$$

This equation states that the higher the covariance of an asset's payoff with consumption, the higher its expected return must be.

We can simplify (748) by considering the return on a risk-free asset. If the payoff to an asset

[100] One implication of this analysis is that individuals should exhibit no particular tendency to hold shares of companies that operate in the individuals' own countries. In fact, because the analysis implies that individuals should avoid assets whose returns are correlated with other sources of risk to their consumption, it implies that their holdings should be skewed against domestic companies. For example, for plausible parameter values it predicts that the typical person in the US should sell US stocks short (Baxter and Jermann 1997). In fact, however, individuals' portfolios are very heavily skewed toward domestic companies (French and Poterba 1991). This pattern is known as home bias.

is certain, then the covariance of its payoff with consumption is zero. Thus, the risk-free rate, $\bar{r}_{t+1}$ satisfies:

$$1+\bar{r}_{t+1} = \frac{(1+\rho)u'(C_t)}{\mathbb{E}_t[u'(C_{t+1})]}. \tag{749}$$

Subtracting (749) from (748) gives:

$$\mathbb{E}_t[r^i_{t+1}] - \bar{r}_{t+1} = \frac{a\,\mathrm{Cov}_t(1+r^i_{t+1}, C_{t+1})}{\mathbb{E}_t[u'(C_{t+1})]}. \tag{750}$$

This states that the expected return premium that an asset must offer relative to the risk-free rate is proportional to the covariance of its return with consumption.

This model of the determination of expected asset returns is known as the consumption capital-asset pricing model, or consumption CAPM[101]. The coefficient from a regression of an asset's return on consumption growth is known as its consumption beta. Thus the central prediction of the CAPM is that the premiums that assets offer are proportional to their consumption betas (Breeden 1979; Merton 1973; Rubinstein 1976).[102]

16.4 The term structure

Equation (743) can also be generalised to assets which involve more than one period of investment (i.e. multiyear bonds). Consider a consumer who is deciding whether to invest in a k period bond which will earn return $R_{k,t} = 1 + r_{k,t}$ over the next k periods. The same logic as used for the consumption Euler equation tells us that the consumer will equate the lost utility from lower consumption this period with higher consumption gained in k periods' time so that the Euler equation is:

$$\mathbb{E}_t\left[\frac{1}{(1+\rho)^k}\frac{u'(C_{t+k})}{u'(C_t)}(1+r_{k,t})\right] = 1. \tag{751}$$

[101] Since we're all doing macroeconomics, I will refer to this as simply CAPM.

[102] The original CAPM in the finance literature assumes that investors are concerned with the mean and variance of the return on their portfolio rather than the mean and variance of consumption. That version of the model therefore focuses on market betas—that is, coefficients from regressions of assets' returns on the returns on the market portfolio—and predicts that expected-return premiums are proportional to market betas (Lintner 1965; Sharpe 1965).

At any moment in time, the term structure is defined as $\{R_{1,t}, R_{2,t}, ..., R_{n,t}\}$, in other words a sequence of interest rates on bonds of different maturities. The term structure is extremely important to policy makers and financial markets as it reveals what the market expects the future interest rate to be. For example, the difference between $R_{2,t}$ (the return on a two year bond) and $R_{1,t}$ (the return on a one year bond) must contain information on what interest rates will be in year 2. This idea forms the basis of the expectations theory of the term structure. This theory basically says $R_{2,t} = R_{1,t}\mathbb{E}_t R_{1,t+1}$ – in other words, investors must yearn the same return from investing in a two year bond as they expect to earn from investing in a one year bond now and then reinvesting the proceeds in a one year bond next year. This theory has been thoroughly tested and its strict implications found not to hold. While the term structure does tell us something about future rates the correlation is not perfect. To see this consider equation (751) where $j = 2$ and we have slightly rewritten the equation so that:

$$\frac{1}{1+r_{2,t}} = \mathbb{E}_t \frac{1}{1+\rho}\frac{u'(C_{t+1})}{u'(C_t)}\frac{1}{1+\rho}\frac{u'(C_{t+2})}{u'(C_{t+1})}. \tag{752}$$

An important statistical result is the Law of Iterated Expectations or LIE.[103] This says that $\mathbb{E}_t\mathbb{E}_{t+1}X_{t+1+j} = \mathbb{E}_t X_{t+1+j}$, While LIE looks foreboding it is actually a very simple result. It says that if we are to forecast today what we think our forecast will be of a variable in the future, then our best forecast of tomorrow's forecasts is simply our current forecast. We can use this result to rewrite (752) as:

$$\begin{aligned}
\frac{1}{1+r_{2,t}} &= \mathbb{E}_t \frac{1}{1+\rho}\frac{u'(C_{t+1})}{u'(C_t)}\frac{1}{1+\rho}\frac{u'(C_{t+2})}{u'(C_{t+1})} \\
\frac{1}{1+r_{2,t}} &= \mathbb{E}_t \frac{1}{1+\rho}\frac{u'(C_{t+1})}{u'(C_t)}\mathbb{E}_{t+1}\frac{1}{1+\rho}\frac{u'(C_{t+2})}{u'(C_{t+1})} \\
\frac{1}{1+r_{2,t}} &= \mathbb{E}_t \frac{1}{1+\rho}\frac{u'(C_{t+1})}{u'(C_t)}\frac{1}{1+r_{1,t+1}},
\end{aligned} \tag{753}$$

[103] Quick refresher of LIE:

$$\mathbb{E}_B\left[\mathbb{E}_{A|B}[A|B]\right] = \mathbb{E}[A],$$

and the Law of Total Variance (for good measure):

$$\mathrm{Var}(\mathbf{u}) = \mathbb{E}\left[\mathrm{Var}(\mathbf{u}|\mathbf{X})\right] + \mathrm{Var}(\mathbb{E}[\mathbf{u}|\mathbf{X}]).$$

where we have again used (751) for $k = 1$ in the last line. This equation is a generalised version of the expectations theory of the term structure. If the covariance term is zero (which would happen if $u(\cdot)$ were linear), then this equation is exactly the standard expectations model of the term structure – the return on a two year bond equals the return on a one year bond times the expected return on a one year bond next period. However, more generally, there is a covariance term reflecting the fact that the consumer dislikes uncertainty. The intuition behind the covariance term is not surprisingly similar to that in the CAPM. If $u'(C_{t+1})/u'(C_t)$ and $1 + r_{1,t+1}$ are negatively correlated then the one period bonds tends to pay a high rate of return when the marginal utility of consumption is high (when consumption is low). If this is the case then the return on a two year bond must be greater than from simply investing in two consecutive one year bonds, to compensate the consumer from losing this insurance effect. This is exactly what (753) says. If the covariance term is negative then:

$$\frac{1}{1 + r_{2,t}} < \frac{1}{1 + r_{1,t}} \mathbb{E}_t \frac{1}{1 + r_{1,t+1}},$$

implying that $1 + r_{2,t}$ is greater.

We need to make some more specific assumptions if we are able to say anything more precise about the slope of the term structure – that is, do interest rates on bonds increase or decrease with maturity? Let's assume that the utility function is given by CRRA preferences, so we have

$$u(C) = \frac{C^{1-\sigma}}{1 - \sigma}.$$

In this case, we can write (751) as:

$$\mathbb{E}_t \left[\frac{1}{(1 + \rho)^k} \left(\frac{C_{t+k}}{C_t} \right)^{-\sigma} (1 + r_{k,t}) \right] = 1. \tag{754}$$

Before we can say anything precise about the term structure, we need to make one more assumption, and that is that consumption growth and interest rates are distributed jointly log normal. This is a standard trick in modern macro and often leads to very tractable analytical expressions. What does it mean? If X is distributed log normal then $\log(\mathbb{E}_t X) = \mathbb{E}_t \log X + \mathrm{Var}(\log X)/2$. It should be stressed that this assumption of joint log-normality does have economic implications. We are essentially assuming something about tastes and technology of the economy and the type of economic fluctuations they produce. If we apply this formula to (754) and rearrange, we have:

$$\ln(1 + r_{k,t}) = \sigma \mathbb{E}_t \ln \left(\frac{C_{t+k}}{C_t} \right) - k \ln \left(\frac{1}{1+\rho} \right) - \frac{1}{2} \operatorname{Var} \left[-\sigma \ln \left(\frac{C_{t+k}}{C_t} \right) \right]. \tag{755}$$

Assuming that on average consumption grows by η per period we can then we can use this equation to calculate an average one year interest rate associated with a k period bond:

$$\frac{\ln(1 + r_{k,t})}{k} = \theta\sigma - \ln \left(\frac{1}{1+\rho} \right) - \frac{1}{2k} \operatorname{Var} \left[-\sigma \ln \left(\frac{C_{t+k}}{C_t} \right) \right] \tag{756}$$

Equation (756)says that the average yield on a k period bond depends on three terms: mean consumption growth, the discount factor, and a variance term. Since $\beta = \frac{1}{1+\rho} < 1$, $\ln \beta < 0$, and so the yield on a k period bond is increasing in the discount factor. The intuition behind this is simple: consumers discount the future and so place less weight on the future marginal utility of consumption. Therefore in order to persuade the consumer to hold a bond the rate of return needs to at least match the rate of time preference. However, this effect is the same for bonds of all maturities and so does not affect the slope of the term structure (the term structure being a plot of essentially $\ln(1 + r_{k,t})/k$ against k.

The term to consider is the expected average consumption growth over the next k periods. If consumption is expected to grow strongly over the next k periods then the ratio of marginal utility in k periods time and now will be less than 1. Therefore the greater this consumption growth, the higher the k period interest rate needs to be to persuade agents to give up even more consumption today in return for higher $t + k$ consumption. We have assumed that consumption growth is expected to be the same 1, 2, or k years out. However, this need not be the case. For instance, at the bottom of a recession consumption growth over the next few years can be expected to be higher than over the next 20 years. Therefore from (756), the average yield on short term bonds should exceed that on 20 year bonds in the depth of a recession.

The final term to consider is the variance of consumption growth. If consumers are not characterised by certainty equivalence then increases in uncertainty affect their behaviour. The CRRA utility function does display certainty equivalence and so uncertainty has an important role to play. Here an increase in uncertainty causes the bond rate to fall, because the greater the uncertainty the more that consumers value the certain payoff provided by the bond. Whether the term structure is upward or downward sloping depends on whether the numerator or denominator of the third term increases the most with maturity k.

Since consumption growth in US data is positively autocorrelated, we would expect the numerator to rise faster than the denominator. In other words, this model delivers a downward sloping term structure. This is contrary to the upward sloping term structure usually observed in data. Intuitively, the term structure slopes downwards in the model for insurance reasons. Suppose that consumption growth is subject to a negative shock sometime between periods t and $t + k$. The worsening outlook for consumption growth will cause interest rates to fall, which implies that the long bond will increase in price. Consumers will therefore have a capital gain with which to offset their reduced consumption. Therefore bonds offer a hedge against consumption risk. From (756) we can see that the greater the uncertainty there is about k period ahead consumption, the lower the return on a k period bond. Once again this is because consumers are willing to earn a lower return on bonds because of the hedging characteristics they offer. Those of you who are interested in how well DSGE models succeed in reproducing the observed behaviour of the term structure would do well to read "The Term Structure of Interest Rates in Real and Monetary Economies" den Haan (1995).

16.5 The Lucas (1978) asset pricing model

We now briefly turn to the model of R. E. Lucas (1978) – an extremely influential paper both then and now. A lot of the asset pricing strategies and formulas we used stem from Lucas (1978) (as we shall soon see). It offers a very abstract model which looks at asset pricing in a general equilibrium context. In the Lucas model the only form of capital are trees which bear fruit. Unfortunately the fruit produced by these trees can only be used for consumption and not investment purposes. Therefore in this economy output must equal consumption (output is simply the crop of fruit). From period to period, the crop varies randomly (presumably because of weather). The idea here is to interpret the tree as an asset which yields a dividend stream for all future periods (the dividends being the crop) and the question is what price to attach to the asset. The questions Lucas tries to answer are actually more ambitious than this explanation might suggest. What Lucas was trying to arrive at were asset pricing formulae. That is, given certain information about the economy (e.g. value of productivity shocks, capital stock, and so on) could one convert these into a formula for determining asset prices? Further, Lucas was interested in asset pricing rules which formed a Rational Expectations equilibrium. That is, if everyone use these asset pricing rules then everyone would choose appropriate capital stocks and consumption such that the prices predicted by these pricing rules actually materialised. However, we shall consider only a very simple example from the Lucas paper.

The return to holding a tree is:

$$1 + r_{t+1} = \frac{P_{t+1} + d_t}{P_t}, \tag{757}$$

where d_t is the period t dividend (crop) and P_t is the price of the tree. Our usual Euler equation holds so that:

$$\mathbb{E}_t \frac{1}{1+\rho} \left(\frac{P_{t+1} + d_t}{P_t} \right) \frac{u'(C_{t+1})}{u'(C_t)} = 1.$$

Because the fruit is perishable it must be the case that each period the crop is consumed ($d_t = c_t$) so that we can rewrite this equation as:

$$P_t = \mathbb{E}_t \sum_{k=1}^{\infty} \frac{1}{(1+\rho)^k} \frac{u'(d_{t+k})}{u'(d_t)} d_{t+k}. \tag{758}$$

This should be looking familiar, and not by coincidence – have a look at (742) and (746). The asset price here is equal to the discounted sum of future dividends, where the consumer uses a discount rate which depends upon the marginal utility of consumption. In this model when dividends are higher they give a lower weight (since $u'(d)$ is low) because consumption is already high and the high output is not valued so highly compared to a low output situation. If we make the strong assumption of log utility, $u(C) = \ln C$, then the Lucas asset pricing equation (758) becomes:

$$\begin{aligned} P_t &= \mathbb{E}_t \sum_{k=1}^{\infty} \frac{1}{(1+\rho)^k} d_t \\ \Leftrightarrow P_t &= \frac{\beta}{1-\beta} d_t, \end{aligned} \tag{759}$$

In words, the asset price simply depends upon the dividend today. This is an extreme case, but it gives an example of an asset pricing function (i.e. feed in today's dividend and out comes equity price), and also illustrates how this function crucially depends upon the utility function. The reason why (759) depends only on current dividends is due to the fact that future dividends are discounted completely. Announcements of future dividends have two effects: Firstly, they increase the price of the share, secondly they increase future discount rates. In this simple logarithmic model, these two faces cancel out leaving the share price to depend only on current dividends. Notice that even though the equity price depends only the current dividend, the model is completely forward looking and characterised by Rational Expectations. Therefore, even though most of the underlying model is the same, we arrive at a very different result from (742) and (746).

16.6 The Equity Premium Puzzle

One of the most important applications of this analysis of assets' expected returns concerns the case where the risky asset is a broad portfolio of stocks. To see the issues involved, it is easiest to return to the consumption Euler equation (743), and to assume that individuals have CRRA utility rather than quadratic utility. With this assumption, the consumption Euler equation becomes:

$$C_t^{-\sigma} = \frac{1}{1+\rho} \mathbb{E}_t \left[(1 + r_{t+1}^i) C_{t+1}^{-\sigma} \right], \tag{760}$$

where σ is the coefficient of relative risk aversion. If we divide both sides by the marginal utility of consumption in t, $C_t^{-\sigma}$, and multiply both sides by $1+\rho$, this expression becomes:

$$1 + \rho = \mathbb{E}_t \left[(1 + r_{t+1}^i) \frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \right]. \tag{761}$$

Finally, it is convenient to let η_{t+1} denote the growth rate of consumption from t to $t+1$, $(C_{t+1}/C_t)-1$, and to omit the time subscripts. Thus, we have:

$$\mathbb{E}\left[(1+r^i)(1+\eta)^{-\sigma}\right] = 1+\rho. \tag{762}$$

To see the implications of (762), we take a second-order Taylor approximation[104] of the LHS about $r^i = \eta = 0$. Computing the relevant derivatives yields:

$$(1+r^i)(1+\eta)^{\sigma} \approx 1 + r^i - \sigma\eta - \sigma\eta r^i + \frac{1}{2}\sigma(\sigma+1)\eta^2. \tag{763}$$

Thus, we can write (762) as:

$$\mathbb{E}[r^i] - \sigma\mathbb{E}[\eta] - \sigma\left\{\mathbb{E}[r^i]\mathbb{E}[\eta] + \mathrm{Cov}(r^i,\eta)\right\} + \frac{1}{2}\sigma(\sigma+1)\left\{\mathbb{E}[\eta]^2 + \mathrm{Var}(\eta)\right\} \approx \rho,$$

and when the time period involved is short, the $\mathbb{E}[r^i]\mathbb{E}[\eta]$ and $\mathbb{E}[\eta]^2$ terms are small relative to the others. Omitting these terms and solving the resulting expression for $\mathbb{E}[r^i]$ yields:

$$\mathbb{E}[r^i] \approx \rho + \sigma\mathbb{E}[\eta] + \sigma\,\mathrm{Cov}(r^i,\eta) + \frac{1}{2}\sigma(\sigma+1)\,\mathrm{Var}(\eta). \tag{764}$$

This equation implies that the difference between the expected returns on two assets, i and j, satisfies:

$$\begin{aligned}\mathbb{E}[r^i] - \mathbb{E}[r^j] &= \sigma\,\mathrm{Cov}(r^i,\eta) - \sigma\,\mathrm{Cov}(r^j,\eta)\\ &= \sigma\,\mathrm{Cov}(r^i - r^j,\eta).\end{aligned} \tag{765}$$

[104] Recall that in the multivariate case, a second order Taylor expansion is given by:

$$\begin{aligned}f(x,y) &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)\\ &\quad + \frac{1}{2!}\left[f_{xx}(a,b)(x-a)^2 + f_{yy}(a,b)(y-b)^2 + 2f_{xy}(a,b)(x-a)(y-b)\right],\end{aligned}$$

or more compactly, in vector notation:

$$f(\mathbf{x}) = f(\mathbf{a}) + [(\mathbf{x}-\mathbf{a})\nabla f(\mathbf{a})] + [(\mathbf{x}-\mathbf{a})(\mathbf{H}(\mathbf{x})(\mathbf{x}-a)].$$

In a famous paper, Mehra and Prescott (1985) show that it is difficult to reconcile the observed returns on stocks and bonds with equation (765). Mankiw and Zeldes (1991) report a simple calculation that shows the essence of the problem. For the US during the period 1890-1979, the difference between the average return on the stock market and return on short-term government debt – the equity premium – is about 6 percent. Over the same period, the standard deviation of the growth of consumption (as measured by real purchases of nondurables and services) is 3.6 percent, and the standard deviation of the excess return on the market is 16.7 percent; the correlation between these two quantities is 0.40. These figures imply that the covariance of consumption growth and the excess return on the market is 0.4(0.036)(0.167), or 0.0024.

Equation (765) therefore implies that the coefficient of relative risk aversion needed to account for the equity premium is the solution to:

$$\begin{aligned} 0.06 &= \theta(0.0024) \\ \implies \sigma &= 25. \end{aligned}$$

This is an extraordinary level of risk aversion; it implies, for example, that individuals would rather accept a 17 percent reduction in consumption with certainty than risk a 50-50 chance of a 20 percent reduction. As Mehra and Prescott describe, other evidence suggests that risk aversion is much lower than this. Among other things, such a high degree of risk aversion to variations in consumption makes it puzzling that the average risk-free rate is close to zero despite the fact that consumption is growing over time.

The large equity premium, particularly when coupled with the low risk-free rate, is thus difficult to reconcile with household optimisation. This Equity Premium Puzzle (EPP) has stimulated a large amount of research, and many explanations for it have been proposed. No clear resolution of the puzzle has been provided, however.[105] We will look at some of the explanations in the remainder of this chapter.

Furthermore, the EPP has become more severe in the period since Mehra and Prescott identified

[105] For example, see Kocherlakota (1996, *Journal of Economic Literature*).

it. From 1979 to 2008, the average equity premium is 7 percent, which slightly high than in Mehra and Prescott's sample period. More importantly, consumption growth has become more stable and less correlated with returns: the standard deviation of consumption over this period 1.1 percent, the standard deviation of the excess market return is 14.2 percent, and the correlation between these two quantities is 0.33. These figures imply a coefficient of relative risk aversion of 0.07/[0.33(0.011)(0.142)], or about 140!

Table 8: Consumption and Rates of Return

	Cons growth		Return T bills		Return stocks		Equity premium	
	Mean	S.D.	Mean	S.D.	Mean	S.D.	Mean	S.D.
1889-1978	1.83	3.57	0.80	5.67	6.98	16.54	6.18	16.67
1889-1898	2.30	4.90	5.80	3.23	7.58	10.02	1.78	11.57
1899-1908	2.55	5.31	2.62	2.59	7.71	17.21	5.08	16.86
1909-1918	0.44	3.07	-1.63	9.02	-0.14	12.81	1.49	9.18
1919-1928	3.00	3.97	4.30	6.61	18.94	16.18	14.64	15.94
1929-1938	-0.25	5.28	2.39	6.50	2.56	27.90	0.18	31.63
1939-1948	2.19	2.52	-5.82	4.05	3.07	14.67	8.89	14.23
1949-1958	1.48	1.00	-0.81	1.89	17.49	13.08	18.30	13.20
1859-1968	2.37	1.00	1.07	0.64	5.58	10.59	4.50	10.17
1969-1978	2.41	1.40	-0.72	2.06	0.03	13.11	0.75	11.64

Source: Mehra and Prescott (1985)

16.6.1 An alternative illustration of the Equity Premium Puzzle

With all the background theory and exposition done, we can illustrate the EPP in an alternative method. The end result is the same, but we can use some shorthand techniques now. Consider the following simple optimisation problem with financial assets:

$$\max \; u(C_1) + \mathbb{E}_1 \beta u(C_2),$$

subject to:

$$\begin{aligned} C_1 + P_1 S_1 + B_1 &\leq R_1 B_0 + (P_1 + d_1) S_0, \\ \mathbb{E}_1 C_2 + \mathbb{E}_1 P_2 S_2 + \mathbb{E}_1 B_2 &\leq \mathbb{E}_1 R_2 B_1 + (\mathbb{E}_1 P_2 + \mathbb{E}_1 d_2) S_1, \end{aligned}$$

where R is the gross interest rate, S denotes the quantity of shares, P is the share price, B are no-coupon one period bonds, and d are dividends. The FOC to this problem is the – surprise surprise – consumption Euler equation:

$$u'(C_1) = \beta \mathbb{E}_1 R_2 u'(C_2),$$

and we can use this to derive the present value discount model:

$$\begin{aligned} P_1 &= \beta \mathbb{E}_1 \frac{u'(C_2)}{u'(C_1)} (P_2 + d_2) \\ \Leftrightarrow P_1 &= \beta \mathbb{E}_1 \frac{u'(C_2)}{u'(C_1)} x_2 \\ \Leftrightarrow P_1 &= \frac{x_2}{R_2}. \end{aligned}$$

In the infinite horizon case, the present value discount model is:

$$\begin{aligned}
P_1 &= \frac{P_2 + d_2}{R_2} \\
&= \frac{\frac{P_3+d_3}{R_3} + d_2}{R_2} \\
&= \frac{\frac{\frac{\frac{P_4+d_4}{R_4}+d_2}{R_2}+d_3}{R_3} + d_2}{R_2} \\
&= \frac{d_2}{1+R_2} + \frac{d_3}{R_2R_3} + \frac{d_4}{R_2R_3R_4} \dots .
\end{aligned}$$

In the finance literature, this is usually written as:

$$P_t = \mathbb{E}[M_{t,t+1}x],$$

where $M_{t,t+1}$ is the stochastic discount, or pricing kernel:

$$M_{t,t+1} = \beta \mathbb{E}_t \frac{u'(C_{t+1})}{u'(C_t)}.$$

Any financial asset can be priced by this stochastic discount factor as long we know the asset's pay-offs.[106]

[106]Consider the case of pricing housing. Assume that agents derive utility from housing stocks, H_t:

$$\max_{C_t, B_t, H_t} \ \mathbb{E}_t \left[\sum_{s=0}^{\infty} \beta^s u(C_{t+s}, H_{t+s}) \right],$$

subject to:

$$C_t + B_t + q_t H_t = R_{t-1}B_{t-1} + q_t H_{t-1} + W_t,$$

where q_t is the relative price of housing and W_t is exogenous transfer income. The Euler equation for housing is:

$$q_t = \frac{u_h'(C_t, H_t)}{u_c'(C_t, H_t)} + \underbrace{\beta \mathbb{E}_t \left[\frac{u_c'(C_{t+1}, H_{t+1})}{u_c'(C_t, H_t)} \right]}_{M_{t,t+1}} q_{t+1}.$$

We will touch on this further when discussing financial frictions and the Iacoviello (2005) model. For further reading see Piazzesi and Schneider (2016).

Now, define the [gross] risk free rate as:

$$R_t^f = \frac{1}{\mathbb{E}[M_{t,t+1}]}.$$

This implies:

$$1 = \mathbb{E}[M_{t,t+1}]R_t^f = \mathbb{E}\left[M_{t,t+1}R_t^f\right],$$

which is an asset with the price being unity and the payoff being $R_t^f = 1 + r_t^f$. With log utility we can write:

$$\begin{aligned}
M_{t,t+1} &= \beta\mathbb{E}_t\frac{u'(C_{t+1})}{u'(C_t)} \\
\Leftrightarrow M_{t,t+1} &= \beta\mathbb{E}_t\frac{C_t}{C_{t+1}} \\
\Leftrightarrow M_{t,t+1} &= \beta\mathbb{E}_t\frac{1}{1+\eta_{t+1}},
\end{aligned}$$

where η_{t+1} is the net growth rate of consumption between t and $t+1$. Hence:

$$R_t^f = 1 + r_t^f = \frac{\mathbb{E}_t[1+\eta_{t+1}]}{\beta}.$$

In words, the expected growth rate divided by the discount factor is the [gross] risk free rate.

Furthermore, we can transform the following:

$$P_t = \mathbb{E}[M_{t,t+1}x],$$

into:

$$\begin{aligned}
P_t &= \mathbb{E}[M_{t,t+1}]\mathbb{E}[x] + \mathrm{Cov}(M_{t,t+1}, x), \\
\Leftrightarrow P_t &= \frac{\mathbb{E}[x]}{R_t^f} + \mathrm{Cov}(M_{t,t+1}, x), \\
\Leftrightarrow P_t &= \frac{\mathbb{E}[x]}{R_t^f} + \frac{\mathrm{Cov}\left(\beta u'(C_{t+1}), x\right)}{u'(C_t)}.
\end{aligned}$$

We see two important features: The asset price is the sum of the discounted expected value and risk adjustment term (the covariance); and, the asset which is highly positively correlated with consumption is considered risky and risk premium of such an asset is higher.

With CRRA preferences,

$$u(C) = \frac{C^{1-\sigma}}{1-\sigma},$$

when we solve the following problem:

$$\max \; u(C_1) + \mathbb{E}_1 \beta u(C_2),$$

subject to:

$$C_1 + S_1 + B_1 \leq R_1^S S_0 + R_1^b B_0,$$
$$\mathbb{E}_1 C_2 + \mathbb{E}_1 S_2 + \mathbb{E}_1 B_2 \leq \mathbb{E}_1 R_2^S S_1 + \mathbb{E}_1 R_2^b B_1,$$

we have the following optimality condition (absent of any arbitrage) for bond and stock holdings:

$$\mathbb{E}_1 \left[\left(\frac{C_2}{C_1} \right)^{-\sigma} \left(R_2^S - R_2^b \right) \right] = 0. \tag{766}$$

But if we look at the data, this condition does not hold. This is the EPP.

16.7 Explaining the Equity Premium Puzzle

A number of proposals have been made to explain the equity premium. Here we discuss a few.

16.7.1 Non-expected utility theory

A model of time varying risk aversion, where the risk aversion depends on the level of consumption may resolve the EPP and the risk free rate puzzles. What happens, say, if we become more risk averse when the level of consumption decreases in a recession? When we assume CRRA, the intertemporal elasticity of substitution (ρ, say) is the inverse of the coefficient of relative risk aversion (which we have denoted with σ in this section):

$$\rho = \frac{1}{\sigma}.$$

More generally, whenever the utility function satisfies expected utility theory, there is an inverse relationship between the intertemporal elasticity and risk aversion. This is an unfortunate restriction as risk aversion and the intertemporal elasticity measure two different things. Risk aversion is about how agents compare consumption in different states of the world whereas intertemporal substitution is about how agents compare consumption at different points in time. In response to this a number of researchers[107] have investigated non-expected utility function which do not impose this inverse relationship between risk aversion and intertemporal substitution. While non-expected utility has gone some way to solving the EPP, its success has been limited. Firstly, while estimates of risk aversion from this approach are higher than with standard expected utility models, they are still not high enough to explain the extent of the equity premium. Explaining the equity premium simply requires counterfactually high risk aversion. Secondly, estimates of the intertemporal elasticity of substitution are approximately the same regardless of whether you use expected or non-expected utility.

16.7.2 Habits

Recall our discussion of habits when we originally tried to improve performance of the RBC model. Constantinides (1990) shows that the EPP and risk free rate puzzles can be explained by assuming habits in the utility function and without recourse to very high levels of risk aversion. The effect of introducing habits is that utility depends not just upon current consumption but also recent consumption. Consider the following form of utility proposed by Abel (1990) and Galí (1994):

$$u_t = \frac{c_t^{1-\sigma} C_t^{\delta} C_{t-1}^{\lambda}}{1-\sigma},$$

where C_t is the aggregate consumption and taken as given when an agent optimises. We assume that agents tend to be jealous and so $\delta, \lambda < 0$. Under these preferences, the optimality condition for bonds

[107] Most notably Epstein and Zin (1989). See the section below on an explanation of EZ preferences.

and stock holdings (766) can be written as:

$$\mathbb{E}_1 \left[\left(\frac{C_2}{C_1} \right)^{\delta - \sigma} \left(\frac{C_1}{C_0} \right)^{\lambda} \left(R_2^S - R_2^b \right) \right] = 0.$$

With reasonable parameters, the model solves the EPP. When δ, which is supposed to be negative, is large in absolute value, marginal utility of an agent's own consumption is highly sensitive to fluctuations in aggregate consumption and therefore strongly negatively correlated to stock returns.

Habit-based explanations seem like an excellent candidate for explaining the EPP then, right? Unfortunately, as shown in Boldrin et al. (1991) this is only partly the case. In the case of an endowment economy (no capital) habit-based utility functions can explain the EPP and risk free puzzle. However, once production and a labour supply choice is introduced, this is no longer the case. The reason why this is the case quite straightforward. In the production model with capital and a labour supply decision, agents have additional ways of smoothing their marginal utility. For instance, when output is high they can choose to investment more rather than raise consumption and similarly if consumption is high they can work harder by taking less leisure. All of these actions serve to reduce the volatility of the marginal rate of substitution, and so go against explaining the equity premium.

16.7.3 Market structure

The Mehra and Prescott study examines a general equilibrium model where all markets are open. Therefore one reason why the model predictions might fail is that some markets are not open. For instance, some consumers may be unable to borrow. If this is the case then a consumer's consumption will be correlated with their income in every period, and as a result there will be some individual specific income risks which will influence an individual's consumption. If there existed perfect borrowing opportunities or insurance possibilities then these idiosyncratic income risks would not influence consumption.

The introduction of borrowing constraints can explain both the low risk free rate and the high equity premium. The risk free rate is the interest rate which ensures equilibrium in the deposit/loan market, that is where savings equals loans. However, if an economy is characterised by borrowing constraints then loans made are very small and so to ensure equilibrium in the deposit market it must be the case that savings are also small. The only way this can be achieved is by having very low interest rates. Therefore in an economy with borrowing constraints the risk free rate is very low.[108] Borrowing constraints can also explain high values of the equity premium. Because of borrowing constraints individual consumption is more volatile than it otherwise would have been. This is because individual specific income risks cannot be diversified away through borrowing. Therefore consumers are already bearing more risk than they would like to if there were complete markets. Therefore in order to take on even more risk by holding equity they need to be rewarded with very high rates of return.

Borrowing constraints/incomplete markets have therefore always been seen as the most likely explanation for the equity premium puzzle. However, this claim has been questioned. Telmer (1993) and D. D. Lucas (1994) both examine the effect that various incomplete market assumptions have on the risk free rate and the equity premium. They find that only if borrowing opportunities are completely absent is it possible to explain the equity premium puzzle. Basically these papers find that agents only need access to one asset which they can sell short (borrow) over some range (i.e. there is still a borrowing constraint) for them to be able to avoid large amounts of diversifiable risk. In other words, markets need to be seriously incomplete to explain the equity premium puzzle. If only a few asset markets are open this still enables asset prices to approximate very closely those predicted by a complete markets representative agent model.

16.7.4 Immobile factors of production

Boldrin et al. (1991) argue that habits combined with immobile factors of production can explain the EPP. To understand why is the case, it is useful to return to the model with habits in the endowment economy. There are two features which any model must possess in order to explain asset market puzzles. The first is that consumers must have frequent motivation to buy and sell assets in order to smooth consumption. The second is that for some reason consumers desire to trade assets is restricted. In the endowment economy with habits both these features are present. Because of habit formation, marginal utility is very volatile and so for a given consumption variability the stronger are habits the

[108] See Hugget (1993).

more consumers wish to trade in assets. However, in an endowment economy there is a fixed supply of capital. As a consequence, variations in the demand for assets lead to large changes in asset prices. As a consequence this model can explain asset price puzzles. However as soon as we introduce production into the model the supply of capital becomes perfectly elastic and asset prices hardly change at all in response to demand variations. Hence the production model with habits cannot explain the asset market puzzles.

Armed with this intuition Boldrin et al. argue that the way to explain the asset market puzzles in the context of a production economy with habits is to introduce some rigidities which frustrate the desire of consumers to trade in assets. In order to do this they introduce a two sector economy: one sector produces capital goods and the other consumer goods. To introduce rigidities they assume that the capital employed in each sector needs to be chosen in advance. As a consequence, capital cannot move between sectors immediately in the aftermath of a shock. To introduce additional frictions they also assume that the labour employed in each sector has to be fixed in advance. Their simulations suggest that these modifications go a significant way to explaining asset price puzzles.

16.8 Epstein-Zin recursive preferences

It's actually worth delving into recursive preferences[109] as it's becoming fairly standard in the literature.[110] Epstein and Zin (1989),[111] following work by Kreps and Porteus (1978), introduced a class of preferences which allow to break the link between risk aversion and intertemporal substitution. To understand the formulation, recall the standard expected utility time-separable preferences which are defined as:

$$V_t = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s u(C_{t+s}).$$

We can also define them recursively as:

$$V_t = u(C_t) + \beta \mathbb{E}_t V_{t+1},$$

[109] Sometimes referred to as "recursive utility" or more simply "Epstein-Zin preferences".
[110] And, handily, I have a couple working papers with a recursive preferences setup.
[111] See also Epstein and Zin (1991).

or equivalently:

$$V_t = (1-\beta)u(C_t) + \beta \mathbb{E}_t V_{t+1}. \tag{767}$$

Epstein-Zin (EZ) preferences generalise this: they are defined recursively over current (known) consumption and a certainty equivalent $R_t(V_{t+1})$ of tomorrow's utility V_{t+1}:

$$V_t = F(C_t, R_t(V_{t+1})),$$

where:

$$R_t(V_{t+1}) = G^{-1}(\mathbb{E}_t G(V_{t+1})),$$

with F and G increasing and concave, and F is HOD1. Note that $R_t(V_{t+1}) = V_{t+1}$ if there is no uncertainty on V_{t+1}, and the more concave G is, and the more uncertain V_{t+1} is, the lower is $R_t(V_{t+1})$. This should be fairly intuitive: we want some kind relationship between uncertainty, risk aversion, and future return on value.

Most papers use a simple CES-like setup for functional forms of F and G, for example:

$$\begin{aligned} F(C,z) &= \left((1-\beta)C^{1-\rho} + \beta z^{1-\rho}\right)^{\frac{1}{1-\rho}}, \ \rho > 0, \\ G(x) &= \frac{x^{1-\sigma}}{1-\sigma}, \ \sigma > 0, \end{aligned}$$

which gives:

$$V_t = \left((1-\beta)C_t^{1-\rho} + \beta\left(\mathbb{E}_t V_{t+1}^{1-\sigma}\right)^{\frac{1-\rho}{1-\sigma}}\right)^{\frac{1}{1-\rho}}. \tag{768}$$

So, in the limits we have:

$$\begin{aligned} F(C,z)|_{\rho=1} &= C^{1-\beta} z^{\beta}, \\ G(x)|_{\sigma=1} &= \log x, \end{aligned}$$

and hence:

$$R_t(V_{t+1}) = \begin{cases} \mathbb{E}_t \left(V_{t+1}^{1-\sigma}\right)^{\frac{1}{1-\sigma}}, & \sigma > 0, \\ \exp\left\{\mathbb{E}_t \log V_{t+1}\right\}, & \sigma = 1. \end{cases}$$

To prove this, define $f(x)$ as:

$$F(C, z) = Cf(x),$$

where $x = \frac{z}{C}$, and:

$$f(x) = \left(1 - \beta + \beta x^{1-\rho}\right)^{\frac{1}{1-\rho}}.$$

So:

$$\frac{f'(x)}{f(x)} = \frac{\beta x^{-\rho}}{1 - \beta + \beta x^{1-\rho}},$$

and:

$$\lim_{\rho \to 1} \frac{f'(x)}{f(x)} = \frac{\beta}{X}.$$

Since f is continuous, this implies:

$$\lim_{\rho \to 1} f(x) = X^\beta,$$

which is really just the proof that a CES function converges to a Cobb-Douglas as $\rho \to 1$. Alternatively:

$$\begin{aligned} \left((1-\beta)c^{1-\rho} + \beta z^{1-\rho}\right)^{\frac{1}{1-\rho}} &= \exp\left\{\frac{1}{1-\rho} \log\left[1 + (1-\rho)\left((1-\beta)\frac{C^{1-\rho}-1}{1-\rho} + \beta\frac{z^{1-\rho}-1}{1-\rho}\right)\right]\right\} \\ &\approx \exp\left\{(1-\beta)\frac{C^{1-\rho}-1}{1-\rho} + \beta\frac{z^{1-\rho}-1}{1-\rho}\right\} \\ &= \exp\left\{(1-\beta)\ln C + \beta \ln z\right\} \\ &= C^{1-\beta} z^\beta. \end{aligned}$$

In general, σ is the relative risk aversion coefficient for static gambles and ρ is the inverse of the intertemporal elasticity of substitution for deterministic variations. As mentioned, the key appeal of EZ preferences is that it breaks the link between the two. To see this, suppose that consumption today is C and the consumption stream for tomorrow is uncertain: $\{C_L, \bar{C}, \bar{C}, ...\}$ or $\{C_H, \bar{C}, \bar{C}, ...\}$, and each has a probability of $\frac{1}{2}$. Utility today would then be:

$$V = F\left(C, G^{-1}\left(\frac{1}{2}G(V_L) + \frac{1}{2}G(V_H)\right)\right),$$

where $V_L = F(C_L, \bar{C})$ and $V_H = F(C_H, \bar{C})$. Curvature of G determines how adverse you are to the uncertainty. If G is linear, you only care about the expected value; if it is non-linear, this is the same as the definition of a certainty equivalent:

$$G(\hat{V}) = \frac{1}{2}G(V_L) + \frac{1}{2}G(V_H).$$

16.8.1 Special case: Deterministic consumption

If consumption is deterministic then we have the usual standard time-separable expected discounted utility with discount factor β, IES $= \frac{1}{\rho}$, and risk aversion $\sigma = \rho$. To see this, consider the case of no uncertainty in which $R_t(V_{t+1}) = V_{t+1}$ and $V_t = F(C_t, V_{t+1})$. With a CES functional form for F, we get CRRA preferences:

$$\begin{aligned} V_t &= \left((1-\beta)C_t^{1-\rho} + \beta V_{t+1}^{1-\rho}\right)^{\frac{1}{1-\rho}}, \\ W_t &= (1-\beta)C_t^{1-\rho} + \beta W_{t+1} \\ &= (1-\beta)\sum_{s=0}^{\infty} \beta^s C_{t+s}^{1-\rho}, \end{aligned}$$

where $W_t = V_t^{1-\rho}$.

16.8.2 Special case: $\sigma = \rho$

Similarly, if $\sigma = \rho$, then the formula:

$$V_t = \left((1-\beta)C_t^{1-\rho} + \beta\left(\mathbb{E}_t V_{t+1}^{1-\sigma}\right)^{\frac{1-\rho}{1-\sigma}}\right)^{\frac{1}{1-\rho}},$$

simplifies to:

$$V_t^{1-\rho} = (1-\beta)C_t^{1-\rho} + \beta \mathbb{E}_t V_{t+1}^{1-\sigma}. \tag{769}$$

If we define $W_t = V_t^{1-\rho}$, we have:

$$W_t = (1-\beta)C_t^{1-\rho} + \beta \mathbb{E}_t W_{t+1},$$

which is just expected utility (767).

16.8.3 Simple example with two lotteries

Suppose we have two lotteries, A and B. A pays in each period $t = 1, 2, ...,$ with an amount of c_H or c_L with probability $\frac{1}{2}$ and the outcome is IID across periods; and B pays starting at $t = 1$ either c_H at all future dates for sure, or c_L at all future dates for sure, and there is a single draw at time $t = 1$.

With expected utility, you are indifferent between these lotteries,[112] but with EZ preferences lottery B is preferred if and only if $\sigma > \rho$. In general, early resolution of uncertainty is preferred if and only if $\sigma > \rho$. In other words, if risk aversion is greater than the inverse of the IES. This is another way to motivate these preferences, since early resolution seems intuitively preferable.

For lottery A, the utility once you know your consumption is either C_H or C_L since:

$$V_H = F(C_H, V_H) = \left((1-\beta)C_H^{1-\rho} + \beta V_H^{1-\rho}\right)^{\frac{1}{1-\rho}}.$$

The certainty equivalent before playing the lottery is:

$$G^{-1}\left(\frac{1}{2}G(C_H) + \frac{1}{2}G(C_L)\right) = \left(\frac{1}{2}C_H^{1-\sigma} + \frac{1}{2}C_L^{1-\sigma}\right)^{\frac{1}{1-\sigma}}.$$

[112] Ljungqvist and Sargent (2018) goes through this in extensive detail.

For lottery B, the values satisfy:

$$\begin{aligned}
W_H^{1-\rho} &= (1-\beta)c_H^{1-\rho} + \beta\left(\frac{1}{2}W_H^{1-\sigma} + \frac{1}{2}W_L^{1-\sigma}\right)^{\frac{1-\rho}{1-\sigma}}, \\
W_L^{1-\rho} &= (1-\beta)c_L^{1-\rho} + \beta\left(\frac{1}{2}W_H^{1-\sigma} + \frac{1}{2}W_L^{1-\sigma}\right)^{\frac{1-\rho}{1-\sigma}}.
\end{aligned}$$

We want to compare $G^{-1}\left(\frac{1}{2}G(W_H) + \frac{1}{2}G(W_L)\right)$ to $G^{-1}\left(\frac{1}{2}G(C_H) + \frac{1}{2}G(C_L)\right)$.

Note that the function $x \to x^{\frac{1-\rho}{1-\sigma}}$ is concave if $1-\rho < 1-\sigma$, i.e., $\rho > \sigma$; and convex otherwise. As a result, if $\rho > \sigma$:

$$\begin{aligned}
\left(\frac{1}{2}W_H^{1-\sigma} + \frac{1}{2}W_L^{1-\sigma}\right)^{\frac{1-\rho}{1-\sigma}} &\geq \frac{1}{2}(W_H^{1-\sigma})^{\frac{1-\rho}{1-\sigma}} + \frac{1}{2}(W_L^{1-\sigma})^{\frac{1-\rho}{1-\sigma}} \\
&= \frac{1}{2}W_H^{1-\rho} + \frac{1}{2}W_L^{1-\rho}.
\end{aligned}$$

Also:

$$\begin{aligned}
W_H^{1-\rho} &\geq (1-\beta)C_H^{1-\rho} + \beta\left(\frac{1}{2}W_H^{1-\rho} + \frac{1}{2}W_L^{1-\rho}\right), \\
W_L^{1-\rho} &\geq (1-\beta)C_H^{1-\rho} + \beta\left(\frac{1}{2}W_H^{1-\rho} + \frac{1}{2}W_L^{1-\rho}\right).
\end{aligned}$$

These results imply that if $\rho > \sigma$, then:

$$\frac{W_H^{1-\rho} + W_L^{1-\rho}}{2} \geq \frac{C_H^{1-\rho} + C_L^{1-\rho}}{2},$$

in which case the certainty equivalent of lottery A is higher than the certainty equivalent of lottery B and agents prefer late to early resolution of uncertainty. Technically, EZ preferences are an extension of expected utility which relaxes the independence axiom. Recall from micro that the independence axiom is: if $x \succeq y$, then for any z, σ: $\sigma x + (1-\sigma)z \succeq \sigma y + (1-\sigma)z$. With EZ preferences, "intertemporal composition of risk matters" – we cannot reduce compound lotteries.

16.8.4 The stochastic discount factor (again) and returns

We have:

$$V_t = \left((1-\beta) C_t^{1-\rho} + \beta R_t(V_{t+1})^{1-\rho} \right)^{\frac{1}{1-\rho}},$$

where:

$$R_t(V_{t+1}) = \left(\mathbb{E}_t V_{t+1}^{1-\sigma} \right)^{\frac{1}{1-\sigma}}.$$

Since V_t is HOD1, Euler's Theorem implies:

$$V_t = \underbrace{\frac{\partial V_t}{\partial C_t}}_{V_{C,t}} C_t + \mathbb{E}_t \underbrace{\frac{\partial V_t}{\partial R_t(V_{t+1})} \frac{\partial R_t(V_{t+1})}{\partial V_{t+1}}}_{V_{V,t+1}} V_{t+1}. \tag{770}$$

Taking derivatives, we get:

$$V_{C,t} = \frac{\partial V_t}{\partial C_t} = (1-\beta) V_t^{\rho} C_t^{-\rho},$$

and

$$\begin{aligned} \frac{\partial V_t}{\partial R_t(V_{t+1})} &= V_t^{\rho} \beta R_t(V_{t+1})^{-\rho}, \\ \frac{\partial R_t(V_{t+1})}{\partial V_{t+1}} &= R_t(V_{t+1})^{\sigma} V_{t+1}^{-\sigma}, \\ \implies V_{V,t+1} = \frac{\partial V_t}{\partial R_t(V_{t+1})} \frac{\partial R_t(V_{t+1})}{\partial V_{t+1}} &= \beta V_t^{\rho} R_t(V_{t+1})^{\sigma-\rho} V_{t+1}^{-\sigma}. \end{aligned}$$

Define the intertemporal marginal rate of substitution – i.e., the stochastic discount factor – as:

$$M_{t,t+1} = \frac{V_{V,t+1} V_{C,t+1}}{V_{C,t}} \tag{771}$$

$$= \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{V_{t+1}}{R_t(V_{t+1})} \right)^{\rho-\sigma}. \tag{772}$$

The first term is familiar. The second term is next period's value relative to its certainty equivalent. If $\rho = \sigma$, or there is no uncertainty, so that $V_{t+1} = R_t(V_{t+1})$ this term equals unity. The problem with the stochastic discount factor expression (772) is that it depends on future utilities $\frac{V_{t+1}}{R_t(V_{t+1})}$, which are not observed. The first possible empirical implementation of these preferences uses the fact that the return on wealth can be substituted in instead of future utility (the second term of the discount factor). This allows us to do GMM estimation, or to use log-linear-log-normal approximations.

The derivation is as follows. Consider household wealth, and start with the value function (770):

$$V_t = V_{C,t} C_t + \mathbb{E}_t V_{V,t+1} V_{t+1},$$

then divide by $V_{C,t}$:

$$\begin{aligned}
\frac{V_t}{V_{C,t}} &= C_t + \mathbb{E}_t \frac{V_{V,t+1}}{V_{C,t}} V_{t+1} \\
&= C_t + \mathbb{E}_t \left(\underbrace{\frac{V_{V,t+1} V_{C,t+1}}{V_{C,t}}}_{M_{t,t+1}} \right) \frac{V_{t+1}}{V_{C,t+1}},
\end{aligned}$$

and define $W_t = V_t / V_{C,t}$, then:

$$W_t = C_t + \mathbb{E}_t M_{t,t+1} W_{t+1}, \tag{773}$$

which is the present-discounted value of [household] wealth. A key result is that:

$$W_t = \frac{V_t}{V_{C,t}},$$

which can proven by a guess-and-verify method for this recursion:

$$\begin{aligned}
W_t &= C_t + \mathbb{E}_t M_{t,t+1} W_{t+1} \\
&= C_t + \mathbb{E}_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{V_{t+1}}{R_t(V_{t+1})} \right)^{\rho-\sigma} \frac{V_{t+1}}{V_{C,t+1}} \right] \\
&= C_t + \mathbb{E}_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{V_{t+1}}{R_t(V_{t+1})} \right)^{\rho-\sigma} \frac{V_{t+1}}{(1-\beta) C_{t+1}^{-\rho} V_{t+1}^{\rho}} \right] \\
&= C_t + \mathbb{E}_t \left[\beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} V_{t+1}^{1-\sigma} R_t(V_{t+1})^{\sigma-\rho} \frac{1}{(1-\beta) C_{t+1}^{-\rho}} \right],
\end{aligned}$$

and rearrange and guess the solution for W_t to write:

$$\begin{aligned}
\frac{V_t}{V_{C,t}}(1-\beta)C_t^{-\rho}R_t(V_{t+1})^{\rho-\sigma} &\stackrel{?}{=} (1-\beta)C_t^{1-\rho}R_t(V_{t+1})^{\rho-\sigma} + \beta\mathbb{E}_t V_{t+1}^{1-\sigma} \\
\frac{V_t}{(1-\beta)C_t^{-\rho}V_t^{\rho}}(1-\beta)C_t^{-\rho}R_t(V_{t+1})^{\rho-\sigma} &\stackrel{?}{=} (1-\beta)C_t^{1-\rho}R_t(V_{t+1})^{\rho-\sigma} + \beta\mathbb{E}_t V_{t+1}^{1-\sigma} \\
V_t^{1-\rho}R_t(V_{t+1})^{\rho-\sigma} &\stackrel{?}{=} (1-\beta)C_t^{1-\rho}R_t(V_{t+1})^{\rho-\sigma} + \beta\mathbb{E}_t V_{t+1}^{1-\sigma} \\
V_t^{1-\rho} &\stackrel{?}{=} (1-\beta)C_t^{1-\rho} + \beta\mathbb{E}_t \frac{V_{t+1}^{1-\sigma}}{R_t(V_{t+1})^{\rho-\sigma}} \\
V_t^{1-\rho} &\stackrel{?}{=} (1-\beta)C_t^{1-\rho} + \beta R_t(V_{t+1})^{1-\rho},
\end{aligned}$$

which is true, and confirms our guess.

Next, consider the return on wealth. Define the return on the wealth portfolio as:

$$R_{t,t+1} = \frac{W_{t+1}}{W_t - C_t}.$$

Note that:

$$W_{t+1} = \frac{V_{t+1}}{V_{C,t+1}} = \frac{V_{t+1}^{1-\rho}C_{t+1}^{\rho}}{1-\beta},$$

hence:

$$R_{t,t+1} = \frac{V_{t+1}^{1-\rho}C_{t+1}^{\rho}}{V_t^{1-\rho}C_t^{\rho} - C_t} = \left(\frac{C_{t+1}}{C_t}\right)^{\rho}\left(\frac{V_{t+1}^{1-\rho}}{V_t^{1-\rho} - (1-\beta)C_t^{1-\rho}}\right).$$

Now, use the fact that:

$$V_t^{1-\rho} = (1-\beta)C_t^{1-\rho} + \beta R_t(V_{t+1})^{1-\rho},$$

to get:

$$R_{t,t+1} = \left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\left(\frac{R_t(V_{t+1})}{V_{t+1}}\right)^{1-\rho}\right]^{-1}. \tag{774}$$

To show this, start with:

$$
\begin{aligned}
R_{t,t+1} &= \frac{V_{t+1}}{V_{C,t+1}} \frac{1}{\left(\frac{V_t}{V_{C,t}} - C_t\right)} \\
&= \frac{V_{t+1}}{V_{C,t+1}} \frac{V_{C,t}}{V_t - C_t V_{C,t}} \\
&= \frac{V_{t+1}}{(1-\beta)C_{t+1}^{-\rho} V_{t+1}^{\rho}} \frac{(1-\beta)C_t^{-\rho} V_t^{\rho}}{\beta R_t(V_{t+1})^{1-\rho} V_t^{\rho}} \\
R_{t,t+1} &= \left[\beta \left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \left(\frac{R_t(V_{t+1})}{V_{t+1}}\right)^{1-\rho}\right]^{-1}.
\end{aligned}
$$

Use this equation to solve for the value function relative to the certainty equivalent:

$$
\begin{aligned}
R_{t,t+1}^{-1} &= \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \left(\frac{R_t(V_{t+1})}{V_{t+1}}\right)^{1-\rho} \\
\frac{V_{t+1}}{R_t(V_{t+1})} &= \left[\beta R_{t,t+1} \left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\right]^{\frac{1}{1-\rho}}.
\end{aligned}
\tag{775}
$$

We can use this to directly evaluate the cost of uncertain returns and consumption.

Hence, the stochastic discount factor can be expressed as a function of the return:

$$
\begin{aligned}
M_{t,t+1} &= \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \left(\frac{V_{t+1}}{R_t(V_{t+1})}\right)^{\rho-\sigma} \\
&= \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\rho} \left[\frac{R_{t,t+1}^{-1}}{\beta \left(\frac{C_{t+1}}{C_t}\right)^{-\rho}}\right]^{\frac{\rho-\sigma}{\rho-1}} \\
&= \beta^{\frac{1-\sigma}{1-\rho}} \left(\frac{C_{t+1}}{C_t}\right)^{-\rho+\rho\frac{\rho-\sigma}{\rho-1}} R_{t,t+1}^{\frac{\rho-\sigma}{1-\rho}} \\
&= \beta^{\frac{1-\sigma}{1-\rho}} \left(\frac{C_{t+1}}{C_t}\right)^{-\rho\left(1+\frac{\rho-\sigma}{1-\rho}\right)} R_{t,t+1}^{\frac{\rho-\sigma}{1-\rho}}.
\end{aligned}
$$

Define $\theta = \frac{1-\sigma}{1-\rho}$ and $\psi = \frac{1}{\rho}$, so we can write:

$$\begin{aligned} M_{t,t+1} &= \beta^{\theta} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\sigma}{1-\rho}} R_{t,t+1}^{\frac{\rho-\sigma+1-1}{1-\rho}} \\ &= \beta^{\theta} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{\theta}{\psi}} R_{t,t+1}^{\theta-1}. \end{aligned} \tag{776}$$

This allows us to implement this empirically since the stochastic discount factor can now be measured.[113] Epstein and Zin proxy the return to wealth by the return on a broad stock market return. An obvious criticism is that a lot of wealth is not traded on the stock market (private firms, human capital, housing, and so on) but these returns may be correlated with the stock market returns.

Now, take logs of the stochastic discount factor:

$$\log M_{t,t+1} = \theta \log \beta - \frac{\theta}{\psi} \Delta c_{t+1} - (1-\theta) r_{t,t+1},$$

where lower case variables denote log values. The return on an i-th asset satisfies:

$$\mathbb{E}_t M_{t,t+1} R_{t,t+1}^i = 1.$$

Note that assets' risk are measured as the covariance with the stochastic discount factor, so the EZ utility rationalises a formula which is a mix of the CAPM and the Consumption CAPM:

$$\begin{aligned} \log \left(\frac{\mathbb{E}_t R_{t,t+1}^i}{R_{t+1}^f} \right) &= -\operatorname{Cov}\left(\log M_{t,t+1}, \log R_{t,t+1}^i\right) \\ &= \frac{\theta}{\psi} \left[\operatorname{Cov}\left(\Delta c_{t+1}, r_{t+1}^i\right) \right] + (1-\theta) \operatorname{Cov}\left(r_{t+1}^m, r_{t+1}^i\right). \end{aligned} \tag{777}$$

Empirically, the extra free parameter of EZ preferences leads to an improvement over CRRA. However, the solutions of asset pricing puzzles with EZ utility require high risk aversion (with some exceptions).

[113] Note that if $\rho = \sigma$, we have:

$$M_{t,t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho}.$$

It is worth noting on the consumption-wealth ratio that:

$$\begin{aligned} W_t &= \frac{V_t}{V_{C,t}} \\ &= \frac{V_t}{(1-\beta)C_t^{-\rho}V_t^{\rho}} \\ W_t &= \frac{V_t^{1-\rho}}{(1-\beta)C_t^{-\rho}} \\ \implies C_t &= (1-\beta)W_t\frac{C_t^{1-\rho}}{V_t^{1-\rho}}, \end{aligned}$$

so that if $\rho = 1$ (log utility in intertemporal elasticity of substitution), the consumption-wealth ratio is constant. Empirical work in Lettau and Ludvigson (2001) gives an important role to this variable.

Moving on, for the market return we have:

$$\log\left(\frac{\mathbb{E}R^m}{R^f}\right) = \frac{\theta}{\psi}\operatorname{Cov}\left(\Delta c, r^m\right) + (1-\theta)\sigma_m^2,$$

where σ_m^2 is the volatility/variance of m. We can rewrite the above as:

$$r^m + \frac{\sigma_m}{2} = r^f + \frac{\theta}{\psi}\operatorname{Cov}\left(\Delta c, r^m\right) + (1-\theta)\sigma_m^2.$$

If $\rho = 1$, then we have:

$$R_{t,t+1} = \left(\beta\left(\frac{C_{t+1}}{C_t}\right)^{-1}\right)^{-1},$$

and $\sigma_{\Delta c} = \sigma_m$, which implies that:

$$\log\left(\frac{\mathbb{E}R_m}{R^f}\right) = \sigma_m^2.$$

For the risk free rate, we have that:

$$\log R_{t+1}^f = \log\left(\mathbb{E}_t \exp\left\{-\log M_{t,t+1}\right\}\right),$$

which in log-form is:

$$r^f_{t+1} = -\theta \log \beta + \frac{\theta}{\psi}\mathbb{E}_t \Delta c_{t+1} + (1-\theta) r^m \\ - \left(\frac{\theta}{\psi}\right)^2 \frac{\sigma^2_{\Delta c}}{2} - (1-\theta)^2 \frac{\sigma^2_m}{2} - \frac{\theta(1-\theta)}{\psi} \operatorname{Cov}(\Delta c, r^m).$$

Substitute in for the market return to obtain:

$$(1-\theta) r^m = (1-\theta) r^f - \frac{(1-\theta)\sigma_m}{2} + (1-\theta)^2 \sigma_m + \frac{\theta(1-\theta)}{\psi} \operatorname{Cov}(\Delta c, r^m).$$

Simplify:

$$r^f_t = -\log \beta + \frac{1}{\psi}\mathbb{E}_t \Delta c_{t+1} - \frac{\theta}{\psi^2}\frac{\sigma^2_{\Delta c}}{2} - (1-\theta)\frac{\sigma^2_m}{2}. \tag{778}$$

Again, if $\rho = \sigma$ then $\theta = 1$, so we have the standard risk-free rate equation. If $\alpha > \rho$ then $\theta < 1$ and the volatility from the market return reduces the real interest rate.

16.8.5 IID consumption and the stochastic discount factor

Let

$$\Delta C_{t+1} = g + \sigma_c \epsilon_{t+1},$$

and let $v_t = \frac{V_t}{C_t}$ and write the value function as:

$$v_t = \left\{ 1 - \beta + \beta \mathbb{E}_t \left[v^{1-\sigma}_{t+1} \left(\frac{C_{t+1}}{C_t} \right)^{1-\sigma} \right]^{\frac{1-\rho}{1-\sigma}} \right\}^{\frac{1}{1-\rho}},$$

since consumption is IID, v is constant.

With $v_t = v$ we have:

$$\begin{aligned} M_{t,t+1} &= \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\rho} \left(\frac{V_{t+1}}{R_t(V_{t+1})} \right)^{\rho-\sigma} \\ &= \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \left[\frac{1}{\mathbb{E}_t \left(\frac{C_{t+1}}{C_t} \right)^{1-\sigma}} \right]^{-(1-\theta)} . \end{aligned}$$

Taking logs, we get:

$$\begin{aligned} \log M_{t,t+1} &= \log \beta - \sigma \Delta c_{t+1} + (1-\theta) \log \left(\mathbb{E} \exp \left\{ (1-\sigma) \Delta c_{t+1} \right\} \right) \\ &= \log \beta - \sigma \Delta c_{t+1} + (\sigma - \rho) g + (1-\theta)(1-\sigma)^2 \frac{\sigma_c}{2}. \end{aligned}$$

The risk free rate is:

$$\begin{aligned} r^f &= -\log \mathbb{E}_t M_{t,t+1} \\ &= -\left(\mathbb{E}_t \log M_{t,t+1} + \frac{\sigma_m^2}{2} \right) \\ &= -\log \beta + \rho g - \left[(1-\theta)(1-\sigma) + \sigma^2 \right] \frac{\sigma_c^2}{2}. \end{aligned}$$

If $\rho = \sigma$, this is the standard expression:

$$r^f = -\log \beta + \rho g - \rho^2 \frac{\sigma_c^2}{2}.$$

16.8.6 Log-linearisation

This is based on the by Campbell, where he uses some log-linear approximations to derive implications even when the intertemporal elasticity of substitution is not one. Some of these log-linear approximations can be useful also with CRRA preferences, or for empirical work.

Just like Campbell and Shiller (1988) did a log-linear approximation of the return, Campbell writes a log-linear approximation to the budget constraint. He writes the budget constraint as:

$$W_{t+1} = R^m_{t+1}(W_t - C_t).$$

Now, do all the steps similar to Campbell and Shiller:

$$\begin{aligned} \frac{W_{t+1}}{W_t} &= R^m_{t+1}\left(1 - \frac{C_t}{W_t}\right) \\ \implies \Delta \log W_{t+1} = w_{t+1} - w_t &= r^m_{t+1} + \log\left(1 - \frac{C_t}{W_t}\right). \end{aligned}$$

Where the last term can be broken down to:

$$\begin{aligned} \log\left(1 - \frac{C_t}{W_t}\right) &= \log\left(1 - \exp\left\{c_t - w_t\right\}\right) \\ &\approx k + \left(1 - \frac{1}{\rho}\right)(c_t - w_t). \end{aligned}$$

Where if:

$$\rho = \frac{W - C}{W} < 1,$$

and this yields:

$$\Delta w_{t+1} = r^m_{t+1} + k + \left(1 - \frac{1}{\rho}\right)(c_t - w_t). \tag{779}$$

We can rewrite this equation as:

$$\begin{aligned} \Delta c_{t+1} + (c_t - w_t) - (c_{t+1} - w_{t+1}) &= r^m_{t+1} + k + \left(1 - \frac{1}{\rho}\right)(c_t - w_t) \qquad (780) \\ \frac{1}{\rho}(c_t - w_t) - (c_{t+1} - w_{t+1}) &= r^m_{t+1} + k - \Delta c_{t+1} \\ (c_t - w_t) - \rho(c_{t+1} - w_{t+1}) &= \rho(r^m_{t+1} + k - \Delta c_{t+1}), \end{aligned}$$

and iterating forward yields:

$$\begin{aligned} c_t - w_t &= \sum_{j\geq 1} \rho^j (r^m_{t+j} + k - \Delta c_{t+j}) \\ &= \frac{k\rho}{1-\rho} + \sum_{j=1}^{\infty} \rho^j (r^m_{t+j} - \Delta c_{t+j}). \end{aligned} \tag{781}$$

This is just an accounting identity, which holds ex-post as well as ex-ante. this holds also in expectation:

$$c_t - w_t = \frac{k\rho}{1-\rho} + \mathbb{E}_t \sum_{j=1}^{\infty} \rho^j (r^m_{t+j} - \Delta c_{t+j}).$$

When the consumption-wealth ratio is high, it means that either future returns will be high or future consumption growth will be low (so that the C/W ratio returns to its average).

Another way to state this equality is to apply the operator $\mathbb{E}_{t+1} - \mathbb{E}_t$ to equation (780) (this operator cancels all terms known at time t):

$$\begin{aligned} (\mathbb{E}_{t+1} - \mathbb{E}_t)\Delta c_{t+1} &= (\mathbb{E}_{t+1} - \mathbb{E}_t)(c_{t+1} - w_{t+1}) + (\mathbb{E}_{t+1} - \mathbb{E}_t) r^m_{t+1} \\ c_{t+1} - \mathbb{E}_t c_{t+1} &= (\mathbb{E}_{t+1} - \mathbb{E}_t) \sum_{j=1}^{\infty} \rho^j (r^m_{t+1+j} - \Delta c_{t+j}) + r^m_{t+1} + \mathbb{E}_t r^m_{t+1} \\ &= (\mathbb{E}_{t+1} - \mathbb{E}_t) \sum_{j=0}^{\infty} \rho^j r^m_{t+1+j} - (\mathbb{E}_{t+1} - \mathbb{E}_t) \sum_{j=1}^{\infty} \rho^j \Delta c_{t+1+j}. \end{aligned} \tag{782}$$

Up to now, we have only log-linearised the budget constraint. We can use this empirically to measure what explains changes in consumption (just like we did with returns), but the interesting part is to note that consumption, wealth, and returns are also tied by the optimality of consumer choice. To do this, we assume that all second moments (variances and covariances) are constant.[114]

Recall the stochastic discount factor formula with EZ utility when we substitute out the return on wealth:

$$M_{t,t+1} = \beta^{\frac{1-\sigma}{1-\rho}} \left(\frac{C_{t+1}}{C_t} \right)^{-\rho \frac{1-\sigma}{1-\rho}} R_{t,t+1}^{\frac{\rho-\sigma}{1-\rho}} \Leftrightarrow \beta^{\theta} \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{\theta}{\psi}} R_{t,t+1}^{\theta-1}.$$

[114]Note the consistency problem since we are interested in explaining changing returns.

When we write out the Euler equation for the market return, we obtain:

$$\mathbb{E}_t\left[\beta^\theta\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{\theta}{\psi}} R_{m,t+1}^{\theta-1}\right] = 1.$$

Logging this leads to:

$$\theta\log\beta - \frac{\theta}{\psi}\mathbb{E}_t\Delta\log C_{t+1} + (\theta-1)\mathbb{E}_t\log R_{t+1}^m + \zeta = 0,$$

where ζ is a constant regrouping the (constant) conditional variances and covariances. Hence, a first result is:

$$\mathbb{E}_t\Delta\log C_{t+1} = k + \frac{1}{\rho}\mathbb{E}_t r_{t+1}^m,$$

where k is a constant. Expected consumption growth moves proportionally to the expected stock return, with the IES $\frac{1}{\rho}$ governing the proportionality.

We now derive the equation for the excess returns. Start from the Euler Equation for any return and for the risk-free rate:

$$\begin{aligned}
1 &= \mathbb{E}_t(M_{t,t+1}R_{t+1}^i)\\
0 &= \mathbb{E}_t\log M_{t,t+1} + \frac{1}{2}\operatorname{Var}_t\left(\log M_{t,t+1}\right) + \mathbb{E}_t\log R_{t+1}^i + \frac{1}{2}\operatorname{Var}_t\left(\log R_{t+1}^i\right) + \operatorname{Cov}_t\left(\log M_{t,t+1}, \log R_{t+1}^i\right),\\
0 &= \mathbb{E}_t\log M_{t,t+1} + \frac{1}{2}\operatorname{Var}_t\left(M_{t,t+1}\right) + \mathbb{E}_t\log R_{t+1}^f.
\end{aligned}$$

Subtracting these equations yields:

$$\begin{aligned}
\log\mathbb{E}_t\frac{R_{t+1}^i}{R_{t+1}^f} &= \mathbb{E}_t\log\frac{R_{t+1}^i}{R_{t+1}^f} + \frac{1}{2}\operatorname{Var}_t\left(\log R_{t+1}^i\right)\\
&= -\operatorname{Cov}_t\left(\log M_{t,t+1}, \log R_{t+1}^i\right)\\
&= \frac{\theta}{\psi}\operatorname{Cov}_t\left(\Delta\log C_{t+1}, \log R_{t+1}^i\right) + (\theta-1)\operatorname{Cov}_t\left(\log R_{t+1}^m, \log R_{t+1}^i\right). \qquad (783)
\end{aligned}$$

We can see how both consumption growth and the market return are risk factors in this equation.[115]

[115] Campbell uses different notation, but he has the same results.

Campbell's last setup: Use the consumption Euler Equations $\mathbb{E}_t \Delta C_{t+1} = k + \frac{1}{\rho}\mathbb{E}_t r^m_{t+1}$ in the present value budget constraint found above:

$$C_{t+1} - \mathbb{E}_t C_{t+1} = (\mathbb{E}_{t+1} - \mathbb{E}_t) \sum_{j=0}^{\infty} \rho^j r^m_{t+j} - (\mathbb{E}_{t+1} - \mathbb{E}_t) \sum_{j=0}^{\infty} \rho^j \Delta C_{t+1+j}.$$

Thus:

$$\begin{aligned} C_{t+1} - \mathbb{E}_t C_{t+1} &= (\mathbb{E}_{t+1} - \mathbb{E}_t) \sum_{j=0}^{\infty} \rho^j r^m_{t+1+j} - \frac{1}{\rho}(\mathbb{E}_{t+1} - \mathbb{E}_t) \sum_{j=1}^{\infty} \rho^j \Delta r^m_{t+1+j} \\ &= r^m_{t+1} - \mathbb{E}_t r^m_{t+1} + \left(1 - \frac{1}{\rho}\right) (\mathbb{E}_{t+1} - \mathbb{E}_t) \sum_{j=1}^{\infty} \rho^j r^m_{t+1+j}. \end{aligned} \tag{784}$$

Campbell likes to se this equation in (783) to substitute out consumption because he believes consumption is a bad proxy for marginal utility and is poorly measured. Hence we get what he calls the "CAPM+" formula:

$$\begin{aligned} \log \mathbb{E}_t \frac{R^i_{t+1}}{R^f_{t+1}} &= \left(\frac{\theta}{\psi} + (\theta - 1)\right) \operatorname{Cov}_t \left(r^m_{t+1}, r^i_{t+1}\right) + \frac{\theta}{\psi}(1 - \psi) \operatorname{Cov}_t \left((\mathbb{E}_{t+1} - \mathbb{E}_t) \sum_{j=1}^{\infty} \rho^j r^m_{t+1+j}, r^i_{t+1} \right) \\ &= \sigma \operatorname{Cov}_t \left(r^m_{t+1}, r^i_{t+1}\right) + (\sigma - 1) \operatorname{Cov}_t \left((\mathbb{E}_{t+1} - \mathbb{E}_t) \sum_{j=1}^{\infty} \rho^j r^m_{t+1+j}, r^i_{t+1} \right). \end{aligned} \tag{785}$$

The novelty is that expectations of future returns now matter. Investors dislike assets that do badly when the market does badly (the first term, which is just the usual CAPM effect), but they also like/dislike assets which do badly when expected future returns are bad. Whether this is a like or dislike depends on whether σ is greater or smaller than 1, respectively.

This formula can be implemented if you use a VAR to measure the news to future market returns, $(\mathbb{E}_{t+1} - \mathbb{E}_t) \sum_{j=0}^{\infty} \rho^j r^m_{t+1+j}$.

16.8.7 The price-dividend ratio

Consider the log dividend price ratio. Conjecture a constant q:

$$q = \mathbb{E}_t M_{t,t+1} \frac{C_{t+1}}{C_t}(1+q),$$

where

$$\begin{aligned}\log M_{t,t+1} + \Delta c_{t+1} &= \log\beta + (1-\sigma)\Delta c_{t+1} + (1-\theta)\log\mathbb{E}_t \exp\left\{(1-\sigma)\Delta c_{t+1}\right\} \\ &= \log\beta + (1-\sigma)\Delta c_{t+1} + (\sigma-\rho)g + (1-\theta)(1-\sigma)^2\frac{\sigma_c}{2}.\end{aligned}$$

So the price dividend ratio satisfies:

$$\begin{aligned}\log\frac{q}{1+q} &= \log\beta + (1-\rho)g - (1-\sigma)^2\theta\frac{\sigma_c^2}{2} \\ &= -r^f + \left(g + \frac{\sigma_c^2}{2}\right) - \sigma\sigma_c^2,\end{aligned}$$

where the term in brackets is expected consumption growth, $\log(\mathbb{E}_t C_{t+1}/C_t)$. Hence, this is a risk-adjusted Gordon growth formula.

The risk premium on a consumption claim is then:

$$\log\mathbb{E}_t R_{t+1} = \log\mathbb{E}_t \frac{q+1}{q}\frac{C_{t+1}}{C_t},$$

so that:

$$r^m + \frac{\sigma_m}{2} - r^f = \sigma\sigma_c^2.$$

16.8.8 The consumption-wealth ratio (again)

We covered a lot of this in the log-linearisation part. But, in any case, start with the identity:

$$W_{t+1} = R^m_{t+1}(W_t - C_t),$$

to obtain the log-linear equation:

$$\Delta w_{t+1} = r^m_{t+1} + k + \left(1 - \frac{1}{\rho}\right)(c_t - w_t),$$

where $\rho = 1 - \exp(c - w)$. Rearrange to obtain:

$$\begin{aligned}(1-\rho)(c_t - w_t) &= \rho r^m_{t+1} - \rho \Delta w_{t+1} + \rho k \\ &= \rho r^m_{t+1} + \rho\left[\Delta(c_{t+1} - w_{t+1}) - \Delta c_{t+1}\right] + \rho k.\end{aligned}$$

We then get the present value relationship:

$$\begin{aligned}c_t - w_t &= \rho(r^m_{t+1} - \Delta c_{t+1}) + \rho(c_{t+1} - w_{t+1}) + \rho k \\ &= \sum_{s=1}^{\infty} \rho^s \left[r^m_{t+s} - \Delta c_{t+s}\right] + \frac{\rho}{1-\rho} k.\end{aligned}$$

Now, combine the risk free rate and market rate Euler equations to obtain:

$$r^m_{t+s} - \Delta c_{t+s} = (1-\psi) r^m_{t+s} - \mu_m,$$

where μ_m is a constant that depends on conditional covariances. Thus we get:

$$c_t - w_t = (1-\psi)\mathbb{E}_t \sum_{s=1}^{\infty} \rho^s r^m_{t+s} + \frac{\rho(\kappa - \mu_m)}{1-\rho}.$$

The consumption-wealth ratio is an increasing function of expected future returns if the IES<1. Note, we started with an identity and combined with the Euler equation for safe vs risky returns for a given IES. Thus, these expressions are general and do not depend specifically on EZ preferences.

Now use:

$$\begin{aligned} c_{t+1} - \mathbb{E}_t c_{t+1} &= W_{t+1} - \mathbb{E}_t W_{t+1} + (1-\psi)(\mathbb{E}_{t+1} - \mathbb{E}_t) \sum_{s=1}^{\infty} \rho^s r^m_{t+s+1} \\ &= r^m_{t+1} - \mathbb{E}_t r^m_{t+1} + (1-\psi)(\mathbb{E}_{t+1} - \mathbb{E}_t) \sum_{s=1}^{\infty} \rho^s r^m_{t+s+1}. \end{aligned}$$

Unexpected returns increases consumption growth, and unexpected future returns increase current consumption growth if the IES<1.

Not that if returns are not forecastable, the consumption-wealth ratio is a constant. In this case, consumption volatility equals the volatility of wealth, or equivalently the market return. In the data this is obviously not true – hence returns must be predictable.

16.8.9 Asset pricing implications

We can now compute:

$$\mathrm{Cov}\left(r^i_{t+1}, \Delta c_{t+1}\right) = \sigma_{ic} = \sigma_{im} + (1-\psi)\sigma_{ih},$$

where σ_{ih} is the covariance of $r_{i,t+1}$ with the surprise in future market returns:

$$\sigma_{ih} = \mathrm{Cov}\left(r^i_{t+1}, (\mathbb{E}_{t+1} - \mathbb{E}_t) \sum_{s=1}^{\infty} \rho^s r^m_{t+s+1} \right).$$

Using EZ preferences, the risk premium is:

$$\mathbb{E}_t r^i_{t+1} - r^f_{t+1} + \frac{\sigma_i^2}{2} = \theta \frac{\sigma_{ic}}{\psi} + (1-\theta)\sigma_{im}.$$

The risk premium for asset i depends on its covariance between current returns and its covariance with news about future market returns:

$$\mathbb{E}_t r^i_{t+1} - r^f_{t+1} + \frac{\sigma_i^2}{2} = \sigma\sigma_{im} + (\sigma - 1)\sigma_{ih}.$$

Note we don't need to know the IES or consumption growth to price risk in this framework.

For EZ preferences we can now write the risk premium on the market return as:

$$\mathbb{E}_t r_{t+1}^m - r_{t+1}^f + \frac{\sigma_m^2}{2} = \sigma\sigma_m^2 + (\sigma - 1)\sigma_{mh}.$$

If returns are unforecastable, $\sigma_{mh} = 0$. Given $\sigma_m = 0.17$, from the studies by Mehra and Prescott (1985) and Mankiw and Zeldes (1991), we would need $\sigma \approx 2.5$ to obtain a risk premium of 6 percent. So we succeed in matching the risk premium with low relative risk aversion but fail on the fact that the consumption-wealth ration will be a constant, and consumption volatility should equal wealth volatility. If there is a mean reversion and future returns are negatively correlated with current returns then $\sigma_{mh} < 0$ and we would need a higher σ. Since mean-reversion is difficult to determine, the estimate could be substantially higher.

16.9 Disaster risk and business cycles (Gourio, 2012)

One more method of resolving the EPP that has garnered quite a lot of attention recently[116] is one proposed in "Disaster Risk and Business Cycles" Gourio (2012): low frequency, high impact disaster risk. In the paper, Gourio presents quite a basic RBC model but which features large, volatile, and countercyclical risk premia, that are driven by a small probability, exogenously time-varying risk of large disaster. The disaster shock is based on the idea presented by Barro (2006), but Gourio goes one step further by embedding it in a RBC model and matching observed business cycle moments. The key mechanism is that agents undertake precautionary savings, which drives the demand for safe assets, leading their yields to fall, but then also increases the spreads on relatively risky securities.

16.9.1 A simple analytical example in an AK economy

To highlight the key mechanism of the paper, consider a simple economy with a representative consumer who has power utility:

$$U_t = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \frac{C_{t+s}^{1-\sigma}}{1-\sigma},$$

[116] Also, coincidentally again, I have a draft paper on.

where C_t is consumption and σ is the risk aversion coefficient (and also the inverse of the intertemporal elasticity of substitution of consumption). This consumer operates an AK technology:

$$Y_t = AK_t,$$

where Y_t is output, K_t is capital, and A is productivity, which is assumed to be constant.[117] The aggregate resource constraint is:

$$C_t + I_t \leq AK_t.$$

The economy is randomly hit by disasters. A disaster destroys a share b_k of the capital stock. This could be due to a war which physically destroys capital, but there are alternative interpretations.[118] For instance, b_k could reflect expropriation of capital holders (if the capital is take away and then not used effectively), or it could be a "technological revolution" that makes a large share of the capital worthless. It could also be that even though physical capital is not literally destroyed, some intangible capital (such as matches between firms, employees, and customers is lost). Finally, one can imagine a situation where the demand for some goods falls sharply, rendering worthless the factories which produce them.

Finally, the probability of a disaster varies over time. To maintain tractability, we assume in this section that it is IID: p_t, the probability of disaster at time $t+1$, is drawn at the beginning of time t from a constant cumulative distribution function F. The law of motion for capital is thus:

$$K_{t+1} = \begin{cases} (1-\delta)K_t + I_t, & \text{if } x_{t+1} = 0 \text{ w.p. } p_t, \\ ((1-\delta)K_t + I_t)(1-b_k), & \text{if } x_{t+1} = 1 \text{ w.p. } 1-p_t, \end{cases}$$

where x_{t+1} is a binomial variable which is 1 with probability p_t and 0 with probability $1-p_t$. A disaster does not affect productivity A.[119] Finally, we assume that the two random variables p_{t+1} and x_{t+1} are independent.

[117] As noted by Gourio, it is easy to extend this example to the case where A is stochastic; this does not affect the results.
[118] Such as the current COVID-19 pandemic.
[119] Gourio does relax this assumption in the "full model" in this paper.

The model has one endogenous state K and one exogenous state p, and there is one control variable C. There are two shocks: the realisation of disaster $x' \in \{0, 1\}$, and the draw of a new probability disaster p'. The Bellman equation for the representative consumer is:

$$V(K, p) = \max_{C,I} \left\{ \frac{C^{1-\sigma}}{1-\sigma} + \beta \mathbb{E}_{p',x'}(V(K', p')) \right\},$$

subject to:

$$\begin{aligned} C + I &\leq AK, \\ K' &= ((1-\delta)K + I)(1 - x' b_k). \end{aligned}$$

The assumptions made ensure that V is homogeneous, i.e., we can guess and verify that V is of the form $V(K, p) = \frac{K^{1-\sigma}}{1-\sigma} g(p)$, where g is defined through the Bellman equation:

$$g(p) = \max_{i} \left\{ \frac{(A-i)^{1-\sigma}}{1-\sigma} + \beta \frac{(1-\delta+i)^{1-\sigma}(1-p+p(1-b_k)^{1-\sigma}}{1-\sigma} \mathbb{E}_{p'} g(p') \right\}, \tag{786}$$

where $i = \frac{I}{K}$ is the investment rate. This implies that consumption and investment are both proportional to the current stock of capital, but they typically depend on the probability of disaster as well:

$$\begin{aligned} C_t &= f(p_t) K_t, \\ I_t &= h(p_t) K_t. \end{aligned}$$

As a result, when a disaster occurs and the capital stock falls by a factor b_k, both consumption and investment also fall by a factor b_k. Given that there are no adjustment costs, the value of capital is equal to the quantity of capital, and hence it falls also by a factor b_k in a downturn. Finally, the return on an all-equity financed firm is:

$$R^e_{t,t+1} = (1 - \delta + A)(1 - x_{t+1} b_k).$$

In other words, it is $1 - \delta + A$ if there is no disaster, and $(1 - \delta + A)(1 - b_k)$ if there is a disaster. Clearly, the equity premium will be high, since the equity return and consumption are correlated and are affected by large shocks. Moreover, the equity premium is larger when p_t is higher, since risk is higher.

Finally, consider the effect of p on the consumption-savings decision. Using (786), the FOC with respect to i yields, after rearranging:

$$\left(\frac{A - i}{1 - \delta + i}\right)^{-\sigma} = \beta(1 - p + p(1 - b_k)^{1-\sigma})\mathbb{E}_{p'} g(p').$$

Given that p is IID, the expectation of g on the RHS is independent of the current p. The LHS is an increasing function of i. The term $(1 - b_k)^{1-\sigma}$ is greater than unity if and only if $\sigma > 1$. Hence, i is increasing in p if $\sigma > 1$ (save more), it is decreasing in p if $\sigma < 1$ (save less), and it is independent of p if $\sigma = 1$ (income and substitution effects cancel as we have log-utility).

The intuition is as follows. If p goes up, investment in physical capital becomes more risky and hence less attractive, i.e., the risk adjusted return[120] goes down. The effect of a change in the return on the consumption-savings choice depends on the value of the IES, because of offsetting income and substitution effects. The if the IES is unity (log utility), savings are unchanged and thus the savings or investment rate does not respond to a change in the probability of disaster. But if the IES is larger than unity, $\sigma < 1$, the substitution effect dominates, and i is decreasing in p. Hence, an increase in the probability of disaster leads initially, in this model, to a decrease in investment, and an increase in consumption, since output is unchanged on impact. Next period, the decrease in investment leads to decrease in the capital stock and hence in output. This simple analytical example thus shows that a change in the perceived probability of disaster can lead to a decline in investment and output. The key mechanism is the effect of rate-of-return uncertainty on the optimal savings decision.

[120] By risk adjusted return we mean $\mathbb{E}(R^{1-\sigma})^{\frac{1}{1-\sigma}}$, where R is the physical return on capital.

16.9.2 Extension to Epstein-Zin preferences

To illuminate the respective role of risk aversion and the IES, it is useful to extend the preceding example to the case of Epstein-Zin utility. Assume, then, that the utility V_t satisfies the recursion:

$$V_t = \left((1-\beta) C_t^{1-\rho} + \beta \left(\mathbb{E}_t V_{t+1}^{1-\sigma} \right)^{\frac{1-\rho}{1-\sigma}} \right)^{\frac{1}{1-\rho}}, \tag{787}$$

where σ measures risk aversion towards static gambles, ρ is the inverse of the IES, and β reflects time preference. It is straightforward to extend the results above; the FOC now reads:

$$\left(\frac{A-i}{1-\delta-i} \right)^{-\rho} = \frac{\beta}{1-\beta} \left(1-p+p(1-b_k)^{1-\sigma} \right)^{\frac{1-\rho}{1-\sigma}} \left(\mathbb{E}_{p'} g(p')^{\frac{1-\sigma}{1-\rho}} \right)^{\frac{1-\rho}{1-\sigma}},$$

and we can apply the same argument as above, in the realistic case where $\sigma \geq 1$: the now risk-adjusted return on capital is $(1-p+p(1-b_k)^{1-\sigma})^{\frac{1}{1-\sigma}}$; it falls as p rises; an increase in the probability of disaster will hence reduce investment if and only if the IES is larger than unity. Hence, the parameter which determines the sign of the response is the IES, and the risk aversion coefficient (as long as it is greater than unity) determines the magnitude of the response only. While this example is revealing, it has a number of simplifying features.

16.9.3 An RBC model with time-varying risk of disasters

This section briefly introduces Gourio's RBC model with time-varying risk of disaster and studies its implications. The model extends the simple example of the previous section by: a) the probability of disaster is persistent instead of IID; b) the production function is neoclassical and affected by standard TFP shocks; c) labour is elastically supplied; d) disasters may affect total factor productivity as well as capital; and, e) there can be capital adjustment costs.

The representative consumer has Epstein-Zin preferences, and the utility index incorporates hours worked N_t as well as consumption C_t:

$$V_t = \left(u(C_t, N_t)^{1-\rho} + \beta \mathbb{E}_t \left(V_{t+1}^{1-\sigma} \right)^{\frac{1-\rho}{1-\sigma}} \right)^{\frac{1}{1-\rho}}, \tag{788}$$

where[121] the per period utility $u(C, N)$ is assumed to have the following form:

$$u(C, N) = C^v (1 - N)^{1-v}.$$

Note that ρ is the inverse of the IES, and σ measures risk aversion towards static gambles, because u is HOD1. But this is risk aversion over the bundle of consumption and leisure.

There is a representative firm, which produces output using a standard Cobb-Douglas technology:

$$Y_t = K_t^\alpha (z_t N_t)^{1-\alpha},$$

where z_t is total factor productivity (TFP), to be described below. The firm accumulates capital subject to adjustment costs:

$$K_{t+1} = \begin{cases} (1-\delta)K_t + \phi\left(\frac{I_t}{K_t}\right) K_t, & \text{w.p. } 1 - p_t, \\ \left[(1-\delta)K_t + \phi\left(\frac{I_t}{K_t}\right) K_t\right](1 - b_k), & \text{w.p. } p_t, \end{cases}$$

where $\phi(\cdot)$ is an increasing and concave function, with curvature that captures adjustment costs, and $x_{t+1} = 1$ if there is a disaster at time $t + 1$ (with probability p_t) and 0 otherwise (probability $1 - p_t$). At this stage b_k is a parameter, which could be zero – i.e., a disaster only affects TFP.

The aggregate resource constraint is:

$$C_t + I_t \leq Y_t.$$

Aggregate investment cannot be negative:

$$I_t \geq 0.$$

Finally, we describe the shock processes. TFP is affected by the "normal shocks" ϵ_t as well as the

[121] It's common to have a $(1 - \beta)$ factor in front of the period t utility function, but this is merely a normalisation, which it is useful to forgo in this case.

disasters. A disaster reduces TFP by a permanent amount b_{tfp}:

$$\log z_{t+1} = \begin{cases} \log z_t + \mu + \gamma \epsilon_{t+1}, & \text{w.p. } 1 - p_t, \\ \log z_t + \mu + \gamma \epsilon_{t+1} + \log(1 - b_{tfp}) & \text{w.p. } p_t, \end{cases}$$

where μ is the drift of TFP, and γ is the standard deviation of "normal shocks". For simplicity assume that p_t follows a stationary Markov process with transition function Q (in the numerical simulations, Gourio assumes an AR(1) process for p_t).

We assume that p_{t+1}, ϵ_{t+1}, and x_{t+1} are independent conditional on p_t. This assumption requires that the occurrence of a disaster today does not affect the probability of a disaster tomorrow. This assumption could be wrong either way: A disaster today may indicate that the economy is entering a phase of low growth or is less resilient than though, leading agents to revise upward the probability of disaster, following the occurrence of a disaster; on the other hand, if a disaster occurred today, and capital or TFP fell by a large amount, it is unlikely that they will fall again by a large amount next year. Rather, historical evidence suggests that the economy is likely to grow above trend for a while (Gourio 2008; Barro et al. 2013).

The model has three states: capital K, technology z, and the probability of disaster p; two independent controls: consumption C and labour supply N; and, three shocks: the realisation of disaster $x^{'} \in \{0, 1\}$, the new probability of disaster p', and the "normal shock" ϵ'. The first welfare theorem holds, hence the competitive equilibrium is equivalent to a social planner problem, which is easier to solve. Denote $V(K, z, p)$ as the value function, and define $W(K, z, p) = V(K, z, p)^{1-\rho}$. The Ramsey social planner's problem can be formulated as:[122]

$$W(K, z, p) = \max_{C,I,N} \left\{ \left(C^{\upsilon}(1 - N)^{1-\upsilon}\right)^{1-\rho} + \beta \left(\mathbb{E}_{p',z',x'} W(K', z', p')^{\frac{1-\sigma}{1-\rho}}\right)^{\frac{1-\rho}{1-\sigma}} \right\}, \tag{789}$$

[122] Because we take a power of $1 - \rho$ of the value function, if $\rho > 1$, the max must be transformed into a min.

subject to:

$$
\begin{aligned}
C + I &\leq z^{1-\alpha} K^{\alpha} N^{1-\alpha}, \\
K' &= \left[(1-\delta)K + \phi\left(\frac{I}{K}\right) K \right] (1 - x' b_k), \\
\log z' &= \log z + \mu + \gamma \epsilon' + x' \log(1 - b_{tfp}).
\end{aligned}
$$

A standard homogeneity argument implies that we can write $W(K, z, p) = z^{\upsilon(1-\rho)} g(k, p)$, where $k = \frac{K}{z}$, and g satisfies the associated Bellman equation:

$$
g(k,p) = \max_{c,i,N} \left\{ \begin{array}{c} c^{\upsilon(1-\rho)}(1-N)^{(1-\upsilon)(1-\rho)} \\ +\beta \exp(\mu\upsilon(1-\rho)) \left[\mathbb{E}_{p',\epsilon',x'} \exp\left(\gamma\epsilon'\upsilon(1-\sigma)\right) \left(1 - x' + x'(1-b_{tfp})^{\upsilon(1-\sigma)}\right) g(k',p')^{\frac{1-\sigma}{1-\rho}} \right]^{\frac{1-\rho}{1-\sigma}} \end{array} \right\}, \tag{790}
$$

subject to:

$$
\begin{aligned}
c &= k^{\alpha} N^{1-\alpha} - i, \\
k' &= \frac{(1 - x' b_k)\left((1-\delta)k + \phi\left(\frac{i}{k}\right) k\right)}{\exp\left\{\mu + \gamma\epsilon'\right\} (1 - x' b_{tfp})}.
\end{aligned}
$$

Here $c = \frac{C}{z}$ and $i = \frac{I}{z}$ are consumption and investment detrended by the stochastic technology trend z. This simplification will leads to analytical results in the Gourio paper, and can be further be studied using standard numerical methods since k is stationary.

16.9.4 Asset prices in the RBC model

It is straightforward to compute asset prices in this economy. The stochastic discount factor is given by the formula:

$$
M_{t,t+1} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{\upsilon(1-\rho)-1} \left(\frac{1-N_{t+1}}{1-N_t}\right)^{(1-\upsilon)(1-\rho)} \left(\frac{V_{t+1}}{\mathbb{E}_t\left[V_{t+1}^{1-\sigma}\right]^{\frac{1}{1-\sigma}}}\right)^{\rho-\sigma}. \tag{791}
$$

The price of a one-period risk-free bond is:

$$P_{rf,t} = \mathbb{E}_t M_{t,t+1} \equiv P_{rf}(k,p).$$

This risk-free asset may not have an observable counterpart. Following Barro (2006), Gourio assumes that government bonds are not risk-free but are subject to default risk during disasters. More precisely, if there is a disaster, then with probably q the bonds will default and the recovery rate will be r. The T-Bill price can then be easily computed as:

$$P_{1,t} = \mathbb{E}_t \left[M_{t,t+1}(1 - x_{t+1}q(1-r)) \right] \equiv P_1(k,p).$$

The ex-dividend value of the firm assets F_t is defined through the value recursion:

$$F_t = \mathbb{E}_t \left[M_{t,t+1}(D_{t+1} + F_{t+1}) \right],$$

where $D_t = F(K_t, z_t N_t) - w_t N_t - I_t$ is the payout of the representative firm, and w_t is the wage rate, given by the marginal rate of substitution of the representative consumer between consumption and leisure. The equity return is then:

$$R_{t,t+1} = \frac{D_{t+1} + F_{t+1}}{F_t}.$$

There is an alternative derivation of firm value and returns. Using the Q-theory, we see that:

$$F_t = \frac{(1-\delta)K_t + \phi\left(\frac{I_t}{K_t}\right)K_t}{\phi'\left(\frac{I_t}{K_t}\right)},$$

where $(1-\delta)K_t + \phi\left(\frac{I_t}{K_t}\right)K_t = \frac{K_{t+1}}{1-x_{t+1}b_k}$ is the capital if no disaster occurs. In the standard model, $p_t = 0$, but here the amount of capital available tomorrow is unknown, since some capital is destroyed in the event of a disaster. As a result, we can find an equivalent expression for the equity return, often known as the investment return, which holds as long as investment is positive:

$$
\begin{aligned}
R_{t,t+1} &= \frac{F_{t+1} + D_{t+1}}{F_t} \\
&= \frac{\frac{(1-\delta)K_{t+1} + \phi(I_{t+1}/K_{t+1})K_{t+1}}{\phi'(I_{t+1}/K_{t+1})} + D_{t+1}}{\frac{(1-\delta)K_t + \phi(I_t/K_t)K_t}{\phi'(I_t/K_t)}} \\
&= (1 - x_{t+1}b_k)\phi'\left(\frac{I_t}{K_t}\right)\left[\frac{1 - \delta + \phi\left(\frac{I_{t+1}}{K_{t+1}}\right)}{\phi'\left(\frac{I_{t+1}}{K_{t+1}}\right)} + \frac{\alpha K_{t+1}^{\alpha} z_{t+1}^{1-\alpha} N_{t+1}^{1-\alpha} - I_{t+1}}{K_{t+1}}\right].
\end{aligned}
$$

This expression is similar to that in Jermann (1998) or Kaltenbrunner and Lochstoer (2010), but for the presence of the term $(1 - x_{t+1}b_k)$, which reflects the capital destruction following a disaster.

Finally, following Abel (1999), Gourio computes the price of a leveraged claim on consumption, defined by its payoff C_t^{λ}, where λ is a leverage parameter. The motivation is that the dividend process implied by the model does not match well the dividend process in the data. In the real world, firms have financial leverage and operating leverage (e.g. fixed costs and labour contracts). This is a substantial source of profit volatility, which is not present in the model. Under some conditions, the only effect of this leverage is to modify the payout process.

16.9.5 Some key analytical results

Following equation (790), Gourio establishes two simple, yet important analytical results:

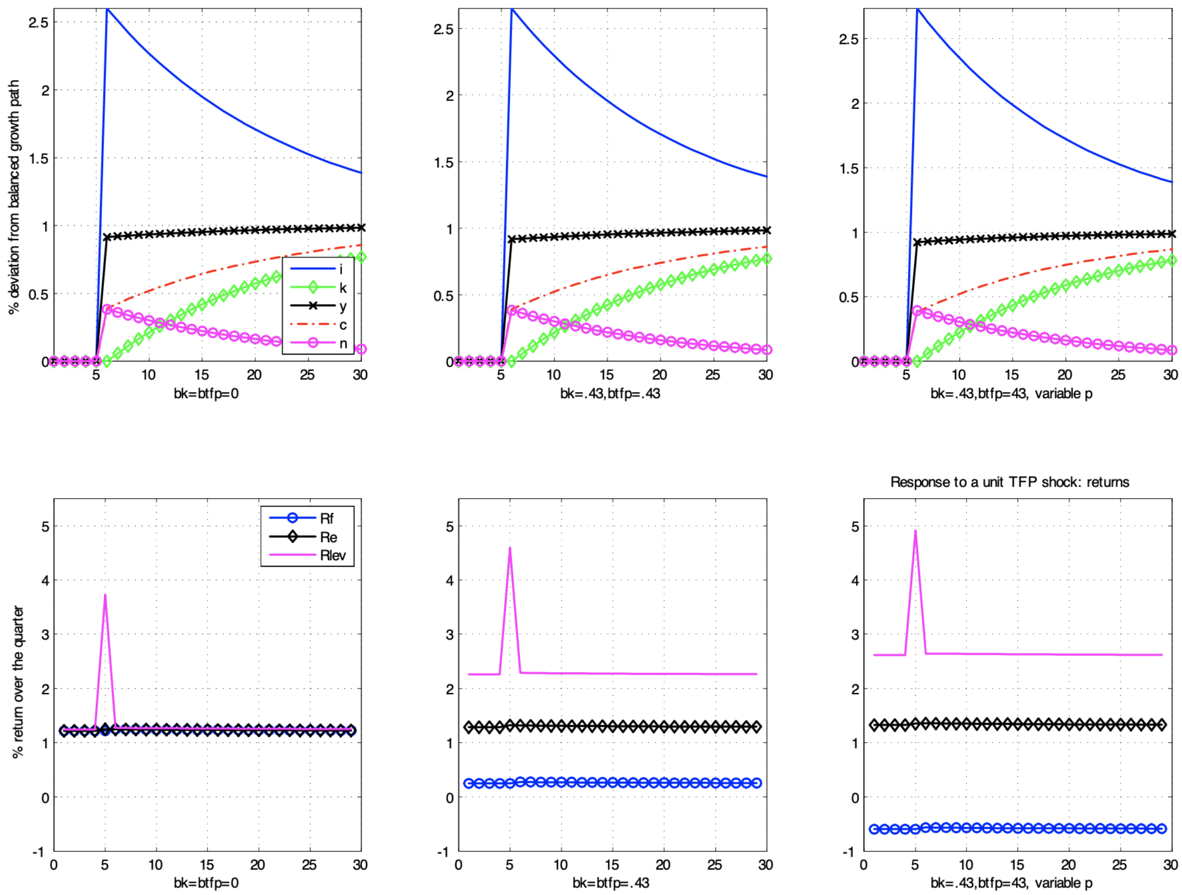
- **Proposition 1**: Assume that the probability of a disaster p is constant, and that $b_k = b_{tfp}$ – i.e., productivity and capital fall by the same amount if there is a disaster. Then, in a sample without disasters ,the quantities implied by the model (consumption, investment, hours, output, and capital) are the same as those implied by a model with no disasters ($p = 0$), but a different time discount factor $\beta^* = \beta(1 - p + p(1 - b_k)^{\upsilon(1-\sigma)})^{\frac{1-\rho}{1-\sigma}}$. Assuming $\sigma \geq 1$, we have $\beta^* \leq \beta$ if and only if $\rho < 1$. asset prices, however, will be different under the two models; in particular, let $\bar{R}$ be the gross return on equity in normal times, and let $\bar{d}$ be the dividend-capital ratio, then in a disaster, the return is $\bar{R}(1 - b_k) + b_k\bar{d}$, which is low, leading to a large equity premium.

- **Proposition 2**: Assume still that $b_k = b_{tfp}$, but that p follows a stationary Markov process.

> Then, in a sample without disaster, the quantities implied by the model are the same as those implied by a model with no disasters, but with stochastic discounting (i.e., β follows a stationary Markov process).

We won't cover the proofs here – they're quite accessible in Gourio's paper – but we can discuss the two results, starting with the first. The result is in the spirit of Tallarini (2000): fixing the asset pricing properties of a RBC model need not change the quantity dynamics. An economy with a high equity risk premium due to disasters ($p > 0$) is observationally equivalent to the standard stochastic growth model ($p = 0$), with a different β. One possible calibration of the model without disasters (e.g. Cooley and Prescott (1995)) is to pick β to match the observed return on stocks. This calibration would pick β^* and hence yield exactly the same implication as the model with disasters. Without the adjustment of β, the quantity implications are very slightly different. This is illustrated in the top panel of Figure 127 which depicts the IRFs of quantities to a TFP shock in three models: a) the model with $p = 0$, b) the model with constant positive p, and c) the benchmark calibration with time varying p. The differences can be seen in the scale (y-axis), but they are tiny. For this calibration, we have $\beta = 0.993$ and $\beta^* = 0.9924$.

Figure 127: Response to a Unit TFP Shock: Quantities



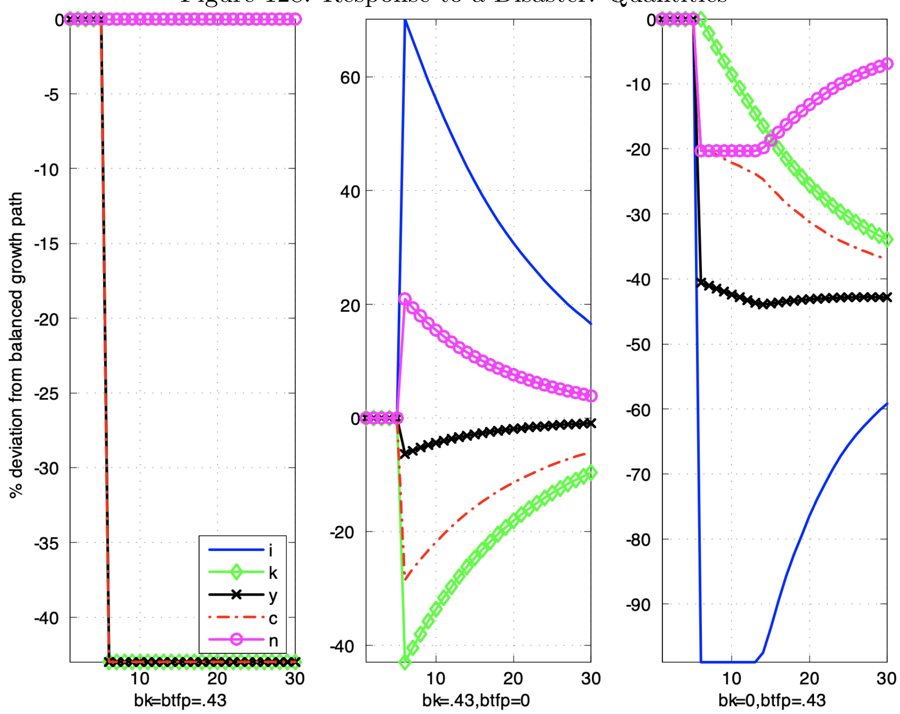
Observational equivalence of quantity dynamics to a TFP shock. The figure plots the IRFs of quantities (C, I, N, K, Y) and returns (risk-free rate, equity return, levered equity) to a permanent TFP shock at $t = 6$. Left panel: model without disasters. Middle panel: model with constant probability of disasters. Right panel: model with time-varying probability of disaster (benchmark). All other parameters (including β) are kept constant across the three panels.

Of course, asset prices will be different, and in particular the equity premium will be higher, as seen in the bottom panel of Figure 127 – the average returns are very different across the three models. The observational equivalence would also be broken in a long enough sample since disasters must occur.

The assumption $b_k = b_{tfp}$ simplifies the analysis substantially: the steady state of the economy shifts due to a change in z, but the ratio of capital to productivity is unaffected by the disaster, i.e., the economy is in the same position relative to its steady state after the disaster and before the disaster.

As a result, a disaster will simply reduce investment, output, and consumption by a factor $b_k = b_{tfp}$, and hours will be unaffected. The economy jumps from one steady state to another steady state, and there no further transitional dynamics. Obviously, the possibility of disaster affects the choice of how much to save, and hence it changes β, but the response to a standard TFP shock is not affected. As emphasised by **Cochrane2005**, in a RBC model there is little that agents can do to increase or decrease the amount of uncertainty that they face.

Figure 128: Response to a Disaster: Quantities



Different types of disasters. Response of quantities (C, I, K, N, Y) to a disaster at $t = 6$, in percent deviation from balanced growth path. left panel: $b_k = b_{tfp}$; middle panel: $b_k = 0.43, b_{tfp} = 0$; and right panel: $b_k = 0, b_{tfp} = 0.43$.

If risk aversion σ is greater than unity, and IES is above unity, then $\beta^* < \beta$, leading people to save less and the steady state capital stock is lower than in a model without disasters.

While this first result is interesting, it is not fully satisfactory, however, since the constant probability of disaster implies (nearly) constant risk premia, and hence P-D ratios are too smooth, and returns not volatile enough. This motivates the second proposition fora time-varying p.

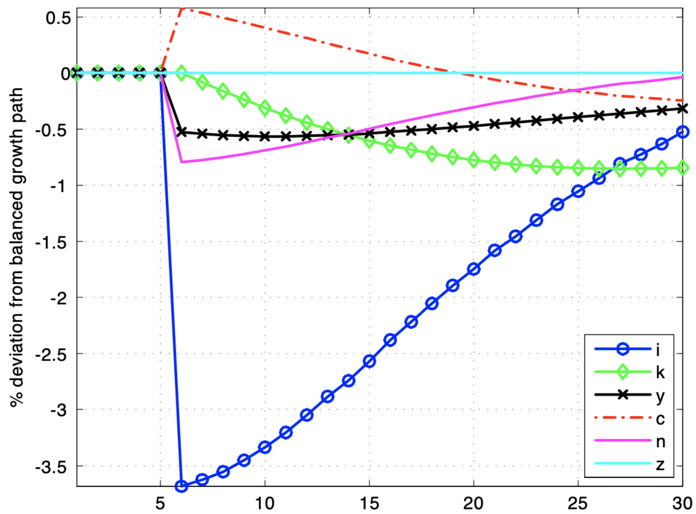
The second proposition shows that the time-varying risk of disaster has the same implications for quantities as a preference shock. It is well known that these shocks have significant effects on macroeconomic quantities. In a sense, this version of the model breaks the "separation theorem" of Tallarini (2000): the source of time-varying risk premia in the model will affect quantity dynamics.

This result is interesting in light of the empirical literature which suggests that "preference shocks" or "equity premium shocks" may be important (Smets and Wouters (2003) and the many papers that followed). Chari et al. (2009) complain that these shocks lack microfoundations. Gourio's model provides a simple microfoundation, which allows to tie these shocks to asset prices precisely. Of course, the model is much "smaller" than the medium-scale New Keynesian models of Smets and Wouters (2003) or Christiano et al. (2005).

Interestingly, this suggests that it is technically feasible to make DSGE models consistent with risk premia. A full non-linear solution of a medium-scale DSGE model is daunting. But under this result, we can solve the quantities of the model for $p = 0$ and a shock process for β, which we know is well approximated in a log-linear approximation.

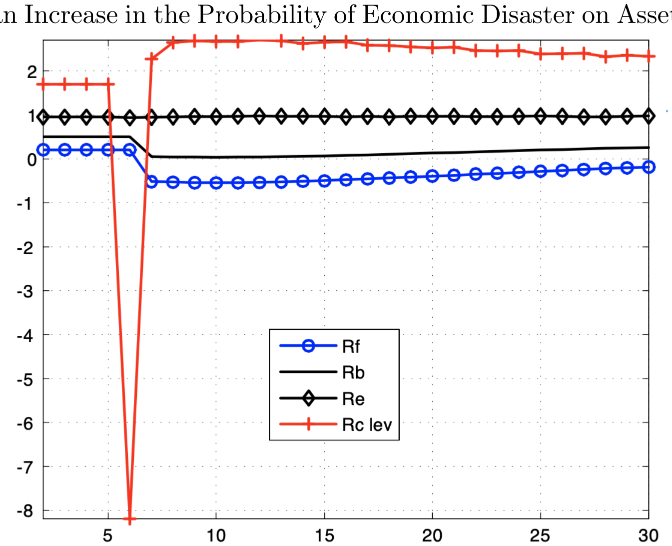
Note that Propositions 1 and 2 require that $b_k = b_{tfp}$; analytical results are impossible without this assumption. Numerical results suggest that result 1 is robust to this assumption, in that the dynamic response to a TFP shock is largely unaffected by the presence of disasters. Result 2, however, relies on this assumption more heavily. If disasters affect only TFP, then an increase in p will lead people to want to hold more capital, for standard precautionary savings reasons. This is true regardless of the IES.

Figure 129: Effect of an Increase in the Probability of Economic Disaster on Quantities



IRFs to a shock to the probability of disaster at $t = 6$. The probability of disaster goes from its long-run average (0.425% per quarter) to twice its long-run average then mean-reverts according to its AR(1) law of motion. For clarity, this figure assumes that there is no shock to TFP, and no disaster realised.

Figure 130: Effect of an Increase in the Probability of Economic Disaster on Asset Returns and Spreads



IRFs of asset returns to a shock to the probability of disaster at $t = 6$. The probability of disaster doubles at $t = 6$, starting from its long-run average. The figure plots the risk-free rate, the short-term government bond return, the equity return, and the levered equity return.

16.10 Comments and key readings

We have shown how the neoclassical model links consumption (not output) and rates of return on different assets and how particular importance is placed on risk and covariance. These are extremely elegant theories which have been widely used in the finance literature. However, as was the case for the neoclassical model's inability to explain non-financial variables, the model fails on a number of important empirical dimensions. Understanding these failures is the subject of much research but as yet no clear consensus regarding how to proceed has been achieved.

Proposed explanations include incomplete markets and transaction costs (Mankiw 1986; Mankiw and Zeldes 1991; Heaton and D. Lucas 1996; and Luttmer 1999); habit formation (Constantinides 1990; Campbell and Cochrane 1999); non-expected utility (Weil 1989; Epstein and Zin 1991; Bekaert et al. 1997); concern about equity returns for other reasons other than just their implications for consumption (Benartzi and Thaler 1995; Barberis et al. 2001); gradual adjustment of consumption (Gabaix and Laibson 2002; Parker 2001); and a small probability of catastrophic decline in consumption and equity prices (Barro 2006; Gourio 2012; Gourio 2013).

Since initially writing these notes, I've added notes on Epstein-Zin preferences and disaster risk. Both these mechanisms have become quite popular in the macro-finance literature, and together, can go a long way in explaining the equity premium puzzle. It should be noted that the application of EZ preferences are much wider than simply explaining the EPP.

Further, while it may be tempting to simply declare the EPP as being solved with EZ preferences and disaster risk, there is one obvious limitation to disaster risk which Gourio himself points out: the probability of disaster is hard to observe. So it's somewhat analogous to the identification of instrumental variables in econometrics. Gourio's paper goes into much more detail than what we've shown here, and I recommend you read his paper if possible. It's actually quite accessible, and the mathematics are fairly straightforward, so long as you're familiar with recursive dynamics notation.

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17 Financial Frictions

17.1 Introduction

Previously, we saw that the standard representative agent macroeconomic model runs into problems when trying to explain the simultaneous existence of a high equity premium and a low risk free rate – the Equity Premium Puzzle. There have been many attempts to amend the representative model to rectify these problems. In this chapter we focus on the role of financial intermediaries and financial frictions. Compared to the standard representative agent model where financial intermediation happens costlessly and perfectly, we will be interested in models where there are imperfections in the efficiency with which financial intermediaries channel funds from savers to investors. Whilst the models are different in terms of the mechanisms they highlight, they all stress the importance of the evolution and holdings of net worth in the economy. This is important when we want to have interactions between the macroeconomy and financial markets. In the representative agent models we typically only have causality from the macroeconomy to financial variables, but in models with financial frictions we have an important feedback from financial variables to the macroeconomy. By the end of this chapter we will have explored various mechanisms whereby a fall in asset values leads to a drop in financial intermediation and jump in interest rate spreads.

As an aside, we only formally covered four models in class: Gertler and Karadi (2011)/Gertler and Kiyotaki (2010), Christiano and Ikeda (2016), Mankiw (1986), and Bernanke, Gertler, and Gilchrist (1999). However, in addition to these four models – and thanks to a set of amazing notes by Eric Sims[123] – in this chapter we will also look at Iacoviello (2005) and Kiyotaki and Moore (1997). In addition we will expand our initial coverage of Bernanke, Gertler, and Gilchrist (1999). The reason being is that these models introduce some well-used financial frictions in the context of a New Keynesian DSGE framework – well, Kiyotaki and Moore (1997) is a bit more abstract, and needn't be restricted to New Keynesian models, but it is probably the most important paper in the financial frictions/macro-finance literature, and will undoubtedly earn the authors a Nobel prize in due course. As such, this chapter is a bit more hefty than usual.

[123] I encourage anyone interested to read Sims' notes. They're brilliant. My only contribution here is to clean up some expressions, typos, and notational quirks.

17.2 Technical aside: The log-normal distribution and Leibniz Rule

Many papers in the financial frictions literature make use of the log-normal distribution. It's worth going over some properties of the distribution which will come in handy.

As Eric Sims also mentions, I'm not very good with probability and statistical theory – and I suspect most macroeconomists aren't either. Integrals are always daunting, and I'm only good with handling them in certain scenarios (like in the New Keynesian model). I'm sure we know that integrals are just sums over a continuous variable, but let's just briefly review continuous and discrete random variables.

17.2.1 Continuous random variables

Let x be some random variable. For now, suppose its support is $(-\infty, \infty)$, so it's continuous over negative infinity to positive infinity. Let the density (or, probability distribution function (PDF)) be given by $f(x)$. Let the cumulative distribution function (CDF) – which we will refer to as simply the distribution – be $F(x)$. Provided that the distribution is differentiable, we have:

$$f(x) = F'(x).$$

In words, the density is the first derivative of the distribution. The distribution measures the probability that the random variable is less than or equal to some cutoff value, $\bar{x}$. This is given by:

$$F(\bar{x}) = \int_{-\infty}^{\bar{x}} f(x)dx.$$

Although, that's not precisely correct in the continuous case. We can think of the density evaluated at a point, $\bar{x}$, as giving the probability that the random variable equals that realisation: $f(\bar{x}) = \Pr(x = \bar{x})$. This isn't quite right in the continuous case because the probability of any single realisation is zero – remember, across the real number line, any point is infinitesimally small. This will be clearer in the discrete case. So the distribution is effectively just the sum of the probabilities that x takes on possible values up to $\bar{x}$. Since x must take on some value, we have:

$$\int_{-\infty}^{\infty} f(x)dx = 1,$$

this means that:

$$1 - F(\bar{x}) = \int_{\bar{x}}^{\infty} f(x)dx.$$

The unconditional expectation of the continuous random variable is:

$$\mathbb{E}(x) = \int_{-\infty}^{\infty} xf(x)dx.$$

Again, this isn't precisely right in the continuous case, but this is essentially the probability-weighted sum of all possible realisations of the random variable x. The partial expectation of the random variable is defined as:

$$\begin{aligned} g(\bar{x}) &= \int_{-\infty}^{\bar{x}} xf(x)dx \\ &= \mathbb{E}[x|x \leq \bar{x}] \Pr(x \leq \bar{x}) \\ &= \mathbb{E}[x|x \leq \bar{x}]F(\bar{x}). \end{aligned}$$

Hence, we can write the conditional expectation as the partial expectation divided by the probability that x is in a region, which is given by the distribution function:

$$\mathbb{E}[x|x \leq \bar{x}] = g(\bar{x})F(\bar{x})^{-1}.$$

17.2.2 Discrete random variables

Discrete random variables are lot easier for macroeconomists to work with – probably since most of the models we've worked with up until now have been in discrete time. Let's consider a particularly simple discrete distribution: Uniform over the support $[1, 10]$, or $x \sim U(1, 10)$. The probability mass function (technically, we should use the term mass not density for the discrete case, but it is functionally the same thing) is just the inverse of the number of potential realisations. Letting there be n possible realisations, we would have:

$$f(x) = \frac{1}{n}.$$

The unconditional expectation is just the probability weighted sum of potential outcomes. For this particular example:

$$\begin{aligned}
\mathbb{E}[x] &= \sum_{x=1}^{n} x f(x) \\
&= \frac{1}{n} \sum_{x=1}^{n} x \\
&= \frac{1}{10} \sum_{x=1}^{10} x \\
&= \frac{1}{10}(1 + 2 + 3 + ... + 10) \\
&= 5.5.
\end{aligned}$$

The cumulative distribution is just the probability of $x \leq \bar{x}$. So, for example, we'd have for this distribution:

$$F(5) = \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} + \frac{1}{10} = \frac{1}{2}.$$

In other words, there is a 50 percent chance of drawing 5 or less from a $U(1, 10)$ discrete distribution. Makes sense.

The partial expectation of $x \leq 5$ is:

$$\begin{aligned}
g(5) &= \frac{1}{10} \sum_{x=1}^{5} x \\
&= \frac{1}{10}(1 + 2 + 3 + 4 + 5) = \frac{3}{2},
\end{aligned}$$

but the conditional expectation effectively re-weights the probabilities – if you condition on knowing

$x \leq 5$, there is a 20 percent chance of each realisation, not a 10 percent chance. So:

$$\begin{aligned}\mathbb{E}[x|x \leq 5] &= \frac{1}{5}\sum x_{x=1}^{5} \\ &= \frac{1}{5}(1+2+3+4+5) = 3.\end{aligned}$$

As above in the continuous case, we can say that the conditional expectation is just the partial expectation divided by the distribution function:

$$\mathbb{E}[x|x \leq 5] = \frac{g(5)}{F(5)} = \frac{3}{2}\frac{2}{1} = 3.$$

17.2.3 The log-normal

Let ω be a random variable. We assume that:

$$\ln \omega \sim N(\mu, \sigma^2).$$

Note that the support for ω must be $(0, \infty)$, since you can't take the log of something negative. $N(\cdot)$ is the normal distribution, μ is the mean, and σ^2 is the variance.

Let $\Phi(\cdot)$ and $\phi(\cdot)$ be the CDF and PDF for a standard normal distribution (i.e., $N(0,1)$). Then, we have the CDF and PDF of the log-normal random variable satisfying:

$$\begin{aligned}F(\omega) &= \Phi\left(\frac{\ln \omega - \mu}{\sigma}\right), \\ f(\omega) = F'(\omega) &= \phi\left(\frac{\ln \omega - \mu}{\sigma}\right)\frac{1}{\omega\sigma}.\end{aligned}$$

Note that $\phi(\cdot) = \Phi'(\cdot)$; the multiplication by the inverse of $\omega\sigma$ is effectively the "derivative of the inside" part of the chain rule. Note that the log-normal density is given by:

$$f(\omega) = \frac{1}{\omega}\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(\ln \omega - \mu)^2}{2\sigma^2}\right).$$

As above, the CDF, $F(\cdot)$, is just the probability that $\omega \leq \bar{\omega}$, for some $\bar{\omega}$. That is:

$$F(\bar{\omega}) = \int_0^{\bar{\omega}} f(\omega) d\omega.$$

The expected value satisfies:

$$\mathbb{E}[\omega] = \int_0^{\infty} \omega f(\omega) d\omega.$$

Again, as in the discrete case, this is just the weighted average realisations, ω, times the probabilities, $f(\omega)$. For this particular distribution, the mean works out to:

$$\mathbb{E}[\omega] = \exp\left(\mu + \frac{1}{\sigma^2}\right).$$

For most of the applications we deal with, we will need $\mathbb{E}[\omega] = 1$. This means that $\mu + \frac{1}{2}\sigma^2 = 0$, so we have:

$$\mu = -\frac{1}{2}\sigma^2.$$

Once again, we might be interested in the partial expectation – i.e., the expected value of ω conditional on being in some region, times the probability of being in that region:

$$\begin{aligned} g(\bar{\omega}) &= \mathbb{E}[\omega | \omega \leq \bar{\omega}] \Pr(\omega < \bar{\omega}) \\ &= \int_0^{\bar{\omega}} \omega f(\omega) d\omega. \end{aligned}$$

In words, this is the expected value of ω, conditional on $\omega \leq \bar{\omega}$, times the probability that $\omega \leq \bar{\omega}$. For the log-normal distribution where $\mathbb{E}[\omega] = 1$, this works out to be:

$$\int_0^{\bar{\omega}} \omega f(\omega) d\omega = \Phi\left(\frac{\ln \bar{\omega} - \mu - \sigma^2}{\sigma}\right),$$

where, again, $\Phi(\cdot)$, is the CDF of a normal distribution. Similarly, we have:

$$\int_{\bar{\omega}}^{\infty} \omega f(\omega) d\omega = \Phi\left(\frac{\mu + \sigma^2 - \ln \bar{\omega}}{\sigma}\right).$$

17.2.4 Leibniz Rule and differentiating with respect to an integral bound

There will be some instances in this literature where we are interested in some function of a cutoff value, $\bar{\omega}$, where this cutoff value appears as an integral bound. For example, suppose we are interested in differentiating the partial expectation with respect to $\bar{\omega}$:

$$g(\bar{\omega}) = \int_{\bar{\omega}}^{\infty} \omega f(\omega) d\omega.$$

Most of us are not used to differentiating with respect to an integral bound – we kind of intuitively know that we move a derivative through an integral (since an integral is effectively a sum, and the derivative is a linear operator), but we get scared when we see the variable we are differentiating with respect to not inside the integral but rather as one of the bounds. Well, we can use Leibniz Rule. In the general form it looks scary. Suppose we have:

$$\int_{a(x)}^{b(x)} f(x,z) dz.$$

The derivative of this with respect to x is:

$$\frac{d}{dx}\left[\int_{a(x)}^{b(x)} f(x,z) dz\right] = f(x,b(x))b'(x) - f(x,a(x))a'(x) + \int_{a(x)}^{b(x)} f_x(x,z) dz.$$

In words, this is the function evaluated at the upper bound, times the derivative of the upper bound with respect to x; minus the function evaluated at the lower bound, evaluated at the lower bound, times the derivative of the lower bound with respect to x; plus the integral of the derivative of the function inside the integral.

This may look very confusing. But you have functionally probably used this rule many times in your life when the bounds of the integral are constants and you simply differentiate inside the integral. Let's take a look at the following equation again:

$$g(\bar{\omega}) = \int_{\bar{\omega}}^{\infty} \omega f(\omega) d\omega.$$

What is $g'(\bar{\omega})$? Using the general formula, we have:

$$g'(\bar{\omega}) = f(\infty)\infty\frac{d\infty}{d\bar{\omega}} - f(\bar{\omega})\bar{\omega}\frac{d\bar{\omega}}{d\bar{\omega}} + \int_{\bar{\omega}}^{\infty}(\omega f'(\omega) + f(\omega))\frac{d\omega}{d\bar{\omega}}d\omega.$$

This looks messy. But it's not. Why? Because ω and $\bar{\omega}$ are different – one has nothing to do with the other in some sense. So $d\infty/d\bar{\omega} = 0$, $d\bar{\omega}/d\bar{\omega} = 1$, and $d\omega/\bar{\omega} = 0$. But this means that only the middle term is left:

$$\begin{aligned} g'(\bar{\omega}) &= -\bar{\omega}f(\bar{\omega}) \\ &= -\bar{\omega}F'(\bar{\omega}). \end{aligned}$$

Via similar logic, suppose we were interested with a different partial expectation, say:

$$h(\bar{\omega}) = \int_{0}^{\bar{\omega}} \omega f(\omega)d\omega.$$

We would then have:

$$h'(\bar{\omega}) = f(\bar{\omega})\bar{\omega}\frac{d\bar{\omega}}{d\bar{\omega}} - f(0)0\frac{d0}{d\bar{\omega}} + \int_{0}^{\bar{\omega}}(\omega f'(\omega) + f(\omega))\frac{d\omega}{d\bar{\omega}}d\omega,$$

which is just:

$$\begin{aligned} h'(\bar{\omega}) &= \bar{\omega}f(\bar{\omega}) \\ &= \bar{\omega}F'(\bar{\omega}). \end{aligned}$$

To get some intuition for this, return to the discrete uniform distribution, and remember that a derivative is basically just the change. This won't be exact give the discrete nature and the fact that derivatives are relevant for small changes and continuous variables, but it'll give us an idea. Suppose we have a uniform distribution over 1 to 5. Suppose we are interested in the partial expectation from

2 to 5:

$$g(2) = \sum x = 2^5 \frac{x}{5} = \frac{1}{5}(2+3+4+5) = \frac{14}{5}.$$

Now, calculate the partial expectation from 3 to 5:

$$g(3) = \sum x = 3^5 \frac{x}{5} = \frac{1}{5}(3+4+5) = \frac{12}{5}.$$

The difference is:

$$g(3) - g(2) = -\frac{2}{5}.$$

But this is of course just the negative density, $\frac{1}{5}$, times the starting point, $\bar{x} = 2$, which is what we have done.

We could also do this in reverse. Suppose we are interest in:

$$h(3) = \sum_{x=1}^{3} \frac{x}{5} = \frac{1}{5}(1+2+3) = \frac{6}{5}.$$

Now calculate the partial expectation where the upper bound is 4:

$$h(4) = \sum_{x=1}^{4} \frac{x}{5} = \frac{1}{5}(1+2+3+4) = \frac{10}{5}.$$

The difference is:

$$h(4) - h(3) = \frac{4}{5}.$$

This just the new point of evaluation, 4, times the density. The starting point would be 3, however, which would us $\frac{3}{5}$, not $\frac{4}{5}$. That's an issue with the discrete nature and not mapping perfectly into the derivative of a continuous random variable. But you can see that the formula Leibniz Rule gives us in this case where we a re differentiating with respect to an integral bound actually makes sense – we are calculating the change in the partial expectation when we change the bound.

17.3 Moral hazard and absconding (Gertler and Karadi, 2011; Gertler and Kiyotaki, 2011)

The first model we examine focuses on moral hazard problems in financial intermediaries following Gertler and Karadi (2011) and Gertler and Kiyotaki (2010). The key assumption is that financial intermediaries can do a runner, absconding with the money that depositors have placed in their care. This provides an incentive for financial intermediaries to default on their obligations to depositors. To make sure this does not happen all the time, it is assumed that financial intermediaries have some net worth of their own which they commit investment projects alongside the funds of depositors. Then, if the financial intermediary does default can only expropriate a proportion of its own net worth. This provides an incentive for financial intermediaries not to default on their obligations. To give a simple example, a bank could default after taking in deposits but in doing so would not be able to abscond with the worth of the real estate (branches, head office, and so on) owned by the bank.

17.3.1 Households

Households in the model consist of some bankers and workers, with perfect insurance within the household such that both bankers and workers have the same level of consumption in each of two periods. The behaviour of the household is then standard, with a first period budget constraint:

$$c_1 + d \leq y,$$

that restricts period one consumption c_1 plus deposits d to be less than an endowment per household member of y goods. The endowment has to be either consumed in the first period or placed on deposit – it cannot be used by bankers directly. In the second period the budget constraint is:

$$c_2 \leq Rd + \pi,$$

where R is the return paid on deposits and π is the profit brought home by the bankers. We assume that the household treats R as given and π as a lump sum transfer. In other words, neither the worker nor the banker takes into account that their individual actions will affect the rate of return or bank profitability. We assume throughout that bankers are randomly matched to households so there is infinitesimal probability that a household will be matched with exactly a banker from their own household. Combining the budget constraints defines the household intertemporal budget constraint:

$$c_1 + \frac{c_2}{R} \leq y + \frac{\pi}{R}. \tag{792}$$

The utility function of the household for two periods is:

$$u(c_1) + \beta u(c_2),$$

where we assume that $u(\cdot)$ is of CRRA form, where γ is the coefficient of relative risk aversion. The interest case for us requires $0 < \gamma < 1$, since this ensures in equilibrium that the substitution effect dominates the income effect such that where the return R increase there is an increase in deposits d. The solution of the household optimisation problem is characterised by the consumption Euler equation for consumption:

$$u'(c_1) = \beta R u'(c_2).$$

In full, the household problem implies:

$$\begin{aligned} c_1 &= \frac{y + \frac{\pi}{R}}{1 + \frac{(\beta R)^{1/\gamma}}{R}}, \\ d &= y - c_1, \\ c_2 &= Rd + \pi, \end{aligned}$$

for given y, β, R, and π. The first two of these variables are exogenous to the model, but R and π are to be determined in equilibrium in the financial market.

17.3.2 Firms

The role of firms in the model is to produce goods for consumption in period 2. To do this, they sell securities s to bankers and use the proceeds to buy goods in the first period which they turn into capital and produce sR^k goods in period 2, where R^k is the return on privately issued securities that is fixed exogenously. Firms are perfectly competitive and make no profit so they pass sR^k revenue back to the bankers.

17.3.3 Bankers

Bankers are endowed with N goods in the first period. We will refer to this as the net worth of bankers. In the simplified 2-period model we discuss here it is treated as an exogenous variable, but in a multi-period model it will change over time as the banker makes profits or losses each period. In a fully specified DSGE model and with many periods, N becomes an important state variable that means financial markets play a role in propagating shocks in the economy. We do not get that feature in our 2-period setup, but we can exogenously change N and see what effect this will have. The bankers accept deposits d and purchase securities s from firms. The banker takes return to deposits R and the return to securities R^k as given, and chooses the amount of deposits to accept from depositors to solve:

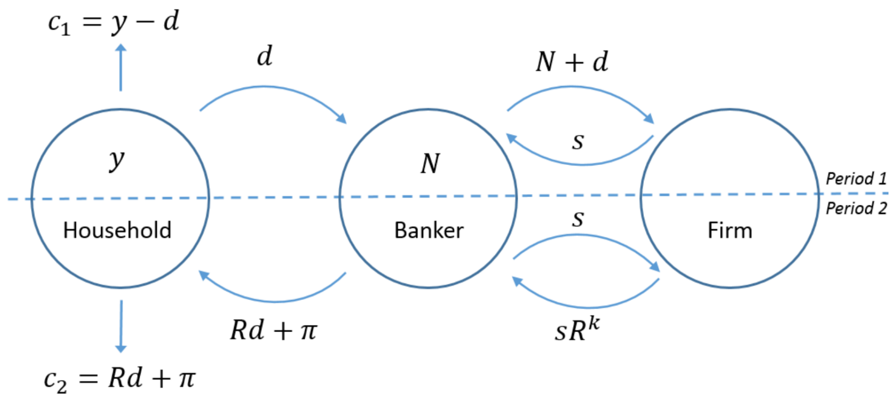
$$\pi = \max_{d} \left(sR^k - Rd \right).$$

The banker will always purchase the maximum quantity of securities possible, given its own net worth and the deposits it takes in, because securities pay a positive return with certainty. In other words:

$$N + d = s.$$

An overview of the model setup is shown in the figure below:

Figure 131: Model Overview



17.3.4 Equilibrium

If there are no financial frictions in the economy then equilibrium is characterised by:

- The household solving its utility maximisation problem;
- The banker solving its profit maximisation;
- Market clearing; and
- Non-negativity constraints $c_1, c_2 > 0$.

The benchmark equilibrium with no financial frictions is easy to characterise since it requires $R = R^k$. Otherwise, if $R > R^k$, the banker would set $d = 0$, or if $R < R^k$ the banker would take in an infinite amount of deposits. the condition $R = R^k$ is sufficient to characterise the equilibrium allocation c_1, c_2, R, and π. Note that this is indeed the first-best allocation with the optimal amount of deposits d.

We now make the equilibrium more interesting by introducing a moral hazard problem. In particular, we follow Gertler and Karadi (2011) and Gertler and Kiyotaki (2010) in assuming that the banker can default after receiving the payment sR^k from firms in period 2. If the banker chooses not to default then the behaviour of the equilibrium is as in the no financial frictions equilibrium as before.

If they do default, then a banker can abscond with a fraction θ of their total assets. The fraction $1-\theta$ is returned to depositors. The allocation of assets to the banker and depositor on default is therefore:

$$\text{Bankers: } \theta R^k(N+d),$$
$$\text{Depositors: } (1-\theta)R^k(N+d).$$

The banker will not default if the profits when not defaulting exceed the profits when defaulting, i.e., if:

$$\underbrace{(N+d)R^k - Rd}_{\text{No default}} \geq \underbrace{\theta R^k(N+d).}_{\text{Default}} \tag{793}$$

The above no-default condition identifies the nature of the trade off faced by the bankers. If they default then they gain because do not have to pay Rd to the depositors, but they lose because they only obtain a fraction θ of the return to securities $(N+d)R^k$. The fundamental tension is between getting the full benefit of financing firms (when not defaulting) and avoiding paying depositors (when defaulting). Note the key role here of the net worth of the bankers N. An increase in N will increase the LHS of (793) proportionally more than the RHS. So increasing N means that a banker is less likely to default. This is because the banker loses some of their net worth when defaulting – the more of their own assets they commit to the project, the less likely they are to want to default.

We are interested in symmetric equilibria where default does not happen even though bankers face a moral hazard problem.[124] As before in the case of no financial imperfections, the banker chooses the level of deposits d to take in and takes the returns R^k and R as given. Since the banker is in a symmetric equilibrium with no default, it cannot accept a level of deposits that would give it an incentive to default. It it did so, then depositors would instantly withdraw their deposits and take them to another bank. In other words, the banker is subject to a no default condition and faces the following problem:

$$\pi = \max_d \left\{(N+d)R^k - Rd\right\},$$

[124] We assume restrictions on R^k and θ that guarantee (793) holds in equilibrium.

subject to:

$$(N+d)R^k - Rd \geq \theta R^k (N+d).$$

The equilibrium is characterised as before by:

- The household solving its utility maximisation problem;
- The banker solving its profit maximisation;
- Market clearing; and
- Non-negativity constraints $c_1, c_2 > 0$.

The problem of the banker can be solved by setting up a Lagrangian:

$$\mathcal{L} = (N+d)R^k - Rd + \lambda \left((N+d)R^k - Rd - \theta R^k (N+d) \right),$$

which has a FOC:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial d} &= R^k - R + \lambda R^k - \lambda R - \lambda \theta R^k = 0 \\ \implies (R^k - R)(1+\lambda) &= \lambda \theta R^k \\ R^k - R &= \frac{\lambda \theta R^k}{1+\lambda} > 0. \end{aligned} \tag{794}$$

If the no default condition does not bind in equilibrium then $\lambda = 0$, and so $R^k = R$ as in the equilibrium with no financial imperfections. However, when the moral hazard problem is sufficiently large the constraint starts to bind and $\lambda > 0$, and so $R^k - R > 0$. Thus, a spread opens up between the exogenous return to securities R^k and the endogenous return to deposits R. The return R is less than socially optimal so deposits d are less than socially optimal and the household does not save as much as they should.[125] In this way, the presence of frictions in financial markets imposes real costs on the economy and a reduction in social welfare.

[125] This result relies on the assumption that $0 < \gamma < 1$, although even if this was not true, the equilibrium in an economy with financial frictions would not have the socially optimal level of deposits.

An increase in the net worth of bankers N helps to ameliorate the problems caused by the moral hazard friction. As we said before, increasing net worth makes it less likely that the banker will default. In the Lagrangian, this means that the no default constraint binds less tightly so through the FOC an increase in N causes a fall in λ and R rises closer to R^k.

Two things are worth noting here. Firstly, if we did have a model with more than 2 periods, then we would have a model with a dynamic financial accelerator with negative shocks causing a fall in net worth in one periods being propagated to the following period by depressed net worth of bankers. Secondly, the focus on net worth of bankers partly explains why central bankers pay so much attention to the health of bank balance sheets and spent so many resources helping bankers to repair their balance sheets after the recent global financial crisis.

17.4 Moral hazard and effort (Christiano and Ikeda, 2016)

The second model we look at is based on Christiano and Ikeda (2016) and it introduces a moral hazard problem between bankers and firms, rather than between bankers and depositors as in the first model. In particular, it is assumed that a banker can choose how much effort to make when buying securities from firms. If the banker makes a lot of effort then they can identify good quality securities and be pretty confident that the securities will pay a high return. In contrast, if the banker makes only little effort then they are likely to end up holding bad securities that only pay a low return. The banker's resulting incentive to make an effort to identify good securities is tempered by effort being costly, which can be thought of as the time cost of the effort of identifying good securities.

Households in the model have the same endowment process as in the previous model, such that optimal consumption decisions in periods 1 and 2 must satisfy the Euler equation for consumption $u'(c_1) = \beta R u'(c_2)$. We continue to assume that $0 < \gamma < 1$ when worked ing a utility function of the CRRA form, to restrict attention to the interesting case whereby an increase in the return on deposits leads to an increase in funds deposited. To get the moral problem running, it is necessary depositors and bankers cannot sign contracts that condition on the level of effort made by bankers. We satisfy this condition by assuming that effort is unobservable, an assumption that does not sit easily in the framework of the first model where there is one-to-one matching between a depositor and a banker. In such a world it is difficult to argue that the depositor cannot observe the effort made by the banker. To circumvent this problem, we assume that depositors place their deposits with mutual funds, which then pass on the deposits to all bankers in the economy. With mutual funds diversified across all bankers, it is easier to argue that it is difficult for depositors to monitor bankers. Mutual funds are completely diversified across bankers so are not in a position to monitor the effort of each and every banker.

17.4.1 Bankers

Bankers have an endowment N of goods in the first period. they receive deposits d from the mutual funds and combine them with their endowment to purchase securities $s = N + d$ from firms. The quality of securities can either be good or bad, with the probability that the securities purchased by the banker being food depending on the amount of effort e made by the banker. For simplicity, it is assumed that the probability of purchased securities being good $P(e)$ is linear in effort, i.e.:

$$P(e) = a + be,$$

where $b > 0$ so $P'(e) = b > 0$ and $P''(e) = 0$. The parameters of the model are restricted so that $0 < P(e) < 1$ in equilibrium. Good securities pay a certain return R^g and bad securities pay a certain return R^b so mean return on bank assets is:

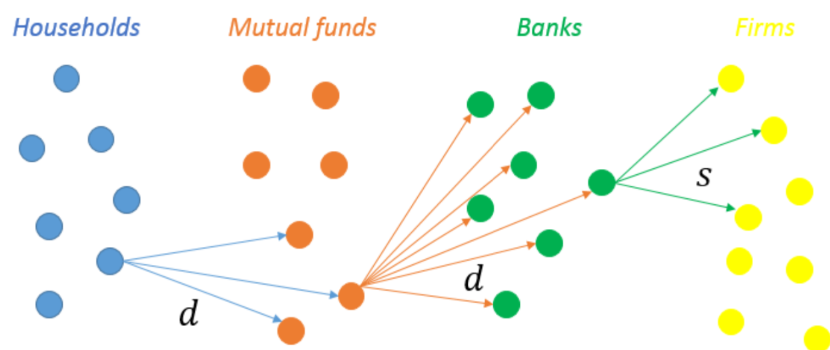
$$P(e)R^g + (1 - P(e))R^b,$$

and the variance of the return on bank assets is:

$$P(e)(1 - P(e))(R^g - R^b)^2.$$

We assume $P(e) > \frac{1}{2}$ so that the variance of the return falls when the banker makes more effort.

Figure 132: Model Overview



17.4.2 Observable effort benchmark

The natural benchmark against which to assess how the second moral hazard problem affects equilibrium is a model in which the effort of banks is observable. In that case, the deposit contract between mutual funds and bankers can be conditioned on the effort of the banker and there will be no financial market imperfections. The loan contract between the banker and mutual fund stipulates (d, e, R_g^d, R_b^d), where R_g^d and R_b^d are conditional returns paid out if the securities purchased by the bank turn out to be good or bad, respectively. The mutual funds themselves take deposits d from household and pay a return R which it treats as given. They are competitive and so any contract between mutual funds and bankers must satisfy a zero profit condition:

$$P(e)R_g^d d + (1 - P(e))R_b^d d = Rd,$$

otherwise if profits were positive mutual funds would set $d \to \infty$ which would exhaust the deposits of households, or if they were negative they would set $d = 0$ and achieve zero profit. The banker needs to ensure they have enough resources to pay the mutual funds irrespective of whether the securities they purchase are good or bad. In other words, there are two cash constraints:

$$R^g(N+d) - R^d_g d \geq 0, \tag{795}$$

$$R^b(N+d) - R^d_b d \geq 0, \tag{796}$$

which have to be satisfied by the optimal contract. It is possible to show that these cash constraints in practice either never bind or if they do bind they bind in the bad state of nature when securities turn out to be bad. In defining the problem of the banker it is then sufficient to only consider the second cash constraint. If the banker has enough resources to cover their commitments when the securities it has purchased are bad, then it will automatically have sufficient resources to cover commitments if it ends up with good securities.

The maximisation problem of the banker/mutual fund involves choosing (d, e, R^d_g, R^d_b) to maximise its return (less effort cost) subject to the zero profit condition for mutual funds and the cash constraint to have sufficient resources to cover its commitments in the bad state of the world:

$$\max_{d,e,R^d_g,R^d_b} \lambda \left[P(e)(R^g(N+d) - R^d_g d) + (1-P(e))(R^b(N+d) - R^d_b d)\right] - \frac{1}{2}e^2,$$

subject to:

$$P(e)R^d_g d + (1-P(e))R^d_b d = Rd,$$
$$R^b(N+d) - R^d_b d \geq 0,$$

where the constant λ is the marginal utility of consumption in the household of the banker, take as given by the banker. The cost of effort is modelled as a quadratic increasing function. The Lagrangian of the problem is:

$$\begin{aligned}\mathcal{L} = \lambda \left[P(e)(R^g(N+d) - R^d_g d) + (1-P(e))(R^b(N+d) - R^d_b d)\right] - \frac{1}{2}e^2 \\ + \mu \left(P(e)R^d_g d + (1-P(e))R^d_b d - Rd\right) + \upsilon \left[R^b(N+d) - R^d_b d\right].\end{aligned}$$

The FOCs determine the nature of the optimal contract. We look at the FOCs with respect to R_g^d, R_b^d, and e:

$$0 = -\lambda P(e) + \mu P(e), \tag{797}$$

$$0 = -\lambda(1 - P(e)) + \mu(1 - P(e)) - \upsilon, \tag{798}$$

$$0 = \lambda P'(e)\left[(R^g - R^b)(N + d) - (R_g^d - R_b^d)d\right] - e + \mu P'(e)(R_g^d - R_b^d)d. \tag{799}$$

The first two of these conditions imply $\mu = \lambda$ and $\upsilon = 0$ so the cash constraint is not binding in the equilibrium where effort is observable. The conditional payments R_g^d and R_b^d are indeterminate in equilibrium. There is an equilibrium where payments are state contingent $R_g^d = R^g$ and $R_b^d = R^b$, but there may also be an equilibrium where deposit rates are not state contingent so that $R_g^d = R_b^d = R$ if N is sufficiently large. The third FOC determines the optimal level of effort:

$$e = \lambda b(R^g - R^b)(N + d). \tag{800}$$

17.4.3 Unobservable effort

If the effort of the bank is unobservable then it is no longer possible to condition the contract between mutual funds and bankers on effort. Instead, we imagine a situation where the mutual funds draw up a contract on (d, R_g^d, R_b^d) and the banker chooses effort e. In choosing effort, the banker has the same objective as before except they do not worry about the constraints that profits for the mutual funds have to be zero and that there is a cash constraint that has to be satisfied in the bad state of the world. We assume that the mutual funds worry about these things and only offer contracts that satisfy those constraints. The banker takes d, R_g^d, R_b^d as given and chooses e to solve the maximisation problem:

$$\max_e \ \lambda\left[P(e)(R^g(N + d) - R_g^d d) + (1 - P(e))(R^b(N + d) - R_b^d d)\right] - \frac{1}{2}e^2,$$

where the FOC is:

$$0 = \lambda P'(e)\left[(R^g - R^b)(N + d) - (R_g^d - R_b^d)d\right] - e, \tag{801}$$

which is the same as (799) except $\mu = 0$ because the banker does not worry about the zero profit condition for mutual funds.

We now turn to the problem of the mutual funds, who choose (d, R_g^d, R_b^d) to maximise the same objective as the bankers but taking into account that the effort of bankers will be determined by their FOC for optimal effort. The Lagrangian of the problem of mutual funds is then:

$$\mathcal{L} = \max_{d, R_g^d, R_b^d, e} \left\{ \begin{matrix} \lambda \left[P(e)(R^g(N+d) - R_g^d d) + (1 - P(e))(R^b(N+d) - R_b^d d) \right] - \frac{1}{2}e^2 \\ +\mu \left[P(e)R_g^d d + (1 - P(e))R_b^d d - Rd \right] \\ +\eta \left[\lambda P'(e) \left[(R^g - R^b)(N+d) - (R_g^d - R_b^d)d \right] - e \right] \\ +\upsilon \left[R^b(N+d) - R_b^d d \right] \end{matrix} \right\},$$

which is identical to that in the observable effort case apart from the additional constraint which is indexed by the Lagrange multiplier η. The FOCs for R_g^d, R_b^d, and e are:

$$0 = -\lambda P(e) + \mu P(e) - \eta \lambda P'(e), \tag{802}$$

$$0 = -\lambda(1 - P(e)) + \mu(1 - P(e)) + \eta \lambda P'(e) - \upsilon, \tag{803}$$

$$0 = \left\{ \begin{matrix} \lambda P'(e) \left[(R^g - R^b)(N+d) - (R_g^d - R_b^d)d \right] - e \\ +\mu P'(e)(R_g^d - R_b^d)d + \eta \left[\lambda P''(e) \left[(R^g - R^d)(N+d) - (R_g^d - R_g^d)d \right] - 1 \right] \end{matrix} \right\}. \tag{804}$$

The first two of these conditions combine to give $\mu = \lambda + \upsilon$, $\upsilon P(e) = \eta \lambda b$, $P'(e) = b$, and $P''(e) = 0$, as $P(e)$ is linear in e. The effort constraint (801) can be used to substitute out for e in the FOC (804), and by doing the appropriate substitutions for μ, $P'(e)$, and $P''(e)$, we get:

$$(R_g^d - R_b^d) = \frac{\eta}{(\lambda + \upsilon)bd}. \tag{805}$$

We distinguish between two different cases. The first characterises "normal times" in that the bankers are assumed to have sufficient net worth N that the cash constraint (796) does not bind in equilibrium. In this case, $\upsilon = 0$, because the cash constraint does not bind and $\upsilon P(e) = \eta \lambda b$ implies $\eta = 0$ as well. It is clear that if $\upsilon = \eta = 0$ then from the above equation it must be the case that

$R_g^d - R_b^d = 0$, and the equilibrium is characterised by non-contingent payments. Imposing further that the mutual funds make zero profits we have that

$$R = R_g^d = R_b^d.$$

To summarise, equilibrium in normal times is characterised by bankers making non-contingent payments to the mutual funds. The degree of effort made by bankers in normal times is defined by imposing $R_g^d = R_b^d$ on the FOC for effort (804) to obtain:

$$e = \lambda b(R^g - R^b)(N + d).$$

This level of effort is the same as that which resulted in the model with observable effort (800) so it will be socially optimal. Intuitively, in normal times bankers have sufficient incentives to make an effort when purchasing securities because they are investing enough of their own net worth N into securities. The moral hazard problem is not present as the bankers choose the optimal amount of effort anyway. In normal times they have sufficient funds that it is in their own self-interest.

The second case is on one of "bad times" where the cash constraint is binding. Since in equilibrium and we have that neither Lagrange multiplier is zero, it follows that (805) must hold. The returns R_g^d and R_b^d to the mutual funds become conditional. We therefore see that financial markets where bankers have low net worth are characterised by a spread between returns. To see the effect of this on the effort made by bankers, return the bankers' FOC for effort (801) and write, using (800):

$$e = \lambda b \left[(R^g - R^b)(N + d) - (R_g^d - R_b^d)d \right] < \lambda b \left[(R^g - R^b)(N + d) \right],$$

to see that the level of effort when effort is not observed is lower than that in the model where effort is observed. This is the key cost imposed by the moral hazard problem in this model.

In an ideal world, it is wise to allow the banker to be the residual claimant on the project. In other words, it is good to allow the agent choosing how much effort to make to reap the full benefit of making their effort. In normal times when the equilibrium is characterised by non-contingent returns, the banker has to pay $R = R_g^d = R_b^d$ to the mutual funds irrespective of whether they purchase good or bad securities. The incentive for the banker to make the effort to find good securities is fully preserved as the banker receives the full benefit of purchasing good securities. However, in bad times it is not possible to support equilibrium with non-contingent returns. Loosely speaking, one can think of the banker as not having sufficient funds to pay out a non-contingent return to the mutual funds in a bad state of the world. In such a situation the optimal contract has to be adjusted so that the banker pays less in the bad state of the world and more in the good state of the world – i.e., returns are contingent. Whilst this is good for ensuring the banker is always able to satisfy their cash constraint, it does cause problems because the banker is no longer the "residual claimant". Put simply, there is less of an incentive for the banker to make the effort to find good securities if they know that conditional on purchasing good securities they will have to pay a higher return to mutual funds. Some of the return to effort is lost to the banker as the additional contingent return they have to pay mutual funds. The differential $R_g^d - R_b^d$ weakens the incentive for effort and the financial friction has real welfare costs in the economy.

17.5 A model of adverse selection (Mankiw, 1986)

The third model we discuss highlights adverse selection problems in investment decisions and is by Mankiw (1986). The key margin of interest is the number of bankers who decide to make investments in risky projects. As we shall see, the optimal number of projects to invest in depends as usual on the production technology (the marginal rate of transformation). however, the number of projects invested in when there is an adverse selection problem depends not only on the production technology but also the net worth of bankers. This leads to too few projects being invested in and returns to deposits being too low when there is an adverse selection problem.

The household in this model consists of workers and bankers, with the measure of bankers being e. Household members perfectly insure each other such that the consumption of worker and banker members of the household is equal. The household receives an endowment y of goods in the first period as in the previous two models, so it faces a first period budget constraint of:

$$c_1 + d = y,$$

where d are deposits at mutual funds. Bankers earn profits per capital of π, so the second period budget constraint of the household is:

$$c_2 \leq Rd + e\pi.$$

The household has CRRA preferences and their Euler equation for consumption is:

$$c_1^{-\gamma} = \beta R c_2^{-\gamma}, \ \gamma > 0.$$

17.5.1 Bankers

Bankers have an endowment $N < 1$, which they can either invest in a mutual fund or invest in a risky project. If the banker invests their endowment in a mutual fund they receive a certain return RN in the next period. The risky project available is indivisible, needing an investment of one unit of goods in period 1. Since the endowment $N < 1$ it follows that a banker wanting to make the risky investment needs to borrow $1 - N$ from the mutual fund. The rate of interest on loans is R^L, to be determined in equilibrium. Bankers take the deposit rate R and the loan rate R^L as given.

The risky investment project on offer to a banker in the first period pays a random return $\theta > 0$ with random probability p in the second period, and with probability $1 - p$ it pays nothing:

$$\text{Return} = \begin{cases} \theta, & \text{w.p. } p, \\ 0, & \text{w.p. } 1 - p. \end{cases}$$

The random variables θ and p are drawn from a distribution $F(\theta, p)$ and are private information to the banker and the household to which the bank belongs. The mutual funds know the distribution function $F(\cdot)$ but do not know the values of θ and p for a particular project. In other words, the banker knows the return and risk associated with their own project, whereas the mutual funds only know the return and risk associated with projects in general.

To derive analytical results, Mankiw imposes a strong restriction on the distribution function $F(\cdot)$. In particular, he imposes the condition $\theta p = \bar{\theta}$ which means that all projects have the same expected return:

$$p\theta + (1-\theta)0 = \bar{\theta}.$$

There are some projects that are very risky yet pay a high return and some projects that are low risk but pay a low return. Reducing the number of random variables in this way means we can consider only the probability of success of the project p to be random and let the payoffs satisfy $\theta = \bar{\theta}/p$. We assume for simplicity that p is distributed according to a uniform distribution over support $[0,1]$.

The choice of the banker is between investing in their project or not. The expected return of investing is:

$$p(\theta - R^L(1-N)) + (1-p)0 = \bar{\theta} - pR^L(1-N), \tag{806}$$

which the banker compares against the certain return RN which they receive if they do no invest in their project. The banker will choose to invest if:

$$\bar{\theta} - pR^L(1-N) > RN, \tag{807}$$

so the projects that are invested in will be those with a low probability of success. We define a critical probability $\bar{p}(R^L)$ so all projects that are invested in satisfy:

$$0 < p < \bar{p}(R^L),$$

where

$$\bar{p}(R^L) = \frac{\bar{\theta} - RN}{R^L(1-N)}. \tag{808}$$

$\bar{p}(R^L)$ is by definition the fraction of bankers activating their projects. Since p is a uniform distribution:

$$e\bar{p}(R^L) = e\int_0^{\bar{p}(R^L)} dp,$$

is also the total quantity of banker investment in the economy. The average value:

$$\Pi(R^L) = \frac{1}{2}\bar{p}(R^L),$$

is the average value of p amongst bankers investing in their projects.

17.5.2 Mutual funds

Mutual funds operate in a competitive market so they make zero profits in equilibrium. They invest in all bankers wanting to invest in their projects, so are fully diversified and have costs and revenues that are non-stochastic. The cost of a unit of funds to mutual funds is the return R they pay to depositors. The income from a unit of funds is the loan rate R^L times the average value $\Pi(R^L) = \frac{1}{2}\bar{p}(R^L)$ of p amongst bankers investing in their project. The zero profit condition is:

$$\Pi(R^L)R^L = R, \tag{809}$$

so the spread between loans and deposit rates is:

$$\frac{R^L}{R} = \frac{2}{\bar{p}(R^L)} > 2.$$

The definition of the critical probability $\bar{p}(R^L)$ at which the bankers invest in projects implies:

$$R = \frac{\bar{\theta}}{2 - N},$$

so the deposit rate is determined by the investment technology $\bar{\theta}$ and the net worth of bankers N. The loan rate is also determined by the same variables as $\Pi(R^L)R^L = R$. The adverse selection problem is at the heart of both these results because the revenue:

$$\Pi(R^L)R^L = \frac{1}{2}\frac{\bar{\theta} - RN}{1 - N},$$

of the mutual fund is completely independent of the rate of interest it charges on its loans. Intuitively, the mutual funds gains extra revenue from increasing R^L as bankers have to pay back more but the high quality projects (in terms of their probability of paying back their loan) are no longer invested in so the average probability $\bar{p}(R^L)$ of the paying also falls. In the adverse selection equilibrium these two effects exactly offset each other. This implies that deposit rates and loan rates are completely determined by the features of the deposit and loan market, not for example by preferences as is usual in first best allocations.

17.5.3 Equilibrium

To fully characterise equilibrium with adverse selection, start with loan market clearing:

$$e\bar{p}(R^L) = d + eN,$$

where $e\bar{p}(R^L)$ is investment, d is deposits, and eN is net worth of bankers. The income of bankers is:

$$\pi = \int_0^{\bar{p}(R^L)} \left[\bar{\theta} - pR^L(1-N)\right] dp + \int_{\bar{p}(R^L)}^1 NRdp = \bar{p}(R^L)\left[\bar{\theta} - \Pi(R^L)R^L(1-N)\right] + (1-\bar{p}(R^L))NR,$$

so total household income in the second period is:

$$Rd + e\bar{p}(R^L)\left[\bar{\theta} - \Pi(R^L)R^L(1-N)\right] + e(1-\bar{p}(R^L))NR = e\bar{p}(R^L)\bar{\theta},$$

which is a particularly neat expression because $\Pi(R^L)R^L = R$ and $d = e\bar{p}(R^L) - eN$. The rest of the equilibrium is:

$$c_1 = \frac{y + eN}{(\beta R)^{1/\gamma} + \bar{\theta}}, \tag{810}$$

$$c_2 = \frac{y + eN}{(\beta R)^{1/\gamma}\bar{\theta}}(\beta R)^{1/\gamma}\bar{\theta}, \tag{811}$$

$$R = \frac{\bar{\theta}}{2 - N}, \tag{812}$$

$$R^L = 2eR\frac{(\beta R)^{-1/\gamma}\bar{\theta} + 1}{y + eN}. \tag{813}$$

17.5.4 Social optimum

The benchmark against which the equilibrium under adverse selection should be compared to is the first best allocation that would be selected by the Ramsey planner. The Ramsey social planner chooses the mass ep^* of bankers who will invest in their projects, needing $d + eN$ resources to do so. The planner is indifferent between which bankers activate their projects since the expected return on all projects is identical and the Ramsey planner can diversify away the risk in any one project. The optimisation problem of the Ramsey planner is:

$$\max_{c_1^*, c_2^*, p^*, d^*} u(c_1^*) + \beta u(c_2^*),$$

subject to:

$$c_1^* + d^* \leq y,$$

$$ep^* \leq d^* + eN,$$

$$c_2^* \leq ep^*\bar{\theta},$$

which has solution:

Table 9: Comparison of Equilibria

Social optimum	Adverse selection equilibrium
$c_1^* = \frac{y+eN}{(\beta\theta)^{1/\gamma}+\theta}\bar{\theta}$	$c_1 = \frac{y+eN}{(\beta\theta)^{1/\gamma}+\theta}\bar{\theta}$
$c_2^* = c_1(\beta\bar{\theta})^{1/\gamma}$	$c_2 = c_1(\beta R)^{1/\gamma}$
$ep^* = \frac{y+eN}{(\beta\theta)^{1/\gamma}+\theta}(\beta\bar{\theta})^{1/\gamma}$	$e\bar{p}(R^L) = \frac{y+eN}{(\beta R)^{1/\gamma}+\theta}(\beta R)^{1/\gamma}$

The social optimum is different to the allocation in the adverse selection equilibrium because $R = \frac{\bar{\theta}}{2-N} \neq \bar{\theta}$. In particular, the return to deposits R in the adverse selection equilibrium is too low, which temps households to consume too much in the first period and $c_1 > c_1^*$. The resources being passed to the second period are too low, with $e\bar{p}(R^L) < ep^*$ as not enough bankers invest in their projects $\bar{p}(R^L) < p^*$. Second period consumption is then too low $c_2 < c_2^*$. With period consumption too high and second period consumption too low, the adverse selection problem distorts the intertemporal trade-off and social welfare is lower than it should be. Note that the net worth of bankers N plays a crucial role in creating the distortion. If the endowment of bankers was $N = 1$, so they could invest in their own project then $R = \frac{\bar{\theta}}{2-N} = \bar{\theta}$ and the adverse selection equilibrium would be efficient. The market fails under adverse selection because the price mechanism is unable to give sufficient incentives for bankers to activate their projects and drive up the return to deposits. Remember that the revenues of the mutual fund are independent of the loan rate R^L, so there is no incentive for them to lower the loan rate to induce more bankers to invest in their projects.

17.6 The financial accelerator (Bernanke, Gertler, and Gilchrist, 1999)

The fourth model to be considered is from the famous financial accelerator paper of Bernanke, Gertler, and Gilchrist (1999) (BGG). This model has, not surprisingly, been very influential in the policies of former-Chairman Bernanke throughout and after the GFC. The BGG paper is in the tradition of Bernanke and Gertler (1989) and Carlstrom and Fuerst (1997). But it departs in a couple of important ways. First, it is a New Keynesian model with sticky prices (as opposed to the RBC model of Carlstrom and Fuerst and the simplified neoclassical OLG model in Bernanke and Gertler). Second, it applies the agency friction to the financing of the entire capital stock, whereas in Carlstrom and Fuerst it is only new investment that is subject to the agency friction. This has the effect of resulting in more amplification. A third more minor difference is that the loan over which there are agency frictions is intertemporal as opposed to intratemporal in Carlstrom and Fuerst.

17.6.1 Overview

We proceed somewhat non-linearly. BGG don't do a great job of laying out the details of their model, so it is very difficult to recreate from scratch.[126] As such, we're going to simply start with the linearised equilibrium conditions that BGG have. They key equation is as follows:

$$\mathbb{E}_t \hat{r}^k_{t+1} - \hat{r}_t = -\upsilon \left[\hat{N}_t - (\hat{Q}_t + \hat{r}_{t+1}) \right],$$

where r_t is the risk-free real interest rate, N_t is net worth, Q_t is the price of capital, and K_{t+1} is the capital stock accumulated in t available for production in $t+1$. $\mathbb{E}_t r^k_{t+1}$ is the expected return on capital. The LHS can be interpreted as an external finance premium, and the right hand side is the negative of a leverage ratio (i.e., assets, in linearised form, are $\hat{Q}_t + \hat{K}_{t+1}$, relative to equity, $\hat{N}_t$). $\upsilon > 0$ means that there are agency frictions. The key insight, as in the earlier papers, is that increases in borrower net worth, $\hat{N}_t$, reduce agency frictions if $\upsilon > 0$. This lowers the external finance premium and stimulates investment and aggregate demand. The notion of the "accelerator" effect is that expansionary shocks which push asset prices up are more expansionary because they accordingly improve the balance sheet condition of borrowers, which further leads to a boom and more asset price appreciation.

[126] Good explanations can be found in Christiano, Motto, et al. (2014), and Carlstrom, Fuerst, and Paustian (2016).

17.6.2 Linearised model

Using our standard log-linearisation notation, the linearised model is as follows:

$$\hat{Y}_t = \frac{\bar{C}}{\bar{Y}}\hat{C}_t + \frac{\bar{I}}{\bar{Y}}\hat{I}_t + \frac{\bar{G}}{\bar{Y}}\hat{G}_t + \frac{\bar{C}^e}{\bar{Y}}\hat{C}^e_t, \tag{814}$$

$$\hat{C}_t = \mathbb{E}_t\hat{C}_{t+1} - \hat{r}_t, \tag{815}$$

$$\hat{C}^e_t = \hat{N}_t, \tag{816}$$

$$\mathbb{E}_t\hat{r}^k_{t+1} - \hat{r}_t = -\upsilon\left[\hat{N}_t - (\hat{Q}_t + \hat{K}_{t+1})\right], \tag{817}$$

$$\hat{r}^k_t = (1-\epsilon)(\hat{Y}_t - \hat{K}_t - \hat{X}_t) + \epsilon\hat{Q}_t - \hat{Q}_{t-1}, \tag{818}$$

$$\hat{Q}_t = \varphi(\hat{I}_t - \hat{K}_t), \tag{819}$$

$$\hat{Y}_t = \hat{A}_t + \alpha\hat{K}_t + (1-\alpha)\Omega\hat{H}_t, \tag{820}$$

$$\hat{Y}_t - \hat{H}_t - \hat{X}_t - \hat{C}_t = \eta^{-1}\hat{H}_t, \tag{821}$$

$$\hat{\pi}_t = -\kappa\hat{X}_t + \beta\mathbb{E}_t\hat{\pi}_{t+1}, \tag{822}$$

$$\hat{K}_{t+1} = \delta\hat{I}_t + (1-\delta)\hat{K}_t, \tag{823}$$

$$\hat{N}_t = \gamma\frac{\bar{R}\bar{K}}{\bar{N}}(\hat{r}^k_t - \hat{r}_{t-1}) + \hat{r}_{t-1} + \hat{N}_{t-1}, \tag{824}$$

$$\hat{r}^n_t = \rho\hat{r}^n_{t-1} + \zeta\hat{\pi}_{t-1} + s_r\epsilon_{r,t}, \tag{825}$$

$$\hat{r}^n_t = \hat{r}_t + \mathbb{E}_t\hat{\pi}_{t+1}, \tag{826}$$

$$\hat{A}_t = \rho_a\hat{A}_{t-1} + s_a\epsilon_{a,t}, \tag{827}$$

$$\hat{G}_t = \rho_g\hat{G}_{t-1} + s_g\epsilon_{g,t}. \tag{828}$$

A lot to process here: (814) is the aggregate resource constraint; (815) is the linearised consumption Euler equation, assuming log utility; Consumption of entrepreneurs, C^e_t is proportional to net worth, and so we have (816);[127] (817) is the key relationship, showing a positive relationship between leverage, $\hat{Q}_t + \hat{K}_{t+1} - \hat{N}_t$, and the external finance premium, $\mathbb{E}_t\hat{r}^k_{t+1} - \hat{r}_t$;[128] (818) is the ex-post return on capital (note that ϵ here is not an elasticity of substitution); and, (819) is the linearised FOC for investment.

[127] Each period, a fixed fraction of entrepreneurs die and consume their net worth, giving rise to this expression.
[128] Note this is governed by the parameter υ. If $\upsilon = 0$, then there is no financial accelerator.

(814)-(819) is the aggregate demand block of the model.

(820)-(822) is the aggregate supply block of the model: (820) is the production function; (821) is the labour market clearing condition, where X_t is the markup of price over marginal cost (or, $-X_t$ is real marginal cost); and (822) is the New Keynesian Phillips Curve.

Rounding out the model we have: (823) is the law of motion of capital, while (824) is the law of motion for net worth[129] – our state variables in the model; (825) is the central bank's Taylor Rule; (826) is the Fisher equation; (827) is the process for productivity; and (828) is the process for government spending.

Overall, $\left\{\hat{Y}_t, \hat{C}_t, \hat{I}_t, \hat{G}_t, \hat{C}^e_t, \hat{r}_t, \hat{N}_t, \hat{r}^k_t, \hat{Q}_t, \hat{X}_t, \hat{K}_t, \hat{H}_{t,} \hat{\pi}_t, \hat{r}_t, \hat{A}_t\right\}$ constitute a linear system with 15 variables and 15 equations.

There are four agents in the model: households, retailers, wholesale producers, and government (including the central monetary authority and the fiscal agency). The household sector is standard. Retailers are just a trick to introduce Calvo price-setting. The government conducts policy via a Taylor Rule and consumes an exogenous amount of output. The action is really on the wholesale firm side. Each period, the wholesalers have to get a loan to finance the entirety of next period's capital stock, subject to idiosyncratic returns to capital, as captured by a variable ω_t. This is like the setup in Carlstrom and Fuerst (1997), except the agency friction applies to the producers of output, rather than the producers of new investment goods. The loan contract is also intertemporal as opposed to intratemporal. In what follows below, we will briefly describe how to get the linearised conditions above. We will then spend some more time on the formal contracting problem.

[129] γ is the fraction of surviving entrepreneurs.

17.6.3 Households

The household problem is standard. There is an Euler equation for consumption, and an intratemporal labour supply condition:

$$\frac{1}{C_t} = \beta R_t \mathbb{E}_t \frac{1}{C_{t+1}}, \tag{829}$$

$$\frac{\xi}{1 - H_t} = \frac{W_t}{C_t}. \tag{830}$$

Taking logs of these and letting $z_t = d \ln Z_t$ for generic variable Z_t, we have:

$$\begin{aligned} - \ln C_t &= \ln \beta + \ln R_t - \mathbb{E}_t \ln C_{t+1} \\ \hat{C}_t &= \mathbb{E}_t \hat{C}_{t+1} - \hat{r}_t, \end{aligned}$$

which is (815). Notice that because in this model $Y_t \neq C_t$, we cannot boil this down to a simple dynamic IS equation as we do in the canonical New Keynesian model. Next, for the labour supply condition:

$$\begin{aligned} \ln \xi - \ln(1 - H_t) &= \ln W_t - \ln C_t \\ \frac{1}{1 - \bar{H}} dH_t &= \hat{W}_t - \hat{C}_t \\ \frac{\bar{H}}{1 - \bar{H}} \hat{H}_t &= \hat{W}_t - \hat{C}_t, \end{aligned}$$

which is (821) when you define $\eta = \frac{1 - \bar{H}}{\bar{H}}$ as the Frisch elasticity and note the definition of the wage from the wholesale producer problem (see (837)).

17.6.4 Capital accumulation and Tobin's Q

The capital accumulation equation is:

$$K_{t+1} = \Phi \left(\frac{I_t}{K_t} \right) K_t + (1 - \delta) K_t, \tag{831}$$

where the function $\Phi(\cdot)$ is defined where $\Phi(0) = 0$, $\Phi(\delta) = 1$, and $\Phi'(\delta) = 1$. Take logs and totally differentiate to get:

$$\begin{aligned}
\ln K_{t+1} &= \ln \left[\Phi\left(\frac{I_t}{K_t}\right) K_t + (1-\delta)K_t \right] \\
\frac{1}{\bar{K}} dK_t &= \frac{1}{\bar{K}} \left[\Phi\left(\frac{\bar{I}}{\bar{K}}\right) dK_t + \Phi'\left(\frac{\bar{I}}{\bar{K}}\right) dI_t - \Phi'\left(\frac{\bar{I}}{\bar{K}}\right) \frac{\bar{I}}{\bar{K}} dK_t + (1-\delta) dK_t \right] \\
\hat{K}_t &= \delta \hat{K}_t + \frac{\bar{I}}{\bar{K}} \hat{I}_t - \delta \hat{K}_t + (1-\delta)\hat{K}_t \\
\hat{K}_t &= \frac{\bar{I}}{\bar{K}} \hat{I}_t + (1-\delta)\hat{K}_t,
\end{aligned}$$

which gives (823).[130]

Now, consider the Tobin's Q relationship. This comes from the optimal choice of investment by firms subject to the adjustment cost embedded in the accumulation equation above. In nonlinear form:

$$Q_t = \left[\Phi'\left(\frac{I_t}{K_t}\right) \right]^{-1}, \tag{832}$$

and taking logs and totally differentiating, noting that $\bar{Q} = 1$, gives:

$$\begin{aligned}
\ln Q_t &= -\ln \left[\Phi'\left(\frac{I_t}{K_t}\right) \right] \\
\hat{Q}_t &= \frac{1}{\bar{Q}} \left[\Phi''\left(\frac{\bar{I}}{\bar{K}}\right) \left(\frac{dI_t}{\bar{K}} - \frac{\bar{I}}{\bar{K}^2} dK_t \right) \right] \\
\hat{Q}_t &= -\Phi''(\delta)\delta \left[\hat{I}_t - \hat{K}_t \right].
\end{aligned}$$

Then, $\varphi = -\Phi''(\delta)\delta$. Since $\Phi''(\cdot) < 0$, this is positive. BGG's notation is awkward, but gives you (819).

The expected return on holding capital from t to $t+1$ is:

$$\mathbb{E}_t \left[R^k_{t+1} \right] = \mathbb{E}_t \frac{\bar{R} R_{t+1} + (1-\delta) Q_{t+1}}{Q_t}, \tag{833}$$

[130]Eric Sims has excellent notes on investment as to how to differentiate $\Phi\left(\frac{I_t}{K_t}\right)$.

where $\bar{RR}_{t+1}$ is the implicit rental rate on capital/marginal product of capital. In words, if you buy an additional unit of capital available for production tomorrow, K_{t+1}, you pay Q_t today. You get $\bar{RR}_{t+1}$ tomorrow and have $(1-\delta)$ left over, which is valued at Q_{t+1}. Take logs, ignoring expectation operators, and totally differentiate to get:

$$\begin{aligned} \ln R^k_{t+1} &= \ln\left[RR_{t+1} + (1-\delta)Q_{t+1}\right] - \ln Q_t \\ \mathbb{E}_t \hat{r}^k_{t+1} &= \frac{1}{\bar{RR} + (1-\delta)}\left[dRR_{t+1} + (1-\delta)dQ_{t+1}\right] - \hat{Q}_t \\ \mathbb{E}_t \hat{r}^k_{t+1} &= \frac{\bar{RR}}{\bar{RR} + (1-\delta)}\mathbb{E}_t \widehat{rr}_{t+1} + \frac{1-\delta}{\bar{RR} + (1-\delta)}\mathbb{E}_t \hat{Q}_{t+1} - \hat{Q}_t, \end{aligned}$$

define $\epsilon = \frac{1-\delta}{\bar{RR}+(1-\delta)}$, and we have:

$$\begin{aligned} 1-\epsilon &= \frac{\bar{RR} + (1-\delta)}{\bar{RR} + (1-\delta)} - \frac{1-\delta}{\bar{RR} + (1-\delta)} \\ &= \frac{\bar{RR}}{\bar{RR} + (1-\delta)}, \end{aligned}$$

hence we can write:

$$\mathbb{E}_t \hat{r}^k_{t+1} = (1-\epsilon)\mathbb{E}_t \widehat{rr}_{t+1} + \epsilon \mathbb{E}_t \hat{Q}_{t+1} - \hat{Q}_t,$$

which is (818) when you take into account the definition of the rental rate as being the marginal product of capital (see (836)).

17.6.5 The wholesale firm's problem

The wholesale firm optimality conditions for capital and household labour are, respectively, to hire up until the point where the marginal products equal the product of the factor prices and the markup of price over marginal cost, X_t:

$$X_t RR_t = \alpha \frac{Y_t}{K_t}, \tag{834}$$

$$X_t W_t = \Omega(1-\alpha)\frac{Y_t}{H_t}. \tag{835}$$

Log-linearising, we get:

$$\widehat{rr}_t = \hat{Y}_t - \hat{K}_t - \hat{X}_t, \tag{836}$$

$$\hat{W}_t = \hat{Y}_t - \hat{H}_t - \hat{X}_t, \tag{837}$$

Subbing these in for $\widehat{rr}_t$ and $\hat{W}_t$ in the capital demand curve and labour market clearing conditions give the log-linearised conditions in the BGG paper.

17.6.6 Aggregate supply and pricing

The aggregate production function is:

$$d_t Y_t = A_t K_t^{\alpha} H_t^{\Omega(1-\alpha)} (H_t^e)^{(1-\Omega)(1-\alpha)}, \tag{838}$$

where d_t is price dispersion. It satisfies:

$$d_t = (1-\theta)(\Pi_t^*)^{-\epsilon} + \theta \Pi_t^{\epsilon} d_{t-1}, \tag{839}$$

which is going to be second order and can be ignored. Linearising gives (nothing that $H_t^e = 1$ is constant):

$$\hat{Y}_t = \hat{A}_t + \alpha \hat{K}_t + \Omega(1-\alpha)\hat{H}_t,$$

which is (820).

The non-linear price-setting conditions can be written as follows. There is bad notation here in that BGG uses ϵ as the price elasticity of demand but then use it again in the expression for the return on capital – so be aware! The price elasticity of demand ends up being irrelevant anyway for the linearised pricing condition. The optimal relative reset price satisfies:

$$\Pi_t^* = \frac{\epsilon}{\epsilon - 1} \frac{x_{1,t}}{x_{2,t}}, \tag{840}$$

where $x_{1,t}$ and $x_{2,t}$ are our auxiliary variables:

$$x_{1,t} = X_t^{-1} Y_t + \theta \mathbb{E}_t \Lambda_{t,t+1} \Pi_{t+1}^{\epsilon} x_{1,t+1}, \tag{841}$$

$$x_{2,t} = Y_t + \theta \mathbb{E}_t \Lambda_{t,t+1} \Pi_{t+1}^{\epsilon-1} x_{2,t+1}. \tag{842}$$

Here $\Pi_t = P_t / P_{t-1}$ is gross inflation and $\Pi_t^* = P_t^* / P_t$ is relative reset price inflation. The aggregate price level evolves according to:

$$1 = (1-\theta)(\Pi_t^*)^{1-\epsilon} + \theta \Pi_t^{\epsilon-1}. \tag{843}$$

Linearising all of these[131] yields the NKPC (822):

$$\hat{\pi}_t = -\kappa \hat{X}_t + \beta \mathbb{E}_t \hat{\pi}_{t+1},$$

where $\kappa = \frac{(1-\theta)(1-\theta\beta)}{\theta}$.

17.6.7 Net worth and financing premium

The key condition relating net worth to the external finance premium is:

$$\mathbb{E}_t \left[R_{t+1}^k \right] = s \left(\frac{N_t}{Q_t K_{t+1}} \right) R_t. \tag{844}$$

Take logs, ignoring the expectations operator:

$$\ln R_{t+1}^k = \ln \left[s \left(\frac{N_t}{Q_t K_{t+1}} \right) \right] + \ln R_t,$$

and then totally differentiate:

$$\hat{r}_{t+1}^k = \frac{s'(\cdot)}{s(\cdot)} \left[\frac{1}{\bar{Q}\bar{K}} dN_t - \frac{\bar{N}}{\bar{Q}^2 \bar{K}} dQ_t - \frac{\bar{N}}{\bar{Q}\bar{K}^2} dK_{t+1} \right] + \hat{r}_t$$
$$\hat{r}_{t+1}^k = \frac{s'(\bar{N}/\bar{K})}{s(\bar{N}/\bar{K})} \frac{\bar{N}}{\bar{K}} \left[\hat{N}_t - \hat{Q}_t - \hat{K}_{t+1} \right] + \hat{r}_t.$$

[131] This is a massive headache. Follow the steps in the New Keynesian chapter.

Letting $\upsilon = \frac{s'(\bar{N}/\bar{K})}{s(\bar{N}/\bar{K})} \frac{\bar{N}}{\bar{K}}$ yields (817). More on the formal contracting problem is below.

Each period, a fraction $1-\gamma$ entrepreneurs die and consume their net worth. Hence, aggregate consumption of entrepreneurs is:

$$C_t^e = (1-\gamma)V_t, \tag{845}$$

where V_t is entrepreneurial equity from the capital holdings. Ignoring the higher order terms:

$$V_t = \left(R_t^k - R_{t-1}\right)\left(Q_{t-1}K_t - N_{t-1}\right) + R_{t-1}N_{t-1}.$$

Net worth at the middle of the period is:

$$N_t = \gamma V_t + W_t^e,$$

where γ is the probability of survival, and W_t^e is the entrepreneurial wage. In other words, surviving entrepreneurs inherit V_t of equity and earn some additional equity from supplying labour, W_t^e. As noted in above, entrepreneurs who exist just consume their existing equity. Since γ is close to 1 and W_t^e is small, you can treat $V_t \approx N_{t+1}$, which is what gives (816):

$$\hat{C}_t^e = \hat{N}_t.$$

Aggregate net worth evolves according to:

$$N_t = \gamma\left[(R_t^k - R_{t-1})Q_{t-1}K_t + \iota_t(Q_{t-1}K_t - N_t) + R_{t-1}N_{t-1}\right] + W_t^e. \tag{846}$$

This is (4.13) in the BGG paper, with ι_t the term involving the integral:

$$\iota_t = \mu \int_0^{\bar{\omega}_t} \omega_t \phi(\omega_t) R_t^k Q_{t-1} K_t d\omega_t. \tag{847}$$

Take logs:

$$\ln N_t = \ln \left\{ \gamma \left[(R_t^k - R_{t-1}) Q_{t-1} K_t + \iota_t (Q_{t-1} K_t - N_t) + R_{t-1} N_{t-1} \right] + W_t^e \right\},$$

and totally differentiate, and ignore the ι term:

$$\begin{aligned}
\hat{N}_t &= \frac{1}{\bar{N}} \left\{ \begin{matrix} \gamma \bar{Q} \bar{K} (dR_t^k - dR_{t-1}) + \gamma (\bar{R}^k - \bar{R}) \bar{K} dQ_{t-1} + \gamma (\bar{R}^k - \bar{R}) dK_t \\ + \gamma \bar{N} dR_{t-1} + \gamma \bar{R} dN_{t-1} + dW_t^e \end{matrix} \right\} \\
\hat{N}_t &= \frac{\gamma \bar{R} \bar{K}}{\bar{N}} \left(\frac{dR_t^k}{\bar{R}} - \frac{dR_t}{\bar{R}} \right) + \frac{\gamma (\bar{R}^k - \bar{R})}{R} \frac{\bar{R} \bar{K}}{\bar{N}} \hat{Q}_{t-1} + \frac{\gamma (\bar{R}^k - \bar{R})}{\bar{R}} \frac{\bar{R} \bar{K}}{\bar{N}} \hat{K}_{t-1} \\
&\quad + \gamma \bar{R} \hat{r}_{t-1} + \gamma \bar{R} \hat{N}_{t-1} + \frac{\bar{W}^e}{\bar{N}} \hat{W}_t^e \\
\hat{N}_t &= \frac{\gamma \bar{R} \bar{K}}{\bar{N}} (\hat{R}_t^k - \bar{R}_t) - \frac{\gamma \bar{R} \bar{K}}{\bar{N}} \hat{R}_t^k + \frac{\gamma \bar{K}}{\bar{N}} dR_t^k + \gamma \frac{\bar{K}}{\bar{N}} \left(\frac{\bar{R}^k}{\bar{R}} - 1 \right) \hat{Q}_{t-1} \\
&\quad \gamma \frac{\bar{K}}{\bar{N}} \left(\frac{\bar{R}^k}{\bar{R}} - 1 \right) \hat{K}_t + \gamma \bar{R} (\hat{R}_{t-1} + \hat{N}_{t-1}) + \frac{\bar{W}^e}{\bar{N}} \hat{W}_t^e \\
\hat{N}_t &= \frac{\gamma \bar{R} \bar{K}}{\bar{N}} (\hat{r}_t^k - \hat{r}_t) + \gamma \bar{R} (\hat{r}_{t-1} + \hat{N}_{t-1}) + \gamma \frac{\bar{K}}{\bar{N}} \left(\frac{\bar{R}^k}{\bar{R}} - 1 \right) (\hat{r}_t^k - \hat{Q}_{t-1} + \hat{K}_t) + \frac{\bar{W}^e}{\bar{N}} \hat{W}_t^e .
\end{aligned}$$

This is almost exactly as what BGG have in their paper (equation 4.24). They have a coefficient 1 multiplying $\hat{r}_{t-1} + \hat{n}_{t-1}$, whereas Sims has $\gamma \bar{R}$. Sims assumes that BGG are approximating $\gamma \bar{R} \approx 1$. γ will be slightly less than 1, and $\bar{R}$ slightly greater than 1, so it's probably fine.

The other terms relate to the "higher order terms" (which don't actually seem to be higher order but which are nevertheless small). BGG may have a couple typos or errors. If you look at ϕ_t^n on page 1362, this is basically what Sims has but with a few exceptions. First, ti seems there should be a γ multiplying the first term in ϕ_t^n. Second, BGG seem to be missing a parenthesis on the $-\hat{X}_t$ at the end of that expression – it should be weighted by $\bar{W}^e / \bar{N}$. But, again, quantitatively they are not missing much by keeping these terms out. $\bar{R}^k / \bar{R} = 1.02^{0.25}$ (a 200 basis point annualised spread). Hence, $\bar{R}^k / \bar{R} - 1 \approx 0$. So BGG are just dropping these terms, which seems fine. Finally, since $\bar{W}^e$ is very small, $\bar{W}^e / \bar{N} \approx 0$ so the last term drops out as well in a loose approximate sense.

The exogenous processes and policy rule are already log-linear.

17.6.8 The formal contracting problem

Where does the formal contracting problem come from? basically, we want to understand where the condition relating the interest rate spread to firm leverage comes from. For completeness, the linearised condition is below:

$$\mathbb{E}_t \hat{r}^k_{t+1} - \hat{r}_t = -\upsilon \left[\hat{N}_t - (\hat{Q}_t + \hat{K}_{t+1}) \right].$$

The formal problem is not very well laid out by BGG. A better exposition can be found in Christiano, Motto, et al. (2014) or Carlstrom, Fuerst, and Paustian (2016).

Because all firms end up with the same optimality conditions, we are going to drop firm-specific superscripts in what follows so as to ease up the notation a bit. A firm gets a loan from an intermediary to finance the entirety of its next-period stock of capital.[132] The firm has net worth of N_t and wishes to purchase $Q_t K_{t+1}$ of new capital at the end of period t. It hence borrows $Q_t K_{t+1} - N_t$ from the intermediary. Suppose that the [gross] loan rate is Z_{t+1}, After the borrower makes the loan decision, he receives an idiosyncratic shock to the return, ω_{t+1}.[133] Let R^k_{t+1} be the aggregate return on capital, over which there is uncertainty because of aggregate shocks; the borrowers' specific return is $\omega_{t+1} R^k_{t+1}$. Average across firms, the $\omega_{t+1} = 1$. A particular firms gets to keep $\omega_{t+1} R^k_{t+1}$, and has to pay back $Z_{t+1}(Q_t K_{t+1} - N_t)$ in the event of no default. The borrower will default if his net return is negative. This implies a cutoff value of ω_{t+1} (call it $\bar{\omega}_{t+1}$) below which she will choose to default. This is implicitly defined by:

$$Z_{t+1}(Q_t K_{t+1} - N_t) = \bar{\omega}_{t+1} R^k_{t+1} Q_t K_{t+1}. \tag{848}$$

Note that $\bar{\omega}_{t+1}$ depends on the realisation of R^k_{t+1}. It is convenient to write this cutoff in terms of a leverage ratio:

$$L_t = \frac{Q_t K_{t+1}}{N_t}.$$

132 This is different than Carlstrom and Fuerst (1997), where you just finance the production of investment.

133 Note, to be completely correct this should have a firm specific superscript on it, but we are going to ignore that for now.

Then, we see that the loan rate satisfies:

$$Z_{t+1} = \bar{\omega}_{t+1} R^k_{t+1} \frac{L_t}{L_t - 1}. \tag{849}$$

The ω_{t+1} that each entrepreneur draws is distributed log-normal, with CDF $\Phi(\omega_{t+1})$, density $\phi(\omega_{t+1})$, and $\mathbb{E}[\omega_{t+1}] = 1$[134]. Let us calculate the expected shares of the payout from the project the entrepreneur and lender each get to keep, respectively. The expected entrepreneurial income from getting a loan is:

$$\int_{\bar{\omega}_{t+1}}^{\infty} \omega_{t+1} \phi(\omega_{t+1}) d\omega_{t+1} R^k_{t+1} Q_t K_{t+1} - (1 - \Phi(\bar{\omega}_{t+1})) Z_{t+1} (Q_t K_{t+1} - N_t). \tag{850}$$

The first term is the expected payout condition on not defaulting, i.e., drawing $\omega_{t+1} \geq \bar{\omega}_{t+1}$. The second term is the expected repayment, which is the probability of non default, $1 - \Phi(\bar{\omega}_{t+1})$, times the repayment, $Z_{t+1}(Q_t K_{t+1} - N_t)$. But from (848), we can get ride of the Z_{t+1} term:

$$\int_{\bar{\omega}_{t+1}}^{\infty} \omega_{t+1} \phi(\omega_{t+1}) d\omega_{t+1} R^k_{t+1} Q_t K_{t+1} - (1 - \Phi(\bar{\omega}_{t+1})) \bar{\omega}_{t+1} R^k_{t+1} Q_t K_{t+1}. \tag{851}$$

But then this reduces to:

$$\left[\int_{\bar{\omega}_{t+1}}^{\infty} \omega_{t+1} \phi(\omega_{t+1}) d\omega_{t+1} - (1 - \Phi(\bar{\omega}_{t+1})) \bar{\omega}_{t+1} \right] R^k_{t+1} Q_t K_{t+1}. \tag{852}$$

Now define $f(\bar{\omega}_{t+1})$ as the term inside the brackets, which is the share of the returns the firm expects to keep:

$$f(\bar{\omega}_{t+1}) = \int_{\bar{\omega}_{t+1}}^{\infty} \omega_{t+1} \phi(\omega_{t+1}) d\omega_{t+1} - (1 - \Phi(\bar{\omega}_{t+1})) \bar{\omega}_{t+1}.$$

The borrower is exposing her net worth, N_t, to earn (852). The total return is the ratio. Using the definition of leverage above, we can write the firm's expected return as:

$$f(\bar{\omega}_{t+1}) R^k_{t+1} L_t. \tag{853}$$

[134]This expectation is across entrepreneurs; there is no aggregate uncertainty on ω_{t+1}.

Now, let's think about the lender's expected return from the project. It is:

$$\int_0^{\bar{\omega}_{t+1}} \omega_{t+1}(1-\mu)R^k_{t+1}Q_tK_{t+1}\phi(\omega_{t+1})d\omega_{t+1} + (1-\Phi(\bar{\omega}_{t+1}))Z_{t+1}(Q_tK_{t+1}-N_t). \tag{854}$$

The first term is what the lender expects to keep in the event of default. She gets to keep $(1-\mu)R^k_{t+1}Q_tK_{t+1}$ times the expected value of ω_{t+1} conditional on the entrepreneur defaulting, i.e. $\omega_{t+1} < \bar{\omega}_{t+1}$. $\mu \geq 0$ is a bankruptcy cost. The second term is just the probability of no default times the return on making a loan in that case. But again, using (848), we can write this as:

$$\left[(1-\mu)\int_0^{\bar{\omega}_{t+1}} \omega_{t+1}\phi(\omega_{t+1})d\omega_{t+1} + (1-\Phi(\bar{\omega}_{t+1}))\bar{\omega}_{t+1}\right] R^k_{t+1}Q_tK_{t+1}. \tag{855}$$

Define the term in brackets as the lender's expected share of the return:

$$g(\bar{\omega}_{t+1}) = (1-\mu)\int_0^{\bar{\omega}_{t+1}} \omega_{t+1}\phi(\omega_{t+1})d\omega_{t+1} + (1-\Phi(\bar{\omega}_{t+1}))\bar{\omega}_{t+1}. \tag{856}$$

The entrepreneur is exposing $Q_tK_{t+1} - N_t$ (the amount of the loan), to get back (855). The expected return is therefore:

$$\frac{g(\bar{\omega}_{t+1})R^k_{t+1}Q_tK_{t+1}}{Q_tK_{t+1}-N_t}. \tag{857}$$

Using the definition of leverage, this can be written as:

$$g(\bar{\omega}_{t+1})R^k_{t+1}\frac{L_t}{L_t-1}. \tag{858}$$

Now, we can write the formal contracting problem. The entrepreneur wants to pick a leverage ratio, L_t, and cutoff value of $\bar{\omega}_{t+1}$, to maximise her expected return subject to a participation constraint for the lender. The lender is assumed to be risk neutral, and hence faces an opportunity cost of funds of the safe gross interest rate, R_t. Hence, the formal problem for the entrepreneur is:

$$\max_{\bar{\omega}_{t+1}, L_t} \ \mathbb{E}_t R^k_{t+1} f(\bar{\omega}_{t+1})L_t,$$

subject to:

$$R^k_{t+1} g(\bar{\omega}_{t+1}) \frac{L_t}{L_t - 1} \geq R_t.$$

As Carlstrom, Fuerst, and Paustian (2016) emphasise, the lender's return is predetermined. It is R_t, the safe [gross] interest rate. This means that $\bar{\omega}_{t+1}$ is state-contingent – it moves with R^k_{t+1} such that the participation constraint will always hold and the lender gets R_t.

We can characterise the optimum using a Lagrangian. Let Λ_{t+1} be the multiplier on the constraint. The Lagrangian is:

$$\mathcal{L} = \mathbb{E}_t \left\{ R^k_{t+1} f(\bar{\omega}_{t+1}) L_t + \Lambda_{t+1} \left[R^k_{t+1} g(\bar{\omega}_{t+1}) L_t - R_t (L_t - 1) \right] \right\},$$

and the FOCs are:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \bar{\omega}_{t+1}} &= \mathbb{E}_t \left\{ R^k_{t+1} f'(\bar{\omega}_{t+1}) L_t + \Lambda_{t+1} R^k_{t+1} g'(\bar{\omega}_{t+1}) L_t \right\}, \\
\frac{\partial \mathcal{L}}{\partial L_t} &= \mathbb{E}_t \left\{ R^k_{t+1} f(\bar{\omega}_{t+1}) + \Lambda_{t+1} \left[R^k_{t+1} g(\bar{\omega}_{t+1}) - R_t \right] \right\}, \\
\frac{\partial \mathcal{L}}{\partial \Lambda_{t+1}} &= R^k_{t+1} g(\bar{\omega}_{t+1}) L_t - (L_t - 1) R_t.
\end{aligned}$$

Setting these equal to zero and simplifying yields:

$$0 = \mathbb{E}_t \left\{ R^k_{t+1} f'(\bar{\omega}_{t+1}) L_t + \Lambda_{t+1} R^k_{t+1} g'(\bar{\omega}_{t+1}) L_t \right\}, \tag{859}$$

$$0 = \mathbb{E}_t \left\{ R^k_{t+1} f(\bar{\omega}_{t+1}) + \Lambda_{t+1} \left[R^k_{t+1} g(\bar{\omega}_{t+1}) - R_t \right] \right\}, \tag{860}$$

$$R^k_{t+1} g(\bar{\omega}_{t+1}) L_t = (L_t - 1) R_t. \tag{861}$$

Note that (861) holds for all possible realisations of R^k_{t+1} – i.e., $\bar{\omega}_{t+1}$ is state-contingent and adjusts to ensure that the lender's return is always predetermined.

Now, let's linearise these FOCs about the steady state. To a first order, we needn't worry about the expectations operator. Start with (859). Note that we can drop L_t and R^k_{t+1} for now:

$$0 = f''(\bar{\bar{\omega}}) d\bar{\omega}_{t+1} + g'(\bar{\bar{\omega}}) d\Lambda_{t+1} + g''(\bar{\bar{\omega}}) \bar{\Lambda} d\bar{\omega}_{t+1}.$$

Apologies for the awkward double bar notation on the steady state value for $\bar{\omega}_{t+1}$. Using our usual log-linearisation notation, we have:

$$0 = f''(\bar{\bar{\omega}})\bar{\bar{\omega}}\hat{\bar{\omega}}_{t+1} + g'(\bar{\bar{\omega}})\bar{\Lambda}\hat{\Lambda}_{t+1} + g''(\bar{\bar{\omega}})\bar{\Lambda}\bar{\bar{\omega}}\hat{\bar{\omega}}_{t+1}.$$

We know that, in steady state, we must have:

$$\bar{\Lambda} = -\frac{f'(\bar{\bar{\omega}})}{g'(\bar{\bar{\omega}})}, \tag{862}$$

hence:

$$0 = \bar{\bar{\omega}} f''(\bar{\bar{\omega}})\hat{\bar{\omega}}_{t+1} - f'(\bar{\bar{\omega}})\hat{\Lambda}_{t+1} - \bar{\bar{\omega}} f'(\bar{\bar{\omega}})\frac{g''(\bar{\bar{\omega}})}{g'(\bar{\bar{\omega}})}\hat{\bar{\omega}}_{t+1}.$$

Move the second term to the LHS, divide both sides by $f'(\bar{\bar{\omega}})$ to get:

$$\hat{\Lambda}_{t+1} = \left[\frac{\bar{\bar{\omega}} f''(\bar{\bar{\omega}})}{f'(\bar{\bar{\omega}})} - \bar{\bar{\omega}}\frac{g''(\bar{\bar{\omega}})}{g'(\bar{\bar{\omega}})}\right]\hat{\bar{\omega}}_{t+1},$$

and then define $\Psi = \frac{\bar{\bar{\omega}} f''(\bar{\bar{\omega}})}{f'(\bar{\bar{\omega}})} - \bar{\bar{\omega}}\frac{g''(\bar{\bar{\omega}})}{g'(\bar{\bar{\omega}})}$, so we can write the log-linear version of (859) as:

$$\hat{\Lambda}_{t+1} = \Psi\hat{\bar{\omega}}_{t+1}. \tag{863}$$

Before log-linearising (860), combine it with (861), noting that

$$R^l_{t+1} g(\bar{\omega}_{t+1}) = \frac{L_t - 1}{L_t} R_t,$$

in order to write:

$$0 = \mathbb{E}_t\left[R^k_{t+1} f(\bar{\omega}_{t+1}) - \Lambda_{t+1}\frac{R_t}{L_t}\right].$$

But, ignoring the expectations operator, we can write this as:

$$R^k_{t+1} f(\bar{\omega}_{t+1}) = \Lambda_{t+1}\frac{R_t}{L_t}.$$

Take logs:

$$\ln R^k_{t+1} + \ln f(\bar{\omega}_{t+1}) = \ln \Lambda_{t+1} + \ln R_t - \ln L_t,$$

and totally differentiate:

$$\hat{r}^k_{t+1} + \frac{f'(\bar{\bar{\omega}})}{f(\bar{\bar{\omega}})} d\bar{\omega}_{t+1} = \hat{\Lambda}_{t+1} + \hat{r}_t - \hat{L}_t.$$

We can write this as:

$$\hat{r}^k_{t+1} + \bar{\bar{\omega}} \frac{f'(\bar{\bar{\omega}})}{f(\bar{\bar{\omega}})} \hat{\bar{\omega}}_{t+1} = \hat{\Lambda}_{t+1} + \hat{r}_t - \hat{L}_t.$$

Define $\Theta_f = \bar{\bar{\omega}} \frac{f'(\bar{\bar{\omega}})}{f(\bar{\bar{\omega}})}$, so we can write:

$$\hat{\Lambda}_{t+1} = \hat{r}^k_{t+1} - \hat{r}_t + \hat{L}_t + \Theta_f \hat{\bar{\omega}}_{t+1}. \tag{864}$$

Now, let's linearise (861). Take logs first:

$$\ln R^k_{t+1} + \ln g(\bar{\omega}_{t+1}) + \ln L_t = \ln R_t + \ln(L_t - 1),$$

and then totally differentiate:

$$\hat{r}^k_{t+1} + \frac{g'(\bar{\bar{\omega}})}{g(\bar{\bar{\omega}})} d\bar{\omega}_{t+1} + \hat{L}_t = \hat{r}_t + \frac{1}{\bar{L} - 1} dL_t.$$

We can rewrite this as:

$$\hat{r}^k_{t+1} + \bar{\bar{\omega}} \frac{g'(\bar{\bar{\omega}})}{g(\bar{\bar{\omega}})} \hat{\bar{\omega}}_{t+1} + \hat{L}_t = \hat{r}_t + \frac{\bar{L}}{\bar{L} - 1} \hat{L}_t,$$

and define $\Theta_g = \bar{\bar{\omega}} \frac{g'(\bar{\bar{\omega}})}{g(\bar{\bar{\omega}})}$, to get:

$$\frac{\bar{L}}{\bar{L} - 1} \hat{L}_t = \hat{r}^k_{t+1} - \hat{r}_t + \Theta_g \hat{\bar{\omega}}_{t+1}. \tag{865}$$

The three log-linearised FOCs are thus:

$$\hat{\Lambda}_{t+1} = \Psi \hat{\bar{\omega}}_{t+1},$$
$$\hat{\Lambda}_{t+1} = \hat{r}^k_{t+1} - \hat{r}_t + \hat{L}_t + \Theta_f \hat{\bar{\omega}}_{t+1},$$
$$\frac{\bar{L}}{\bar{L}-1}\hat{L}_t = \hat{r}^k_{t+1} - \hat{r}_t + \Theta_g \hat{\bar{\omega}}_{t+1}.$$

Now, we can combine these to eliminate $\hat{\bar{\omega}}_{t+1}$ and $\hat{\Lambda}_{t+1}$. Start by plugging (863) into (864) to get:

$$\hat{r}^k_{t+1} - \hat{r}_t + \hat{L}_t = (\Psi - \Theta_f)\hat{\bar{\omega}}_{t+1}.$$

Now, from (865), we can solve for $\hat{\bar{\omega}}_{t+1}$ as:

$$\hat{\bar{\omega}}_{t+1} = \frac{1}{\Theta_g(\bar{L}-1)}\hat{L}_t - \frac{\hat{r}^k_{t+1} - r_t}{\Theta_g},$$

and combine it with the above expression to get:

$$\hat{r}^k_{t+1} - \hat{r}_t + \hat{L}_t = (\Psi - \Theta_f)\left[\frac{1}{\Theta_g(\bar{L}-1)}\hat{L}_t - \frac{\hat{r}^k_{t+1} - r_t}{\Theta_g}\right],$$

which can be written as:

$$\begin{aligned}
(\hat{r}^k_{t+1} - \hat{r}_t)\left[1 + \frac{\Psi - \Theta_f}{\Theta_g}\right] &= \left[\frac{\Psi - \Theta_f}{\Theta_g(\bar{L}-1)} - 1\right]\hat{L}_t \\
\Leftrightarrow (\hat{r}^k_{t+1} - \hat{r}_t)\frac{\Theta_g - \Theta_f + \Psi}{\Theta_g} &= \frac{\Psi - \Theta_f - \Theta_g(\bar{L}-1)}{\Theta_g(\bar{L}-1)}\hat{L}_t \\
\Leftrightarrow (\hat{r}^k_{t+1} - \hat{r}_t) &= \frac{\Psi - \Theta_f - \Theta_g(\bar{L}-1)}{(\Theta_g - \Theta_f + \Psi)(\bar{L}-1)}\hat{L}_t. \qquad (866)
\end{aligned}$$

Now, before stopping, we can note that there is a relationship between Θ_f and Θ_g. In the steady state, combining (861) with (860), we have:

$$\begin{aligned}
0 &= \bar{R}^k f(\bar{\bar{\omega}}) + \bar{\Lambda}\left[\bar{R}^k g(\bar{\bar{\omega}}) - \bar{R}^k g(\bar{\bar{\omega}})\frac{\bar{L}}{\bar{L}-1}\right] \\
\Leftrightarrow 0 &= f(\bar{\bar{\omega}}) + \bar{\Lambda} g(\bar{\bar{\omega}})\left[1 - \frac{\bar{L}}{\bar{L}-1}\right] \\
\Leftrightarrow 0 &= f(\bar{\bar{\omega}}) - \frac{\bar{\Lambda} g(\bar{\bar{\omega}})}{\bar{L}-1}.
\end{aligned}$$

But we know that $\bar{\Lambda} = -\frac{f'(\bar{\bar{\omega}})}{g'(\bar{\bar{\omega}})}$ from (862), so:

$$0 = f(\bar{\bar{\omega}}) - \frac{f'(\bar{\bar{\omega}}) g(\bar{\bar{\omega}})}{g'(\bar{\bar{\omega}})}\frac{1}{\bar{L}-1},$$

and divide both sides by $f'(\bar{\bar{\omega}})$:

$$0 = \frac{f(\bar{\bar{\omega}})}{f'(\bar{\bar{\omega}})} - \frac{g(\bar{\bar{\omega}})}{g'(\bar{\bar{\omega}})}\frac{1}{\bar{L}-1}.$$

But from (864) we have $\frac{f(\bar{\bar{\omega}})}{f'(\bar{\bar{\omega}})} = \frac{\bar{\bar{\omega}}}{\Theta_f}$ and similarly for the terms involving $g(\cdot)$ and Θ_g, hence:

$$\begin{aligned}
0 &= \frac{\bar{\bar{\omega}}}{\Theta_f} + \frac{\bar{\omega}}{\Theta_g}\frac{1}{\bar{L}-1} \\
\implies -\Theta_f &= \Theta_g(\bar{L}-1),
\end{aligned}$$

so put this into (866) to get:

$$\hat{r}^k_{t+1} - \hat{r}_t = \frac{\Psi}{\Psi(\bar{L}-1) - \Theta_f \bar{L}}\hat{L}_t. \tag{867}$$

Since $\hat{L}_t = \hat{Q}_t + \hat{K}_{t+1} - \hat{N}_t$, this expression is the same as (818), where $\upsilon = \frac{\Psi}{\Psi(\bar{L}-1)-\Theta_f\bar{L}}\hat{L}_t$. The important point here is that there is a positive relationship between entrepreneur leverage, $\hat{L}_t$, and the lending spread, $\hat{r}^k_{t+1} - \hat{r}_t$.

Now, recall from (863) that:

$$\Psi = \bar{\bar{\omega}}\left[\frac{f''(\bar{\bar{\omega}})}{f'(\bar{\bar{\omega}})} - \frac{g''(\bar{\bar{\omega}})}{g'(\bar{\bar{\omega}})}\right].$$

We have that these shares must sum to (this is always, not just at steady state, but we're evaluating it at steady state):

$$f(\bar{\bar{\omega}}) + g(\bar{\bar{\omega}}) = 1 - \mu \int_0^{\bar{\omega}} \omega \phi(\omega) d\omega$$

Note that if $\mu = 0$ (no bankruptcy cost), then we have $f(\bar{\bar{\omega}}) = -g(\bar{\bar{\omega}})$. But this then would mean that $\Psi = 0$. In other words, $\upsilon = 0$, and there would be no relationship between leverage and the external finance premium!

17.6.9 Calibration

We're not going to go into great depth on calibrating the model. For the purposes of the linearisation, all that really matters are a few steady state ratios and a few key parameters (such as υ, the sensitivity of the interest rate spread to leverage).

We're going to follow what BGG reports. Some parameters they don't fully report, so Sims picks values that are reasonable and will look at sensitivity of the model's IRFs to those parameters.

Table 10: Calibration of BGG Model

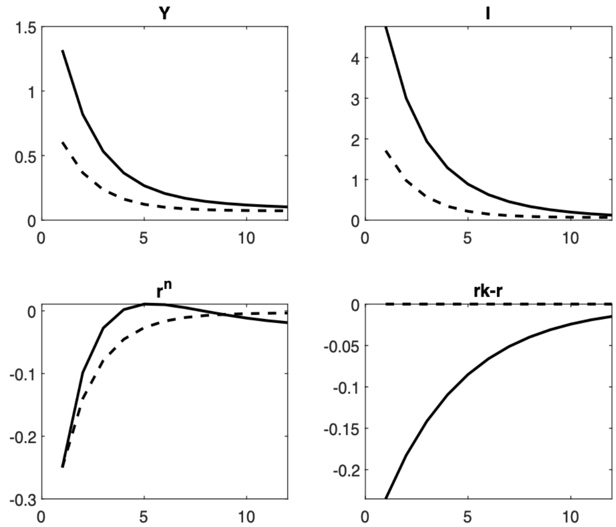
β	0.99	Discount factor	φ	0.25	Capital adjust cost
η	3	Frisch labour elasticity	$1-\gamma$	0.0272	Entrepreneur mortality rate
α	0.35	Capital share of output	θ	0.75	Calvo parameter
$(1-\alpha)(1-\Omega)$	0.64	Entrepreneur labour share is 0.01	ρ	0.9	Interest rate rule
δ	0.025	Depreciation	ζ	0.11	Parameter on inflation[135]
$\frac{\bar{G}}{\bar{Y}}$	0.2	G steady state ratio	ϵ	0.96	Capital Euler equation
$\frac{\bar{C}}{\bar{Y}}$	0.51	C steady state ratio	υ	0.2	External finance premium
$\frac{\bar{I}}{\bar{Y}}$	0.18	I steady state ratio	ρ_g	0.95	AR(1) coefficient
$\frac{\bar{C}^e}{\bar{Y}}$	0.12	C^e steady state ratio	ρ_a	1	AR(1) coefficient

17.6.10 Numerical results

Sims' results broadly matches the results in the BGG paper, albeit not perfectly. First, consider the responses to the monetary policy shock. Note the scale of the shock – Sims shocks the policy rule by 0.25/4 in the model (which is quarterly), which in turn produces an annualised policy rate response of 25 basis points on impact.

[135] So the long-run response of the interest rate to inflation $\zeta/(1-\rho)$, so 1.1, consistent with the Taylor principle.

Figure 133: IRFs to Monetary Policy Shock

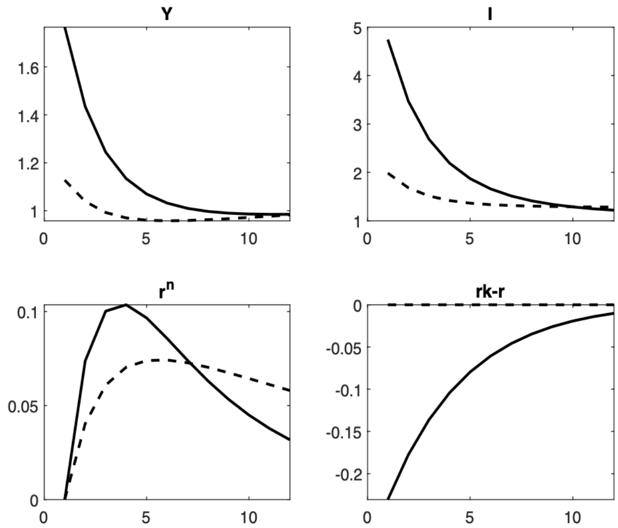


Source: Sims 2020

The responses shown above are very similar (if not exact) to the responses shown in Figure 3 of the BGG paper. Solid lines show responses when $\upsilon = 0.2$, so that there is a financial accelerator mechanism. Dashed lines fix $\upsilon = 0$, so that this mechanism is absent. Output and investment go up (and revert) after an exogenous cut in the policy rate. The financial accelerator in fact amplifies the effects of the policy shock – both output and investment go up significantly more. The interest rate spread, or perhaps more precisely the external finance premium, shown in the bottom right of the figure declines. This is the source of the amplification.

There is a kind of multiplier effect. The stimulative monetary policy raises the demand for capital, which raises investment and the price of capital, $\hat{Q}_t$. This increase in asset prices raises net worth. Higher net worth lowers the external finance premium. But this further stimulates investment and the price of capital, which further stimulates net worth. This is the "accelerator" idea – the change in asset prices lowers the external finance premium, which in turn further stimulates asset prices and real activity.

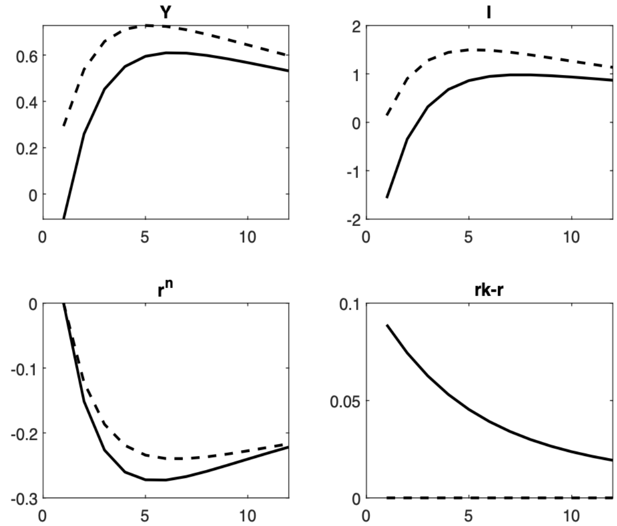
Figure 134: IRFs to Productivity Shock



Source: Sims 2020

Impulse responses to the productivity shock are shown above. These are very similar to what is shown in Figure 4 of the paper. But there is a bit of slight of hand going on. This result turns out to be very sensitive to the assumed autocorrelation of the productivity process. What happens when we assume a more mean-reverting value such as $\rho_a = 0.95$ instead of $\rho_a = 1$ in the BGG paper? The IRFs are shown below:

Figure 135: IRFs to Productivity Shock ($\rho_a = 0.95$ instead of $\rho_a = 1$)

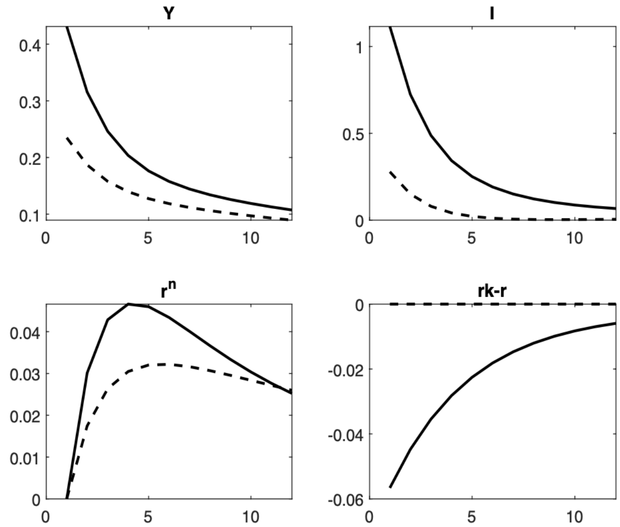


Source: Sims 2020

In this specification, the financial accelerator actually dampens the responses to the productivity shock relative to the unconstrained model. What's driving this is again the price of capital. When $\rho_a \to 1$, the productivity shock is much more of a demand shock than a supply shock, and with sticky prices, output is at least partially demand determined. There is a big demand for output, which puts upward pressure on $\hat{Q}_t$ and net worth, and consequently lowers the external finance premium. But when the shock is (just a little) less persistent, things flip – demand doesn't rise by much, $\hat{Q}_t$ doesn't change by much, and the external finance premium actually goes up, not down. It's not a formal proof, but in lots of these models, you see that financial frictions amplify demand shocks but often weaken supply shocks. This is what we see at play here.

Next, consider the government spending shock. The responses are shown below. These are similar to what BGG report in their paper. There is amplification from the financial accelerator mechanism and the external finance premium.

Figure 136: IRFs to Government Spending Shock



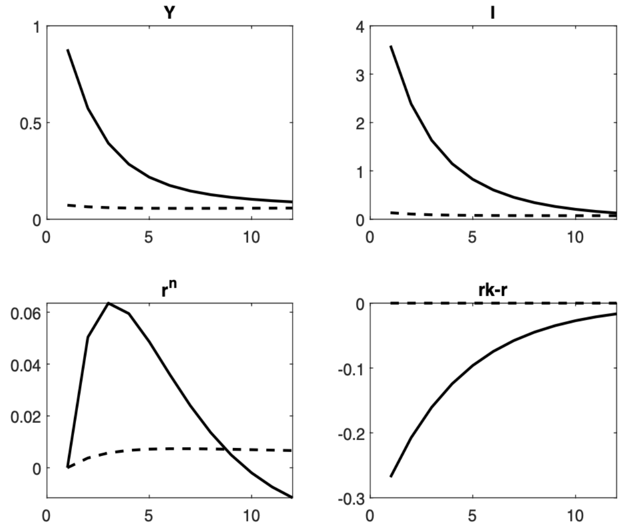
Source: Sims 2020

Finally, consider an exogenous shock to net worth. This is introduced via a shock to the net worth evolution expression (and an offsetting transfer from households, which does not otherwise show up in the linearised equilibrium conditions). In particular:

$$\hat{N}_t = \gamma \frac{\bar{R}\bar{K}}{\bar{N}} (\hat{r}_t^k - \hat{r}_{t-1}) + \hat{r}_{t-1} + \hat{N}_{t-1} + \epsilon_{n,t}.$$

The next figure shows the impulse responses to the net worth shock.

Figure 137: IRFs to Net Worth Shock



Source: Sims 2020

The effects of the shock are pretty easy to understand. When entrepreneurs exogenously get more net worth, agency frictions decline. This lowers the external finance premium and leads to a boom. Note that this redistribution would have small, non-zero effects when when $\upsilon = 0$ (so no financial accelerator mechanism); this is because more net worth stimulates entrepreneurial consumption.

17.6.11 Differences relative to an RBC model (Carlstrom and Fuerst (1997))

In Carlstrom and Fuerst, the agency friction tends to dampen the response to a productivity shock but increases propagation. In the BGG setup, we don't see the hump-shaped propagation but instead see amplification.

There are some differences in the two setups that end up driving these results. For one, BGG have stick prices and a capital adjustment cost (which, even absent agency frictions, would result in a time-variation in the price of capital, $\hat{Q}_t$). But there is another subtle difference. In Carlstrom and Fuerst, the agency friction only applies to entrepreneurs who produce new investment goods. In BGG, the agency friction applies to production firms who own their entire capital stock. A simple way to think about this is that in Carlstrom and Fuerst agency frictions apply to producers of new investment goods, whereas in BGG agency frictions apply to the whole capital stock (which is much bigger than the flow of new investment). See the discussion above about the formal contracting problem. Fluctuations in the supply price of capital therefore have much bigger effects on net worth in the BGG framework and end up being a source of amplification.

17.7 Credit cycles (Kiyotaki and Moore, 1997)

There are two basic ways macroeconomists introduce financial frictions into models – either via the CSV approach (e.g. Bernanke and Gertler (1989), Bernanke, Gertler, and Gilchrist (1999), and so on), or the limited enforcement approach.

We now look at the limited enforcement approach featured in Kiyotaki and Moore (1997).[136] The basic idea of the limited enforcement approach is that borrowers face a binding borrowing constraint, where the constraint is some function of the market-value of their assets. This constraint arises because of limited enforceability – lenders can seize borrower assets in default, but those assets are worth less to the lender than in the hands of the borrower (e.g. there is a bankruptcy cost or the borrower is more efficient at using the underlying asset). Because of this, the lender will limit how much credit he/she will extend to a borrower so that the borrower does not find it optimal to default. In equilibrium, provided certain assumptions (typically on discounting the future) are satisfied, the borrowing constraints will bind. This in turn gives rise to a financial accelerator type effect. Shocks that raise asset prices will ease borrowing constraints. This will allow borrowers more access to credit, which will result in more investment and aggregate demand, and hence even higher asset prices.

This is very similar to, for example, the BGG story. But it turns to be "easier" to work with, as you don't have the heterogeneity of the CSV framework. The drawback is that, in equilibrium, there is no default/bankruptcy in the limited enforcement approach, whereas there is in the CSV approach.

In this chapter, we are going through the "simple" model of Kiyotaki and Moore (Section II). In Section III, they add reproducible capital which has some desirable properties, including generating more persistence. In Section IV, they talk about sectoral spillovers. But the key insights come from Section II.

[136] My personal opinion, but I think Kiyotaki and Moore will win the Nobel prize in economics for this paper.

17.7.1 The simple model

There are two types of agents in the model: farmers and gatherers, with gatherers denoted with a $'$ superscript. Time is discrete and lasts forever, starting in period t. There is a durable asset which is used as a factor of production. It is not reproducible. Think of it as land, and denote the fixed aggregate supply of it as $\bar{K}$. Denote consumption of farmers and gatherers as x_t and x'_t, respectively. Both types of agents are risk neutral. The farmers have discount factor β and the gatherers have discount factor β', with $\beta < \beta'$. This means that farmers are relatively more impatient, and in equilibrium they will be borrowers.

17.7.2 Farmers

Farmers and gatherers have different production technologies. Let k_t be the land held by a farmer in period t. This can be turned into output via the constant returns to scale production technology:

$$y_t = (a + c)k_{t-1}, \tag{868}$$

where ak_t is the amount of output that is tradable, and ck_t is the amount of a farmer's output that is non-tradable, but nevertheless still consumable, but only by the farmer himself. Think of this as being something like bruised fruit – a farmer can't sell it, but he can eat it.

Land trades in a competitive spot market at price q_t (measured in units of the consumption good, called fruit). Let R_t be the gross interest rate on bonds carried from t to $t + 1$. Farmers are subject to a borrowing constraint:

$$R_t b_t \leq q_{t+1} k_t. \tag{869}$$

What is the intuition for (869)? There are two underlying assumptions that give rise to this constraint. First, the farmer has to "work" to produce output (though we are not formally modelling labour at all), but can in principle choose not to work. Second, if the farmer doesn't "work", no one else can use his land to produce trees. If a lender makes a loan to a farmer, he is due back $R_t b_t$ in the subsequent period. If the borrower chooses to not pay back, the lender can't force the borrower to work (hence, "limited enforcement"). Rather, the lender can just confiscate the farmer's land, k_t, which will be worth q_{t+1} in period $t+1$. The lender would never loan to the farmer if $R_t b_t > q_{t+1} k_t$. If this were the case, the farmer would definitely choose to default: Instead of paying back $R_t b_t$, he could just not work and selling his land for $q_{t+1} k_{t+1}$, which would allow him to enjoy more consumption. So $R_t b_t \leq q_{t+1} k_t$ ensures that the farmer never defaults on the interperiod loan.

The farmer's budget constraint looks as follows:

$$q_t(k_t - k_{t-1}) + R_{t-1} b_{t-1} + x_t = (a + c) k_{t-1} + b_t. \tag{870}$$

On the "expenditure side" of (870), the farmer i) purchases new land, $q_t(k_t - k_{t-1})$; ii) pays off interest plus principle on any loans, $R_{t-1} b_{t-1}$; and iii) chooses how much to eat, x_t. On the "income side", the farmer produces output using inherited capital, $(a + c) k_{t-1}$, and can issue more intertemporal debt, b_t.

Thus, the farmer's problem is:[137]

$$\max_{x_{t+s}, k_{t+s}, b_{t+s}} \sum_{s=0}^{\infty} \beta^s x_{t+s},$$

subject to

$$\begin{aligned} q_t(k_t - k_{t-1}) + R_{t-1} b_{t-1} + x_t &= (a + c) k_{t-1} + b_t, \\ R_t b_t &\leq q_{t+1} k_t, \\ x_t &\geq c k_{t-1}. \end{aligned}$$

A Lagrangian for the farmer is as follows, with λ_t and μ_t denoting the multipliers on the budget and borrowing constraints, respectively. We have a third constraint, which is that $x_t \geq c k_{t-1}$, since $c k_{t-1}$ is not tradable, the farmer must eat at least this quantity. Let φ_t be the multiplier on this constraint. We will consider a world with no aggregate uncertainty (though we will consider perfect

[137] We are assuming a very simple linear utility function here.

foresight or "MIT shocks" later), so we can drop expectations operators.

$$\mathcal{L} = \sum_{s=0}^{\infty} \beta^s \left\{ \begin{array}{c} x_{t+s} + \lambda_{t+s} \left[(a+c)k_{t+s-1} + b_{t+s} - q_{t+s}(k_{t+s} - k_{t+s-1}) - R_{t+s-1}b_{t+s-1} - x_{t+s} \right] \\ +\mu_{t+s} \left[q_{t+s+1}k_{t+s} - R_{t+s}b_{t+s} \right] + \varphi_{t+s} \left[x_{t+s} - ck_{t+s-1} \right] \end{array} \right\},$$

and take derivatives of the Lagrangian:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_t} &= 1 - \lambda_t + \varphi_t, \\
\frac{\partial \mathcal{L}}{\partial b_t} &= \lambda_t - \mu_t R_t + \beta \lambda_{t+1} R_t, \\
\frac{\partial \mathcal{L}}{\partial k_t} &= -q_t \lambda_t + \mu_t q_{t+1} + \beta \lambda_{t+1} \left[(a+c) + q_{t+1} \right] - \beta c \varphi_{t+1}.
\end{aligned}$$

Set these equal to zero and eliminate the multiplier on the budget constraint:

$$1 + \varphi_t = (\beta(1 + \varphi_{t+1}) + \mu_t) R_t, \tag{871}$$

$$q_t(1 + \varphi_t) + \beta c \varphi_{t+1} = \beta(1 + \varphi_{t+1})[a + c + q_{t+1}] + \mu_t q_{t+1}. \tag{872}$$

(871) and (872) would be standard asset pricing conditions in the absence of the constraints. The price of the bond (normalised to 1) would just equal the product of the stochastic discount factor (just β with linear preferences) with the bond payout, R_t. The price of the land would equal the product of the stochastic discount factor (again, just β) with the sum of the flow benefit of the land, $a + c$, with the continuation value of land, q_{t+1}. $\mu_t \geq 0$ and $\varphi_t > 0$ throw "wedges" into both FOCs.

17.7.3 Gatherers

Gatherers produce output via:

$$y_t^{'} = G(k_{t-1}'), \tag{873}$$

where $G'(\cdot) > 0$, $G''(\cdot) < 0$, $G'(0) > 0$. There are two other auxiliary assumptions that ensure that, in equilibrium, both farmers and gathers will produce. The first is that all gatherer output is tradable.

A gatherer's budget constraint is:

$$q_t(k_t^{'} - k_{t-1}^{'}) + R_{t-1}b_{t-1}^{'} + x_t^{'} = G(k_{t-1}^{'}) + b_t^{'}. \tag{874}$$

In equilibrium, we will have $b_t^{'} < 0$, so that gatherers are actually saving (positive values would denote borrowing the way the constraint has been written). (874) follows similar ideas as the constraint for the farmer.

The gathers' problem is thus:

$$\max_{x_{t+s}^{'}, k_{t+s}^{'}, b_{t+s}^{'}} \sum_{s=0}^{\infty} (\beta^{'})^s x_{t+s}^{'},$$

subject to

$$q_t(k_t^{'} - k_{t-1}^{'}) + R_{t-1}b_{t-1}^{'} + x_t^{'} = G(k_{t-1}^{'}) + b_t^{'}.$$

The Lagrangian for the gatherer is:

$$\mathcal{L} = \sum_{s=0}^{\infty} (\beta^{'})^s \left\{ x_{t+s}^{'} + \lambda_t^{'} \left[G(k_{t-1}^{'}) + b_t^{'} - q_t(k_t^{'} - k_{t-1}^{'}) - R_{t-1}b_{t-1}^{'} - x_t^{'} \right] \right\}.$$

The first order conditions are:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x_t^{'}} &= 1 - \lambda_t^{'}, \\
\frac{\partial \mathcal{L}}{\partial b_t^{'}} &= \lambda_t^{'} - \beta^{'} R_t \lambda_{t+1}^{'}, \\
\frac{\partial \mathcal{L}}{\partial k_t^{'}} &= -\lambda_t^{'} q_t + \beta^{'} \lambda_{t+1}^{'} G'(k_t^{'}) + \beta^{'} \lambda_{t+1}^{'} q_{t+1}.
\end{aligned}$$

Eliminating the multiplier, we get:

$$1 = \beta^{'} R_t, \tag{875}$$

$$q_t = \beta^{'} [G'(k_t^{'}) + q_{t+1}]. \tag{876}$$

(875) implies that the gross interest rate is constant at $R = 1/\beta^{'}$, and (876) is the standard asset pricing condition.

17.7.4 Equilibrium

The population size of farmers is 1; the population size of gatherers is m. Within type, everyone is identical. So for aggregate market clearing, we can just sum across types. Market clearing requires:

$$b_t + mb_t^{'} = 0, \tag{877}$$

$$k_t + mk_t^{'} = \bar{K}. \tag{878}$$

(877) just requires that one type's saving equals the other type's borrowing. The m just scales the gatherer population relative to the farmers. (878) reflects market clearing for the fixed quantity of land. Now, sum the budget constraints across type, imposing that R is fixed as shown above:

$$q_t k_t - q_t k_{t-1} + Rb_{t-1} + x_t + q_t mk_t^{'} - q_t mk_{t-1}^{'} + Rmb_{t-1}^{'} = (a+c)k_{t-1} + b_t + mG(k_{t-1}^{'}) + mb_t^{'},$$

which may be written as:

$$q_t(k_t + mk_t^{'}) + q_t(k_{t-1} + mk_{t-1}^{'}) + R(b_{t-1} + mb_{t-1}^{'}) + x_t + mx_t^{'} = (a+c)k_{t-1} + mG(k_{t-1}^{'}) + (b_t + mb_t^{'}).$$

But then using the market clearing conditions (zero total debt, fixed supply of capital), we get:

$$x_t + mx_t^{'} = (a+c)k_{t-1} + mG(k_{t-1}^{'}) = y_t + my_{t-1} = Y_t, \tag{879}$$

which is just the standard aggregate resource constraint.

The full set of equilibrium conditions can then be written as:

$$1 + \varphi_t = (\beta(1 + \varphi_{t+1}) + \mu_t)\frac{1}{\beta'}, \tag{880}$$

$$q_t(1 + \varphi_t) + \beta c \varphi_{t+1} = \beta(1 + \varphi_{t+1})[a + c + q_{t+1}] + \mu_t q_{t+1}, \tag{881}$$

$$q_t(k_t - k_{t-1}) + \frac{1}{\beta'} b_{t-1} + x_t = (1 + c)k_{t-1} + b_t, \tag{882}$$

$$b_t \leq \beta^{'} q_{t+1} k_t, \tag{883}$$

$$q_t = \beta^{'} [G'(k_t^{'}) + q_{t+1}], \tag{884}$$

$$x_t + m x_t^{'} = (a + c)k_{t-1} + mG(k_{t-1}^{'}), \tag{885}$$

$$k_t + m k_t^{'} = \bar{K}, \tag{886}$$

$$x_t \geq c k_{t-1}, \tag{887}$$

where we have eliminated R_t as a variable, instead treating it as a parameter, $R = 1/\beta^{'}$. We have also eliminated $b_t^{'}$ using the bond market clearing condition, and the aggregate resource constraint subsumes the budget constraint for gatherers. This leaves just eight equations in eight variables: $\left\{x_t, x_t^{'}, k_t, k_t^{'}, b_t, q_t, \mu_t, \varphi_t\right\}$.

17.7.5 Steady state

Let's suppose that both constraints bind in the steady state. After solving for the steady state using this assumption, we can then check later whether $\varphi > 0$ and $\mu > 0$, thus confirming (or not) our guess.

Go to (882) in the steady state. We have:

$$\frac{b}{\beta'} = ak + b.$$

This makes use of assuming that (887) binds, so $x = ck$. But from (883) we can then eliminate b:

$$
\begin{aligned}
qk &= ak + \beta^{'} qk \\
\Leftrightarrow q &= a + \beta^{'} q \\
\Leftrightarrow q &= \frac{a}{1-\beta^{'}}.
\end{aligned}
\tag{888}
$$

(888) is the same as (13a) in the Kiyotaki and Moore paper (albeit written a bit differently). But once we know q, we can get k' from (884):

$$
\begin{aligned}
q &= \beta^{'} \alpha (z + k^{'})^{\alpha-1} + \beta^{'} q \\
\Leftrightarrow (1-\beta^{'}) q &= \beta^{'} \alpha (z + k^{'})^{\alpha-1}.
\end{aligned}
$$

So:

$$
k^{'} = \left(\frac{\beta^{'} \alpha}{a} \right)^{\frac{1}{1-\alpha}} - z. \tag{889}
$$

But then we can get k from the market clearing condition for capital/land (886):

$$
k = \bar{K} - mk^{'}. \tag{890}
$$

Then we get b from (883):

$$
b = \beta^{'} qk. \tag{891}
$$

Similarly, we now have $x^{'}$ from (885) and $x = ck$:

$$
x^{'} = \frac{ak}{m} + (z + k^{'})^{\alpha}. \tag{892}
$$

Now we need to check the multipliers. From (880), we have:

$$
\mu = (\beta^{'} - \beta)(1 + \varphi).
$$

From (881) we have:

$$q(1+\varphi) + c\beta\varphi = \beta(1+\varphi)(1+c) + \beta(1+\varphi)q + \mu q.$$

Plug in for μ:

$$q(1+\varphi) + c\beta\varphi = \beta(1+\varphi)(a+c) + \beta(1+\varphi)q + q(\beta' - \beta)(1+\varphi),$$

then distribute terms:

$$q + q\varphi + c\beta\varphi = \beta(a+c) + \beta(a+c)\varphi + \beta q + \beta q\varphi + q(\beta^{'} - \beta) + q(\beta^{'} - \beta)\varphi,$$

then isolate terms involving φ on the LHS:

$$[q + c\beta - \beta(a+c) - \beta q - q(\beta^{'} - \beta)]\varphi = \beta(a+c) + \beta q + q(\beta^{'} - \beta) - q.$$

This can written as:

$$[q(1-\beta^{'}) - \beta a]\varphi = \beta(a+c) - q(1-\beta^{'}),$$

but recall from above that $q = a/(1-\beta^{'})$, hence:

$$\begin{aligned} a(1-\beta)\varphi &= \beta(a-1) + \beta c \\ \Leftrightarrow \varphi &= \frac{a(\beta-1)+\beta c}{a(1-\beta)}. \end{aligned} \tag{893}$$

Note, this ties into Assumption 2 of the Kiyotaki and Moore paper. That assumption requires that $\beta c > (1-\beta)a$. That assumption assures that the numerator in (893) is positive, which means that the farmer wants to just eat the non-tradable fruit. Now that we have this, we can solve for μ:

$$\mu = (\beta - \beta)\left(1 + \frac{a(\beta-1)+\beta c}{a(1-\beta)}\right),$$

which simplifies to:

$$\mu = (\beta^{'} - \beta)\frac{\beta c}{a(1-\beta)}.$$

The sign of μ is simply determined by $\beta^{'} - \beta$. As long as gatherers are more patient than borrowers, i.e. $\beta^{'} > \beta$, we will have $\mu > 0$ so that the farmers will be up against their borrowing constraint in the steady state.

17.7.6 Efficient solution

We can characterise the efficient solution to the model by thinking about the Ramsey social planner picking the allocation of capital/land between farmers and gatherers to maximise the aggregate output in each period. Because of the linearity of preferences, we cannot find a planner's solution for the allocation of consumption across the two types of agents. But, we can think about the optimal allocation of capital.

In period t, the planner would want to pick k_t and $k_t^{'}$ to maximise next period's output (current output is predetermined), subject to the constraint that capital sums up to the total available. We could impose the constraint that $k^{'} = \frac{\bar{K}-k}{m}$ and write problem as choosing k_t:

$$\max_{k_t^e} \; (a+c)k_t^e + mG\left(\frac{\bar{K} - k_t^e}{m}\right).$$

The FOC would be:

$$a + c = G'\left(\frac{\bar{K} - k_t^e}{m}\right). \tag{894}$$

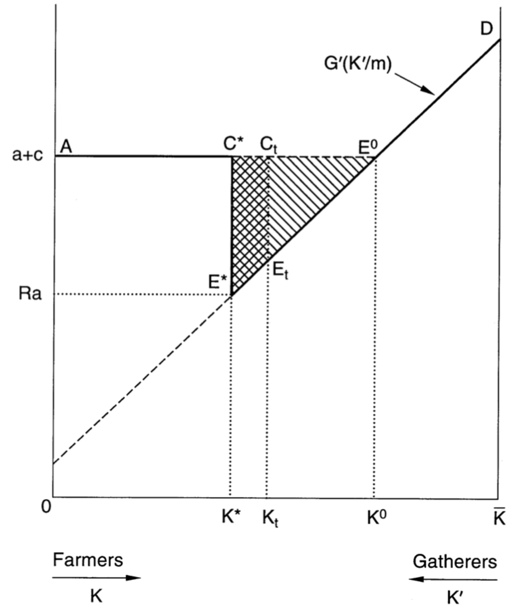
This is pretty simple, really – the Ramsey planner would like to allocate capital so as to equate the marginal products of capital across farmers and gatherers. This is what maximises total output; how that is split amongst the two types is not something we can solve for. With our particular production function, this would be:

$$
\begin{aligned}
a + c &= \alpha \left(z + \frac{\bar{K} - k_t^e}{m} \right)^{\alpha - 1} \\
\implies k_t^e &= mz + \bar{K} - m \left(\frac{\alpha}{a + c} \right)^{\frac{1}{1-\alpha}}.
\end{aligned} \tag{895}
$$

There are couple things to note with this allocation. First, the value of k_t is independent of anything with a t subscript. This means that, in the efficient allocation, the economy just sits in steady state in terms of how capital is allocated across the two types of producers. If there were productivity shocks, it would not affect the allocation of capital across farmers and gatherers. It would be constant.

Not in the parameterisation we're using, we would have $k^e = 0.91$, whereas in the steady state of the competitive equilibrium we have $k = 0.84$. The borrowing constraint distorts the steady state by having too much capital allocated to gatherers.

Figure 138: Equilibrium Characterisation in Kiyotaki and Moore (1997)



On the horizontal axis, farmers' demand for land is measure from the left, gatherers' demand from the right, and the sum of two equals total supply $\bar{K}$. On the vertical axes are the marginal products of land. The farmers' marginal product of land equals $a + c$, indicated by the line AC^*E^0. The gatherers' marginal product is shown by the line DE^0E^*; it falls with their land usage. If there were no debt enforcement problem so that there were no credit constraints, then the first-best allocation would be at the point $E^0 = (K^0, a+c)$, at which the marginal products of the farmers and gatherers would be equalised. The land price would be $q^0 = (a + c)/(R - 1)$, the discounted gross return from farming. In the credit-constrained economy, the steady state equilibrium is at the point $E^* = (K^*, aR)$, where the marginal product of the farmers, $a + c$, exceeds the marginal product of the gatherers, $G'((\bar{K} - k^{ss})/m) = aR$. That is, relative to the first-best, in the constrained equilibrium too little land is used by the farmers.

17.7.7 Solving and simulating the simple model

We solve the model via a first order approximation. To do so, we need to i) specify parameter values and ii) introduce a shock process.

We set $\beta^{'} = 0.99$ and $\beta = 0.98$, and we set $m = 0.5$. For the production technology of farmers, we set $a = 0.7$ and $c = 0.3$. For the production technology for the gatherers we assume $z = 0.01$ and $\alpha = \frac{1}{3}$. We need z to be sufficiently small so that the conditions in (5) in the Kiyotaki and Moore paper are satisfied and both types of households produce in the steady state.

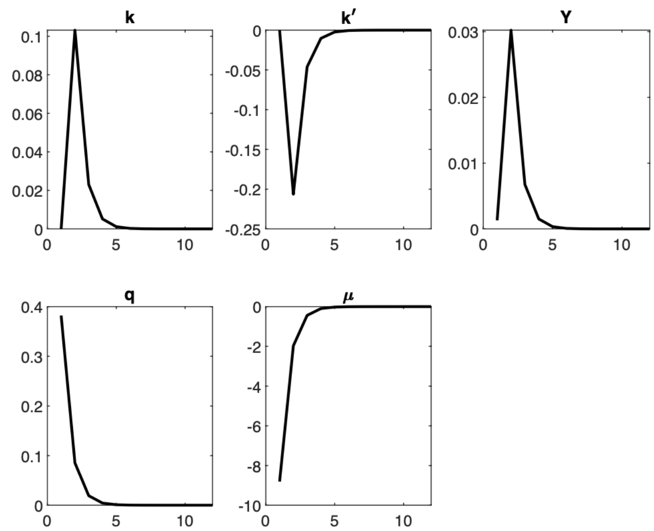
Then we need to introduce a shock. We're going to introduce a one period IID mean zero technology shock. It affects the production technologies as follows:

$$y_t = (1 + \epsilon_t)(a + c)k_{t-1}, \tag{896}$$

$$y_t^{'} = (1 + \epsilon_t)(z + k_t^{'})^{\alpha}. \tag{897}$$

Since it is IID and mean zero, $\mathbb{E}_t \epsilon_{t+j} = 0$ for $j > 0$. Thus, to a first order, we don't need to worry about this in any of the dynamic Euler equations. It will only appear in (882) and (885) multiplying the relevant period t outputs.

Figure 139: IRFs to Productivity Shock



Source: Sims (2020)

The figure above plots the impulse responses of k_t, $k_t^{'}$, $Y_t = y_t + my_t^{'}$, q_t, and μ_t. Focus first on output. Because the capital stock is predetermined, in the period of the IID shock output just reacts proportionally to the shock. But then starting in the next period, it jumps way up, and remains high for about four periods. What is going on? We can see in the first two graphs that capital is being reallocated to the farmers away from the gatherers. Why is this happening? The productivity shock is pushing up the price of land, q_t. This ends up easing the borrowing constraint facing the farmers, as evidenced by the decline in μ. This allows the farmers to borrow more and hence purchase more land.

Because the steady state is distorted relative to the Ramsey planner allocation, reallocating capital to farmers is efficient and gets the economy closer to the efficient outcome. This results in output rising. This effect lasts more than just one period. With more capital, this further eases the borrowing constraint facing farmers in the future, even though productivity has gone back to where it started. But this easing of the constraint pushes up future land prices, and results in land still being allocated predominantly back to farmers. This effect eventually fades out, but the important point here is that a perfectly transitory productivity shock generates a persistent reallocation of capital that results in output rising.

It is worth noting, referencing back to (895), that in the efficient allocation there would be no reallocation of capital between farmers and gatherers. Since the shock is IID, this means there would be no persistence in response to the IID shock in the efficient allocation. But there is persistence here. So, in a sense, the borrowing constraint propagates the IID shock through time, as we can see in the impulse response graph.

17.8 Collateral constraints and monetary policy (Iacoviello, 2015)

The final chapter in this section will go through "*House Prices, Borrowing Constraints, and Monetary Policy in the Business Cycle*" by Iacoviello (2005). In the model, housing is both a source of consumption flows (for households) and also a factor of production (for entrepreneurs/firms). Because of a limited enforcement constraint as in Kiyotaki and Moore (1997), housing serves as collateral for entrepreneurs. In equilibrium, this borrowing constraint being binding results in too little housing being allocated to entrepreneurs.

In addition to these features, the model is a stick price New Keynesian model. Importantly, debt is denominated in nominal terms. This generates a sort of formal "debt-deflation" mechanism. A contractionary monetary policy shock lowers inflation, which, other things equal, tightens borrowers' collateral constraints, and further cramps demand.

The paper has two parts. A base model with adjustment costs and no physical capital, and a more involved model with adjustment costs, capital accumulation, and additional shocks. We work through both parts.

As in the previous section, these notes are based on the amazing set of notes by Eric Sims. My only contribution here is to catch typos and clean up some notational bits and pieces.

17.8.1 Basic model

The basic model is comprised of the following agents: patient households (who consume housing), entrepreneurs (who use housing as a production input), a competitive final goods producer, a continuum of retailers (they repackage entrepreneurial output and this is where price stickiness is included), and a monetary authority that sets nominal interest rates according to a Taylor Rule.

17.8.2 Patient households

Choices made by patient households are indicated with a dash ($'$) notation. They can choose consumption $c_t^{'}$, housing $h_t^{'}$, labour $L_t^{'}$, and borrowing, $B_t^{'}$. The gross nominal return on borrowing is R_t, the nominal wage is W_t, the nominal house price is Q_t, and the price of goods is P_t. Money is included in the model of the paper, but money ends up being irrelevant when policy is set via an interest rate rule.

The household problem is:

$$\max_{c_t^{'},h_t^{'},L_t^{'},B_t^{'}} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left\{ \ln c_{t+s}^{'} + j \ln h_{t+s}^{'} - \frac{(L_t^{'})^{\eta}}{\eta} \right\},$$

subject to:

$$P_t c_t^{'} + Q_t h_t^{'} + R_{t-1} B_{t-1}^{'} \leq B_t^{'} + W_t L_t^{'} + P_t F_t - P_t T_t^{'}.$$

On the expenditure side of the budget constraint, the household can consume goods ($P_t c_t^{'}$), consume housing ($Q_t h_t^{'}$), and pays interest on its outstanding stock of debt, $R_{t-1} B_{t-1}^{'}$. On the income side, the household earns labour income ($W_t L_t^{'}$), has housing valued at $Q_t h_{t-1}$ that it inherited from the previous period, $P_t F_t$ denotes lump sum profits from firms, $P_t T_t^{'}$ denotes transfers from the government/central bank, and it can also issue new debt, $B_t^{'}$.

Equation (1) in the Iacoviello paper re-writes this in real terms. Define $q_t = Q_t/P_t$ and $w_t = W_t/P_t$. Similarly, let $b_t = B_t/P_t$ denote real debt holdings, and $\pi_t = P_t/P_{t-1}$ as gross inflation. Dividing through by P_t and then using these we would get:

$$c_t^{'} + q_t h_t^{'} + \frac{R_{t-1} b_{t-1}^{'}}{\pi_t} \leq b_t^{'} + w_t L_t^{'} + q_t h_{t-1}^{'} + F_t - T_t^{'}. \tag{898}$$

A Lagrangian where we take the budget constraint written in real terms is:

$$\mathcal{L} = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left\{ \begin{array}{c} \ln c_{t+s}^{'} + j \ln h_{t+s}^{'} - \frac{(L_{t+s}^{'})^{\eta}}{\eta} \\ +\lambda_{t+s}^{'} \left[b_{t+s}^{'} + w_{t+s} L_{t+s}^{'} + q_{t+s} h_{t+s-1}^{'} + F_{t+s} - T_{t+s}^{'} - c_{t+s}^{'} - q_{t+s} h_{t+s}^{'} - \frac{R_{t+s-1} b_{t+s-1}^{'}}{\pi_{t+s}} \right] \end{array} \right\},$$

and the FOCs are:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c_t^{'}} &= \frac{1}{c_t^{'}} - \lambda^{'} = 0, \\ \frac{\partial \mathcal{L}}{\partial h_t^{'}} &= \frac{j}{h_t^{'}} - \lambda_t^{'} q_t + \beta \mathbb{E}_t \lambda_{t+1}^{'} q_{t+1} = 0, \\ \frac{\partial \mathcal{L}}{\partial L_t^{'}} &= (L_t^{'})^{\eta-1} - \lambda_t^{'} w_t = 0, \\ \frac{\partial \mathcal{L}}{\partial b_t^{'}} &= \lambda_t^{'} + \beta \mathbb{E}_t \frac{\lambda_{t+1}^{'} R_t}{\pi_{t+1}} = 0. \end{aligned}$$

With a bit of rearranging to eliminate $\lambda_t^{'}$, we get:

$$\frac{q_t}{c_t^{'}} = \frac{j}{h_t^{'}} + \beta \mathbb{E}_t \frac{q_{t+1}}{c_{t+1}^{'}}, \tag{899}$$

$$(L_t^{'})^{\eta-1} = \frac{w_t}{c_t^{'}}, \tag{900}$$

$$\frac{1}{c_t^{'}} = \beta \mathbb{E}_t \frac{1}{c_{t+1}^{'}} \frac{R_t}{\pi_{t+1}}. \tag{901}$$

(900) is a standard labour supply schedule, and (901) is an Euler equation. The intuition for (899) is as follows: Purchasing an additional unit of housing costs q_t units of consumption, which is valued at $1/c_t^{'}$ in terms of utility. Hence, the LHS is the marginal utility cost of purchasing more housing. The first term on the RHS is the marginal utility, in period t, of having more housing. The second term is the extra utility one gets in period $t+1$ from purchasing more housing in t – purchasing more

housing in t generates q_{t+1} additional units of income in $t+1$, which is valued at β/c'_{t+1}. One could alternatively write (899) as:

$$q_t = \frac{jc'_t}{h'_t} + \beta\mathbb{E}_t \frac{c'_t}{c'_{t+1}} q_{t+1},$$

which says that the price of housing, q_t, is equal to the flow benefit of housing, measured in units of consumption, in period t, $\frac{jc''_t}{h'_t}$, plus the expected value of the product of the stochastic discount factor $\beta c'_t/c'_{t+1}$, with the future price, q_{t+1}.

17.8.3 Entrepreneurs

Entrepreneurs produce an intermediate good, $Y_{w,t}$, using their stock of real estate and labour hired from the patient household. In particular:

$$Y_{w,t} = Ah^v_{t-1} L^{1-v}_t. \tag{902}$$

We're changing the notation somewhat by adding a w subscript (for "wholesale"); see below. This intermediate output is sold to retailers before being available for consumption. Intermediate output is sold to retailers at P^w_t. Repacked intermediate output is then sold at the aforementioned retail price, P_t. $X_t = P_t/P^w_t$ is the markup. Really, the w notation stands for "wholesale" rather than retail.

Entrepreneurs do not work. They discount future utility flows at $\gamma < \beta$, so they less patient than patient households. They are also subject to a collateral constraint on their housing. Their objective and budget constraints are:

$$\max_{c_t, h_t, L_t, b_t} \quad \mathbb{E}_t \sum_{s=0}^{\infty} \gamma^s \ln c_{t+s},$$

subject to:

$$P^w_t Ah^v_{t-1} L^{1-v}_t - W_t L_t + B_t + Q_t h_{t-1} \leq P_t c_t + Q_t h_t + R_{t-1} B_{t-1},$$
$$B_t \leq m\mathbb{E}_t \frac{Q_{t+1} h_t}{R_t}.$$

The budget constraint says that entrepreneurial resources (LHS) are the value of output less payments to labour, plus new debt issued, plus the value of the existing housing stock. On the expenditure side, the entrepreneur can consume goods or new housing and pays interest plus principal on its outstanding debt. The borrowing constraint says that borrowing in the present cannot exceed the discounted expected value of future housing. Next period's expected value of housing, $Q_{t+1}h_t$, in effect serves as collateral. m isa parameter between zero and one. If an entrepreneur defaults in $t+1$, the creditor can recovery $(1-m)Q_{t+1}h_t$. Hence, the most an entrepreneur can borrow is $m\mathbb{E}_t Q_{t+1}h_t/R_t$.

We can rewrite the constraints in real terms by dividing by P_t:

$$\frac{Ah_{t-1}^{\upsilon}L_t^{1-\upsilon}}{X_t} - w_t L_t + b_t + q_t h_{t-1} \leq c_t + q_t h_t + \frac{R_{t-1}b_{t-1}}{\pi_t}, \tag{903}$$

$$b_t \leq m\mathbb{E}_t \frac{q_{t+1}h_t\pi_{t+1}}{R_t}, \tag{904}$$

and then form the Lagrangian:

$$\mathcal{L} = \mathbb{E}_t \sum_{s=0}^{\infty} \gamma^s \left\{ \begin{array}{c} \ln_{t+s} + \mu_{t+s}\left[\frac{Ah_{t+s-1}^{\upsilon}L_{t+s}^{1-\upsilon}}{X_{t+s}} - w_{t+s}L_{t+s} + b_{t+s}q_{t+s}h_{t+s-1} - c_{t+s} - q_{t+s}h_{t+s} - \frac{R_{t+s-1}b_{t+s-1}}{\pi_{t+s}}\right] \\ +\lambda_{t+s}\left[m\mathbb{E}_s q_{t+s+1}h_{t+s}\pi_{t+s+1} - b_{t+s}R_{t+s}\right] \end{array} \right\}.$$

The FOCs are:

$$\frac{\partial \mathcal{L}}{\partial c_t} = \frac{1}{c_t} - \mu_t = 0,$$

$$\frac{\partial \mathcal{L}}{\partial L_t} = \mu_t\left[\frac{(1-\upsilon)Ah_{t-1}^{\upsilon}L_t^{-\upsilon}}{X_t} - w_t\right] = 0,$$

$$\frac{\partial \mathcal{L}}{\partial h_t} = -\mu_t q_t + m\lambda_t\mathbb{E}_t q_{t+1}\pi_{t+1} + \gamma\mathbb{E}_t\mu_{t+1}\left[\frac{\upsilon Ah_t^{\upsilon-1}L_{t+1}^{1-\upsilon}}{X_{t+1}} + q_{t+1}\right] = 0,$$

$$\frac{\partial \mathcal{L}}{\partial b_t} = \mu_t - \lambda_t R_t + \gamma\mathbb{E}_t\frac{\mu_{t+1}R_t}{\pi_{t+1}} = 0.$$

Eliminating the multiplier on the budget constraint, we get:

$$X_t w_t = (1-\upsilon) A h_{t-1}^{\upsilon} L_t^{-\upsilon}, \tag{905}$$

$$\frac{q_t}{c_t} = \mathbb{E}_t \left[\frac{\gamma}{c_{t+1}} \left(\frac{\upsilon A h_t^{\upsilon-1} L_{t+1}^{1-\upsilon}}{X_{t+1}} + q_{t+1} \right) + m \lambda_t q_{t+1} \pi_{t+1} \right], \tag{906}$$

$$\frac{1}{c_t} = \gamma \mathbb{E}_t \frac{1}{c_{t+1}} \frac{R_t}{\pi_{t+1}} + \lambda_t R_t. \tag{907}$$

(905)-(907) are the same as in the Iacoviello paper. If you like, you can rearrange (906) to be:

$$q_t = \mathbb{E}_t \left[\frac{\gamma c_t}{c_{t+1}} \left(\frac{\upsilon A h_t^{\upsilon-1} L_{t+1}^{1-\upsilon}}{X_{t+1}} + q_{t+1} \right) + m \lambda_t c_t q_{t+1} \pi_{t+1} \right], \tag{908}$$

which says that the price of real estate is the expectation of the stochastic discount factor with i) the flow payout, which is the marginal product of housing received in $t+1$, plus ii) the continuation value, q_{t+1}. The final term is the amount by which having more housing eases the borrowing constraint; λ_t is the shadow value (in utils) of easing the constraint, so $\lambda_t c_t$ puts this into units of consumption. (905) is a standard labour demand schedule, (907) is a standard Euler equation, except for the $\lambda_t R_t$ term at the end. $\lambda_t R_t$ is effectively how much more you could borrow by relaxing the constraint.

17.8.4 Final goods and retailers

There are a continuum of retailers indexed by $z \in [0,1]$. They costlessly transform wholesale output, $Y_{w,t}$, purchased at P_t^w, into retail output, $Y_t(z)$. They then sell this retail output to a competitive final goods firm at $P_t(z)$. The competitive final goods firm produces final output, which is a CES aggregate using a Dixit-Stiglitz aggregator:

$$Y_t = \left(\int_0^1 Y_t(z)^{\frac{\epsilon-1}{\epsilon}} dz \right)^{\frac{\epsilon}{\epsilon-1}}. \tag{909}$$

Demand for each retail good is:

$$Y_t(z) = \left(\frac{P_t(z)}{P_t} \right)^{-\epsilon} Y_t, \tag{910}$$

and the price index is:

$$P_t^{1-\epsilon} = \int_0^1 P_t(z)^{1-\epsilon} dz. \tag{911}$$

Note that there is bad notation in the paper, and footnote 10 about aggregation is wrong. So we are changing things up a bit.

Retailers can update their price in each period with probability $1-\theta$. They discount future profits via the stochastic discount factor of patient households, $\Lambda_{t,t+k} = \beta^k \frac{c_t^{'}}{c_{t+k}^{'}}$. Flow nominal profit for each intermediary is:

$$F_t(z)^n = P_t(z)Y_t(z) - P_t^w Y_t(z).$$

They produce $Y_t(z)$, and use $Y_{w,t}$ as an input, but this is transformed costlessly into $Y_t(z)$, so we can eliminate $Y_{w,t}$ and just write this in terms of $Y_t(z)$. Plugging in the demand function (910), we get:

$$\begin{aligned} F_t(z)^n &= P_t(z) \left(\frac{P_t(z)}{P_t} \right)^{-\epsilon} Y_t - P_t^w \left(\frac{P_t(z)}{P_t} \right)^{-\epsilon} Y_t \\ &= P_t(z)^{1-\epsilon} P_t^{\epsilon} Y_t - P_t^w P_t(z)^{-\epsilon} P_t^{\epsilon} Y_t, \end{aligned}$$

and then write this in real terms by dividing by the price level, P_t:

$$\begin{aligned} F_t(z) &= P_t(z)^{1-\epsilon} P_t^{\epsilon-1} Y_t - P_t^w P_t(z)^{-\epsilon} P_t^{\epsilon-1} Y_t \\ &= P_t(z)^{1-\epsilon} P_t^{\epsilon-1} Y_t - X_t^{-1} P_t(z)^{-\epsilon} P_t^{\epsilon} Y_t, \end{aligned}$$

where $X_t = P_t/P_t^w$. The problem of a firm getting to reset its prices is therefore to pick $P_t(z)$ to maximise the present discounted value of $F_t(z)$, where discounting is by the stochastic discount factor, $\Lambda_{t,t+k}$, as well as the probability that a price chosen in t is still in place in period $t+k$, θ^k:

$$\max_{P_t(z)} \mathbb{E}_t \sum_{k=0}^{\infty} \theta^k \left\{ \Lambda_{t,t+k} \left[P_t(z)^{1-\epsilon} P_{t+k}^{\epsilon-1} Y_{t+k} - X_{t+k}^{-1} P_t(z)^{-\epsilon} P_{t+k}^{\epsilon} Y_{t+k} \right] \right\}.$$

The FOC is:

$$(\epsilon - 1)P_t(z)^{-\epsilon}\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k \Lambda_{t,t+k} P_{t+k}^{\epsilon-1} Y_{t+k} = \epsilon P_t(z)^{-\epsilon-1}\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k \Lambda_{t,t+k} X_{t+k}^{-1} P_{t+k}^{\epsilon} Y_{t+k},$$

or:

$$P_t^* = \frac{\epsilon}{\epsilon - 1} \frac{\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k \Lambda_{t,t+k} X_{t+k}^{-1} P_{t+k}^{\epsilon} Y_{t+k}}{\mathbb{E}_t \sum_{k=0}^{\infty} \theta^k \Lambda_{t,t+k} P_{t+k}^{\epsilon-1} Y_{t+k}}. \tag{912}$$

We can write the numerator and denominator recursively as:

$$Z_{1,t} = X_t^{-1} P_t^{\epsilon} Y_t + \theta \mathbb{E}_t \Lambda_{t,t+1} Z_{1,t+1},$$

$$Z_{2,t} = P_t^{\epsilon-1} Y_t + \theta \mathbb{E}_t \Lambda_{t,t+1} Z_{2,t+1}.$$

We will need to re-scale these to get rid of price levels. Define $z_{1,t} = Z_{1,t}/P_t^{\epsilon}$ and $z_{2,t} = Z_{2,t}/P_t^{\epsilon-1}$. We then have:

$$z_{1,t} = X_t^{-1} Y_t + \theta \mathbb{E}_t \Lambda_{t,t+1} \pi_{t+1}^{\epsilon} z_{1,t+1}, \tag{913}$$

$$z_{2,t} = Y_t + \theta \mathbb{E}_t \Lambda_{t,t+1} \pi_{t+1}^{\epsilon-1} z_{2,t+1}. \tag{914}$$

Since $Z_{1,t}/Z_{2,t} = \frac{z_{1,t}}{z_{2,t}} P_t$, we can then define $\pi_t^* = P_t^*/P_t$ as the relative reset price. Then we simply have:

$$\pi_t^* = \frac{\epsilon}{\epsilon - 1} \frac{z_{1,t}}{z_{2,t}}. \tag{915}$$

17.8.5 Monetary policy

The monetary policy rule is:

$$R_t = (\bar{rr})^{1-r_R} (R_{t-1})^{r_R} \left(\pi_{t-1}^{1+r_\pi} (Y_{t-1}/\bar{Y})^{r_Y} \right)^{1-r_R} e_{R,t}. \tag{916}$$

Here $\bar{rr}$ is the steady state nominal rate and $\bar{Y}$ is steady state output. r_R is a smoothing parameter, $1 + r_\pi$, with $\bar{r_\pi} > 0$, is the coefficient on lagged inflation, and r_Y is the coefficient on the deviation of output from steady state. $e_{R,t}$ is a shock. Note that the Taylor Rule is purely backward-looking.

17.8.6 Aggregation

The aggregate price level evolves according to (911) taking into account properties of Calvo pricing:

$$P_t^{1-\epsilon} = \theta P_{t-1}^{1-\epsilon} + (1-\theta)(P_t^*)^{1-\epsilon}.$$

Divide both sides by $P_t^{1-\epsilon}$ to write this in terms of inflation rates:

$$1 = \theta \pi_t^{\epsilon-1} + (1-\theta)(\pi_t^*)^{1-\epsilon}.$$

Integrating (910) across z, noting that $Y_t(z) = Ah_{t-1}^{\upsilon} L_t^{1-\upsilon}$, we get:

$$Ah_{t-1}^{\upsilon} L_t^{1-\upsilon} = Y_t v_t^p,$$

where v_t^p os a measure of price dispersion, which can be written as:

$$v_t^p = (1-\theta)(\pi_t^*)^{\epsilon} + \theta \pi_t^{\epsilon} v_{t-1}^p. \tag{917}$$

Sum the budget constraints of the patient household and the entrepreneur together:

$$c_t + c_t^{'} + q(h_t + h_t^{'}) + \frac{R_{t-1}}{\pi_t}(b_{t-1} + b_{t-1}^{'}) = (b_t + b_t^{'}) + q_t(h_{t-1} + h_{t-1}^{'}) + \frac{Ah_{t-1}^{\upsilon} L_t^{1-\upsilon}}{X_t} + F_t - T_t^{'}.$$

Here we have imposed labour market clearing, so that the $w_t L_t^{'}$ and $w_t L_t$ terms cancel (i.e. $L_t = L_t^{'}$). Market clearing for bonds requires that $b_t + b_t^{'} = 0$ (i.e., one lends, one borrows). The aggregate stock of housing is fixed at H, so $h_t + h_t^{'} = H$. But then these terms of the LHS and RHS cancel, leaving:

$$c_t + c_t^{'} = \frac{Ah_{t-1}^{\upsilon} L_t^{1-\upsilon}}{X_t} + F_t - T_t^{'}.$$

Because we have omitted money, $T_t^{'} = 0$ (i.e., there is no transfer/tax from the government). What about F_t? Recall from above that we have:

$$F_t(z) = P_t(z)^{1-\epsilon} P_t^{\epsilon-1} Y_t - X_t^{-1} P_t(z)^{-\epsilon} P_t^{\epsilon} Y_t.$$

Aggregate profits are just profits integrated across retailers:

$$F_t = \int_0^1 F_t(z) dz = P_t^{\epsilon-1} Y_t \int_0^1 P_t(z)^{1-\epsilon} dz - X_t^{-1} Y_t \int_0^1 \left(\frac{P_t(z)}{P_t} \right)^{-\epsilon} dz.$$

Now, note from above that $\int_0^1 P_t(z)^{1-\epsilon} dz = P_t^{1-\epsilon}$, so in the first term the price terms just drop out. In the second term, the term inside the integral, $\int_0^1 \left(\frac{P_t(z)}{P_t} \right)^{\epsilon} dz = v_t^p$. So we have:

$$F_t = Y_t - \frac{Y_t v_t^p}{X_t}.$$

But since $A h_{t-1}^{\upsilon} L_t^{1-\upsilon} = Y_t v_t^p$, the summed budget constraints work out to the typical resource constraint:

$$c_t + c_t^{'} = Y_t. \tag{918}$$

17.8.7 Full set of equilibrium conditions for the simple model

For completeness, the full set of equilibrium conditions for the simple Iacoviello model are:

$$\frac{q_t}{c_t^{'}} = \frac{j}{h_t^{'}} + \beta \mathbb{E}_t \frac{q_{t+1}}{c_{t+1}^{'}}, \tag{919}$$

$$\frac{w_t}{c_t^{'}} = (L_t)^{\eta-1}, \tag{920}$$

$$\frac{1}{c_t^{'}} = \beta \mathbb{E}_t \frac{1}{c_t^{'}} \frac{R_t}{\pi_{t+1}}, \tag{921}$$

$$X_t w_t = (1-\upsilon) A h_{t-1}^{\upsilon} L_t^{-\upsilon}, \tag{922}$$

$$\frac{q_t}{c_t} = \mathbb{E}_t \left[\frac{\gamma}{c_{t+1}} \left(\frac{\upsilon A h_t^{\upsilon-1} L_{t+1}^{1-\upsilon}}{X_{t+1}} + q_{t+1} \right) + m \lambda_t q_{t+1} \pi_{t+1} \right], \tag{923}$$

$$\frac{1}{c_t} = \gamma \mathbb{E}_t \frac{1}{c_{t+1}} \frac{R_t}{\pi_{t+1}} + \lambda_t R_t, \tag{924}$$

$$b_t = m \mathbb{E}_t \frac{q_{t+1} h_t \pi_{t+1}}{R_t}, \tag{925}$$

$$z_{1,t} = X_t^{-1} Y_t + \theta \mathbb{E}_t \Lambda_{t,t+1} \pi_{t+1}^{\epsilon} z_{1,t+1}, \tag{926}$$

$$z_{2,t} = Y_t + \theta \mathbb{E}_t \Lambda_{t,t+1} \pi_{t+1}^{\epsilon-1} z_{2,t+1}, \tag{927}$$

$$\pi_t^* = \frac{\epsilon}{\epsilon-1} \frac{z_{1,t}}{z_{2,t}}, \tag{928}$$

$$R_r = (\bar{rr})^{1-r_R} (R_{t-1})^{r_R} \left[\pi_{t-1}^{1+r_\pi} \left(\frac{Y_t}{\bar{Y}} \right)^{r_Y} \right]^{1-r_R} e_{R,t}, \tag{929}$$

$$1 = \theta \pi^{\epsilon-1} + (1-\theta)(\pi_t^*)^{1-\epsilon}, \tag{930}$$

$$Y_t v_t^p = A h_{t-1}^{\upsilon} L_t^{1-\upsilon}, \tag{931}$$

$$v_t^p = (1-\theta)(\pi_t^*)^{-\epsilon} + \theta \pi_t^{\epsilon} v_{t-1}^p, \tag{932}$$

$$c_t + c_t^{'} = Y_t, \tag{933}$$

$$h_t + h_t^{'} = H, \tag{934}$$

$$b_t = c_t + q_t(h_t - h_{t-1}) + \frac{R_{t-1} b_{t-1}}{\pi_t} + w_t L_t - \frac{Y_t v_t^p}{X_t}. \tag{935}$$

This is 17 variables, $\left\{ c_t^{'}, h_t^{'}, L_t, c_t, h_t, b_t, Y_t, X_t, v_t^p, q_t, w_t, R_t, \pi_t, \pi_t^*, z_{1,t}, z_{2,t}, \lambda_t \right\}$ and 17 equations.

17.8.8 Log linearising the simple model

Let's log linearise these conditions about the steady state. We will use our standard notation here – variables with a bar denote steady state values, and variables with a hat denote log deviations from steady state.

The price-setting conditions (926)-(928) and (930), become[138] the standard New Keynesian Phillips Curve (NKPC):

$$\hat{\pi}_t = -\kappa \hat{X}_t + \beta \mathbb{E}_t \hat{\pi}_{t+1}. \tag{936}$$

The resource constraint is (933):

$$\hat{Y}_t = \frac{\bar{c}}{\bar{Y}} \hat{c}_t + \frac{\bar{c}^{'}}{\bar{Y}} \hat{c}^{'}_t. \tag{937}$$

The patient household's linearised Euler equation (921) is:

$$\hat{c}^{'}_t = \mathbb{E}_t \hat{c}^{'}_{t+1} + \widehat{rr}_t, \tag{938}$$

where $\widehat{rr}_t$ is the real interest rate:

$$\widehat{rr}_t = \hat{R}_t - \mathbb{E}_t \hat{\pi}_{t+1}. \tag{939}$$

The linearised household labour schedule (920) is:

$$(\eta - 1)\hat{L}_t = \hat{w}_t + \hat{c}_t. \tag{940}$$

The linearised labour demand condition is:

$$\upsilon \hat{h}_{t-1} - \upsilon \hat{L}_t = \hat{w}_t + \hat{X}_t \tag{941}$$

Re-write the Euler equation for housing for the patient household (919) as:

$$q_t = j \frac{c^{'}_t}{h^{'}_t} + \beta \mathbb{E}_t \frac{c^{'}_t}{c^{'}_{t+1}} q_{t+1},$$

[138] This is a massive headache. See the section on the New Keynesian DSGE model for a detail of how to derive the NKPC.

and then take logs:

$$\ln q_t = \ln\left[j\frac{c_t^{'}}{h_t^{'}} + \beta\mathbb{E}_t\frac{c_t^{'}}{c_{t+1}^{'}}q_{t+1}\right],$$

then totally differentiate (ignore the expectations operator):

$$\frac{1}{\bar{q}}dq_t = \frac{1}{\bar{q}}\left[j\frac{1}{\bar{h}^{'}}dc_t^{'} - j\frac{\bar{c}^{'}}{(\bar{h}^{'})^2}dh_t^{'} + \beta\bar{q}\frac{1}{\bar{c}^{'}}dc_t^{'} - \beta\bar{q}\frac{\bar{c}^{'}}{(\bar{c}^{'})^2}dc_{t+1}^{'} + \beta\frac{\bar{c}^{'}}{\bar{c}^{'}}dq_{t+1}\right],$$

and simplify to get:

$$\hat{q}_t = j\frac{\bar{c}^{'}}{\bar{q}\bar{h}^{'}}\left[\hat{c}_t^{'} - \hat{h}_t^{'}\right] + \beta\hat{c}_t^{'} - \beta\mathbb{E}_t\hat{c}_{t+1}^{'} + \beta\mathbb{E}_t\hat{q}_{t+1}. \tag{942}$$

Now take logs of the bond Euler equation for entrepreneurs (924):

$$-\ln c_t = \ln\left[\frac{\gamma}{c_{t+1}}\frac{R_t}{\pi_{t+1}} + \lambda_t R_t\right],$$

then totally differentiate:

$$-\frac{1}{\bar{c}}dc_t = \bar{c}\left[-\frac{\gamma}{\bar{c}^2}\bar{R}dc_{t+1} + \frac{\gamma}{\bar{c}}dR_t - \frac{\gamma\bar{R}}{\bar{c}}d\pi_{t+1} + \bar{R}d\lambda_t + \bar{\lambda}dR_t\right],$$

noting that $\bar{\pi} = 1$. How did $\bar{c}$ appear on the RHS instead of $1/\bar{c}$? Because the term in the square brackets on the RHS is equal to $1/\bar{c}$ in the steady state, and when we take the derivative of the log, this term gets inverted. Now do some rearranging to get:

$$-\hat{c}_t = -\frac{\gamma}{\beta}\mathbb{E}_t\hat{c}_{t+1} + \frac{\gamma}{\beta}\widehat{rr}_t + \frac{\bar{\lambda}\bar{c}}{\beta}\left(\hat{\lambda}_t + \hat{R}_t\right). \tag{943}$$

Now focus on the Euler equation for housing for the entrepreneur (923). Re-arrange slightly:

$$q_t = \mathbb{E}_t\left[\frac{\gamma c_t}{c_{t+1}}\left(\frac{\upsilon A h_t^{\upsilon-1}L_{t+1}^{1-\upsilon}}{X_{t+1}} + q_{t+1}\right) + m\lambda_t c_t q_{t+1}\pi_{t+1}\right],$$

and then take logs, ignoring the expectations operator for now:

$$\ln q_t = \ln \left[\frac{\gamma c_t}{c_{t+1}} \left(\frac{\upsilon A h_t^{\upsilon-1} L_{t+1}^{1-\upsilon}}{X_{t+1}} + q_{t+1} \right) + m \lambda_t c_t q_{t+1} \pi_{t+1} \right].$$

Totally differentiate this, noting that $\frac{\upsilon A h_t^{\upsilon-1} L_{t+1}^{1-\upsilon}}{X_{t+1}} = \frac{\upsilon Y_{t+1}}{h_t X_{t+1}}$:

$$\frac{1}{\bar{q}} dq_t = \frac{1}{\bar{q}} \left\{ \begin{array}{c} \left(\frac{\gamma}{\bar{c}} dc_t - \frac{\gamma}{\bar{c}} dc_{t+1} \right) \left(\frac{\upsilon \bar{Y}}{\bar{h}\bar{X}} + \bar{q} \right) + \gamma \left(\frac{\upsilon}{\bar{h}\bar{X}} dY_{t+1} - \frac{\upsilon \bar{Y}}{\bar{h}\bar{X}^2} dX_{t+1} - \frac{\upsilon \bar{Y}}{\bar{h}^2 \bar{X}} dh_t + dq_{t+1} \right) \\ + m\bar{c}\bar{q} d\lambda_t + m\bar{\lambda}\bar{q} dc_t + m\bar{\lambda}\bar{c} dq_{t+1} + m\bar{\lambda}\bar{c}\bar{q} d\pi_{t+1} \end{array} \right\},$$

which simplifies to:

$$\hat{q}_t = \gamma \left(\hat{c}_t - \mathbb{E}_t \hat{c}_{t+1} \right) \left(\frac{\upsilon \bar{Y}}{\bar{q}\bar{h}\bar{X}} + 1 \right) + \frac{\gamma \upsilon \bar{Y}}{\bar{q}\bar{h}\bar{X}} \left(\mathbb{E}_t \hat{Y}_{t+1} - \mathbb{E}_t \hat{X}_{t+1} - \hat{h}_t \right) + \gamma \hat{q}_{t+1} + m\bar{\lambda}\bar{c} \left(\hat{\lambda}_t + \hat{c}_t + \mathbb{E}_t \hat{q}_{t+1} + \mathbb{E}_t \hat{\pi}_{t+1} \right) \tag{944}$$

Now, linearise the borrowing constraint (925):

$$\ln b_t = \ln m + \ln q_{t+1} + \ln h_t + \ln \pi_{t+1} - \ln R_t,$$

or:

$$\hat{b}_t = \mathbb{E}_t \hat{q}_{t+1} + \hat{h}_t - \widehat{rr}_t. \tag{945}$$

The price-setting conditions are already linearised and expressed via the NKPC.

The Taylor Rule (929)is log-linear:

$$\hat{R}_t = r_R \hat{R}_{t-1} + (1 - r_R) \left[(1 + r_\pi) \hat{\pi}_{t-1} + r_Y \hat{Y}_{t-1} \right] + \hat{e}_{R,t}. \tag{946}$$

The production function (931) is log-linear, noting that the price dispersion is constant to a first-order (so we can drop it):

$$\hat{Y}_t = \upsilon \hat{h}_{t-1} + (1 - \upsilon) \hat{L}_t. \tag{947}$$

The resource constraint (933) and house market clearing condition (934) are fairly straightforward:

$$\hat{Y}_t = \frac{\bar{c}}{\bar{Y}}\hat{c}_t + \frac{\bar{c}^{'}}{\bar{Y}}\hat{c}_t^{'}, \tag{948}$$

$$\hat{h}_t^{'} = -\frac{\bar{h}^{'}}{\bar{h}}\hat{h}_t. \tag{949}$$

Now, we need to log-linearise the budget constraint for the entrepreneur (935). First, re-arrange to get c_t isolated on the LHS:

$$c_t = b_t - q_t(h_t - h_{t-1}) - \frac{R_{t-1}b_{t-1}}{\pi_t} - w_t L_t + \frac{Y_t v_t^p}{X_t}.$$

Now take logs:

$$\ln c_t = \ln\left[b_t - q_t(h_t - h_{t-1}) - \frac{R_{t-1}b_{t-1}}{\pi_t} - w_t L_t + \frac{Y_t v_t^p}{X_t}\right],$$

and then totally differentiate:

$$\frac{1}{\bar{c}}dc_t = \frac{1}{\bar{c}}\left[db_t - dq_t(0) - \bar{q}dh_t + \bar{q}dh_{t-1} - \bar{b}dR_{t-1} - \bar{R}db_{t-1} + \bar{R}\bar{b}d\pi_t - \bar{w}dL_t - \bar{L}dw_t + \frac{1}{\bar{X}}dY_t - \frac{\bar{Y}}{\bar{X}^2}dX_t\right],$$

which is:

$$\hat{c}_t = \frac{1}{\bar{c}}\left[\bar{b}\hat{b}_t - \bar{q}\bar{h}\hat{h}_t + \bar{q}\bar{h}\hat{h}_{t-1} - \frac{\bar{b}}{\beta}\left(\hat{R}_{t-1} + \hat{b}_{t-1} - \hat{\pi}_t\right) - \bar{w}\bar{L}(\hat{w}_t + \hat{L}_t) + \frac{\bar{Y}}{\bar{X}}(\hat{Y}_t - \hat{X}_t)\right],$$

or:

$$\bar{c}\hat{c}_t = \bar{b}\hat{b}_t - \bar{q}\bar{h}\hat{h}_t + \bar{q}\bar{h}\hat{h}_{t-1} - \frac{\bar{b}}{\beta}\left(\hat{R}_{t-1} + \hat{b}_{t-1} - \hat{\pi}_t\right) - \bar{w}\bar{L}(\hat{w}_t + \hat{L}_t) + \frac{\bar{Y}}{\bar{X}}(\hat{Y}_t - \hat{X}_t). \tag{950}$$

17.8.9 Simple model full set of linearised conditions

The full set of log-linearised equation for the Iacoviello (2005) model is:

$$\hat{q}_t = j\frac{\bar{c}'}{\bar{q}\bar{h}'}\left[\hat{c}_t' - \hat{h}_t'\right] + \beta\hat{c}_t' - \beta\mathbb{E}_t\hat{c}_{t+1}' + \beta\mathbb{E}_t\hat{q}_{t+1}, \tag{951}$$

$$(\eta - 1)\hat{L}_t = \hat{w}_t + \hat{c}_t', \tag{952}$$

$$\hat{c}_t' = \mathbb{E}_t\hat{c}_{t+1}' + \widehat{rr}_t, \tag{953}$$

$$\upsilon\hat{h}_{t-1} - \upsilon\hat{L}_t = \hat{w}_t + \hat{X}_t, \tag{954}$$

$$\begin{aligned}\hat{q}_t = {} & \gamma\left(\hat{c}_t - \mathbb{E}_t\hat{c}_{t+1}\right)\left(\frac{\upsilon\bar{Y}}{\bar{q}\bar{h}\bar{X}} + 1\right) + \frac{\gamma\upsilon\bar{Y}}{\bar{q}\bar{h}\bar{X}}\left(\mathbb{E}_t\hat{Y}_{t+1} - \mathbb{E}_t\hat{X}_{t+1} - \hat{h}_t\right) \\ & + \gamma\hat{q}_{t+1} + m\bar{\lambda}\bar{c}\left(\hat{\lambda}_t + \hat{c}_t + \mathbb{E}_t\hat{q}_{t+1} + \mathbb{E}_t\hat{\pi}_{t+1}\right),\end{aligned} \tag{955}$$

$$-\hat{c}_t = -\frac{\gamma}{\beta}\mathbb{E}_t\hat{c}_{t+1} + \frac{\gamma}{\beta}\widehat{rr}_t + \frac{\bar{\lambda}\bar{c}}{\beta}\left(\hat{\lambda}_t + \hat{R}_t\right), \tag{956}$$

$$\hat{b}_t = \mathbb{E}_t\hat{q}_{t+1} + \hat{h}_t - \widehat{rr}_t, \tag{957}$$

$$\hat{\pi}_t = -\kappa\hat{X}_t + \beta\mathbb{E}_t\hat{\pi}_{t+1}, \tag{958}$$

$$\hat{R}_t = r_R\hat{R}_{t-1} + (1 - r_R)\left[(1 + r_\pi)\hat{\pi}_{t-1} + r_Y\hat{Y}_{t-1}\right] + \hat{e}_{R,t}, \tag{959}$$

$$\hat{Y}_t = \upsilon\hat{h}_{t-1} + (1 - \upsilon)\hat{L}_t, \tag{960}$$

$$\hat{Y}_t = \frac{\bar{c}}{\bar{Y}}\hat{c}_t + \frac{\bar{c}'}{\bar{Y}}\hat{c}_t', \tag{961}$$

$$\hat{h}_t' = -\frac{\bar{h}}{\bar{h}'}\hat{h}_t, \tag{962}$$

$$\bar{c}\hat{c}_t = \bar{b}\hat{b}_t - \bar{q}\bar{h}\hat{h}_t + \bar{q}\bar{h}\hat{h}_{t-1} - \frac{\bar{b}}{\beta}\left(\hat{R}_{t-1} + \hat{b}_{t-1} - \hat{\pi}_t\right) - \bar{w}\bar{L}(\hat{w}_t + \hat{L}_t) + \frac{\bar{Y}}{\bar{X}}(\hat{Y}_t - \hat{X}_t), \tag{963}$$

$$\widehat{rr}_t = \hat{R}_t - \mathbb{E}_t\hat{\pi}_{t+1}. \tag{964}$$

This is 14 variables $\left\{\hat{c}_t', \hat{L}_t, \hat{h}_t', \hat{c}_t, \hat{h}_t, \hat{b}_t, \hat{Y}_t, \hat{X}_t, \hat{R}_t, \hat{q}_t, \hat{\pi}_t, \hat{w}_t, \widehat{rr}_t, \hat{\lambda}_t\right\}$ and 15 equations. In the paper, Iacoviello lists nine equations (really ten, because the real interest rate is a separate in-text equation), but focuses only on $\left\{\hat{c}_t', \hat{c}_t, \hat{h}_t, \hat{Y}_t, \hat{X}_t, \widehat{rr}_t, \hat{R}_t, \hat{b}_t, \hat{q}_t, \hat{\pi}_t\right\}$. The variables $\hat{L}_t$, $\hat{h}_t'$, $\hat{\lambda}_t$, and $\hat{w}_t$ have been eliminated. Let's eliminate these and see if we can recover what Iacoviello has in the paper (equations L.1-L.9).

First, combine labour supply (952) and demand (954) to eliminate $\hat{w}_t$:

$$(\eta - 1)\hat{L}_t + \hat{c}_t^{'} = \upsilon \hat{h}_{t-1} - \upsilon \hat{L}_t - \hat{X}_t,$$

and solve for $\hat{L}_t$:

$$\hat{L}_t = \frac{\upsilon}{\eta - (1-\upsilon)} \hat{h}_{t-1} - \frac{1}{\eta - (1-\upsilon)} \left(\hat{X}_t + \hat{c}_t^{'} \right).$$

Now plug this into the linearised production function (960) to get:

$$\hat{Y}_t = \upsilon \hat{h}_{t-1} + \frac{\upsilon(1-\upsilon)}{\eta - (1-\upsilon)} \hat{h}_{t-1} - \frac{1-\upsilon}{\eta - (1-\upsilon)} \left(\hat{X}_t + \hat{c}_t^{'} \right),$$

which simplifies to:

$$\hat{Y}_t = \frac{\eta \upsilon}{\eta - (1-\upsilon)} \hat{h}_{t-1} - \frac{1-\upsilon}{\eta - (1-\upsilon)} \left(\hat{X}_t + \hat{c}_t^{'} \right). \tag{965}$$

This is exactly as (L7) in the paper.

Now, if you look at, (919), we can solve for something about the steady state, which is:

$$\frac{j}{\bar{h}^{'}} = \frac{\bar{q}}{\bar{c}^{'}}(1-\beta).$$

But this means that:

$$\frac{j\bar{c}^{'}}{\bar{q}\bar{h}^{'}} = 1 - \beta.$$

Now, use this, along with the fact that $\hat{h}_t^{'} = -\frac{\bar{h}^{'}}{\bar{h}} \hat{h}_t$ from (962), to write (951) as:

$$\hat{q}_t = \hat{c}_t^{'} + (1-\beta)\frac{\bar{h}}{\bar{h}^{'}} \hat{h}_t - \beta \mathbb{E}_t \hat{c}_{t+1}^{'} + \beta \mathbb{E}_t \hat{q}_{t+1}, \tag{966}$$

which is exactly the same as (L1) in the paper with $\iota = (1-\beta)\bar{h}/\bar{h}^{'}$.

Now, let's re-write the budget constraint for entrepreneur (963), subbing out $\hat{w}_t$ and $\hat{L}_t$. First, note that in steady state we have $\bar{w}\bar{L} = \frac{(1-\upsilon)\bar{Y}}{\bar{X}}$. Second, from (954) (960), we can write:

$$\hat{w}_t + \hat{L}_t = \hat{Y}_t - \hat{X}_t,$$

hence, we have the term:

$$\begin{aligned}-\bar{w}\bar{L}(\hat{w}_t + \hat{L}_t) + \frac{\bar{Y}}{\bar{X}}(\hat{Y}_t - \hat{X}_t) &= -\frac{\bar{Y}}{\bar{X}}\left((1-\upsilon)(\hat{Y}_t - \hat{X}_t) - (\hat{Y}_t - \hat{X}_t)\right) \\ &= \frac{\upsilon\bar{Y}}{\bar{X}}(\hat{Y}_t - \hat{X}_t).\end{aligned}$$

But then we can write the entrepreneur budget constraint as:

$$\bar{c}\hat{c}_t = \bar{b}\hat{b}_t - \bar{q}\bar{h}\hat{h}_t + \bar{q}\bar{h}\hat{h}_{t-1} - \frac{\bar{b}}{\beta}\left(\hat{R}_{t-1} + \hat{b}_{t-1} - \hat{\pi}_t\right) + \frac{\upsilon\bar{Y}}{\bar{X}}(\hat{Y}_t - \hat{X}_t). \tag{967}$$

Now, we need to deal with Euler equation for housing for the entrepreneur. It first helpful to start with the steady state. In steady state, we can write:

$$\frac{\bar{q}}{\bar{c}} = \frac{\gamma\bar{q}}{\bar{c}}\left(\frac{\upsilon\bar{Y}}{\bar{q}\bar{h}\bar{X}} + 1\right) + m\bar{\lambda}\bar{q}.$$

But then the $\bar{q}$'s drop out, leaving:

$$\gamma\left(\frac{\upsilon\bar{Y}}{\bar{q}\bar{h}\bar{X}} + 1\right) = 1 - m\bar{\lambda}\bar{c},$$

and consequently:

$$\frac{\gamma\upsilon\bar{Y}}{\bar{q}\bar{h}\bar{X}} = 1 - m\bar{\lambda}\bar{c} - \gamma. \tag{968}$$

This means we can write the Euler equation for using for the entrepreneur as:

$$\hat{q}_t = (1-m\bar{\lambda}\bar{c})(\hat{c}_t - \mathbb{E}_t\hat{c}_{t+1}) + (1-m\bar{\lambda}\bar{c}-\gamma)(\mathbb{E}_t\hat{Y}_{t+1} - \mathbb{E}_t\hat{X}_{t+1} - \hat{h}_t) + \gamma\hat{q}_{t+1} + m\bar{\lambda}\bar{c}(\hat{\lambda}_t + \hat{c}_t + \mathbb{E}_t\hat{q}_{t+1} + \hat{\pi}_{t+1}).$$

Solve for $\hat{\lambda}_t$ from the Euler equation for bonds (956):

$$\bar{\lambda}\bar{c}\hat{\lambda}_t = -\beta\hat{c}_t + \gamma\hat{c}_{t+1} - \gamma\widehat{rr}_t - \bar{\lambda}\bar{c}(\widehat{rr}_t + \hat{\pi}_{t+1}).$$

Now combine these to get:

$$\begin{aligned}\hat{q}_t = (1 - m\bar{\lambda}\bar{c})(\hat{c}_t - \mathbb{E}_t\hat{c}_{t+1}) + (1 - m\bar{\lambda}\bar{c} - \gamma)(\mathbb{E}_t\hat{Y}_{t+1} - \mathbb{E}_t\hat{X}_{t+1} - \hat{h}_t) + \gamma\hat{q}_{t+1} - m\beta\hat{c}_t \\ + m\gamma\hat{c}_{t+1} - m\gamma\widehat{rr}_t - m\bar{\lambda}\bar{c}(\widehat{rr}_t + \hat{\pi}_{t+1}) + m(\beta - \gamma)\hat{q}_{t+1} + m(\beta - \gamma)\hat{\pi}_{t+1},\end{aligned}$$

which can be reduced to:

$$\begin{aligned}\hat{q}_t = (1 - m\bar{\lambda}\bar{c})(\hat{c}_t - \mathbb{E}_t\hat{c}_{t+1}) + (1 - m\bar{\lambda}\bar{c} - \gamma)(\mathbb{E}_t\hat{Y}_{t+1} - \mathbb{E}_t\hat{X}_{t+1} - \hat{h}_t) + \gamma\hat{q}_{t+1} \\ (m\bar{\lambda}\bar{c} - m\beta)(\hat{c}_t - \hat{c}_{t+1}) + m(\beta - \gamma)\hat{q}_{t+1}.\end{aligned}$$

This can be further reduced to:

$$\hat{q} = (1 - m\beta)(\hat{c} - \mathbb{E}_t\hat{c}_{t+1}) + (1 - m\bar{\lambda}\bar{c} - \gamma)(\mathbb{E}_t\hat{Y}_{t+1} - \mathbb{E}_t\hat{X}_{t+1} - \hat{h}_t) - m\beta\widehat{rr}_t + (\gamma(1 - m) + m\beta)\hat{q}_{t+1}.$$

But since $\bar{\lambda}\bar{c} = \beta - \gamma$, we can write:

$$\begin{aligned}(1 - m\bar{\lambda}\bar{c} - \gamma) &= 1 - m(\beta - \gamma) - \gamma \\ &= 1 - m\beta + m\gamma - \gamma \\ &= 1 - m\beta - \gamma(1 - m),\end{aligned}$$

and so we have:

$$\hat{q}_t = (1 - m\beta)(\hat{c}_t - \mathbb{E}_t\hat{c}_{t+1}) + (1 - m\beta - \gamma(1 - m))(\mathbb{E}_t\hat{Y}_{t+1} - \mathbb{E}_t\hat{X}_{t+1} - \hat{h}_t) - m\beta\widehat{rr}_t + (\gamma(1 - m) + m\beta)\hat{q}_{t+1}. \tag{969}$$

This is identical to (L4) in the paper, where he defines $\gamma_e = m\beta + (1 - m)\gamma$, since $1 - \gamma_e = 1 - m\beta -$

$\gamma(1-m)$.

So then the reduced linear system is:

$$\hat{c}_t^{'} = \mathbb{E}_t\hat{c}_{t+1}^{'} - \widehat{rr}_t, \tag{970}$$

$$\hat{b}_t = \mathbb{E}_t\hat{q}_{t+1} + \hat{h}_t - \widehat{rr}_t, \tag{971}$$

$$\hat{\pi}_t = -\kappa\hat{X}_t + \beta\mathbb{E}_t\hat{\pi}_{t+1}, \tag{972}$$

$$\hat{R}_t = r_R\hat{R}_{t-1} + (1-r_R)\left[(1+r_\pi)\hat{\pi}_{t-1} + r_Y\hat{Y}_{t-1}\right] + \hat{e}_{R,t}, \tag{973}$$

$$\hat{Y}_t = \frac{\bar{c}}{\bar{Y}}\hat{c}_t + \frac{\bar{c}^{'}}{\bar{Y}}\hat{c}_t^{'}, \tag{974}$$

$$\begin{aligned}\hat{q}_t &= (1-m\beta)(\hat{c}_t - \mathbb{E}_t\hat{c}_{t+1}) + (1-m\beta-\gamma(1-m))(\mathbb{E}_t\hat{Y}_{t+1} - \mathbb{E}_t\hat{X}_{t+1} - \hat{h}_t) \\ &\quad - m\beta\widehat{rr}_t + (\gamma(1-m)+m\beta)\hat{q}_{t+1},\end{aligned} \tag{975}$$

$$\hat{Y}_t = \frac{\eta\upsilon}{\eta-(1-\upsilon)}\hat{h}_{t-1} - \frac{1-\upsilon}{\eta-(1-\upsilon)}\left(\hat{X}_t + \hat{c}_t^{'}\right), \tag{976}$$

$$\bar{c}\hat{c}_t = \bar{b}\hat{b}_t - \bar{q}\bar{h}\hat{h}_t + \bar{q}\bar{h}\hat{h}_{t-1} - \frac{\bar{b}}{\beta}\left(\hat{R}_{t-1} + \hat{b}_{t-1} - \hat{\pi}_t\right) + \frac{\upsilon\bar{Y}}{\bar{X}}(\hat{Y}_t - \hat{X}_t), \tag{977}$$

$$\hat{q}_t = \hat{c}_t^{'} + \iota\hat{h}_t - \beta\mathbb{E}_t\hat{c}_{t+1}^{'} + \beta\mathbb{E}_t\hat{q}_{t+1}, \tag{978}$$

$$\widehat{rr}_t = \hat{R}_t - \mathbb{E}_t\hat{\pi}_{t+1}. \tag{979}$$

where $\gamma_e = m\beta + (1-m)\gamma$ and $\iota = (1-\beta)\bar{h}/\bar{h}^{'}$, and $\kappa = (1-\theta)(1-\theta\beta)/\theta$, these equations are identical to (L1)-(L9) in the paper (augmented to include the real interest rate expression).

17.8.10 The simple model steady state

With our usual notation for the steady state, and assuming a zero steady state inflation (so $\bar{\pi} = 1$ – recall that in Iacoviello's paper, π refers to gross inflation). This means that $\bar{\pi}^* = 1$ and $\bar{v}^p = 1$. This implies that $\bar{X} = \frac{\epsilon}{\epsilon-1}$.

From (921), we get that $\bar{R} = \beta^{-1} = \bar{rr}$. Since $\gamma < \beta$, this insures that $\bar{\lambda} > 0$ (i.e., the borrowing constraint binds in the steady state). Let's start evaluating the other relationships in the steady state.

We have:

$$\frac{j}{\bar{h}'} = (1-\beta)\frac{\bar{q}}{\bar{c}'}, \tag{980}$$

$$\bar{L}^{\eta-1} = \frac{\bar{w}}{\bar{c}'}, \tag{981}$$

$$\bar{w} = \frac{(1-\upsilon)\bar{Y}}{\bar{X}\bar{L}}, \tag{982}$$

$$\bar{q} = \gamma\left(\frac{\upsilon\bar{Y}}{\bar{X}\bar{h}} + \bar{q}\right) + m\bar{\lambda}\bar{c}\bar{q}, \tag{983}$$

$$\bar{\lambda}\bar{c} = \beta - \gamma, \tag{984}$$

$$\bar{b} = \beta m\bar{q}\bar{h}, \tag{985}$$

$$\bar{Y} = A\bar{h}^{\upsilon}\bar{L}^{1-\upsilon}, \tag{986}$$

$$\frac{\bar{c}'}{\bar{Y}} = 1 - \frac{\bar{c}}{\bar{Y}}, \tag{987}$$

$$\frac{\bar{h}}{H} = 1 - \frac{\bar{h}'}{H}, \tag{988}$$

$$\bar{b} = \bar{c} + \frac{\bar{b}}{\beta} + \bar{w}\bar{L} - \frac{\bar{Y}}{\bar{X}}. \tag{989}$$

We can write these expressions as ratios relative to output or the aggregate housing stock, as in the Iacoviello paper.

We can eliminate $\bar{w}\bar{L}$ from (982), and this allows us to write (989) as:

$$\frac{\beta-1}{\beta}\frac{\bar{b}}{\bar{Y}} = \frac{\bar{c}}{\bar{Y}} - \frac{\upsilon}{\bar{X}}. \tag{990}$$

Now fiddle with (983) and (984):

$$1 = \gamma\left(\frac{\upsilon\bar{Y}}{\bar{q}\bar{X}\bar{h}} + 1\right) + m(\beta-\gamma),$$

which we can use to solve for $\bar{q}\bar{h}/\bar{Y}$. In particular:

$$\frac{1}{\gamma} - \frac{m(\beta-\gamma)}{\gamma} - 1 = \frac{\upsilon\bar{Y}}{\bar{q}\bar{h}\bar{X}},$$

which is:

$$\frac{1-m(\beta-\gamma)-\gamma}{\gamma}=\frac{\upsilon}{\bar{X}}\frac{\bar{Y}}{\bar{q}\bar{h}},$$

or:

$$\frac{\bar{q}\bar{h}}{\bar{Y}}=\frac{\upsilon}{\bar{X}}\frac{\gamma}{1-m(\beta-\gamma)-\gamma}=\frac{\gamma\upsilon}{1-\gamma_e}\frac{1}{\bar{X}}. \tag{991}$$

This expression is identical to the second expression for the steady state in the Appendix. But then we can trivially get $\bar{b}/\bar{Y}$ from (985):

$$\frac{\bar{b}}{\bar{Y}}=\frac{\beta m\gamma\upsilon}{1-\gamma_e}\frac{1}{\bar{X}}, \tag{992}$$

which is identical to the third expression the steady state appendix. But now that we know this, we can solve for $\bar{c}/\bar{Y}$:

$$\frac{\beta-1}{\beta}\frac{\beta m\gamma\upsilon}{1-\gamma_e}\frac{1}{\bar{X}}+\frac{\upsilon}{\bar{X}}=\frac{\bar{c}}{\bar{Y}},$$

which can be simplified further to:

$$\begin{aligned}\frac{\bar{c}}{\bar{Y}}&=\frac{1}{\bar{X}}\left[\upsilon-\frac{(1-\beta)m\upsilon\gamma}{1-\gamma_e}\right]\\&=\frac{1}{\bar{X}}\left[\frac{\upsilon(1-\gamma_e)-(1-\beta)m\upsilon\gamma}{1-\gamma_e}\right].\end{aligned}$$

The numerator inside the brackets can be written as $\upsilon(1-m\beta-(1-m\gamma)-(1-\beta)m\gamma)$. But this equals $1-m\beta-\gamma(1-m\beta)=(1-m\beta)(1-\gamma)$. Hence, we have:

$$\frac{\bar{c}}{\bar{Y}}=\frac{\upsilon}{\bar{X}}\frac{(1-m\beta)(1-\gamma)}{1-\gamma_e}, \tag{993}$$

which is identical to the expression in the appendix. But then we can get $\bar{c}^{'}/\bar{Y}$ as simply one minus this.

We are left with getting $\bar{h}/H$, the loan remaining condition in the steady state appendix. Fiddle with (980) to get:

$$j\bar{c}^{'}=(1-\beta)\bar{q}\bar{h}^{'}.$$

Now plug in that $\bar{h}^{'} = H - \bar{h}$:

$$\begin{aligned} j\bar{c}^{'} &= (1-\beta)\bar{q}(H-\bar{h}) \\ \Leftrightarrow j\frac{\bar{c}^{'}}{\bar{h}} &= (1-\beta)\bar{q}\left(\frac{H}{\bar{h}}-1\right) \\ \Leftrightarrow \frac{j\bar{c}^{'}}{(1-\beta)\bar{q}\bar{h}} &= \frac{H}{\bar{h}} - 1, \end{aligned}$$

so:

$$\frac{H}{\bar{h}} = 1 + \frac{j\bar{c}^{'}}{(1-\beta)\bar{q}\bar{h}}.$$

Now multiply and divide the fraction on the RHS by $\bar{Y}$:

$$\begin{aligned} \frac{H}{\bar{h}} &= 1 + \frac{j\bar{c}^{'}\bar{Y}}{(1-\beta)\bar{q}\bar{h}\bar{Y}} \\ &= 1 + \frac{j}{1-\beta}\frac{\bar{c}^{'}}{\bar{Y}}\frac{\bar{Y}}{\bar{q}\bar{h}}, \end{aligned}$$

but we know everything on the RHS, so we have:

$$\frac{\bar{h}}{H} = \left[1 + \frac{j}{1-\beta}\frac{\bar{c}^{'}}{\bar{Y}}\frac{\bar{Y}}{\bar{q}\bar{h}}\right]^{-1}. \tag{994}$$

That completes everything we need for the steady state.

One final thing: we need to re-write the entrepreneur's budget constraint to be in terms of ratios. We have:

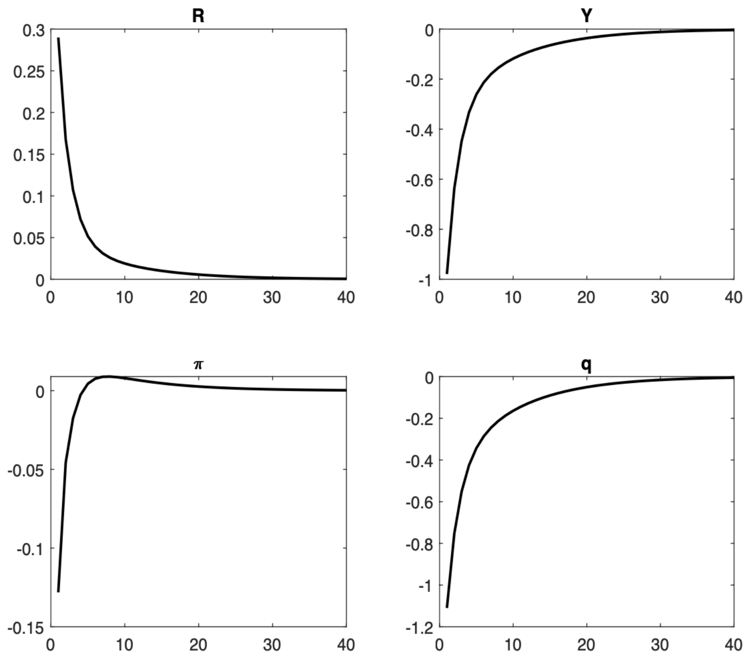
$$\frac{\bar{c}}{\bar{Y}}\hat{c}_t = \frac{\bar{b}}{\bar{Y}}\hat{b}_t - \frac{\bar{q}\bar{h}}{\bar{Y}}\hat{h}_t + \frac{\bar{q}\bar{j}}{\bar{Y}}\hat{h}_{t-1} - \frac{\bar{b}}{\beta\bar{Y}}(\hat{R}_{t-1} + \hat{b}_{t-1} - \hat{\pi}_t) + \frac{\upsilon}{\bar{X}}(\hat{Y}_t - \hat{X}_t). \tag{995}$$

17.8.11 Calibration and policy shocks

Iacoviello sets $\beta = 0.99$, $\gamma = 0.98$, $\upsilon = 0.03$, $j = 0.1$, $m = 0.89$, and $\eta = 1.01$. From this, we can solve for $\iota$and γ_e and all the steady state ratios that we need.

The figure below plots the IRFs to a policy shock. The interest rate exogenously increase. This increases in output and inflation falling. Furthermore, the price of housing falls.

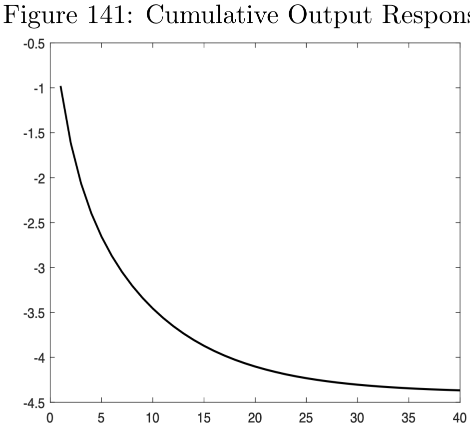
Figure 140: IRFs to Policy Shock



Source: Sims (2020)

The next figure recreates Figure 2 from the Iacoviello paper, which plots the cumulative response of output to a policy shock. This is identical to what he reports in the paper.

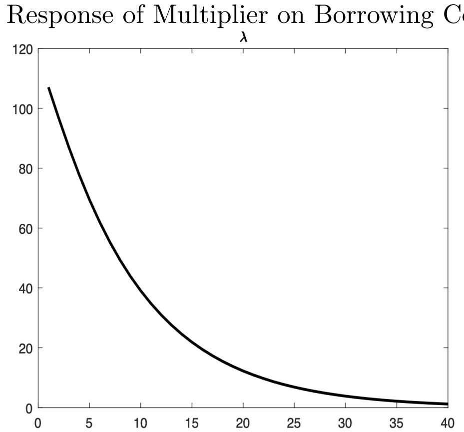
Figure 141: Cumulative Output Response



Source: Sims (2020)

The important insight here is that the borrowing constraint amplifies the response to the monetary policy shock! We're not going to show that explicitly as it is more difficult than just re-parameterising something (i.e. you need to have households and entrepreneurs to have the same discount factor, which makes the borrowing constraint non-binding). But we can think about the logic for why the constraint exacerbates the output effects of the policy shock by focusing on the response of the multiplier facing the entrepreneur.

Figure 142: Response of Multiplier on Borrowing Constraint, λ_t



Source: Sims (2020)

We observe that λ_t goes up quite markedly – i.e., the borrowing constraint gets tighter. There are several reinforcing effects driving this. Recall that the borrowing constraint is given by (925):

$$b_t = m\mathbb{E}_t \frac{q_{t+1} h_t \pi_{t+1}}{R_t}.$$

First, here is direct effect at play – the increase in R_t causes the RHS to get smaller, other factors being equal. This tightens the constraint. Second, there is the effect on house prices. With declining aggregate demand q_{t+1} will decline, also tightening the constraint. And then there is a "debt deflation" channel that occurs because nominal debt is not indexed to inflation (in the paper, Iacoviello talks a decent amount about this). In particular, the decline in inflation also itself makes the constraint tighter. All three of these things work in the same direct – λ_t goes up, which exacerbates the steady state misallocation wherein the entrepreneur has too little housing relative to what would be efficient.

The shock makes it even harder for the entrepreneur to get land, which moves the economy even further from the efficient allocation. And of course, through general equilibrium all these effects on the tightness of the constraint are exacerbated in a "multiplier" or "accelerator" type mechanism – a tighter borrowing constraint for entrepreneurs further reduces the price of housing and inflation, which tightens the constraint, and so on.

17.8.12 Solving the non-linear model

Instead of log-linearising by hand – which is great for intuition of model dynamics, but is incredibly frustrating and time consuming – we can also simply put in the non-linear equations and let Dynare just solve the model for us (via first order or higher approximation). That is, use (919)-(935) without log-linearising by hand and without eliminating static variables.

We have to think a bit about the steady state, though. What matters for the linearisation are steady state rations relative to output – the absolute size of steady state output is irrelevant. To solve the steady state of the non-linearised model, however, we do have to worry about absolute sizes. There is typically a "free" normalisation at play. Most often, we normalise $\bar{A} = 1$. Iacoviello instead normalises $\bar{Y} = 1$. This is absolutely fine, and maps in nicely to the steady state ratio work when constructing the linearised model. But we have to pick A to be consistent with that normalisation, instead of the typical approach of setting $A = 1$. The choice of steady state H (the total available fixed stock of housing) will matter for the requisite normalisation of A in this setup but is otherwise not directly relevant.

(991)-(994) give us $\bar{q}\bar{h}/\bar{Y}$, $\bar{b}/\bar{Y}$, $\bar{c}/\bar{Y}$, and $\bar{h}/H$. Normalising $\bar{Y} = 1$, this then gives us steady state values $\bar{q}\bar{h}$, $\bar{b}$, and $\bar{c}$ (and hence $\bar{c}'$). Let's just set $H = 1$. But then we can use (994) to give us $\bar{h}$:

$$\bar{h} = \left[1 + \frac{j}{1-\beta}\frac{\bar{c}'}{\bar{q}\bar{h}}\right]^{-1}. \tag{996}$$

The FOC for labour supply (920) can be written as:

$$\bar{L}^{\eta} = \frac{\bar{w}\bar{L}}{\bar{Y}}\frac{\bar{Y}}{\bar{c}'},$$

but from the labour demand condition we know:

$$\frac{\bar{w}\bar{L}}{\bar{Y}} = \frac{1-\upsilon}{\bar{X}}.$$

Using this, along with the normalisation of $\bar{Y} = 1$ and the above-found values of $\bar{c}^{'}$, gives us:

$$\bar{L} = \left(\frac{1-\upsilon}{\bar{X}\bar{c}'}\right)^{\frac{1}{\eta}}. \tag{997}$$

But since we know $\bar{L}$ and $\bar{h}$ now, we can determine the A that is consistent with $Y = 1$ from the production function:

$$\bar{A} = \frac{1}{\bar{h}^{\upsilon} L^{1-\upsilon}}. \tag{998}$$

We can put the non-linear equations into Dynare and let it do the linearisation, and we will get virtually identical IRFs and moments as when we do the linearisation by hand. Handy!

17.8.13 Iacoviello's extended model

The extended model is basically the same as the baseline model, with a couple of modifications. First, entrepreneurs can accumulate physical capital. Second, an additional impatient household is added that is also subject to a borrowing constraint. Third, all households face convex housing stock adjustment costs. Fourth, there is now a stochastic productivity shock and a preference shock to housing.

17.8.14 Entrepreneurs

Entrepreneurs produce wholesale output according to:

$$Y_{w,t} = A_t K_{t-1}^{\mu} h_{t-1}^{\upsilon} (L_t^{'})^{\alpha(1-\mu-\upsilon)} (L_t^{''})^{(1-\alpha)(1-\mu-\upsilon)}, \tag{999}$$

where the prime denotes the labour supply of the patient household and the double prime the impatient household. The budget constraint of the entrepreneur is:

$$\frac{Y_{w,t}}{X_t} + b_t + q_t h_{t-1} - w_t^{'} L_t^{'} - w_t^{''} L_t^{''} = c_t + q_t h_t + \frac{R_{t-1} b_{t-1}}{\pi_t} + I_t + \xi_{e,t} + \xi_{k,t}. \tag{1000}$$

On the income side, the entrepreneur earns income from selling to retailers, issues new debt, earns income from its existing stock of housing, and pays labour (to both types of households). On the expenditure side, it consumes, buys new housing, pays interest on its debt, invests in new capital, and pays adjustment costs on housing and capital. These adjustment costs are given by:

$$\begin{aligned}
\xi_{k,t} &= \psi \left(\frac{I_t}{K_{t-1}} - \delta \right)^2 \frac{K_{t-1}}{2\delta}, \\
\xi_{e,t} &= \phi_e \left(\frac{h_t - h_{t-1}}{h_{t-1}} \right)^2 \frac{q_t h_{t-1}}{2}.
\end{aligned}$$

Preferences are the same as before, as is the borrowing constraint. The Lagrangian is:

$$\mathcal{L} = \mathbb{E}_t \sum_{s=0}^{\infty} \gamma^s \left\{ \begin{array}{c} \ln c_{t+s} + \lambda_{t+s} \left[m \mathbb{E}_s q_{t+s+1} h_{t+s} \pi_{t+s+1} - b_{t+s} R_{t+s} \right] \\ +\mu_{1,t+s} \left[\begin{array}{c} A_{t+s} K_{t+s-1}^{\mu} h_{t+s-1}^{\upsilon} (L_{t+s}^{'})^{\alpha(1-\mu-\upsilon)} (L_{t+s}^{''})^{(1-\alpha)(1-\mu-\upsilon)} \frac{1}{X_{t+s}} \\ +b_{t+s} + q_{t+s} h_{t+s-1} - w_{t+s}^{'} L_{t+s}^{'} - w_{t+s}^{''} L_{t+s}^{''} - c_{t+s} - q_{t+s} h_{t+s} \\ -\frac{R_{t+s} b_{t+s-1}}{\pi_{t+s}} - I_{t+s} - \psi \left(\frac{I_{t+s}}{K_{t+s-1}} - \delta \right)^2 \frac{K_{t+s-1}}{2\delta} - \phi_e \left(\frac{h_{t+s} h_{t+s-1}}{h_{t+s-1}} \right)^2 \frac{q_{t+s} h_{t+s-1}}{2} \end{array} \right] \\ +\mu_{2,t+s} \left[I_{t+s} + (1-\delta) K_{t+s-1} - K_{t+s} \right] \end{array} \right\}$$

The FOCs are:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial c_t} &= \frac{1}{c_t} - \mu_{1,t} = 0, \\
\frac{\partial \mathcal{L}}{\partial L_t^{'}} &= \frac{\alpha(1-\mu-\upsilon)Y_t}{L_t^{'} X_t} - w_t^{'} = 0, \\
\frac{\partial \mathcal{L}}{\partial L_t^{''}} &= \frac{(1-\alpha)(1-\mu-\upsilon)Y_t}{L_t^{''} X_t} - w_t^{''} = 0, \\
\frac{\partial \mathcal{L}}{\partial I_t} &= -\mu_{1,t}\left[1 + \frac{\psi}{\delta}\left(\frac{I_t}{K_{t-1}} - \delta\right)\right] + \mu_{2,t} = 0, \\
\frac{\partial \mathcal{L}}{\partial K_t} &= -\mu_{2,t} + \gamma \mathbb{E}_t \mu_{1,t+1}\left[\frac{\mu Y_{t+1}}{K_t X_{t+1}} - \frac{\psi}{2\delta}\left(\frac{I_{t+1}}{K_t} - \delta\right)^2 + \frac{\psi}{\delta}\left(\frac{I_{t+1}}{K_t} - \delta\right)\frac{I_{t+1}}{K_t}\right] + \gamma(1-\delta)\mathbb{E}_t \mu_{2,t+1}, \\
\frac{\partial \mathcal{L}}{\partial b_t} &= -\lambda_t R_t + \mu_{1,t} - \gamma \mathbb{E}_t \frac{\mu_{1,t+1} R_t}{\pi_{t+1}}, \\
\frac{\partial \mathcal{L}}{\partial h_t} &= \lambda_t m \mathbb{E}_t q_{t+1}\pi_{t+1} - \mu_{1,t} q_t \left(1 + \phi_e \left(\frac{h_t - h_{t-1}}{h_{t-1}}\right)\right) \\
&\quad + \gamma \mathbb{E}_t \mu_{1,t+1}\left[\frac{\upsilon Y_{t+1}}{h_t X_{t+1}} + q_{t+1} - \frac{\phi_e}{2}\left(\frac{h_{t+1} - h_t}{h_t}\right)^2 q_t + \phi_e \left(\frac{h_{t+1} - h_t}{h_t}\right)\frac{q_{t+1} h_{t+1}}{h_t}\right].
\end{aligned}
$$

The labour demand schedules are straightforward:

$$
w_t^{'} = \frac{\alpha(1-\mu-\upsilon)Y_t}{L_t^{'} X_t}, \tag{1001}
$$

$$
w_t^{''} = \frac{(1-\alpha)(1-\mu-\upsilon)Y_t}{L_t^{''} X_t}. \tag{1002}
$$

Now, let's deal with the multipliers. From the FOC for investment, we have:

$$
\mu_{2,t} = \frac{1}{c_t}\left(1 + \frac{\psi}{\delta}\left(\frac{I_t}{K_{t-1}} - \delta\right)\right),
$$

which is what Iacoviello calls υ_t in the appendix – so let's use that notation. Then, if you got the capital FOC, we have:

$$
\upsilon_t = \gamma \mathbb{E}_t \frac{1}{c_{t+1}}\left[\frac{\psi}{\delta}\left(\frac{I_{t+1}}{K_t} - \delta\right)\frac{I_{t+1}}{K_t} - \frac{\psi}{2\delta}\left(\frac{I_{t+1}}{K_t} - \delta\right)^2\right] + \gamma \mathbb{E}_t\left[\frac{\mu Y_{t+1}}{c_{t+1} K_t X_{t+1}} + (1-\delta)\upsilon_{t+1}\right]. \tag{1003}
$$

This looks almost just like what is in the appendix, except for the timing terms on the adjustment cost part. This has to be a mistake in the Iacoviello paper – K_{t-1} is predetermined, you are choosing K_t, and that only affects adjustment costs in $t+1$, so i) there should be discount, ii) it should be weighted by $1/c_{t+1}$, not $1/c_t$, and iii) the terms should be I_{t+1} and K_t, not I_t and K_{t-1}. Note if you go the technical appendix on Iacoviello's homepage, he has the right timing consistent with this equation.

The FOC for housing can be re-written as:

$$\frac{q_t}{c_t}\left(1+\phi_e\left(\frac{h_t-h_{t-1}}{h_{t-1}}\right)\right)=\gamma\mathbb{E}_t\frac{1}{c_{t+1}}\left[\frac{\upsilon Y_{t+1}}{h_t X_{t+1}}+q_{t+1}-\frac{\phi_e q_{t+1}}{2}\left(\frac{h_{t+1}-h_t}{h_t}\right)^2+\phi_e\left(\frac{h_{t+1}-h_t}{h_t}\right)\frac{q_{t+1}h_{t+1}}{h_t}\right]$$
$$m\lambda_t\mathbb{E}_t q_{t+1}\pi_{t+1}, \tag{1004}$$

which is just the same as in the paper, just modified to include the terms related to the adjustment cost. The FOC for bonds is the same as in the standard model:

$$\frac{1}{c_t}=\gamma\mathbb{E}_t\frac{1}{c_{t+1}}\frac{R_t}{\pi_{t+1}}+\lambda_t R_t. \tag{1005}$$

17.8.15 Impatient households

The new agents are impatient households, denoted with double primed variables. They discount via $\beta^{''}<\beta$. Their problem looks just like the patent household, modified to include a housing adjustment cost, with the exception that, like the entrepreneurs, they face a borrow constraint. Their budget constraint is:

$$c_t^{''}+q_t h_t^{''}+\frac{R_{t-1}b_{t-1}^{''}}{\pi_t}=b_t^{''}+q_t^{''}h_{t-1}^{''}+w_t^{''}L_t^{''}+T_t^{''}-\frac{\phi_h}{2}\left(\frac{h_t^{''}-h_{t-1}^{''}}{h_{t-1}^{''}}\right)^2 q_t h_{t-1}^{''}.$$

Preferences are the same as in the base model, though modified to include a preference shock for housing, j_t (i.e. j is now stochastic, and will apply to both patient and impatient households). A

Lagrangian for the impatient households' problem is:

$$\mathcal{L} = \mathbb{E}_t \sum_{s=0}^{\infty} (\beta^{''})^s \left\{ \begin{array}{l} \ln c^{''}_{t+s} + j_{t+s} \ln h^{''}_{t+s} - \frac{1}{\eta}(L^{''}_{t+s})^{\eta} + \lambda^{''}_{t+s} \left[m^{''} \mathbb{E}_s q_{t+s+1} h^{''}_{t+s} \pi_{t+s+1} - b^{''}_{t+s} R_{t+s} \right] \\ +\mu^{''}_{t+s} \left[\begin{array}{c} b^{''}_{t+s} + q_{t+s} h^{''}_{t+s-1} + w^{''}_{t+s} L^{''}_{t+s} + T^{''}_{t+s} \\ -\frac{\phi_h}{2} \left(\frac{h^{''}_{t+s} - h^{''}_{t+s-1}}{h^{''}_{t+s-1}} \right)^2 q_{t+s} h^{''}_{t+s-1} - c^{''}_{t+s} - q_{t+s} h^{''}_{t+s} - \frac{R_{t+s-1} b^{''}_{t+s-1}}{\pi_{t+s}} \end{array} \right] \end{array} \right\},$$

and the FOCs are:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial c^{''}_t} &= \frac{1}{c^{''}_t} - \mu^{''}_t = 0, \\ \frac{\partial \mathcal{L}}{\partial L^{''}_t} &= -(L^{''}_t)^{\eta-1} + \mu^{''}_t w^{''}_t = 0, \\ \frac{\partial \mathcal{L}}{\partial b^{''}_t} &= -R_t \lambda^{''}_t + \mu^{''}_t - \beta^{''} \mathbb{E}_t \frac{\mu^{''}_t R_t}{\pi_{t+1}} = 0, \\ \frac{\partial \mathcal{L}}{\partial h^{''}_t} &= \frac{j_t}{h^{''}_t} + m^{''} \lambda_t \mathbb{E}_t q_{t+1} \pi_{t+1} - \mu^{''}_t \left[\phi_h \left(\frac{h^{''}_t - h^{''}_{t-1}}{h^{''}_{t-1}} \right) q_t - q_t \right] \\ &\quad + \beta^{''} \mathbb{E}_t \mu^{''}_t \left[q_{t+1} - \frac{\phi_h q_{t+1}}{2} \left(\frac{h^{''}_{t+1} - h^{''}_t}{h^{''}_t} \right)^2 + \phi_h \left(\frac{h^{'''}_{t+1} - h^{''}_t}{h^{''}_t} \right) q_{t+1} \frac{h^{''}_{t+1}}{h^{''}_t} \right]. \end{aligned}$$

We can eliminate the multiplier and write these as:

$$\frac{w^{''}_t}{c^{''}_t} = (L^{''}_t)^{\eta-1}, \tag{1006}$$

$$\frac{1}{c^{''}_t} = \beta^{''} \mathbb{E}_t \frac{1}{c^{''}_{t+1}} \frac{R_t}{\pi_{t+1}} + R_t \lambda^{''}_t, \tag{1007}$$

$$\begin{aligned} \frac{q_t}{c^{''}_t} \left[1 + \phi_h \left(\frac{h^{''}_t - h^{''}_{t-1}}{h^{''}_{t-1}} \right) \right] &= \frac{j_t}{h^{''}_t} + \beta^{''} \mathbb{E}_t \left[\frac{q_{t+1}}{c^{''}_{t+1}} \left(1 - \frac{\phi_h}{2} \left(\frac{h^{''}_{t+1} - h^{''}_t}{h^{''}_t} \right)^2 + \phi_h \left(\frac{h^{''}_{t+1} - h^{''}_t}{h^{''}_t} \right) \frac{h^{''}_{t+1}}{h^{''}_t} \right) \right] \\ &\quad + m^{''} \lambda^{''}_t q_{t+1} \pi_{t+1}. \end{aligned} \tag{1008}$$

The third FOC here is quite close to what Iacoviello has in the appendix, but he seems to be missing the term involving the square of the difference.

17.8.16 Patient households

The patient households look exactly like the impatient households, except they are not subject to a borrowing constraint. They are subject to the same adjustment cost and same preference shock, j_t. Hence, their FOCs are:

$$\frac{w_t^{'}}{c_t^{'}} = (L_t^{'})^{\eta-1}, \tag{1009}$$

$$\frac{1}{c_t^{'}} = \beta \mathbb{E}_t \frac{1}{c_{t+1}^{'}} \frac{R_t}{\pi_{t+1}}, \tag{1010}$$

$$\frac{q_t}{c_t^{'}} \left[1 + \phi_h \left(\frac{h_t^{'} - h_{t-1}^{'}}{h_{t-1}^{'}} \right) \right] = \frac{j_t}{h_t^{'}} + \beta \mathbb{E}_t \left[\frac{q_{t+1}}{c_{t+1}^{'}} \left(1 - \frac{\phi_h}{2} \left(\frac{h_{t+1}^{'} - h_t^{'}}{h_t^{'}} \right)^2 + \phi_h \left(\frac{h_{t+1}^{'} - h_t^{'}}{h_t^{'}} \right) \frac{h_{t+1}^{'}}{h_t^{'}} \right) \right]. \tag{1011}$$

17.8.17 Other parts

The retailer problem is identical to before. Aggregation related to price-setting and aggregate production is the same. Aggregation related to price-setting and aggregate production is the same. Aggregation on the demand side is a bit trickier. We need to sum the budget constraints of the three agents. We have:

$$\left\{ \begin{array}{c} c_t^{'} + q_t h_t^{'} + \frac{R_{t-1} b_{t-1}^{'}}{\pi_t} + \xi_{h,t}^{'} \\ + c_t^{''} + q_t h_t^{''} + \frac{R_{t-1} b_{t-1}}{\pi_t} + \xi_{h,t}^{''} \\ + c_t + q_t h_t + \frac{R_{t-1} b_{t-1}}{\pi_t} + I_t + \xi_{e,t} + \xi_{k,t} \end{array} \right\} = \left\{ \begin{array}{c} b_t^{'} + w_t^{'} L_t^{'} + q_t h_{t-1}^{'} + F_t + \frac{Y_{w,t}}{X_t} \\ + b_t + q_t h_{t-1} - w_t^{'} L_t^{'} - w_t^{''} L_t^{''} \\ + b_t^{''} + q_t^{''} h_{t-1}^{''} + w_t^{''} L_t^{''} \end{array} \right\}.$$

Bond market clearing requires $b_t^{'} + b_t^{''} + b_t = 0$. House market clearing requires $h_t^{'} + h_t^{''} + h_t = H$. Imposing these things plus labour market clearing gives (using the same fact about remitted profits) gives the constraint:

$$c_t^{'} + c_t^{''} + c_t + I_t + \xi_{e,t} + \xi_{k,t} + \xi_{h,t}^{'} + \xi_{h,t}^{''} = Y_t, \tag{1012}$$

where the ξ terms are shorthand for the adjustment costs.

17.8.18 Full set of equilibrium conditions for the extended model

The full set of equilibrium conditions are:

$$\frac{w_t^{'}}{c_t^{'}} = (L_t^{'})^{\eta-1}, \tag{1013}$$

$$\frac{1}{c_t^{'}} = \beta \mathbb{E}_t \frac{1}{c_{t+1}^{'}} \frac{R_t}{\pi_{t+1}}, \tag{1014}$$

$$\frac{w_t^{''}}{c_t^{''}} = (L_t^{''})^{\eta-1}, \tag{1015}$$

$$\frac{1}{c_t^{''}} = \beta^{''} \mathbb{E}_t \frac{1}{c_{t+1}^{''}} \frac{R_t}{\pi_{t+1}} + R_t \lambda_t^{''}, \tag{1016}$$

$$\frac{1}{c_t} = \gamma \mathbb{E}_t \frac{1}{c_{t+1}} \frac{R_t}{\pi_{t+1}} + \lambda_t R_t, \tag{1017}$$

$$\frac{q_t}{c_t^{'}} \left[1 + \phi_h \left(\frac{h_t^{'} - h_{t-1}^{'}}{h_{t-1}^{'}} \right) \right] = \frac{j_t}{h_t^{'}} + \beta \mathbb{E}_t \left[\frac{q_{t+1}}{c_{t+1}^{'}} \left(1 - \frac{\phi_h}{2} \left(\frac{h_{t+1}^{'} - h_t^{'}}{h_t^{'}} \right)^2 + \phi_h \left(\frac{h_{t+1}^{'} - h_t^{'}}{h_t^{'}} \right) \frac{h_{t+1}^{'}}{h_t^{'}} \right) \right], \tag{1018}$$

$$\begin{aligned} \frac{q_t}{c_t^{''}} \left[1 + \phi_h \left(\frac{h_t^{''} - h_{t-1}^{''}}{h_{t-1}^{''}} \right) \right] &= \frac{j_t}{h_t^{''}} + \beta^{''} \mathbb{E}_t \left[\frac{q_{t+1}}{c_{t+1}^{''}} \left(1 - \frac{\phi_h}{2} \left(\frac{h_{t+1}^{''} - h_t^{''}}{h_t^{''}} \right)^2 + \phi_h \left(\frac{h_{t+1}^{''} - h_t^{''}}{h_t^{''}} \right) \frac{h_{t+1}^{''}}{h_t^{''}} \right) \right] \\ &+ m^{''} \lambda_t^{''} q_{t+1} \pi_{t+1}, \end{aligned} \tag{1019}$$

$$\begin{aligned} \frac{q_t}{c_t} \left(1 + \phi_e \left(\frac{h_t - h_{t-1}}{h_{t-1}} \right) \right) &= \gamma \mathbb{E}_t \frac{1}{c_{t+1}} \left[\begin{array}{c} \frac{v Y_{t+1}}{h_t X_{t+1}} + q_{t+1} - \frac{\phi_e q_{t+1}}{2} \left(\frac{h_{t+1} - h_t}{h_t} \right)^2 \\ + \phi_e \left(\frac{h_{t+1} - h_t}{h_t} \right) \frac{q_{t+1} h_{t+1}}{h_t} \end{array} \right] \\ &+ m \lambda_t \mathbb{E}_t q_{t+1} \pi_{t+1}, \end{aligned} \tag{1020}$$

$$w_t^{'} = \frac{\alpha(1-\mu-\upsilon)Y_t}{L_t^{'}X_t}, \tag{1021}$$

$$w_t^{''} = \frac{(1-\alpha)(1-\mu-\upsilon)Y_t}{L_t^{''}X_t}, \tag{1022}$$

$$\upsilon_t = \frac{1}{c_t}\left(1+\frac{\psi}{\delta}\left(\frac{I_t}{K_{t-1}}-\delta\right)\right), \tag{1023}$$

$$\begin{aligned}\upsilon_t &= \gamma\mathbb{E}_t\frac{1}{c_{t+1}}\left[\frac{\psi}{\delta}\left(\frac{I_{t+1}}{K_t}-\delta\right)\frac{I_{t+1}}{K_t}-\frac{\psi}{2\delta}\left(\frac{I_{t+1}}{K_t}-\delta\right)^2\right] \\ &\quad+\gamma\mathbb{E}_t\left[\frac{\mu Y_{t+1}}{c_{t+1}K_tX_{t+1}}+(1-\delta)\upsilon_{t+1}\right],\end{aligned} \tag{1024}$$

$$b_t = m\mathbb{E}_t\frac{q_{t+1}h_t\pi_{t+1}}{R_t}, \tag{1025}$$

$$b_t^{''} = m^{''}\mathbb{E}_t\frac{q_{t+1}h_t^{''}\pi_{t+1}}{R_t}, \tag{1026}$$

$$b_t = c_t + q_t(h_t-h_{t-1}) + \frac{R_{t-1}b_{t-1}}{\pi_t} + w_t^{'}L_t^{'} + w_t^{''}L_t^{''} + I_t + \xi_{e,t} + \xi_{k,t} - \frac{Y_t\upsilon_t^p}{X_t}, \tag{1027}$$

$$b_t^{''} = c_t^{''} + q_t(h_t^{''}-h_{t-1}^{''}) + \frac{R_{t-1}b_{t-1}^{''}}{\pi_t} - w_t^{''}L_t^{''} + \xi_{h,t}^{''}, \tag{1028}$$

$$z_{1,t} = X_t^{-1}Y_t + \theta\mathbb{E}_t\Lambda_{t,t+1}\pi_{t+1}^{\epsilon}z_{1,t+1}, \tag{1029}$$

$$z_{2,t} = Y_t + \theta\mathbb{E}_t\Lambda_{t,t+1}\pi_{t+1}^{\epsilon-1}z_{2,t+1}, \tag{1030}$$

$$\pi_t^* = \frac{\epsilon}{\epsilon-1}\frac{z_{1,t}}{z_{2,t}}, \tag{1031}$$

$$1 = \theta\pi_t^{\epsilon-1} + (1-\theta)(\pi_t^*)^{1-\epsilon}, \tag{1032}$$

$$\upsilon_t^p = (1-\theta)(\pi_t^*)^{-\epsilon} + \theta\pi_t^{\epsilon}\upsilon_{t-1}^p, \tag{1033}$$

$$R_t = (\bar{rr})^{1-r_R}(R_{t-1})^{r_R}\left[\pi_{t-1}^{1+r_\pi}\left(\frac{Y_{t-1}}{\bar{Y}}\right)^{r_y}\right]^{1-r_R}e_{R,t}, \tag{1034}$$

$$Y_t v_t^p = A_t K_{t-1}^{\mu} h_{t-1}^{\upsilon} (L_t^{'})^{\alpha(1-\mu-\upsilon)} (L_t^{''})^{(1-\alpha)(1-\mu-\upsilon)}, \tag{1035}$$

$$K_t = I_t + (1-\delta) K_{t-1}, \tag{1036}$$

$$Y_t = c_t + c_t^{'} + c_t^{''} + I_t + \xi_{e,t} + \xi_{k,t} + \xi_{h,t}^{'} + \xi_{h,t}^{''}, \tag{1037}$$

$$H = h_t + h_t^{'} + h_t^{''}, \tag{1038}$$

$$\xi_{e,t} = \frac{\phi_e}{2} \left(\frac{h_t - h_{t-1}}{h_{t-1}} \right)^2 q_t h_{t-1}, \tag{1039}$$

$$\xi_{h,t}^{'} = \frac{\phi_h}{2} \left(\frac{h_t^{'} - h_{t-1}^{'}}{h_{t-1}^{'}} \right)^2 q_t h_{t-1}^{'}, \tag{1040}$$

$$\xi_{h,t}^{''} = \frac{\phi_h}{2} \left(\frac{h_t^{''} - h_{t-1}^{''}}{h_{t-1}^{''}} \right)^2 q_t h_{t-1}^{''}, \tag{1041}$$

$$\xi_{k,t} = \frac{\psi}{2\delta} \left(\frac{I_t}{K_{t-1}} - \delta \right)^2 K_{t-1}, \tag{1042}$$

$$j_t = (1-\rho_j) j + \rho_j j_{t-1} + s_j \epsilon_{j,t}, \tag{1043}$$

$$A_t = (1-\rho_A) A + \rho_A A_{t-1} + s_A \epsilon_{A,t}. \tag{1044}$$

This is 32 equations. There are 32 variables:
$\{c_t, c_t^{'}, c_t^{''}, h_t, h_t^{'}, h_t^{''}, L_t^{'}, L_t^{''}, I_t, K_t, Y_t, X_t, b_t, b_t^{''}, q_t, R_t, w_t^{'}, w_t^{''}, \pi_t, \pi_t^*, z_{1,t}, z_{2,t}, v_t^p, v_t, \lambda_t, \lambda_t^{''}, j_t, A_t,$
$\xi_{e,t}, \xi_{k,t}, \xi_{h,t}^{'}, \xi_{h,t}^{''}\}$.

17.8.19 The steady state

We're going to solve for the steady state by hand. First of all, the steady state interest rate is standard:

$$\bar{R} = \beta^{-1}. \tag{1045}$$

We know that $\upsilon = 1/\bar{c}$. But this means we can write the Euler equation for capital as:

$$1 = \left[\frac{\mu \bar{Y}}{\bar{K}\bar{X}} + (1-\delta) \right].$$

Since we are targeting $\bar{Y} = 1$ and have a target value of $\bar{X}$, this allows us to solve for the steady state capital stock:

$$\bar{K} = \frac{\mu}{\bar{X}\left[\frac{1}{\gamma} - (1-\delta)\right]}. \tag{1046}$$

Given $\bar{K}$, we then have $\bar{I} = \delta \bar{K}$. None of the adjustment cost terms will be different than zero outside of steady state, and with zero trend inflation the pricing conditions are straightforward. Let's write out a bunch of other equations in steady state and work rom there, given what we know and given the normalisation of $\bar{Y} = 1$.

$$\begin{aligned}
\frac{\bar{w}^{'}\bar{L}}{\bar{c}^{'}} &= (\bar{L}^{'})^{\eta}, \\
\frac{\bar{q}}{\bar{c}^{'}} &= \frac{j}{\bar{h}^{'}} + \beta \frac{\bar{q}}{\bar{c}^{'}}, \\
\frac{\bar{w}^{''}\bar{L}^{''}}{\bar{c}^{''}} &= (L^{''})^{\eta}, \\
\bar{R}\bar{\lambda}^{''} &= \frac{1}{\bar{c}^{''}}(1 - \beta^{''}\bar{R}), \\
\frac{\bar{q}}{\bar{c}^{''}} &= \frac{j}{\bar{h}^{''}} + \beta^{''}\frac{\bar{q}}{\bar{c}^{''}} + m^{''}\bar{\lambda}^{''}\bar{q}, \\
\bar{b}^{''}\bar{R} &= m^{''}\bar{q}\bar{h}^{''}, \\
\bar{w}^{'}\bar{L}^{'} &= \frac{\alpha(1-\mu-\upsilon)}{\bar{X}}, \\
\bar{w}^{''}\bar{L}^{''} &= \frac{(1-\alpha)(1-\mu-\upsilon)}{\bar{X}} \\
\frac{\bar{q}}{\bar{c}} &= \frac{\gamma}{\bar{c}}\left[\frac{\upsilon}{\bar{h}\bar{X}} + \bar{q}\right] + m\bar{\lambda}\bar{q}, \\
\bar{R}\bar{\lambda} &= \frac{1}{\bar{c}}\left[1 - \gamma\bar{R}\right], \\
\bar{b}\bar{R} &= m\bar{q}\bar{h},
\end{aligned}$$

$$
\begin{aligned}
1 &= A\bar{K}^{\mu}\bar{h}^{\upsilon}(\bar{L}^{'})^{\alpha(1-\mu-\upsilon)}(\bar{L}^{''})^{(1-\alpha)(1-\mu-\upsilon)}, \\
1 &= \bar{c} + \bar{c}^{'} + \bar{c}^{''}, \\
H &= \bar{h} + \bar{h}^{'} + \bar{h}^{''}, \\
\bar{b} &= \bar{c} + \bar{R}\bar{b} + \bar{w}^{'}\bar{L}^{'} + \bar{w}^{''}\bar{L}^{''} + \delta\bar{K} - \frac{\bar{Y}}{\bar{X}}, \\
\bar{b}^{''} &= \bar{c}^{''} + \bar{R}\bar{b}^{''} - \bar{w}^{''}\bar{L}^{''}.
\end{aligned}
$$

Let's start eliminating things. First, we can eliminate the wage terms. We get:

$$
\begin{aligned}
\frac{\alpha(1-\mu-\upsilon)}{\bar{X}} &= \bar{c}^{'}(\bar{L}^{'})^{\eta}, \\
\frac{(1-\alpha)(1-\mu-\upsilon)}{\bar{X}} &= \bar{c}^{''}(\bar{L}^{''})^{\eta}, \\
\bar{b}(1-\bar{R}) &= \bar{c} + \frac{1-\mu-\upsilon}{\bar{X}} + \delta\bar{K} - \frac{1}{\bar{X}}, \\
\bar{b}^{''}(1-\bar{R}) &= \bar{c}^{''} - \frac{(1-\alpha)(1-\mu-\upsilon)}{\bar{X}}.
\end{aligned}
$$

Furthermore, note that we can write:

$$
\begin{aligned}
\bar{\lambda} &= \frac{1}{\bar{c}}(\beta-\gamma), \\
\bar{\lambda}^{''} &= \frac{1}{\bar{c}^{''}}(\beta-\beta^{''}).
\end{aligned}
$$

This means that we can write the housing Euler equations as:

$$
\begin{aligned}
\frac{\bar{q}}{\bar{c}^{''}} &= \frac{j}{\bar{h}^{''}} + \beta^{''}\frac{\bar{q}}{\bar{c}^{''}} + m^{''}\bar{q}\frac{1}{\bar{c}^{''}}(\beta-\beta^{''}), \\
\frac{\bar{q}}{\bar{c}} &= \frac{\gamma}{\bar{c}}\left[\frac{\upsilon}{\bar{h}\bar{X}} + \bar{q}\right] + m\bar{q}\frac{1}{\bar{c}}(\beta-\gamma), \\
\frac{\bar{q}}{\bar{c}^{'}} &= \frac{j}{\bar{h}^{'}} + \beta\frac{\bar{q}}{\bar{c}^{'}}.
\end{aligned}
$$

Focus on the second expression. The c's cancel out, and we can multiply both sides by $\bar{h}$:

$$\begin{aligned}
\bar{q}\bar{h} &= \gamma\left[\frac{\upsilon}{\bar{X}} + \bar{q}\bar{h}\right] + m\bar{q}\bar{h}(\beta - \gamma) \\
\Leftrightarrow \bar{q}\bar{h} &= \frac{\gamma\upsilon}{1 - \gamma - (\beta - \gamma)m}\frac{1}{\bar{X}}.
\end{aligned}$$

Similarly:

$$\begin{aligned}
\bar{q}\bar{h}^{''} &= j\bar{c}^{''} + \beta^{''}\bar{q}\bar{h}^{''} + m^{''}\bar{q}\bar{h}^{''}(\beta - \beta^{''}) \\
\Leftrightarrow \bar{q}\bar{h}^{''} &= \frac{j\bar{c}^{''}}{1 - \beta'' - m''(\beta - \beta'')}.
\end{aligned}$$

And, finally:

$$\begin{aligned}
\bar{q}\bar{h}^{'} &= j\bar{c}^{'} + \beta\bar{q}\bar{h}^{'} \\
\Leftrightarrow \bar{q}\bar{h}^{'} &= \frac{j\bar{c}^{'}}{1 - \beta}.
\end{aligned}$$

Now, we can solve for $\bar{c}$. How? Because we know:

$$\bar{c} = \bar{b}\left(\frac{\beta - 1}{\beta}\right) + \frac{\mu + \upsilon}{\bar{X}} - \delta\bar{K},$$

but we know that:

$$\bar{b} = \beta m\bar{q}\bar{h},$$

so we get:

$$\bar{c} = m\bar{q}\bar{h}(\beta - 1) + \frac{\mu + \upsilon}{\bar{X}} - \delta\bar{K}.$$

Knowing $\bar{q}\bar{h}$, we now have $\bar{c}$. We similarly now can solve for $\bar{c}^{''}$:

$$\bar{c}^{''} = \bar{b}''\left(\frac{\beta - 1}{\beta}\right) + \frac{(1 - \alpha)(1 - \mu - \upsilon)}{\bar{X}}.$$

But $\bar{b}^{''} = \beta m^{''} \bar{q}\bar{h}^{''}$, so:

$$\bar{c}^{''} = m^{''} \bar{q}\bar{h}^{''}(\beta - 1) + \frac{(1-\alpha)(1-\mu-\upsilon)}{\bar{X}},$$

and we know $\bar{q}\bar{h}^{''}$ from above, so:

$$\begin{aligned}
\bar{c}^{''} &= m^{''}(\beta - 1)\frac{j\bar{c}^{''}}{1-\beta'' - m''(\beta-\beta'')} + \frac{(1-\alpha)(1-\mu-\upsilon)}{\bar{X}} \\
\Leftrightarrow \left[1 - \frac{jm^{''}(\beta-1)}{1-\beta''-m''(\beta-\beta'')}\right]\bar{c}^{''} &= \frac{(1-\alpha)(1-\mu-\upsilon)}{\bar{X}} \\
\Leftrightarrow \bar{c}^{''} &= \left[1 - \frac{jm^{''}(\beta-1)}{1-\beta''-m''(\beta-\beta'')}\right]^{-1} \frac{(1-\alpha)(1-\mu-\upsilon)}{\bar{X}}.
\end{aligned}$$

We can then solve for $\bar{c}^{'}$ from the resource constraint:

$$\bar{c}^{'} = 1 - \bar{c} - \bar{c}^{''} - \delta\bar{K},$$

which then gives us $\bar{L}^{'}$ and $\bar{L}^{''}$:

$$\begin{aligned}
\bar{L}^{'} &= \left[\frac{\alpha(1-\mu-\upsilon)}{\bar{c}'\bar{X}}\right]^{\frac{1}{\eta}}, \\
\bar{L}^{''} &= \left[\frac{(1-\alpha)(1-\mu-\upsilon)}{\bar{c}''\bar{X}}\right]^{\frac{1}{\eta}}.
\end{aligned}$$

Then we can solve for $\bar{q}$, but noting that $\bar{q}\bar{h} + \bar{q}\bar{h}^{'} + \bar{q}\bar{h}^{''} = \bar{q}$ (since $\bar{h} + \bar{h}^{'} + \bar{h}^{''} = H = 1$):

$$\bar{q} = \frac{\gamma\upsilon}{1-\gamma-(\beta-\gamma)m}\frac{1}{\bar{X}} + \frac{j\bar{c}^{''}}{1-\beta''-m''(\beta-\beta'')} + \frac{j\bar{c}^{'}}{1-\beta}.$$

But then we can recover $\bar{h}$ and $\bar{h}^{'}$, knowing $\bar{q}$, and then $\bar{h}^{''} = 1 - \bar{h} - \bar{h}^{'}$. But then we also have the $\bar{b}$ and $\bar{b}^{'}$ from the borrowing constraints :

$$\begin{aligned}
\bar{b}^{''} &= \beta m^{''}\bar{q}\bar{h}^{''}, \\
\bar{b} &= \beta m\bar{q}\bar{h}.
\end{aligned}$$

We can also solve for the A consistent with out normalisation:

$$A = \left[\bar{K}^{\mu}\bar{h}^{v}(\bar{L}^{'})^{\alpha(1-\mu-v)}(\bar{L}^{''})^{(1-\alpha)(1-\mu-v)}\right]^{-1}.$$

17.8.20 Parameterisation and IRFs

For the extended model, Iacoviello calibrates some parameters and estimates others. More on the estimation below. The calibrated parameters are $\beta = 0.99$, $\gamma = 0.98$, and $\beta^{''} = 0.95$. The parameter on labour in the utility function is $\eta = 1.01$, and the steady state weight on housing in the utility function is $j = 0.1$. Two fo the three production function parameters are $\mu = 0.3$ and $v = 0.03$. The depreciation rate is $\delta = 0.03$, and the capital adjustment cost parameter is $\psi = 2$. Even though we went to the trouble of writing down the model with the housing adjustment cost, this is set to zero: $\phi_e = \phi_h = 0$.[139] The steady state markup is $\bar{X} = 1.05$, and the probability of non-price adjustment is $\theta = 0.75$.

A subset of other parameters are estimated. The estimation is via impulse response matching. The exercise is a bit weird, so we will describe it in words here. First, Iacoviello is estimating a four variable VAR in the nominal interest rate, inflation, the house price, and output. Then he is identifying impulse responses via recursive Cholesky ordering, with the variables following this ordering (i.e. the interest rate is ordered first, etc). To be able to estimate a four variable VAR, he is adding a fourth shock – an "inflation" shock which appears as a residual in the linearised NKPC.

The important point here is this: this VAR ordering only makes sense from the perspective of the model for the interest rate. Why does this ordering only make sense for the interest rate? Well, the interest rate specification has monetary policy reacting only with a lag to inflation and output. This means that ordering monetary policy "first" means that the reduced-form innovation in the interest rate equation can be interpreted as the monetary policy shock – this affects all other variables immediately, but the interest rate only reacts to other shocks with a lag of one period.[140] The orthogonalised shocks

[139] Go figure...

[140] Interestingly, it is worth noting that this is exactly the opposite from how monetary policy shocks are identified in most empirical VARs. In those VARs (see, for example, Christiano, Eichenbaum, et al. (2005)), monetary policy is ordered "late" in the VAR wherein monetary policy is assumed to react instantaneously to other shocks but only affects other variables with a lag.

from the VAR don't map into the model. In the recursive VAR, inflation does not react within period to the last two orthogonalised shocks. But in the model, inflation will react immediately to all shocks. So, we cannot interpret the VAR impulse responses other than the response to the identified monetary shock in a structural way in terms of the model. It is a well-defined exercise to take the DSGE model, form a reduced-form VAR, and compute IRFs to the Cholesky-identified orthogonal "shocks" – these are, if you will, interesting moments on might hope the model can match. But the IRFs to these shocks do not map one-to-one into the IRFs to the actual shocks in the model (again, other than the interest rate).

Let $\hat{\boldsymbol{\Psi}}$ be a vector collection of impulse responses of variables from the empirical VAR. Let $\hat{\boldsymbol{\Psi}}(\boldsymbol{\zeta})$ be a vector collecting the model impulse responses, where $\boldsymbol{\zeta}$ is the vector of parameters to estimate (see below). But what impulse responses in the model? These are not the impulse responses to different shocks in the model. Rather, they are impulse responses to the Cholesky orthogonalisation of the reduced form VAR representation of these four variables in the model. As we just covered, only the impulse response to the "R shock" will be interpretable as the response to a shock in the model.

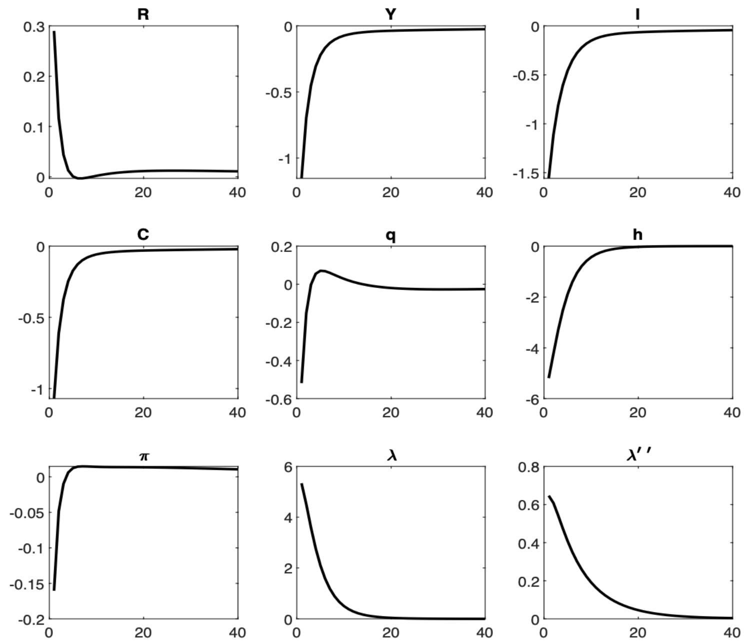
Put differently, the responses Figure 5 (of the paper) are NOT responses to the four shocks in the model; they are responses to the four Choleski-orthogonalised innovations from the reduced-form VAR representation of the model.

The parameters Iacoviello estimates are α (the share on the two kinds of labour), m and $m^{''}$ (the down payment requirements for the entrepreneur and impatient household), and the parameters of the shock processes (but not the monetary policy rule, which is based on estimation of a single equation Taylor Rule). The objective function is to pick $\boldsymbol{\zeta}$ to minimise the distance between the VAR and model impulse responses:

$$\min_{\delta} \; \left[\boldsymbol{\Psi}(\boldsymbol{\zeta}) - \hat{\boldsymbol{\Psi}}\right]^{\top} \boldsymbol{\Phi} \left[\boldsymbol{\Psi}(\boldsymbol{\zeta}) - \hat{\boldsymbol{\Psi}}\right].$$

Here $\boldsymbol{\Phi}$ is the weighting matrix. Typically in these kinds of exercises, the weighting matrix is the inverse variance-covariance matrix of empirical moments that you are targeting (in this case, the IRFs to the Cholesky shocks in the VAR). Iacoviello does something slightly different – see the discussion in the paper. The estimates are in Table 2 of his paper. He essentially estimates the productivity shock to be IID. For what we're going to show below, we use $\rho_A = 0.803$ instead of $\rho_A = 0.03$. Otherwise, we use what Iacoviello reports in the paper. The impulse responses to the three shocks we have in the model are shown below:

Figure 143: IRFs to Policy Shock

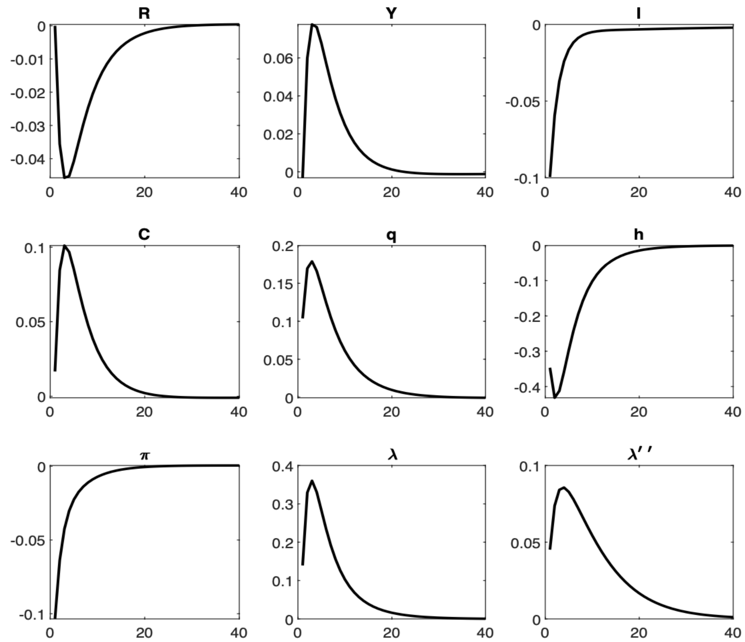


Source: Sims (2020)

The policy shock impulse responses look similar to what shows up in the simple model. They are also basically exactly the same as what he reports in Figure 5 of the paper (solid lines for the model); based on the discussion above, this makes sense, as we actually can interpret the first orthogonalised response as the response to the structural monetary policy shock. The C response we show is aggregate consumption (the sum of consumption of the three types of agents). Investment goes down (by a bit more than output); that's really the only new response relative to the simple model. As shown before, the Lagrange multipliers on the now two borrowing constraints go up – i.e. these constraints become tighter. This amplifies the negative output response to the shock. The decline in the house price, increase in the nominal rate, and decrease in inflation tightens the constraints on patient households and entrepreneurs and amplifies the effects of the exogenous monetary policy disturbance.

Next, we show IRFs to the productivity shock in the model. As noted above, these are not directly comparable to the IRFs he shows in Figure 5; furthermore, we have changed the AR parameter on the productivity process to something more reasonable. These look a little weird relative to most standard models – the increase in productivity is contractionary on impact for output (before going up) and contractionary for investment, whereas in most standard models it would be expansionary for both over all horizons. What is going on? The productivity improvement leads to an increase in the price of housing – patient households end up wanting more of it because their consumption goes up. This price increases causes entrepreneurs to shed their stock of housing, which keeps output from rising much. Note that the entrepreneurs' constraint becomes tighter (i.e. λ_t goes up), even though q_t also goes up – part of what is driving this is that the productivity shock is deflationary. This is kind of a general result in these collateral constraint models. When constraints apply to nominal asset holdings, the constraint tends to amplify the effects of demand shocks (which move inflation in the same direction as output, therefore loosening constraints in periods where demand is high) but does the opposite for supply shocks (because inflation falls, which works to tighten the constraint).

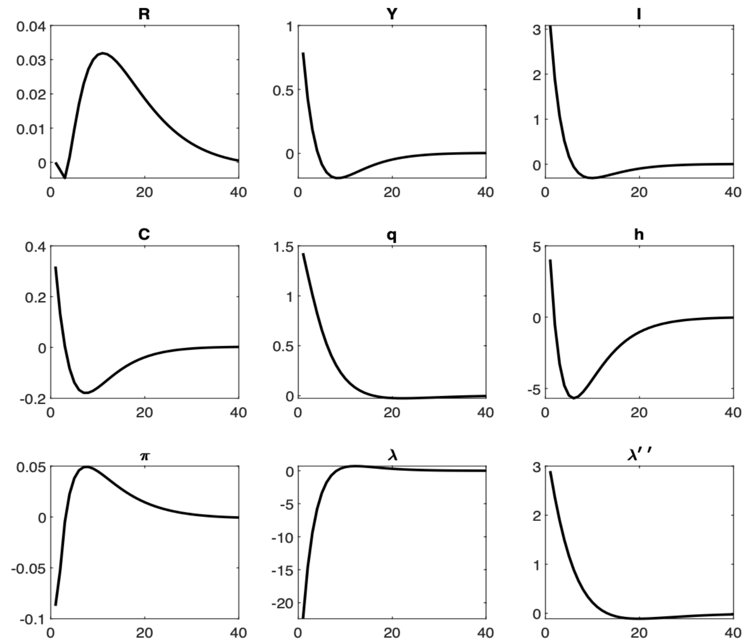
Figure 144: IRFs to Productivity Shocks



Source: Sims (2020)

The responses to the housing preference shock are shown below. Households and patient households decide they like housing more. The immediate impact of this is pushing up the price of housing. On its own, this would cause the entrepreneur to want less housing. But the increase in q_t eases the entrepreneur's borrowing constraint by quite a lot, as evidenced by the decline in λ_t. In spite of the higher price of capital, this actually increases the amount of housing that entrepreneur's have, ht, at least immediately. This triggers an increase in investment because the marginal product of capital is higher. Because of the higher housing initially held by the entrepreneur and the higher investment, the wage goes up initially and labor increases, so we get a temporary output boom. But eventually, the higher price of housing dissuades entrepreneurs from holding housing – basically, the constraint on entrepreneurs is only eased for a while. After that time, housing ends up being consumed by patient and impatient households, which actually results in output eventually falling.

Figure 145: IRFs to Housing Preference Shock



Source: Sims (2020)

17.9 Comments and key readings

The models in this chapter stress different financial imperfections. Most of the discussion was quite self contain to the sub-chapters, so there isn't much more to add here. Besides, this chapter is already long enough. Just very quickly for some additional reading: Christiano and Ikeda (2011) discuss the efficacy of different policy options for the first four models; *Monetary Theory and Policy* by Walsh (2010) also gives a great exposition of these models; and I strongly encourage readers to check out Eric Sims' fantastic set of notes on his website. Like I said, my notes on the Bernanke, Gertler, and Gilchrist; Kiyotaki and Moore; and Iacoviello papers were based on his notes.

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18 Financial Crises

18.1 Introduction

This chapter is inspired by two interviews Tom Sargent gave to the Euro Area Business Cycle Network[141] and the Minneapolis Fed.[142] Sargent offers a strong defence of modern macroeconomics – which came under fire following the fallout of the GFC. In this chapter, we pick up on his thoughts on financial crises in the Minneapolis Fed interview:

> I like to think about two polar models of bank crises and what government lender-of-last-resort and deposit insurance do to arrest them or promote them. Both models had origins in papers written at the Federal Reserve Bank of Minneapolis, one authored by John Kareken and Neil Wallace in 1978 and the other by John Bryant in 1980, then extended by Diamond and Dybvig in 1983. I call them polar models because in the Diamond-Dybvig and Bryant model, deposit insurance is purely a good thing, while in the Kareken and Wallace model, it is purely bad. These differences occur because of what the two models include and what they omit.

The contrast between this chapter and the last on financial frictions is the speed at which financial crises occur. The imperfections in financial markets discussed in the previous chapter cause welfare losses at all times, albeit at some times the constraints are more binding than others. Models of financial crises are more dramatic. In this chapter we will look at two models which offer key insights into financial crises: the models of Diamond and Dybvig (1983), and Kareken and Wallace (1978).

Additionally, since initially writing these notes, I've decided to add a section about the 2007-08 GFC, an overview of Del Negro et al. (2017), and unconventional monetary policy (in particular, quantitative easing). We will cover these topics subsequent to reviewing the Diamond-Dybvig and Kareken-Wallace models – I figured it would be better to get an understanding of these classics before returning to a more contemporary New Keynesian DSGE setting.

[141] http://www.eabcn.org/podcast/andrew-scott-interviews-tom-sargent-nyu
[142] https://www.minneapolisfed.org/article/2010/interview-with-thomas-sargent

18.2 The Diamond-Dybvig model

Financial crises are about runs on short term bank debt. The situation in which short term liability holders "run" en masse to liquidate their savings in financial intermediaries, forcing intermediaries to engage in asset sales that could render them insolvent, is referred to as a bank run. Bank runs, broadly construed, are a recurrent theme in economic history. Some key questions we want to gain some insight into are: i) Why are bank runs so prevalent? ii) Why do people hold short term bank debt (e.g. deposits) if it is nevertheless susceptible to runs? iii) What type of policies can be used to prevent/reduce/mitigate runs?

The Diamond-Dybvig model is a celebrated contribution that:

- Provides a precise definition of liquidity;
- Exposits the benefits of the liquidity transformation that financial intermediaries do;
- Points out the perils of liquidity transformation – susceptibility to runs; and
- Provides a framework to think about policies.

18.2.1 Model basics

There are three periods, indexed by T, $T = \{0, 1, 2\}$. $T = 0$ is the "present" and $T = \{1, 2\}$ measures the "future". There are (many) households who are (ex-ante) identical and are endowed with 1 in $T = 0$ and will need to consume in either $T = 1$ or $T = 2$. There is idiosyncratic uncertainty amongst these households, so the individual household does not know (at $T = 0$) whether she will be type 1 ("impatient" and needs to consume in $T = 1$) or a type 2 ("patient" and can wait to consume until $T = 2$). Each household's type is revealed in $T = 1$. But, there is no aggregate uncertainty: a fixed fraction, $t \in [0, 1]$, of households will be type 1, and a fixed fraction $1 - t$ are type 2.

There are two assets: 1) A costless storage technology (cash), where a household can save 1 unit of endowment in $T = 0$ and have 1 unit available to consume in either $T = 1$ or $T = 2$; and 2) An illiquid investment opportunity where a household can save 1 unit of endowment in $T = 0$ and get R_1 (gross) if liquidated (sold) in $T = 1$, and $R_2 \geq R_1$ if liquidated in $T = 2$.

An individual household has utility:

$$U(c) = 1 - \frac{1}{c},$$

and its expected utility is simply the probability-weighted sum of utility flows depending on which type it ends up being:

$$\mathbb{E}[U] = tU(c_1) + (1-t)U(c_2),$$

where c_1 and c_2 are consumption at each date depending on type. The consumption allocations are $c_1 = c_2 = 1$ if the storage is used, and $c_1 = R_1$ and $c_2 = R_2$ if the investment opportunity is used.

18.2.2 Numerical example

Suppose $R_1 = 1$ and $R_2 = 2$ on the investment technology, and that $t = \frac{1}{4}$. The expected return (gross) from investing is:

$$\mathbb{E}[R] = \frac{1}{4} \times 1 + \frac{3}{4} \times 2 = \frac{7}{4} > 1,$$

and the expected return (gross) on the storage is of course just 1. The expected utility from storage and investing are:

$$\begin{aligned} \mathbb{E}[U]_{\text{storage}} &= \frac{1}{4} \times 0 + \frac{3}{4} \times 2 = 0, \\ \mathbb{E}[U]_{\text{invest}} &= \frac{1}{4} \times 0 + \frac{3}{4}\left(1 - \frac{1}{2}\right) = \frac{3}{8}, \end{aligned}$$

and thus the household prefers investment to storage.

We can think about the liquidity of an asset as the discount one has to pay for "early" liquidation:

$$L = \frac{R_1}{R_2},$$

Since $R_2 \geq R_1$ (by assumption), $L \leq 1$. The further L is from 1, the less liquid is the asset. Cash is, of course, perfectly liquid at $L = 1$. In other words, you get 1 regardless of when you access it.

The investment opportunity is less liquid, with $L = \frac{1}{2}$. Though, in this example, you still prefer to hold the less liquid asset.

18.2.3 Alternative example with less liquid investment

Suppose that early liquidation of the investment incurs a cost of $1 - \tau$, where $\tau \geq 0$. So you get $(1 - \tau)R_1$ for early liquidation. The liquidity of investment in above example is then:

$$L = (1 - \tau)\frac{1}{2} \leq \frac{1}{2}.$$

How big must τ be for the household to not want to do the investment?

$$\mathbb{E}[U]_{\text{invest}} = \frac{1}{4}\left(1 - \frac{1}{1 - \tau}\right) + \frac{3}{4}\left(1 - \frac{1}{2}\right) < 0.$$

We can show $\tau > \frac{3}{5}$ makes investment undesirable relative to storage.

Now consider the case where $\tau = \frac{2}{3}$. The expected utility from storage verses investment is:

$$\begin{aligned} \mathbb{E}[U]_{\text{store}} &= 0, \\ \mathbb{E}[U]_{\text{invest}} &= \frac{1}{4}\left(1 - \frac{1}{\frac{1}{3}}\right) + \frac{3}{4}\left(1 - \frac{1}{2}\right) = -\frac{1}{8}, \end{aligned}$$

and now the household prefers storage to investment. This is despite the expected (gross) return to investment is higher:

$$\begin{aligned} \mathbb{E}[R]_{\text{store}} &= 1, \\ \mathbb{E}[R]_{\text{invest}} &= \frac{1}{4}\frac{1}{3} + \frac{3}{4}2 = \frac{19}{12} > 1. \end{aligned}$$

Thus, if a project is sufficiently illiquid and/or the household is sufficiently risk averse (i.e. $u''(C) < 0$), then the household may not want to directly invest in positive net return projects.

18.2.4 Banks and liquidity transformation

A mutual bank (no equity, just trying to make profit for itself) can potentially step in and make households better off regardless of whether households would directly find the investment project or not. How? In essence, by exploiting a law of large numbers and engaging in what amounts to provision of insurance.

An individual household is uncertain about when she will need to consume: this gives rise to a preference for liquidity. But in the aggregate, there is no uncertainty – exactly the fraction t of households will be type 1 and $1 - t$ will be type 2. The bank can pool the resources from many households exploiting this lack of aggregate uncertainty and offer households an asset that is more liquid than the investment project that the household prefers to both direct investment and storage.

Assuming the same setup as before: $R_1 = 1$, $R_2 = 2$, $t = \frac{1}{4}$, and $\tau = 0$, suppose that the bank offers households an asset with the following payout structure:

$$R^d = \begin{cases} 1.28 & \text{in period 1,} \\ 1.813 & \text{in period 2.} \end{cases}$$

This is more liquid than the investment opportunity:

$$L^d = \frac{R_1^d}{R_2^d} = \frac{1.28}{1.813} = 0.706 > \frac{1}{2}.$$

How does this work? Suppose there are 100 households and exactly 25 will need to withdraw in period $T = 1$. The bank takes 100 in period $T = 0$, and puts it into 100 units of the investment (assume R_1 and R_2 are independent of amount invested). The bank will need to liquidate $25 \times 1.28 = 32$ units of the investment to raise necessary funds in $T = 1$, leaving 68 invested. These 68 will generate 136 in income in $T = 2$, which can be distributed to the remaining 75 deposit holders for $R_2 = \frac{136}{75} = 1.813$.

What does the household prefer? An individual household has three options: storage ($\mathbb{E}[R] = 1$), deposits ($\mathbb{E}[R] = 1.68$), or direct investment ($\mathbb{E}[R] = 2$). Which does it prefer? Its expected utilities are:

$$\begin{aligned}\mathbb{E}[U]_{\text{store}} &= 0,\\ \mathbb{E}[U]_{\text{invest}} &= \frac{3}{8},\\ \mathbb{E}[U]_{\text{deposit}} &= \frac{1}{4}\left(1-\frac{1}{1.28}\right)+\frac{3}{4}\left(1-\frac{1}{1.813}\right) = 0.391 > \frac{3}{8},\end{aligned}$$

and so it prefers deposits. A household's willing to tolerate a lower expected return on deposits because of higher liquidity of deposits relative to direct investment. We can make this even starker if $\tau > 0$.

18.2.5 Consumption smoothing and preference for liquidity

We have assumed that households are risk averse and uncertain about when they will need to consume. Given risk aversion ($U''(c) < 0$), the household has incentive to smooth consumption across states (i.e., type 1 or type 2). If it directly invests in the investment opportunity, marginal utility $U'(c)$ is high if type 1 (gets comparatively low return) and low if type 2 (gets comparatively high return).

The household would like to potentially reallocate some consumption from type 2 state (low marginal utility) to type 1 state (higher marginal utility) – i.e., it would like something more liquid. The household would even be willing to sacrifice some expected return to get this.

The bank is engaging in liquidity transformation:

- It is creating an asset (deposit, which is a liability to the bank but asset to the household) that is more liquid than the underlying asset it is investing in;
- In so doing, it can make households better off.

This is essentially functioning just like insurance: Give up some consumption in "good states" (low marginal utility, type 2) by paying a "premium" to get some extra consumption in "bad states" (high marginal utility, type 1). The bank can offer this, just like an insurance. If we had aggregate uncertainty, things would be more complicated but the basic gist would be the same.

18.2.6 Nash Equilibrium

With many households and a mutual bank, the outcome described above is a Nash equilibrium. Everyone is behaving optimally given beliefs about how others are going to play which implies that there is no incentive to deviate. Suppose I wake up in $T = 1$ and am revealed to be type 2, I do worse by withdrawing in $T = 1$ ($R_1^d = 1.28$) than by waiting until $T = 2$ ($R_2^d = 1.813$) – provided I think other type 2's are going to wait, it's optimal to wait, all type 2's will do this, and then beliefs are self-fulfilling.

When would it make sense to withdraw in $T = 1$ even if I don't have to? Only if I think I will get back less than 1.28 in $T = 2$. For example, if I think [enough] other type 2's are going to withdraw "early", or if I'm worried the bank's investments are going to go bad. But casting concerns over the bank's investments going bad (which would require some aggregate uncertainty[143]), let's focus on multiplicity of equilibria with no aggregate uncertainty: i) The good equilibrium (which is what we just described); and ii) The bad equilibrium, where type 2's withdraw early in $T = 1$ because they expect other type 2's to withdraw early as well, which will cause the bank to fail and make everyone [weakly] worse off.

Let $\hat{f}$ be the expectation of each household about what f will be (i.e., the fraction who will withdraw in $T = 1$). Suppose $\hat{f} = \frac{1}{2}$, so it's believed that half the population is going to withdraw early. Is this expectation self-fulfilling? If $\hat{f} = \frac{1}{2}$, then:

$$
\begin{aligned}
\hat{R}_2^d &= \frac{(1 - \hat{f} R_1^d) R}{1 - \hat{f}} \\
&= \frac{(1 - \frac{1}{2} 1.28) 2}{1 - \frac{1}{2}} \\
&= 1.44.
\end{aligned}
$$

This is less than what was promised $R_2^d = 1.813$, but nevertheless better than what you get by withdrawing today. So it cannot be optimal for type 2's to withdraw early given this forecast (they're better off waiting), so $\hat{f} = \frac{1}{2}$ is not self-fulfilling.

[143] This is, however, an important concern in the real world.

In any equilibrium, at least a fraction t of deposits will be withdrawn, or $f \geq t$, because type 1 households will always withdraw at $T = 1$. The type 2 households will choose to withdraw at $T = 1$ as well if $R_2^d(f) < R_1^d$. In an example economy with $t = \frac{1}{4}$, if just the type 1 households withdraw, $f = t = \frac{1}{4}$, and $R_1^d = 1.28$, then $R_2^d = 1.183 > R_1^d$, and the type 2 households will choose to wait until $T = 2$ to withdraw. Thus, $\hat{f} = f = t = \frac{1}{4}$ is a Nash Equilibrium.

Now suppose instead that $\hat{f} = \frac{3}{4}$. Then people will believe they will get:

$$\hat{R}_2^d = \frac{(1 - \frac{3}{4}1.28)2}{1 - \frac{3}{4}} = 0.32.$$

This is significantly worse than R_1^d. Given this belief, it's best to "get out now". But then $\hat{f} = \frac{3}{4}$ is not self-fulfilling: If that's what everyone believes, then everyone should withdraw. So $\hat{f} = f = 1$ is another Nash Equilibrium. Note it is completely rational (from the perspective of a type 2 household) to withdraw in $T = 1$ given this belief.

If everyone withdraws, then the bank will fail. It can at most come up with N in $T = 1$, where N is the mass of households who each hold 1 unit of the endowment. Suppose that $N = 100$, and everyone chooses to withdraw their deposits, then the bank can't even meet the promised R_1^d. Typically, there is a "first come, first served" aspect – the first 78 people to line up ($100/1.28 \approx 78$) are "made whole" and get $R_1^d = 1.28$ they were promised, but the last 22 get nothing. This increases the incentive to withdraw and withdraw early – you lose out by not being first in line. So there are two equilibria: good (no run) and bad (run).

How do we know which equilibrium will be "played"? We don't – there will exist a cutoff $\bar{f}$ above which any $\hat{f} \to 1$ (run) and below which $\hat{f} \to t$ (no run). In the above example $\bar{f} = 0.5625$. As long as this is pretty far above t, we will spend most of our time in the "good" equilibrium.

It would take a big event that is widely observed to move beliefs enough to switch to the run equilibrium. These shocks are referred to as sunspots – big and easily observed by all agents.

18.2.7 Dealing with runs

Financial intermediation (i.e. "borrow short, lend long") is structurally subject to runs because of liquidity transformation. Given that runs occur, what kind of policies can be instituted to deal with runs once they start? This is the key point from Diamond and Dybvig: A policy which effectively deals with runs ought not to really need to be used in practice. Basically, common knowledge of an effective policy once a run has started decreases the likelihood of a run happening in the first place. If we know our deposits are safe no matter how many type 2's withdraw early, we have no reason to withdraw early ourselves, and we all stay in the "good" equilibrium.

Prior to a well organised central bank in the US, private banks dealt with [recurrent] runs internally via clearinghouses (consortium of banks in a location, e.g. New York). Principle means by which this was done was suspension of convertibility. Basically, the bank would simply refuse (temporarily) to honour demands for conversion of bank debt (e.g. deposits) into cash. Banks would do this together (effectively banding together as one large bank rather than many small banks for the duration of the crisis), until the panic was over. In practice, this was economically costly and didn't stop runs from happening, but it was pretty effective at preventing liquidity crises to force banks into insolvency.

18.2.8 Lender of last resort

The key difficulty is that some people really do need their funds at short notice. How do you decide how much conversion to do before suspending? How do you make sure the cash gets into the appropriate hands? The Federal Reserve was in large part brought into existence to attempt to more efficiently deal with crises and subsequent suspensions that had plagued US banking for much of the 19th century. The idea being that a central bank can create all the reserves it wants to, and if banks ran out of cash to meet withdrawal demands, it could go to the central bank to get requisite cash (Bagehot's rule). Policy makers thought this would put an end to crises.

But it didn't (e.g. US Great Depression), for a variety of a reasons. First, there was a stigma attached to going to the Fed – banks didn't want to borrow from the Fed for fear of exposing themselves as weak and losing future customers. Secondly, the Fed itself didn't understand its role and powers.

In response to the bank failures of the early 1930s, the Federal Deposit Insurance Corporation

(FDIC) was established in 1933. It promised the full value of deposits at member institutions up to a certain limiting value (originally \$2,500, now \$250,000) in the event that the bank failed. In practice this has more or less eliminated traditional banking panics – people know deposits are safe, so no need to run, and we stay in the good equilibrium.

18.3 The Kareken-Wallace model

In stark contrast to Diamond and Dybvig, Kareken and Wallace paper argues that the provision of deposit insurance may be problematic because it gives financial intermediaries incentives to take too risky positions. A policymaker contemplating introducing deposit insurance should therefore brace themselves to also regulate the portfolio positions of financial intermediaries. The argument is a simple one, in that deposit insurance creates a moral hazard problem and too much risk taking. If financial intermediaries do not bare the full consequences of their actions (because they are insured) then profit maximising portfolios will be too risky.

The Kareken and Wallace paper is a difficult read so we look at a simple model to highlight the dangers of insurance when agents select the riskiness of their portfolios. The same intuition carries to the Kareken and Wallace results.

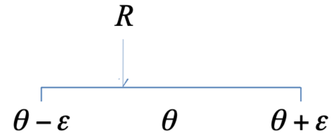
18.3.1 Households and optimal portfolios

Households have a wealth endowment of 1 in period 1 which they wish to transfer to period 2 for consumption. They have access to two assets: i) Safe assets; and ii) Risky assets. The safe asset pays a certain [gross] return of $R > 1$ in period 2. The risky asset pays a return of:

$$R^r = \begin{cases} \theta + \epsilon & \text{w.p. } \frac{1}{2}, \\ \theta - \epsilon & \text{w.p. } \frac{1}{2}, \end{cases}$$

where ϵ is small. We assume that $\theta > R$ so the risky asset has a higher expected return than the safe asset. To ensure neither asset dominates we also assume $R + \epsilon > \theta > R$, so the asset returns are as follows:

Figure 146: Payoff Schedule



The household invests a proportion $1-\mu$ of their endowment in the safe asset and proportion μ in the risky asset. Assuming a quadratic utility function:

$$U(c) = -(c-\bar{c})^2,$$

where c is consumption in period 2, the maximisation problem of the household in the absence of deposit insurance is:

$$\max_{\mu} \; -\frac{1}{2}\left[(1-\mu)R + \mu(\theta+\epsilon) - \bar{c}\right]^2 - \frac{1}{2}\left[(1-\mu)R + \mu(\theta-\epsilon) - \bar{c}\right]^2.$$

The FOC gives the following optimal portfolio share:

$$\mu = \frac{(\bar{c}-R)(\theta-R)}{\epsilon^2 + (\theta-R)^2} < 1,$$

if $\bar{c}-R$ is small.

18.3.2 Deposit insurance

We now introduce deposit insurance so that the household is guaranteed a return of at least R whatever the state of the world. The insurance will only be invoked if the return on the risky asset turns out to be bad, $\theta-\epsilon$. In this case the insurance mechanism kicks and the household receives R. The optimisation problem under insurance is:

$$\max_{\mu'} \; -\frac{1}{2}((1-\mu)R + \mu R - \bar{c})^2 - \frac{1}{2}\left[(1-\mu^{'})R + \mu^{'}(\theta+\epsilon) - \bar{c}\right]^2,$$

and the optimal share of risky assets in the portfolio is:

$$\mu^{'} = 1.$$

Comparing the portfolio shares with and without insurance, we find that $\mu^{'} > \mu$. The introduction of deposit insurance therefore incentivises the households to take riskier portfolio decisions. Kareken and Wallace extend this intuition to a model in which deposit insurance induces bankers to take positions that lead to bankruptcy with positive probability in equilibrium. It is thus necessary for policymakers to regulate the portfolio positions of banks. The focus on Basel II and III on risk-adjusted capital requirements can be seen as a response to this problem. Note that in the Kareken and Wallace framework there is no moral hazard problem until deposit insurance is introduced – banks take positions that do not admit bankruptcy in equilibrium if they face the full consequences of their actions. In this sense, we see that deposit insurance is unambiguously bad in the Kareken and Wallace framework.

18.4 The 2007-08 Global Financial Crisis

A lot has been said and written about the 2008 GFC. I'm sure you can find editorials on *The Financial Times*, *The Economist*, *The Wall Street Journal*, and economists such as Nassim Taleb and Raghuran Rajan[144] have written extensively on the crisis in an effort to explain it to the general public. So, we won't focus so much on the narrative of the GFC, but rather some of the key insights gained from the crisis. In particular, we will focus on: i) The rise of the shadow banking system; ii) the mechanism of spread of the crisis; and iii) monetary policy and its interaction with the crisis.

18.4.1 The rise of shadow banking

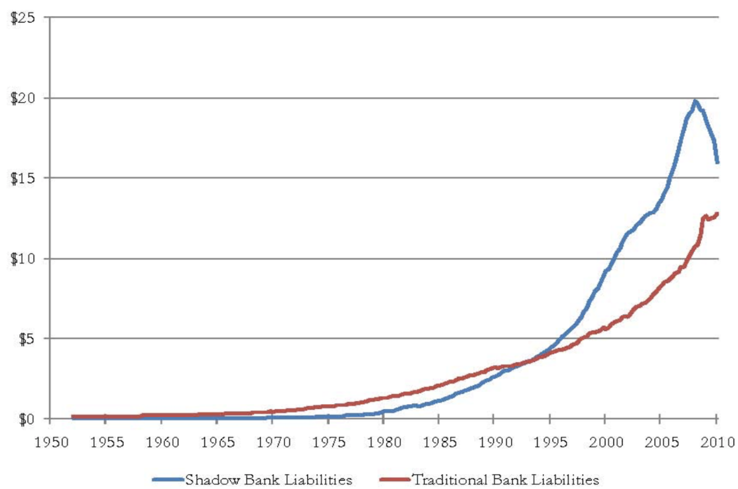
The seeds of the GFC were sewn with the rise of the shadow banking industry. The term "shadow banking"[145] refers to non-bank financial intermediaries which buy, sell, and create credit. Credit is

[144] I would actually recommend reading *Fault Lines: How Hidden Fractures Still Threaten the World Economy* if you have the time.

[145] There is an excellent article by the New York Fed which explains what this means: https://www.newyorkfed.org/medialibrary/media/research/epr/2013/0713adri.pdf

intermediated through a wide range of securitisation and secured financing techniques, including asset-backed commercial paper (CP), asset-backed securities (ABS) – such as mortgage backed securities (MBS) – collateralised debt obligations (CDOs), and repurchase agreements (repos). These are all terms I'm sure many have become familiar with after the GFC, but they existed well before the collapse of Lehman Brothers. Financial deregulation which began in the 1970s, culminating in the 1990s, spurred the growth of the shadow banking sector. In particular, regulations which enforced a strict separation between commercial and investment banks, and geographical restrictions on bank branches were rolled back.

Figure 147: Shadow Bank Liabilities vs Traditional Bank Liabilities ($ trillion)

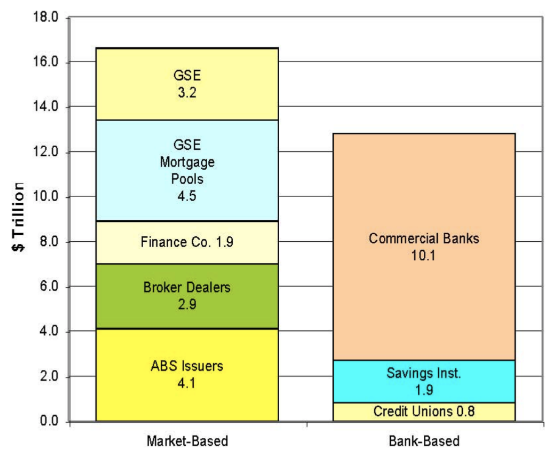


The perceived benefits of this regulatory rollback were increased profitability of the [then booming] financial sector, lower operational costs and loan losses, higher rates of interest being paid to deposit holders, lower borrowing costs for borrowers, and the diversification of local risk. By the 1980s this new system had gained prominence on Wall Street, as traditional banks become less central. Investment banks and government sponsored enterprises (GSEs) (such as Fannie Mae and Freddie Mac) had become more involved, interacting with insurance companies, mutual funds, pension funds, hedge funds, venture capital, and private equity funds.

The result was a systematic increase in competition between all these financial institutions; yet, almost paradoxically, an increase in risk-taking within the banking sector at large. The finance sector had become increasingly hungry to drive profits, and so they sold more originated risk (which was a clear example of moral hazard), and more illiquid activities became profitable. In particular the shadow banks had taken an aggressively active role in providing financial services to customers, despite their inability to raise their own deposits (as raising deposits would involve regulatory oversight). Not only did they take on more risk than traditional banks, but they were also less transparent and were thus more difficult to regulate. Compounding this problem was of course the continued systematic watering-down of regulations and regulators. As the former-CEO of Citibank Group once infamously stated:

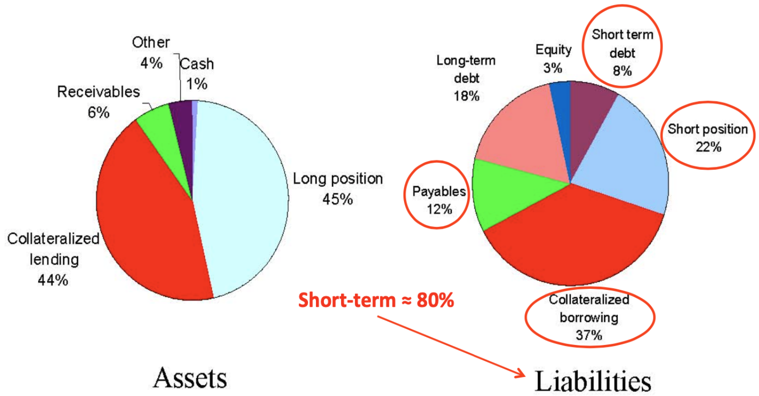
> "As long as the music is playing, you've got to get up and dance. We're still dancing."
>
> – Charles "Chuck" Prince (July 2007)[146]

Figure 148: Total Assets at 2007Q2



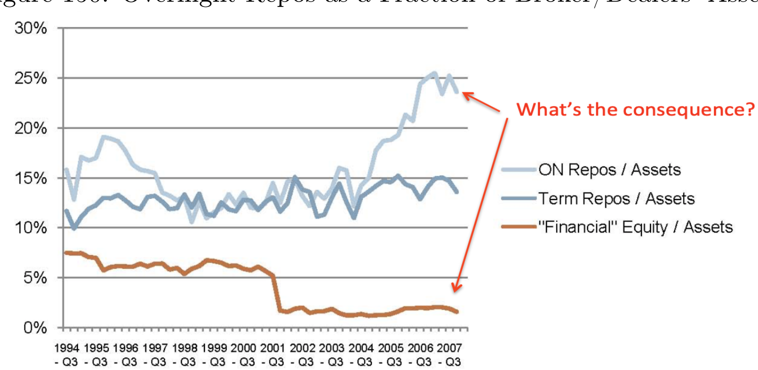
[146] https://www.ft.com/content/80e2987a-2e50-11dc-821c-0000779fd2ac

Figure 149: Lehman's Balance Sheet (2007)



Not only did financial intermediaries specialise in liquidity transformation, but they were also in the game of maturity transformation – raise short term funds using CP markets to finance long-term assets and sell them off. This shortening of maturity can best be observed by repo balances and leverage. Consider the balance sheet of Lehman Brothers in 2007 – short-term liabilities made up around 80 percent of Lehman's balance sheet obligations!

Figure 150: Overnight Repos as a Fraction of Broker/Dealers' Assets



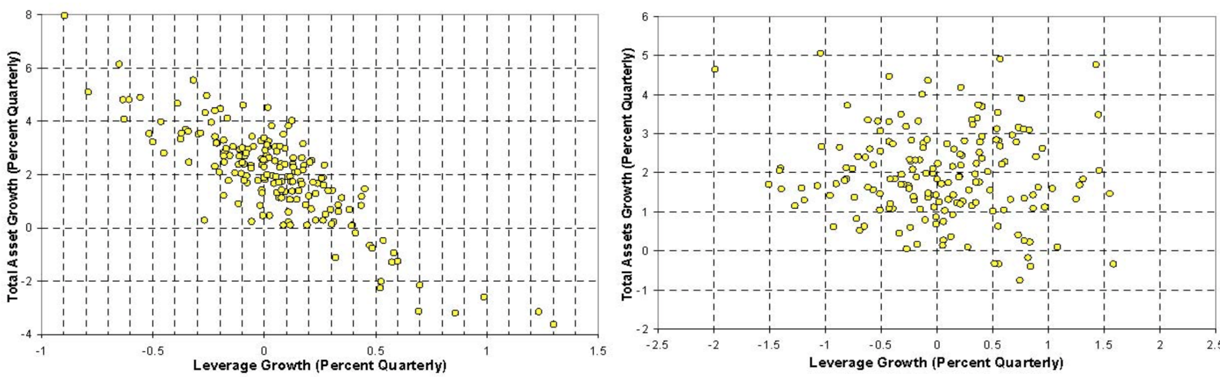
Leverage is defined as:

$$\text{Leverage} = \frac{\text{Assets}}{\text{Equity}} = \frac{\text{Assets}}{\text{Assets} - \text{Liabilities}}.$$

Now, typically, leverage should be inversely related to asset valuations. Take the example of a household. The house price goes up, home owners build more equity in their property, and so their leverage goes down – assuming of course that liabilities remain the same. This is of course intuitive. You take out credit, make an investment, and if your investment pays off (asset prices increase) you reduce your leverage position in that investment.

So how about leverage and asset growth in the US for households and corporations?

Figure 151: Household (left) and Corporate (right) Leverage (1994Q3-2007Q3)



So far so good. We see that higher asset growth is accompanied by reductions in leverage for US households. For the corporate sector we basically see little to no relationship – not too bad. How about banks and brokers?

Figure 152: Commercial Bank (left) and Broker/Dealer (right) Leverage (1994Q3-2007Q3)

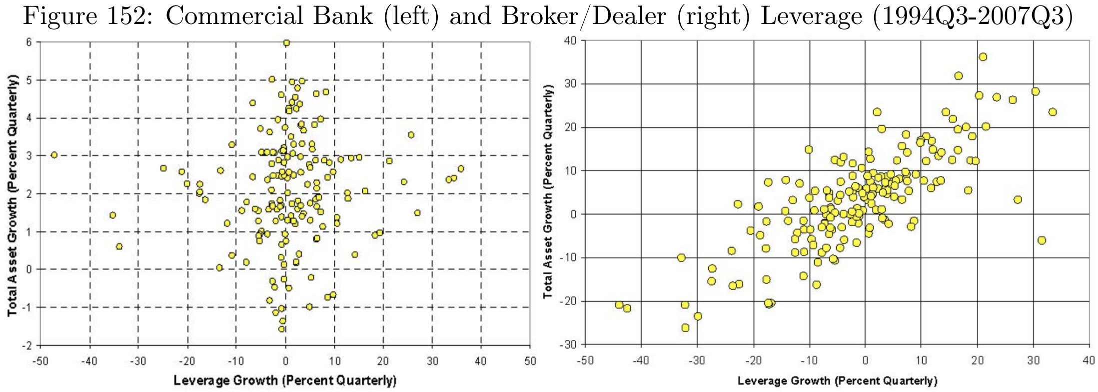
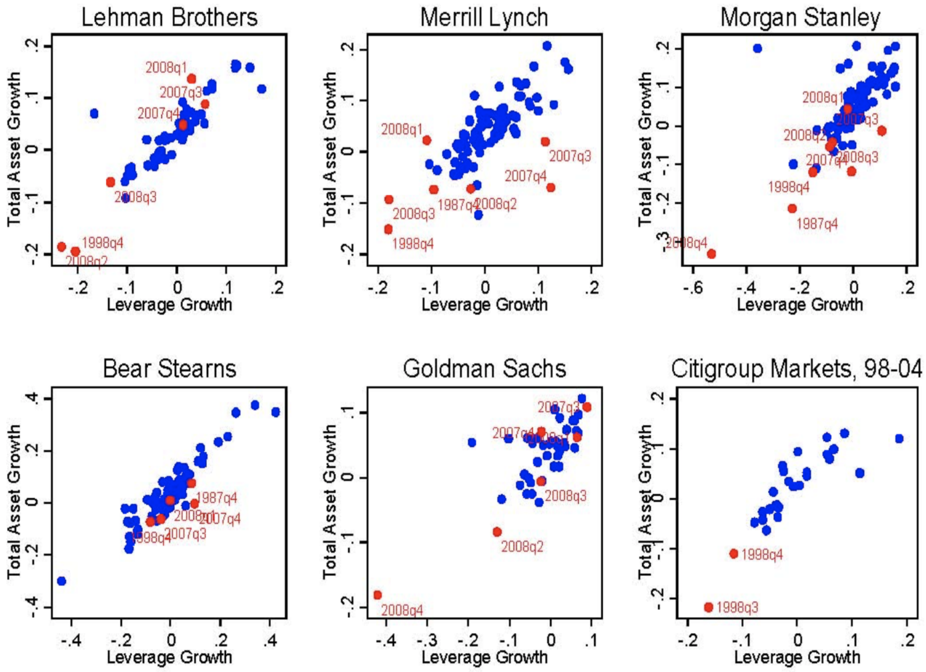


Figure 153: Leverage of US Investment Banks



...and this is where the problem arises. Rather than seeing a countercyclical leverage, we see a procyclical leverage relationship. So as asset valuations increased, the banking sector doubled down on their leveraged positions. These are potential great amplifiers for the crisis.

18.4.2 Targeting a leverage ratio

Suppose a bank wants to target a leverage ratio of 10 (in reality this was more like 30 or 40, but 10 is easier to work for the sake of illustration), where assets are marked to market, and the bank's balance sheet initially looks like:

Assets		Liabilities	
Securities	100	Debt	90
		Equity	10

Now, suppose there is a 1 percent increase in asset prices, holding liabilities constant:

Assets		Liabilities	
Securities	101	Debt	90
		Equity	11

where the leverage ratio is $101/11 = 9.18$. To hit the target leverage ratio, the bank will take on additional debt D such that:

$$\frac{101 + D}{11} = 10.$$

The bank's final balance sheet would thus be:

Assets		Liabilities	
Securities	110	Debt	99
		Equity	11

So the bank's 1 unit increase in asset values, with a constant leverage ratio, allows the bank to increase its holdings by 10 units (and its debt by 9 units). You can imagine just what kind of multiplier-like effects a leverage ratio of 30 or 40 would do. But we just saw that leverage ratios were procyclical in the US in the lead up to the crisis. So we saw even greater amplification of leverage on bank balances sheets.

This effect also works in reverse, as shown by Adrian and Shin (2010). Suppose the bank balance sheet is initially:

Assets		Liabilities	
Securities	100	Debt	90
		Equity	10

But, this time, suppose there is a 1 percent decrease in asset prices, holding liabilities constant:

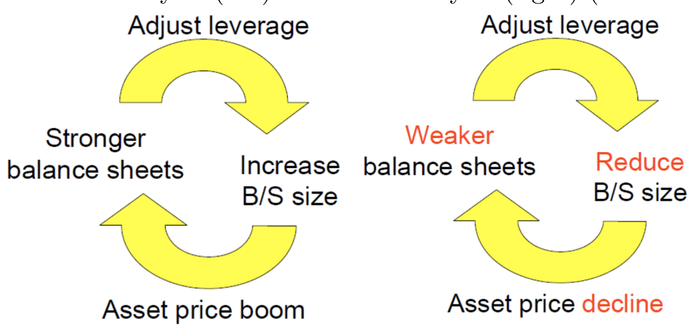
Assets		Liabilities	
Securities	99	Debt	90
		Equity	9

Leverage is now $99/9 = 11$. To get back to target, they sell 9 units of assets and use it to pay off debt (deleverage). The balance sheet becomes:

Assets		Liabilities	
Securities	90	Debt	81
		Equity	9

The fall in asset prices triggers selling of assets. With liquidity and fire-sale effects, this can lead to a nasty downward spiral:

Figure 154: The Virtuous Cycle (left) and Reverse Cycle (right) (Adrian and Shin, 2012)



The point of capital rules are to keep individual institutions solvent. Indeed, these rules are called prudential regulation: They are there to maintain stability by encouraging prudence. However, rules put in place to encourage each institution to be prudent can lead to the whole financial system becoming unstable (think back to the Kareken-Wallace model).

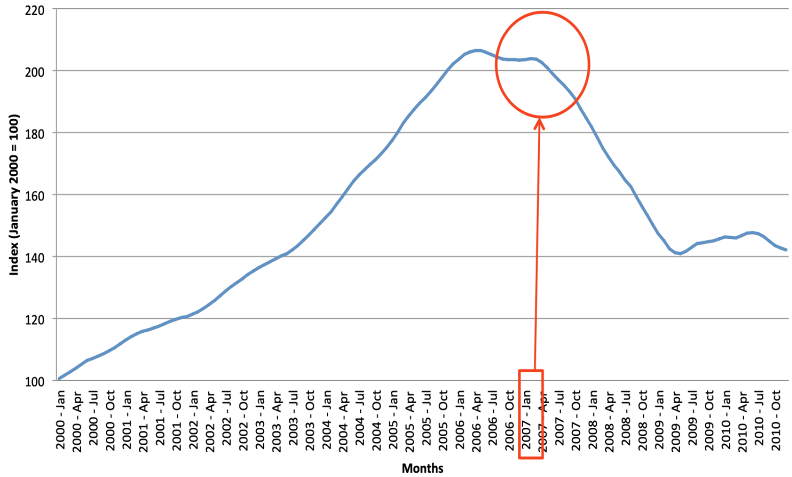
In upswings, asset prices rise, loans are paid back, and this increases equity for banks. because of the increase in equity, the regulatory capital rules allow banks to expand their operations by acquiring new assets. With lots of demand, nobody worries about liquidity or risk. Assets boom further.

But, booms don't last forever. Eventually, cycles play out and recession arrives. Now asset prices fall and loans default, eroding equity. Banks worry about meeting their capital requirements and so they sell off assets. These sales drive down asset prices and erode equity across the system.

18.4.3 The trigger to the crisis

A picture is worth a thousand words, so:

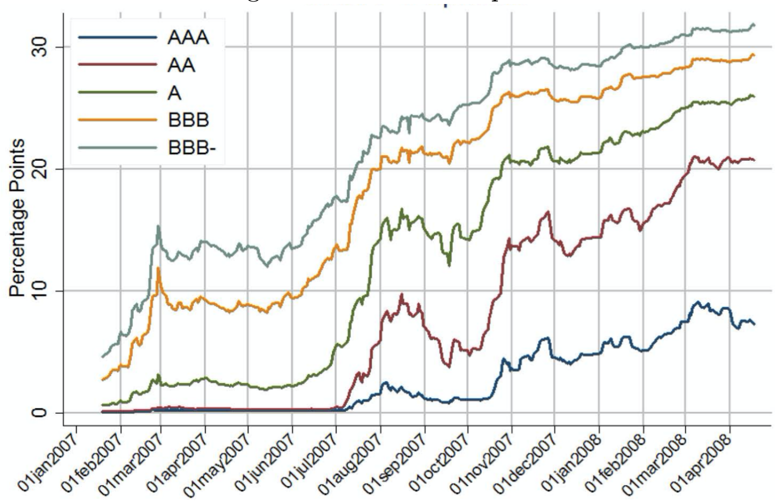
Figure 155: S&P/Case-Schiller Composite 20 House Price Index



Don't just look at the peak and decline, just look at the growth leading up to the peak! It's quite accurate to say that the US financial sector was built (pun not intended) on a housing boom. The critical problem – the dynamite to this keg – was that the housing boom itself was fuelled by subprime mortgages. Again, we won't go over subprime mortgages here, but recall that mortgages were packaged into CDOs – some were made of prime and subprime mortgages, while others were mostly subprime – and the CDO's stream of payments were classed and sold under three tranches: junior, mezzanine, and safe. These tranches were graded by the rating agencies as unrated, BBB, and AAA, respectively.

The idea of a CDO itself is not that controversial. The problem was the quality of the underlying asset (the mortgages) going into them. Wall Street's appetite for CDOs (and associated financial products such as credit default swaps (CDSs) and other insurance-like products) continued to grow in the lead up to the crisis, and the effect on "Main Street" was that brokers were offering more and more mortgages to NINJA (no income no job or assets) applicants, further degrading the quality of the CDOs. The CDOs had become so wide-spread and entangled with other financial products that systematic risk had risen across the entire financial sector as the flows from the junior, mezzanine, and safe tranches dried:

Figure 156: ABX 7-1 Spreads



How big was the subprime mortgage segment? Well it was believed that approximately 15 percent of the $10 trillion US mortgage market were subprime. Assuming that half of these mortgages default, and only half of those defaults are recoverable, we're looking at a loss of roughly $375 billion.

18.4.4 Repo runs on investment banks

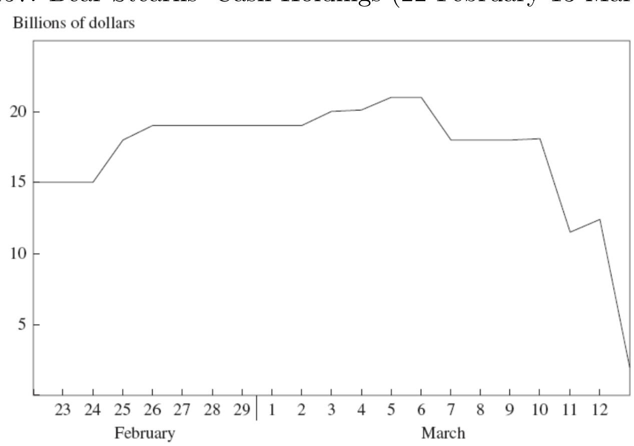
The deterioration of the underlying subprime mortgage market led to a domino effect of chaos. A key amplification mechanism was a run on financial institutions. As we saw in the Diamond-Dybvig model, bank runs benefit first movers. As panic and concerns began to spread in financial markets, there were runs in different sectors: commercial banks (e.g. Northern Rock), hedge funds (prime brokers), and investment banks (e.g. Bear Stearns and Lehman Brothers), which were subject to Repo runs.

A Repo (sale and repurchase agreement) is a deposit of cash at a "bank" which is short-term, receives interest, and is backed by collateral. Depositors take physical possession of collateral and can use it (rehypothecate it) in other transactions. By September 2007, it was clear that some major banks and non-bank institutions were going to incur large losses that would threaten their solvency, Many of them had funded their operations with very short-term borrowing such as Repos, which began to flow out.

Why? See Diamond and Rajan (2009): "Given the complexity of bank risk-taking, and the potential breakdown in internal control processes, investors would have demanded a very high premium for financing the bank long term. By contrast, they would have been far more willing to hold short-term claims on the bank, since that would give them the option to exit – or get a higher premium – if the bank appeared to be getting into trouble." In other words, they took this option because it was cheap and profitable.

> "By mid-afternoon the dam was breaking. One by one, repo lenders began to jump ship. As word spread of the withdrawals, still more repo lenders turned tail .... A full $30 billion or so of repo loans would not be rolled over the next morning. They might be able to replace maybe half that in the next day's market, but that would still leave Bear $15 billion short of what it needed to make it through the day ... By four o'clock the firm's reserves, which had been $18 billion that Monday, had dwindled to almost nothing."
>
> – "Bringing Down Bear Stearns" (Vanity Fair, 2008)[147]

Figure 157: Bear Stearns' Cash Holdings (22 February-13 March, 2008)



Source: Letter from SEC Chairman Christopher Cox to the Chairman of the Basel Committee on Banking Supervision, 20 March, 2008.

[147]https://www.vanityfair.com/news/2008/08/bear_stearns200808-2

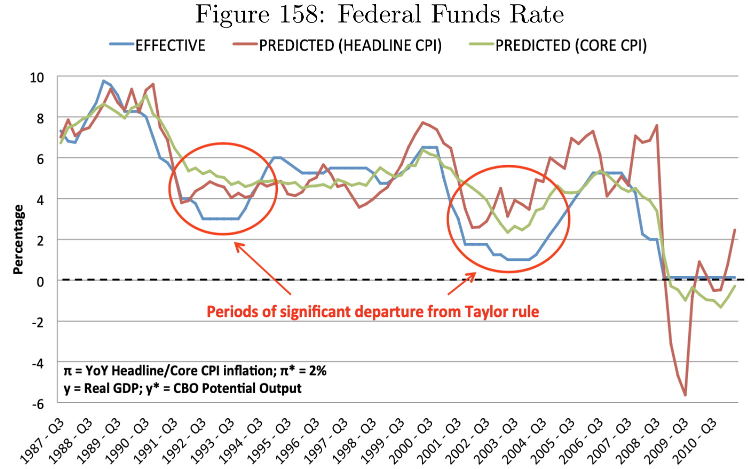
18.4.5 Monetary policy and the housing boom

We will look at the Fed's response to the GFC in the next section, but it's worth going through some key points here first as it's essential that future monetary policy design avoids repeating the same mistakes. Several hypotheses were presented by the Fed and its Board as to the root causes of the GFC:

- Deregulation and financial innovation;
- The "global savings glut"[148] (Bernanke, 2005);
- Irrational exuberance[149] (Greenspan, 1996); and
- Interest rates being "too low for too long" (Taylor, 2007).

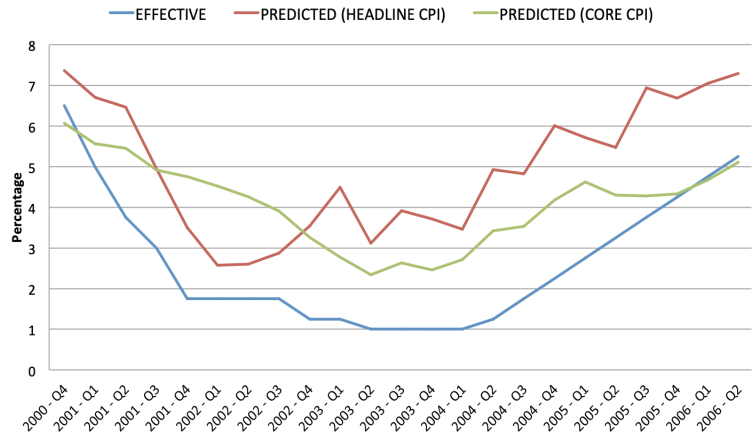
According to "Housing and Monetary Policy" Taylor (2007), the Federal Funds Rate (FFR) was too low for too long, contributing to house price bubble:

Figure 158: Federal Funds Rate



[148] https://www.federalreserve.gov/boarddocs/speeches/2005/200503102/default.htm
[149] https://www.federalreserve.gov/boarddocs/speeches/1996/19961205.htm

Figure 159: Federal Funds Rate



This argument was of course rebutted by then-Chairman of Fed, Ben Bernanke:

> "The aggressive monetary policy response in 2002 and 2003 was motivated by two principal factors. First, [...]the recovery remained quite weak and "jobless" into the latter part of 2003[...] Second, the FOMC's policy response also reflected concerns about a possible unwelcome decline in inflation.
>
> FOMC decisions during this period were informed by a strong consensus among researchers that, when faced with the risk of hitting the zero lower bound, policymakers should lower rates preemptively, thereby reducing the probability of being constrained by the lower bound on the policy interest rate."
>
> – Ben Bernanke, Speech at the American Economic Association (January 2010).

Does his argument stack up? You be the judge:

Figure 160: US Prices and Inflation

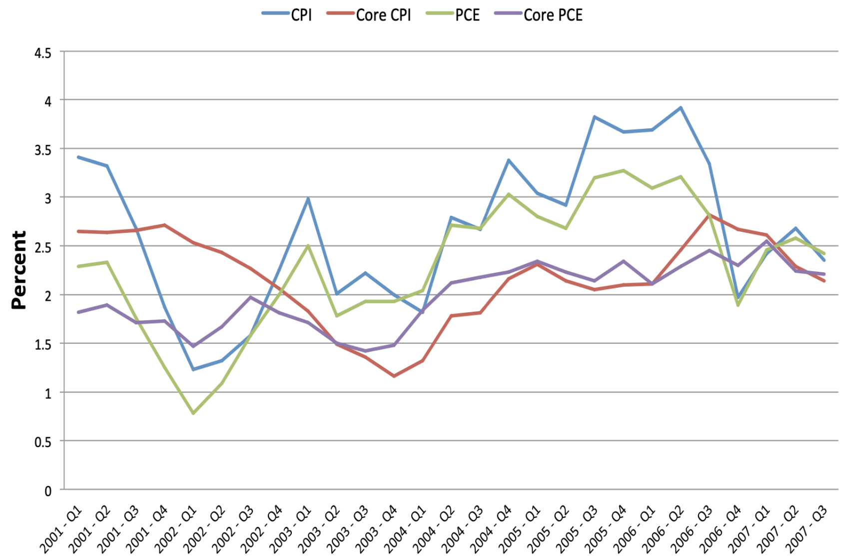
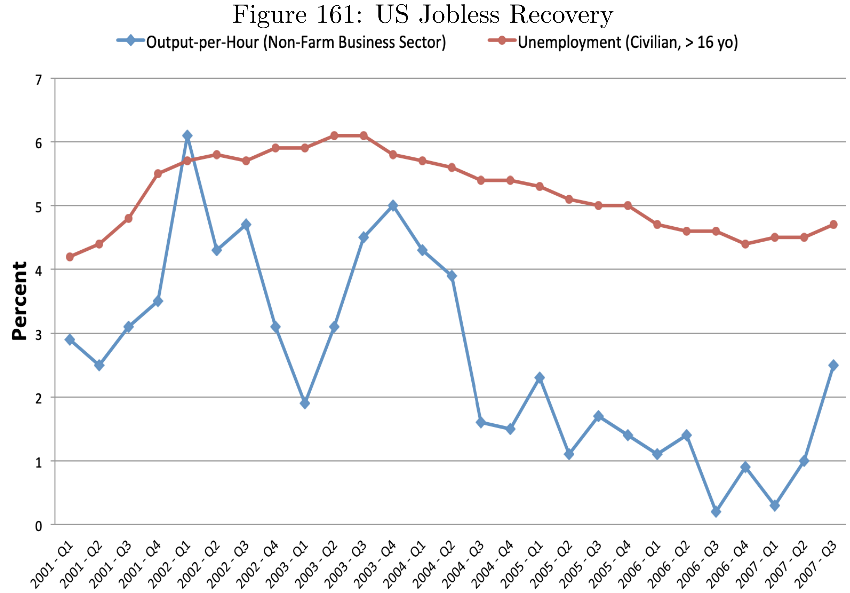


Figure 161: US Jobless Recovery



Broadly, it seems like both Taylor and Bernanke's arguments hold some merit – and the aforementioned factors contributed to the GFC too. Ultimately, however, one could blame greed and ignorance for the GFC, and there is a beautiful quote from Alan Greenspan I would like to share:

> "Those of us who have looked to the self-interest of lending institutions to protect shareholder's equity — myself especially — are in a state of shocked disbelief."
>
> – Alan Greenspan, former-Chairman of the Fed to the House Committee on Oversight and Government Reform (October, 2008).[150]

But, as this is a macroeconomics course, our focus will be on policy. So in the next sections we will undertake a closer examination of the Fed's response to the GFC and unconventional monetary policy.

18.5 The Great Escape? (Del Negro, Eggertsson, Ferrero, and Kiyotaki, 2017)

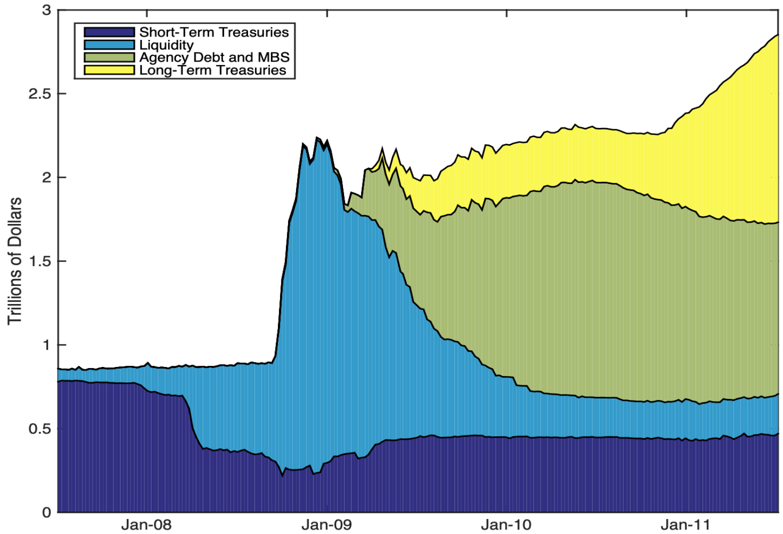
In this section we will look at the paper "The Great Escape? A Quantitative Evaluation of the Fed's Liquidity Facilities" Del Negro et al. (2017) (DEFK). The title is quite self explanatory – it explores the role of liquidity frictions in a New Keynesian DSGE model and the response of monetary policy in exchanging liquidity for illiquid toxic assets. The paper finds that liquidity shocks in a theoretical model do generate the kind of collapse in interest rates and output that we saw in the GFC, and that the Fed's liquidity facilitation programs helped prevent a repeat of the Great Depression in 2008-09.

18.5.1 Introduction

In December 2008, the FFR collapsed to zero. Standard monetary policy through interest rate cuts had reached its limit. Around the same time, the Fed started to expand its balance sheet. By January 2009, the overall size of the Fed's balance sheet exceeded $2 trillion, an increase of more than $1 trillion compared to a few months earlier. This expansion mostly involved the Fed exchanging liquidity for private financial assets through direct purchases or collateralised short term loans.

[150] https://www.nytimes.com/2008/10/24/business/economy/24panel.html

Figure 162: Composition of the Fed's Balance Sheet



Source: Del Negro et al. (2017)

The interventions by the Fed in private credit markets can be thought of as non-standard open market operations. Alternatively, one can think of them as non-standard discount window lending, which provides government liquidity using private assets collateral. The DEFK paper studies the quantitative effects of these liquidity policies on macroeconomic and financial variables.

Wallace (1981), using an irrelevance result, showed that non-standard open market operations in private assets are irrelevant. Eggertson and Woodford (2003) showed that this result extends to standard open market operations in models with nominal frictions and money in the utility function, provided that the nominal interest rate is zero.

DEFK depart from the Wallace irrelevance result by incorporating a form of credit frictions proposed by Kiyotaki and Moore (2019) (KM): i) a firm that faces an investment opportunity can borrow only up to a fraction of the value of its current investment (much like in Kiyotaki and Moore (1997)); and ii) a firm that faces an investment opportunity can sell only up to a certain fraction of the "illiquid" assets on its balance sheet in each period. In the model, these illiquid assets correspond to equity holdings of other firms. This secondary friction is a less standard "resaleability" constraint.

Relative to other related literature, unique to DEFK is that shocks are purely financial. Similar to "haircuts" in Gorton and Metrick (2010), and similar to "margin requirements" in Ashcraft et al. (2010) and Gârleanu and Pedersen (2011), who focus on asset pricing implications. Factors of production are not directly affected in DEFK – there is no "quality of capital shock" as in Gertler and Karadi (2011).

18.5.2 Households

There is a continuum of households, indexed by $j \in [0, 1]$. Each period, households receive an IID draw that determines whether they are entrepreneurs or workers:

$$\text{Household} = \begin{cases} \text{Entrepreneur}, & \text{w.p. } \varkappa : j \in [0, \varkappa), \\ \text{Worker}, & \text{w.p. } 1 - \varkappa : j \in [\varkappa, 1]. \end{cases}$$

Entrepreneurs have an opportunity to invest, but do not work, and workers supply differentiated labour of type j but do not invest. The friction in the DEFK model involves the transfer of funds from those who do not have an investment opportunity (the workers) to those who do (the entrepreneurs).

$C_t(j)$ denotes the amount of the consumer good each member of the household purchases in the market place in period t. Utility is thus an aggregate of these consumption bundles:

$$C_t = \int_0^1 C_t(j) dj. \tag{1047}$$

Let $H_t(j)$ denote hours worked by worker member j. The household's objective is thus:

$$U_t(C_t, H_t) = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left[\frac{C_{t+s}^{1-\sigma}}{1-\sigma} + \frac{\omega}{1+\upsilon} \int_{\varkappa}^{1} H_{t+s}(j)^{1+\upsilon} dj \right], \tag{1048}$$

where $\beta \in (0, 1)$ is the subjective discount factor, $\sigma > 0$ is the coefficient of relative risk aversion, $\upsilon > 0$ is the inverse Frisch elasticity of labour supply, and $\omega > 0$ is a parameter that pins down the steady-state level of hours.

At the end of each period, the household shares all the assets accumulated during the period amongst its members. Entering the next period, each member holds an equal share of the household's assets. An important assumption is that, after the idiosyncratic shock is realised and member knows its type, the household cannot reshuffle the allocation of resources among its members. Instead, those household members who would like to obtain more funds need to seek the money from other sources. The assets available to household members are described in the table below, which summarises the household balance sheet at the beginning of period t (before interest payments):

Table 11: Household Balance Sheet (Tradable Assets)

Assets		Liabilities	
Nominal bonds	B_t/P_t	Equity issued	$q_t N_t^I$
Others' equity	$q_t N_t^O$		
Capital stock	$q_t K_t$	Net worth	$q_t N_t + B_t/P_t$

Households own government-issued nominal bonds, B_t, where P_t is the price level, K_t is physical capital, and N_t^O represents claims on other households' capital. Household liabilities consist of claims on own capital sold to other households N_t^I, and net equity N_t is defined as:

$$N_t = N_t^O + K_t - N_t^I. \tag{1049}$$

Capital is homogeneous, earns per-unit rental income r_t^k, and has a unit value q_t in terms of consumption goods. A fraction δ of capital depreciates in each period. Bonds pay a gross nominal interest rate R_t. Note that all household liabilities – all claims to the assets of the private sector in the model – are in the form of equity.

Owners of capital receive the rental income as well as profits of intermediate goods producers and capital goods producers as dividend in proportion of capital ownership. Define per-period real profits of all the intermediated goods producers and capital good producers as $D_t = \int_0^1 D_t(i)di$ and D_t^I, respectively. The dividend per unit of capital ownership is:

$$R_t^k = r_t^k + \frac{D_t + D_t^I}{K_t}.$$

Finally, households pay lump-sum taxes τ_k to the government.

The budget constraint for a household is thus:

$$C_t(j) + p_t^I I_t(j) + q_t \left[N_{t+1}(j) - I_t(j)\right] + \frac{B_{t+1}(j)}{P_t} = \left[R_t^k + (1-\delta)q_t\right] N_t + \frac{R_{t-1}B_t}{P_t} + \frac{W_t(j)}{P_t} H_t(j) - \tau_t, \tag{1050}$$

where $H_t(j) = 0$ for entrepreneurs ($j \in [0, \varkappa)$) and $I_t(j) = 0$ for workers ($j \in [\varkappa, 1]$), $W_t(j)$ is the nominal wage for type-j labour, and p_t^I the price of new capital in terms of the consumption good, which differs from 1 due to capital adjustment costs.

Most of the action in the model is a consequence of the financial frictions, which translate into constraints on the financing of new investment projects by entrepreneurs and on the evolution of the balance sheet. The key frictions proposed by KM that DEFK adopt are of two firms. First, a borrowing constraint implies that any entrepreneur can only issue new equity up to a fraction θ of her investment. Second, a resaleability constraint implies that in any given period a household member can sell only a fraction ϕ_t of her existing equity holdings. An important simplification in KM is that the equity issued by the other households is a perfect substitute for the equity position in the household's own business (capital stock minus equity issued) and thus subject to exactly the same resaleability constraint. As a consequence, the borrowing constraint and the two resaleability constraints (on claims on capital of other households and on claims on own capital) can be consolidated.

So, the evolution of inside equity is:

$$N_{t+1}^I(j) - (1-\delta)N_t^I \leq \underbrace{\phi_t^I(1-\delta)\left(K_t - N_t^I\right)}_{\text{Resaleability constraint}} + \underbrace{\theta I_t(j)}_{\text{Borrowing constraint}},$$

and the evolution of outside equity is:

$$-\left[N_{t+1}^O(j) - (1-\delta)N_t^O\right] \leq \underbrace{\phi_t^O(1-\delta)N_t^O}_{\text{Resaleability constraint}},$$

and with the assumption of KM, $\phi_t^I = \phi_t^O = \phi_t$, this implies that the evolution of total equity is:

$$N_{t+1}(j) \geq (1-\theta)I_t(j) + (1-\phi_t)(1-\delta)N_t. \tag{1051}$$

The first term of the RHS, $(1-\theta_t)I_t(j)$, represents a constraint on borrowing to finance new investment for those agents who have an investment opportunity. If $\theta = 1$, the entrepreneur would be able to finance the entire investment by selling equity in financial markets. When $\theta < 1$, the entrepreneur is forced to retain $1-\theta$ fraction of investment as her own equity and use her own funds to partly finance the investment cost. The second term on the RHS, $(1-\phi_t)(1-\delta)N_t$, represents the resaleability constraint. In period t, household members can sell only a fraction ϕ_t of their existing equity.

DEFK also follows KM in that they interpret changes in ϕ_t as "liquidity shocks". These shocks capture, in reduced form, changes in market liquidity. Alternatively, ϕ_t can be thought of as one minus the haircut in the repo market – a measure of how much liquidity entrepreneurs can obtain for one dollar worth of collateral. Under this interpretation, shocks to ϕ_t would capture changes in funding conditions in the repo market.

Another significant feature of the model is that the asset B_t is not subject to any resaleability constraint and is therefore "liquid". Obviously, household members for whom constraint (994) is binding would like to acquire resources from the market by issuing liquid assets. DEFK rule out this possibility by assuming that only the government can issue the liquid asset while households can only take a long position in it:

$$B_{t+1}(j) \geq 0. \tag{1052}$$

Broadly speaking, equity in the DEFK model is comprised of all claims on private assets, which reality take the form of equity or debt, while B_t represents any form of government paper. The two constraints (1051) and (1052) are central to the analysis, and in equilibrium, both constraints are binding for entrepreneurs.

At the end of the period, household equity, bond holdings, and capital are given, respectively, by:

$$N_{t+1} = \int_0^1 N_{t+1}(j)dj, \tag{1053}$$

$$B_{t+1} = \int_0^1 B_{t+1}(j)dj, \tag{1054}$$

$$K_{t+1} = (1-\delta) + \int_0^1 I_t(j)dj. \tag{1055}$$

18.5.3 Entrepreneurs

The flow of funds for entrepreneur $j \in [0, \varkappa)$ is given by (1050) with $H_t(j) = 0$:

$$C_t(j) + p_t^I I_t(j) + q_t \left[N_{t+1}(j) - I_t(j)\right] + \frac{B_{t+1}(j)}{P_t} = \left[R_t^k + (1-\delta)q_t\right] N_t + \frac{R_{t-1}B_t}{P_t} - \tau_t.$$

The constraint clarifies that, as long as the market price of equity q_t is greater than the price of newly produced capital p_t^I, entrepreneurs trying to maximise the household's utility will use all available resources to create new capital. DEFK restricts their analysis of equilibria in which the condition $q_t > p_t^I$ is satisfied. In these equilibria, entrepreneurs sell all holdings of government bonds because the expected return on new investment dominates the return on the liquid asset. The entrepreneur sells as much existing equity as possible and issues the maximum amount of new equity to take full advantage of the investment opportunity. As a consequence (1051) and (1052) are both binding, and entrepreneurs spend no resources on consumption goods:

$$N_{t+1}(j) = (1-\theta)I_t(j) + (1-\phi_t)(1-\delta)N_t(j), \tag{1056}$$

$$B_{t+1}(j) = 0, \tag{1057}$$

$$C_t(j) = 0. \tag{1058}$$

Substituting these values into (1050) and setting $H_t(j) = 0$, we get:

$$I_t(j) = \frac{\left[R_t^k + (1-\delta)q_t\phi_t\right] N_t + \frac{R_{t-1}B_t}{P_t} - \tau_t}{p_t^I - \theta q_t}. \tag{1059}$$

Therefore, aggregate investment in the economy equals:[151]

$$I_t = \int_0^{\varkappa} I_t(j)dj = \varkappa \frac{\left[R_t^k + (1-\delta)q_t\phi_t\right] N_t + \frac{R_{t-1}B_t}{P_t} - \tau_t}{p_t^I - \theta q_t}. \tag{1060}$$

[151] I'm skipping ahead and stating that this condition holds with equality. See the paper and its appendix for full details.

The denominator represents the liquidity needs for one unit of investment – the gap between the investment goods price and the amount the entrepreneur can finance by issuing equity (θq_t). The numerator measures the amount of liquidity available to entrepreneurs. Clearly, a drop in ϕ_t reduces the amount of liquidity available to finance investment.

18.5.4 Workers

The flow of funds for worker $j \in [\varkappa, 1]$ is given by (1050), with $I_t(j) = 0$:

$$C_t(j) + q_t N_{t+1}(j) + \frac{B_{t+1}(j)}{P_t} = \left[R_t^k + (1-\delta)q_t\right] N_t + \frac{R_{t-1}B_t}{P_t} + \frac{W_t(j)}{P_t} H_t(j) - \tau_t$$

Workers do no choose hours directly. Rather, the union who represents each type of worker member sets wages on a staggered basis. As a consequence, the household supplies labour as demanded by firms at the posted wages.

In order to find the workers' decisions in terms of asset and consumption choices, we drive the household's decisions for N_{t+1}, B_{t+1}, and C_t as a whole, taking wages and hours as given. Since we know the solution for entrepreneurs from the last section (that is $N_{t+1}(j)$, $B_{t+1}(j)$, and $C_t(j)$ for $j \in [0, \varkappa)$), constraints (1047), (1053), and (1054) determine $C_t(j)$, $N_{t+1}(j)$, and $B_{t+1}(j)$ for workers. We then check that these choices satisfy (1051) and (1052) for workers.

The aggregation of workers' and entrepreneurs' budget constraints yields

$$C_t + p_t^I I_t + q_t(N_{t+1} - I_t) + \frac{B_{t+1}}{P_t} = \left[R_t^k + (1-\delta)q_t\right] N_t + \frac{R_{t-1}B_t}{P_t} + \int_{\varkappa}^{1} \frac{W_t(j)H_t(j)}{P_t} dj - \tau_t. \quad (1061)$$

Households choose C_t, N_{t+1}, and B_{t+1} in order to maximise utility (1048) subject to (1060) and (1061). As long as $q_t > p_t^I$, the FOCs (Euler equations) for bonds and equity are, respectively:

$$C_t^{-\sigma} = \beta \mathbb{E}_t \left\{ C_{t+1}^{-\sigma} \frac{R_t}{\Pi_{t+1}} \left[1 - \frac{\varkappa(q_{t+1} - p_{t+1}^I)}{p_{t+1}^I - \theta q_{t+1}} \right] \right\}, \quad (1062)$$

where Π_t is the gross inflation rate, and:

$$C_t^{-\sigma} = \beta \mathbb{E}_t \left\{ C_{t+1}^{-\sigma} \left[\frac{R_{t+1}^k + (1-\delta) q_{t+1}}{q_t} + \frac{\varkappa (q_{t+1} - p_{t+1}^I)}{p_{t+1}^I - \theta q_{t+1}} \frac{R_{t+1}^k + (1-\delta)\phi_{t+1} q_{t+1}}{q_t} \right] \right\}. \tag{1063}$$

Equations (1060), (1062), and (1063) describe the household's choice of investment, consumption, and portfolio for a given price process. Payoffs from holding either bonds or equity consist of two parts. The first is a standard return R_t/Π_{t+1} for bonds and $\frac{R_{t+1}^k + (1-\delta) q_{t+1}}{q_t}$ for equity. The second part is the premium associated with the fact that this paper, when in the hand of entrepreneurs, relaxes their investment constraint. The value of this premium is:

$$\frac{\varkappa (q_t - p_t^I)}{p_t^I - \theta q_t}.$$

The quantity $\frac{\varkappa}{p_t^I - \theta q_t}$ measures the increase in investment afforded by an extra dollar of liquidity, where $\varkappa$ and $\frac{1}{p_t^I - \theta q_t}$ capture the fraction of liquidity going to entrepreneurs and the extend to which the investment increases by an extra unit of liquidity, respectively. The value $q_t - p_t^I$ measures the marginal value to the household of relaxing the constraint. The larger the difference between q_t and p_t^I, the more valuable for the household to acquire capital by investment and pay p_t^I per unit, rather than pay q_t on the market. This premium for liquidity applies to the entirety of bond returns, but only to the liquid part of the equity return:

$$\frac{R_{t+1}^k + (1-\delta)\phi_{t+1} q_{t+1}}{q_t},$$

if $\phi_{t+1} < 1$. Hence, equity pays a premium in the expected rate of return relative to bonds because of its lower liquidity.

18.5.5 The convenience yield

At the heart of the DEFK model is the idea that government paper is more liquid than privately issued papers: agents are willing to pay a premium for holding T-Bills – what Krishnamurthy and Vissing-Jorgensen (2012) (KVJ) call the convenience yield. In the DEFK model, the convenience yield arises because liquid assets relax the financing constraint in the next period. It is the natural to define it as:

$$CY_t = \mathbb{E}_t \left[\frac{\varkappa(q_{t+1} - p^I_{t+1})}{p^I_{t+1} - \theta q_{t+1}} \right], \tag{1064}$$

where the term inside the expectations operator is the premium due to the relaxation of the investment constraint.

DEFK expresses CY_t as a spread. The gross nominal interest rate R_t on a perfectly liquid one-period T-Bill satisfies Euler equation (1062). The Euler equation for an otherwise identical security offering no convenience services is:

$$C_t^{-\sigma} = \beta \mathbb{E}_t \left[C_t^{-\sigma} \frac{R_t^0}{\Pi_{t+1}} \right], \tag{1065}$$

where R_t^0 is its gross nominal interest rate. The spread between these two securities is given by:

$$\bar{CY}_t = \left[R_t^0 - R_t \right] \mathbb{E}_t \left[\frac{1}{\Pi_{t+1}} \right].$$

DEFK show (in their appendix) that $CY_t \approx \bar{CY}_t$.

18.5.6 Household optimality conditions

The household chooses C_t, I_t, N_{t+1}, and B_{t+1} to maximise utility (1048) subject to the budget constraint (1061) and the financing constraint of investment (1060). Let ξ_t and η_t be the Lagrange multipliers attached to (1061) and (1060). The FOCs for consumption, investment, equity, and bonds are respectively:

$$C_t^{-\sigma} = \xi_t, \tag{1066}$$

$$\xi_t (q_t - p_t^I) = \eta_t, \tag{1067}$$

$$q_t \xi_t = \beta \mathbb{E}_t \left[\xi_{t+1} \left[R^k_{t+1} + (1-\delta) q_{t+1} \right] + \eta_{t+1} \frac{\varkappa \left[R^k_{t+1} + (1-\delta) \phi_{t+1} q_{t+1} \right]}{p^I_{t+1} - \theta q_{t+1}} \right], \tag{1068}$$

$$\xi_t = \beta \mathbb{E}_t \left[\frac{R_t}{\Pi_{t+1}} \left(\xi_{t+1} + \eta_{t+1} \frac{\varkappa}{p^I_{t+1} - \theta_{t+1} q_{t+1}} \right) \right]. \tag{1069}$$

DEFK focus on equilibria in which the financing constraint on investment is sufficiently tight so that the equity price is bigger than its installation cost, i.e. $q_t > p_t^I$. Therefore, the Lagrangian multiplier η_t on the financing constraint on investment equation (1060) is always positive. This implies that each entrepreneur satisfies the financing constraints on equity holdings (1051) and bond holdings (1052) with equality, and her consumption is zero. Getting rid of the Lagrangian multipliers from the FOCs gives the Euler equations for bonds and equity that characterise the household portfolio decisions (1062) and (1063). We first define the premium of liquidity from relaxing the investment constraint as:

$$\Lambda_t = \varkappa \frac{q_t - p_t^I}{p_t^I - \theta q_t}. \tag{1070}$$

The convenience yield in the DFK model is then defined as:

$$CY_t = \mathbb{E}_t \Lambda_{t+1}. \tag{1071}$$

The Euler equations (1062) and (1063) become:

$$C_t^{-\sigma} = \beta \mathbb{E}_t \left[C_{t+1}^{-\sigma} \frac{R_t}{\Pi_{t+1}} (1 + \Lambda_{t+1}) \right], \tag{1072}$$

$$C_t^{-\sigma} = \beta \mathbb{E}_t \left\{ C_{t+1}^{-\sigma} \frac{R_{t+1}^k + (1-\delta) q_{t+1}}{q_t} \left[1 + \Lambda_{t+1} \frac{R_{t+1}^k + \phi_{t+1}(1-\delta) q_{t+1}}{R_{t+1}^k + (1-\delta) q_{t+1}} \right] \right\}. \tag{1073}$$

Let us denote L_{t+1}, the real value of liquid assets at the end of the period, as:

$$L_{t+1} = \frac{B_{t+1}}{P_t}. \tag{1074}$$

Together with the expression for dividends, aggregate investment (1060) can be rewritten as:

$$I_t = \varkappa \frac{\left[R_t^k + (1-\delta) q_t \phi_t \right] N_t + \frac{R_{t-1} L_t}{\Pi_t} - \tau_t}{p_t^I - \theta q_t}. \tag{1075}$$

The rest of the model is fairly standard, and follows the lines of Christiano et al. (2005) and Smets and Wouters (2007).

18.5.7 Labour markets and wage setting

Competitive labour unions combine j-specific labour inputs into a homogenous composite H_t, according to:

$$H_t = \left[\left(\frac{1}{1-\varkappa} \right)^{\frac{\epsilon_w}{1+\epsilon_w}} \int_{\varkappa}^{1} H_t(j)^{\frac{1}{1+\epsilon_w}} dj \right]^{1+\epsilon_w},$$

where $\epsilon_w > 0$. Firms hire the labour input from the labour agencies at the wage W_t, which in turn remunerate the household for the labour actually provided. The zero-profit condition for labour agencies implies that:

$$W_t H_t = \int_{\varkappa}^{1} W_t(j) H_t(j) dj.$$

The demand for the j-th labour input is:

$$H_t(j) = \frac{1}{1-\varkappa} \left[\frac{W_t(j)}{W_t} \right]^{-\frac{1+\epsilon_w}{\epsilon_w}} H_t, \tag{1076}$$

where $W_t(j)$ is the wage specific to type j and W_t is the aggregate wage index that comes out of the zero profit condition for labour agencies:

$$W_t = \left[\frac{1}{1-\varkappa} \int_{\varkappa}^{1} W_t(j)^{-\frac{1}{\epsilon_w}} dj \right]^{-\epsilon_w}. \tag{1077}$$

Labour unions representing workers of type j set wages on a staggered basis, taking as given the demand for their specific labour input. In each period, with probability $1 - \zeta_w$, a union is able to reset the wage $W_t(j)$, while with the complementary probability the wage remains fixed. Workers are committed to supply whatever amount of labour is demanded at that wage. In the event of a wage change at time t, unions choose the wage $W_t^{\#}$ to minimise the present discounted value of the disutility from work condition on not changing the wage in the future:

$$\max_{W_t^{\#}(j)} \mathbb{E}_t \sum_{s=0}^{\infty} (\beta\zeta_w)^s \left[C_{t+s}^{-\sigma} \frac{W_{t+s}^{\#} H_{t+s}(j)}{P_{t+s}} - \frac{\omega}{1-\upsilon} H_{t+s}(j)^{1+\upsilon} \right],$$

subject to (1076) and (1061). The optimal wage setting condition is:

$$\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\zeta_w)^s C_{t+s}^{-\sigma} \left[\frac{H_{t+s}(j)}{P_{t+s}} W_t^{\#} - (1+\epsilon_w)\omega \frac{H_{t+s}(j)^{\upsilon}}{C_{t+s}^{-\sigma}} \right] = 0.$$

Let $w_t = W_t/P_t$ denote the real wage, and re-write the FOC for optimal wage setting in terms a symmetric equilibrium (since all labour unions will choose the same reset wage):

$$\mathbb{E}_t \sum_{t=0}^{\infty} (\beta\zeta_w)^s C_{t+s}^{-\sigma} \left\{ \frac{w_t^{\#}}{\Pi_{t,t+s}} - (1+\epsilon_w) \frac{\omega \left[\left(\frac{w_t^{\#}}{\Pi_{t,t+s} w_{t+s}} \right)^{-\frac{1+\epsilon_w}{\epsilon_w}} H_{t+s} \right]^{\upsilon}}{C_{t+s}^{-\sigma}} \right\} \left(\frac{w_t^{\#}}{\Pi_{t,t+s} w_{t+s}} \right)^{-\frac{1+\epsilon_w}{\epsilon_w}} H_{t+s} = 0, \tag{1078}$$

where $\Pi_{t,t+s} = \frac{P_{t+s}}{P_t}$.

By the law of large numbers (and the Calvo assumptions), the probability of changing the wage corresponds to the fraction of types who actually do change their wage. Consequently, from expression (1077), the real wage evolves according to:

$$w_t^{-\frac{1}{\epsilon_w}} = (1-\zeta_w) \left(w_t^{\#} \right)^{-\frac{1}{\epsilon_w}} + \zeta_w \left(\frac{w_{t-1}}{\Pi_t} \right)^{-\frac{1}{\epsilon_w}}. \tag{1079}$$

Defining the wage inflation as $\Pi_t^w = W_t/W_{t-1}$ and using (1079), (1078) becomes:

$$\left(\frac{1-\zeta_w \left(\Pi_t^w \right)^{\frac{1}{\epsilon_w}}}{1-\zeta_w} \right)^{-\epsilon_w + (1+\epsilon_w)\upsilon} = \frac{X_{1,t}^w}{X_{2,t}^w}, \tag{1080}$$

where the auxiliary variables $X_{1,t}^w$ and $X_{2,t}^w$ are the expected present value of marginal disutility of work and real marginal wage revenue, respectively:

$$X_{1,t}^w = \frac{\omega}{(1-\varkappa)^{\upsilon}} H_t^{1+\upsilon} + \beta\zeta_w \mathbb{E}_t \left[\left(\Pi_{t+1}^w \right)^{\frac{(1+\epsilon_w)(1+\upsilon)}{\epsilon_w}} X_{1,t+1}^w \right], \tag{1081}$$

$$X_{2,t}^w = \frac{1}{1+\epsilon_w} C_t^{-\sigma} w H_t + \beta\zeta_w \mathbb{E}_t \left[\left(\Pi_{t+1}^w \right)^{\frac{1}{\epsilon_w}} X_{2,t+1}^w \right]. \tag{1082}$$

18.5.8 Final and intermediate goods producers

Competitive final goods producers combine intermediate goods $Y_t(i)$, where $i \in [0,1]$ indexes intermediate goods producing firms, to sell a homogenous final good Y_t according to the technology:

$$Y_t = \left[\int_0^1 Y_t(i)^{\frac{1}{1+\epsilon_p}} \, di \right]^{1+\epsilon_w}, \tag{1083}$$

where $\epsilon_w > 0$. The competitive final goods producers' problem is:Their demand for the generic i-th intermediate good is:

$$Y_t(i) = \left[\frac{P_t(i)}{P_t} \right]^{-\frac{1+\epsilon_w}{\epsilon_w}} Y_t, \tag{1084}$$

where $P_t(i)$ is the nominal price of good i. The zero-profit condition for competitive final goods producers implies that the aggregate price level is:

$$P_t = \left[\int_0^1 P_t(i)^{-\frac{1}{\epsilon_w}} \, di \right]^{-\epsilon_p}. \tag{1085}$$

The intermediate goods firm uses $K_t(i)$ units of capital and $H_t(i)$ units of composite labour to produce output $Y_t(i)$ according to the production technology:

$$Y_t(i) = A_t K_t(i)^{\gamma} H_t(i)^{1-\gamma} - \Gamma, \tag{1086}$$

where $\gamma \in (0,1)$ is the share of capital, $\Gamma > 0$ is a fixed cost of production, and A_t is an aggregate productivity shock. Intermediate goods firms operate in monopolistic competition and set prices on a staggered basis (a la Calvo (1983)) taking the real wage $w_t = W_t/P_t$ and the rental rate of capital r_t^k as given. With probability $1 - \zeta_p$, the firm resets its price, while with the same complementary probability the price remains fixed. In the event of a price change at time t, the firm chooses the price $P_t^{\#}$ to maximise the present discounted value of profits, conditional on not changing prices in the future subject to the demand for its on good. DEFK assume that the profit is zero in the deterministic steady state.

Competitive final goods producers choose $Y_t(i)$ to maximise profits:

$$P_tY_t - \int_0^1 P_t(i)Y_t(i)di.$$

The solution to the profit maximisation problem yields the demand for good i as (1084). The zero profit condition for the competitive final goods producers implies that the aggregate price level is (1085).

The problem for the monopolistically competitive intermediate goods producers is solved in two steps. First, solve for the optimal amount of inputs (capital and labour) demanded. For this purpose, intermediate goods producers minimise costs:

$$r_t^k K_t(i) + w_t H_t(i),$$

subject to (1086). Let $mc_t(i)$ be the Lagrange multiplier on the constraint, the real marginal cost. The FOC implies that the capital-labour ratio at the firm level is independent of firm-specific variables as:

$$\frac{K_t(i)}{H_t(i)} = \frac{K_t}{H_t} = \frac{\gamma}{1-\gamma}\frac{w_t}{r_t^k}. \tag{1087}$$

Then the marginal cost is independent of firm-specific variables as:

$$mc_t(i) = mc_t = \frac{1}{A_t}\left(\frac{r_t^k}{\gamma}\right)^{\gamma}\left(\frac{w_t}{1-\gamma}\right)^{1-\gamma}. \tag{1088}$$

The second step consists of characterising the optimal price setting decisions in the even that firm i can adjust its price. Recall that this adjustment occurs in each period with probability $1-\zeta_w$ independent of previous history. If a firm can reset its price, it chooses $P_t^{\#}$ to maximise:

$$\mathbb{E}_t\sum_{s=0}^{\infty}(\beta\zeta_p)^s C_{t+s}^{-\sigma}\left[\frac{P_t^{\#}}{P_{t+s}} - mc_{t+s}\right]Y_{t+s}(i),$$

subject to (1084). The FOC for this problem is:

$$\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\zeta_p)^s C_{t+s}^{-\sigma} \left[\frac{P_t^{\#}}{P_{t+s}} - (1+\epsilon_p) mc_{t+s} \right] Y_{t+s}(i) = 0.$$

Again, we look at a symmetric equilibrium as all firms with the opportunity to change their price will select the same reset price $P_t^{\#}$. The FOC for optimal price setting becomes:

$$\mathbb{E}_t \sum_{s=0}^{\infty} (\beta\zeta_p)^s C_{t+s}^{-\sigma} \left[\frac{p_t^{\#}}{\Pi_{t,t+s}} - (1+\epsilon_p) mc_{t+s} \right] \left(\frac{p_t^{\#}}{\Pi_{t,t+s}} \right)^{-\frac{1+\epsilon_p}{\epsilon_p}} Y_{t+s} = 0, \tag{1089}$$

where $p_t^{\#} = P_t^{\#}/P_t$ is optimal relative price.

By a LLN-type argument, the probability of changing the price coincides with the fraction of firms who actually do change the price in equilibrium. Therefore, from expression (1085), inflation depends on the optimal reset price according to:

$$1 = (1-\zeta_p) \left(p_t^{\#} \right)^{-\frac{1}{\epsilon_p}} + \zeta_p \left(\frac{1}{\Pi_t} \right)^{-\frac{1}{\epsilon_p}}. \tag{1090}$$

Using (1090), the price setting rule (1089) becomes:

$$\left(\frac{1-\zeta_p \Pi_t^{\frac{1}{\epsilon_p}}}{1-\zeta_p} \right)^{-\epsilon_p} = \frac{X_{1,t}^p}{X_{2,t}^p}, \tag{1091}$$

where the auxiliary variables $X_{1,t}^p$ and $X_{2,t}^p$ are expected present value of real marginal cost and real marginal revenue as:

$$X_{1,t}^p = C_t^{-\sigma} Y_t mc_t + \beta\zeta_p \mathbb{E}_t \left(\Pi_{t+1}^{\frac{1+\epsilon_p}{\epsilon_p}} X_{1,t+1}^p \right), \tag{1092}$$

$$X_{2,t}^p = \frac{1}{1+\epsilon_p} C_t^{-\sigma} Y_t + \beta\zeta_p \mathbb{E}_t \left(\Pi_{t+1}^{\frac{1}{\epsilon_p}} X_{2,t+1}^p \right). \tag{1093}$$

The evolution of real wage is given by:

$$\frac{w_t}{w_{t-1}} = \frac{\Pi_t^w}{\Pi_t}. \tag{1094}$$

The fact that the capital output ratio is independent of firm-specific factors implies that we can obtain an aggregate production function:

$$\begin{aligned} A_t K_t^\gamma H_t^{1-\gamma} - \Gamma &= \int_0^1 Y_t(i) di \\ &= \sum_{s=0}^{\infty} \zeta_p (1-\zeta_p)^{t-s} \left(\frac{p_{t-s}^{\#}}{\Pi_{t-s,t}} \right)^{-\frac{1+\epsilon_p}{\epsilon_p}} Y_t, \end{aligned}$$

where $K_t = \int_0^1 K_t(i)di$ and $H_t = \int_0^1 H_t(i)di$. Defining the effect of price dispersion as:

$$\Delta_t = \sum_{s=0}^{\infty} \zeta_p (1-\zeta_p)^{t-s} \left(\frac{p_{t-s}^{\#}}{\Pi_{t-s,t}} \right)^{-\frac{1+\epsilon_p}{\epsilon_p}},$$

the aggregate production function becomes:

$$A_t K_t^\gamma H_t^{1-\gamma} - \Gamma = \Delta_t Y_t. \tag{1095}$$

Using (1090), we can define Δ_t recursively as:

$$\Delta_t = \zeta_p \Delta_{t-1} \Pi_t^{\frac{1+\epsilon_p}{\epsilon_p}} + (1-\zeta_p) \left(\frac{1 - \zeta_p \Pi_t^{\frac{1}{\epsilon_p}}}{1-\zeta_p} \right)^{1+\epsilon_p}. \tag{1096}$$

18.5.9 Capital producers

Perfectly competitive capital producers produce investment goods, sold to the entrepreneurs at price p_t^I, under decreasing returns to scale technology. The total cost of producing I_t investment goods equals $I_t[1 + S(I_t/\bar{I})]$, where $\bar{I}$ is investment in steady state. DEFK assume $S(1) = S'(1) = 0$ and $S''(I_t/\bar{I}) > 0$ so that the price of investment goods differs from the price of consumption goods in the short-run. Their problem is choose the amount of investment goods produced I_t to maximise profits:

$$D_t^I = \left\{ p_t^I - \left[1 + S\left(\frac{I_t}{\bar{I}}\right) \right] \right\} I_t,$$

taking the price of investment goods p_t^I as given. The FOC for this problem is:

$$p_t^I = 1 + S\left(\frac{I_t}{\bar{I}}\right) + S'\left(\frac{I_t}{\bar{I}}\right)\frac{I_t}{\bar{I}}. \tag{1097}$$

18.5.10 Dividend of equity

The dividend per unit of equity is the sum of the rental rate of capital and the profits of intermediate goods producers and capital goods producers per unit of capital as:

$$R_t^k = r_t^k + \frac{Y_t - w_t H_t - r_t^k K_t + p_t^I I_t - I_t \left[1 + S\left(\frac{I_t}{\bar{I}}\right)\right]}{K_t}. \tag{1098}$$

18.5.11 Government

The government conducts conventional monetary policy, unconventional monetary policy, and fiscal policy. Conventional monetary policy consists of the central bank setting the nominal interest rate following a standard feedback rule subject to the ZLB:

$$R_t = \max\left\{ \bar{R}\Pi_t^{\psi_\pi}\left(\frac{Y_t}{\bar{Y}}\right)^{\psi_y}, 1 \right\}, \tag{1099}$$

where $\psi_\pi > 1$ and $\psi_y > 1$. Unconventional monetary policy corresponds to government purchases of private paper (denoted by N_{t+1}^g) as a function of its liquidity:

$$N_{t+1}^g = \psi_k(\phi_t - \phi), \tag{1100}$$

where $\psi_k < 0$. Rule (1099) captures the behaviour of the Fed in terms of the liquidity facilities. According to this rule, the government intervenes when the liquidity of private paper is abnormally low. When the liquidity returns to normal, the facilities are discontinued. DEFK consider a crisis state as low resaleability ϕ_t state, and believe that this description of the intervention captures the behaviour of the Fed during the GFC.

Because the government intervenes in the open market, the intervention does not directly relax any agents' resaleability constraint (1051).

The government budget constraint is:

$$q_t N^g_{t+1} + \frac{R_{t-1} B_t}{P_t} = \tau_t + \left[R^k_t + (1-\delta) q_t \right] N^g_t + \frac{B_{t+1}}{P_t}. \tag{1101}$$

The government purchase of equity and debt repayment is financed by a net tax (primary surplus), returns on equity holdings, and the new debt issuances. DEFK assume that the government ensures intertemporal solvency by following a fiscal rule, written in deviations from steady state, according to which net taxes are proportional to the beginning-of-period government net debt position:

$$\tau_t - \bar{\tau} = \psi_\tau \left[\left(\frac{R_{t-1} B_t}{P_t} - \frac{\bar{R}\bar{B}}{\bar{P}} \right) - q_t N^g_t \right], \tag{1102}$$

where $\psi_\tau > 0$, and where $\bar{N}^g = 0$ by assumption.

18.5.12 Market clearing and equilibrium

The market clearing conditions for composite labour and capital use are:

$$H_t = \int_0^1 H_t(i) di,$$

and:

$$K_t = \int_0^1 K_t(i) di.$$

The aggregate capital stock evolves according to:

$$K_{t+1} = (1-\delta) K_t + I_t, \tag{1103}$$

and the capital is owned by either households or government:

$$K_{t+1} = N_{t+1} + N^g_{t+1}. \tag{1104}$$

Finally, the aggregate resource constraint requires that:

$$Y_t = C_t + \left[1 + S\left(\frac{I_t}{\bar{I}}\right)\right] I_t. \tag{1105}$$

The total factor productivity and resaleability (A_t, ϕ_t) follow an exogenous Markov process. In addition to these, we have five endogenous state variables $(K_t, N^g_t, R_{t-1}L_t, w_{t-1}\Delta_{t-1})$ – aggregate capital stock, government ownership of capital, a real liquidity measure, the real wage rate, and the effect of price dispersion from the previous period. The recursive competitive equilibrium is given by nine quantities $(C_t, I_t, H_t, Y_t, \tau_t, K_{t+1}, N_{t+1}, N^g_{t+1}, L_{t+1})$ and 15 prices $(R_t, q_t, p^I_t, w_t, R^k_t, mc_t, \Lambda_t, \Pi_t, \Pi^w_t, X^p_{1,t}, X^p_{2,t}, X^w_{1,t}, X^w_{2,t}, \Delta_t)$ as a function of the state variables $(K_t, N^g_t, R_{t-1}L_t, w_{t-1}, \Delta_{t-1}, A_t, \phi_t)$ which satisfy the 24 equilibrium conditions: (1099), (1100), (1103), (1104), (1105), (1070), (1072), (1073), (1075), (1080)-(1082), (1087), (1088), (1091)-(1098), (1101), and (1102). Once all the market clearing conditions and government budget constraints are satisfied, the household budget constraint (1061) is satisfied by Walras' Law. Additionally, DEFK define $R^q_t = \mathbb{E}_t\left[\frac{R^k_{t+1} + (1-\delta)q_{t+1}}{q_t}\right]$, the expected rate of return on equity.

The calibration used in the DEFK model is as follows:

Table 12: DEFK Model Calibration

$\sigma = 1$	Risk aversion
$\upsilon = 1$	Inverse Frisch elasticity
$\zeta_p, \zeta_w = 0.75$	Nominal rigidities
$\epsilon_p, \epsilon_w = 0.1$	Steady state markups
$S''(1) = 0.75$	Investment adjustment cost
$\psi_\pi = 1.5$	Taylor rule response to inflation
$\psi_y = 0.125$	Taylor rule response to output
$\psi_\tau = 0.1$	Tax rule response to net liabilities

Parameter	Value	Moment	Data	Model
β	0.993	Real interest rate	2.2%	2.2%
γ	0.34	Labour share	0.65	0.66
δ	0.024	Investment/GDP	0.26	0.26
ϕ	0.31	b_2	0.548	0.548
θ	0.79	CY	0.455	0.455
$\varkappa$	0.01	Liquidity share	12.55	12.55

Model is based on a quarterly frequency. Sample period is 1953:1-2008:III. Liquidity share is defined as $LS_t \equiv \frac{B_{t+1}}{B_{t+1}+P_t q_t K_{t+1}}$.

Another note is that the calibration for the convenience yield is based on Krishnamurthy and Vissing-Jorgensen (2012):

$$\bar{CY} = b_1 \max\left\{b_2 - \frac{\bar{B}}{\bar{P}\bar{Y}}, 0\right\},$$

and the model replicates KVJ's convenience yield curve quite well ($\bar{B}/(\bar{P}\bar{Y}) = 0.40$).

Figure 163: Two-Part KVJ Demand Curve

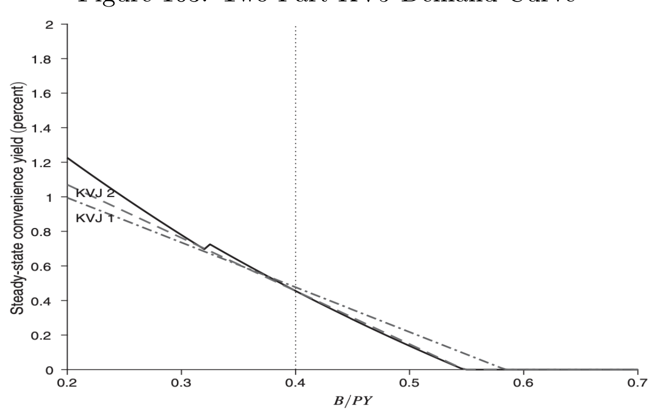


Figure 2. Two-Part KVJ Demand Curve

18.5.13 Numerical experiments

In this section we plot the responses of key macro variables to the following:

- Shock to resaleability constraint (sudden drop in ϕ) – dry up of secondary markets. The shock is calibrated to reflect a spike in the convenience yield after the collapse of Lehman Brothers.
- Since the data is also a function of government intervention, DEFK calibrate liquidity response ψ_k to match \$1.4 trillion increase in the Fed's balance sheet.
- Shock is large enough to push FFR to ZLB.
  - Survey evidence on expected duration of ZLB in 2008-09: 4-5 quarters.
  - Predictions of estimated Taylor rule: 2 years.
  - Choose $\rho_\phi = 0.953$ so that ZLB lasts 6 quarters:

$$\phi_t = \rho_\phi \phi_{t-1} + \epsilon_{\phi,t}.$$

Figure 164: Response of Key Macro Variables

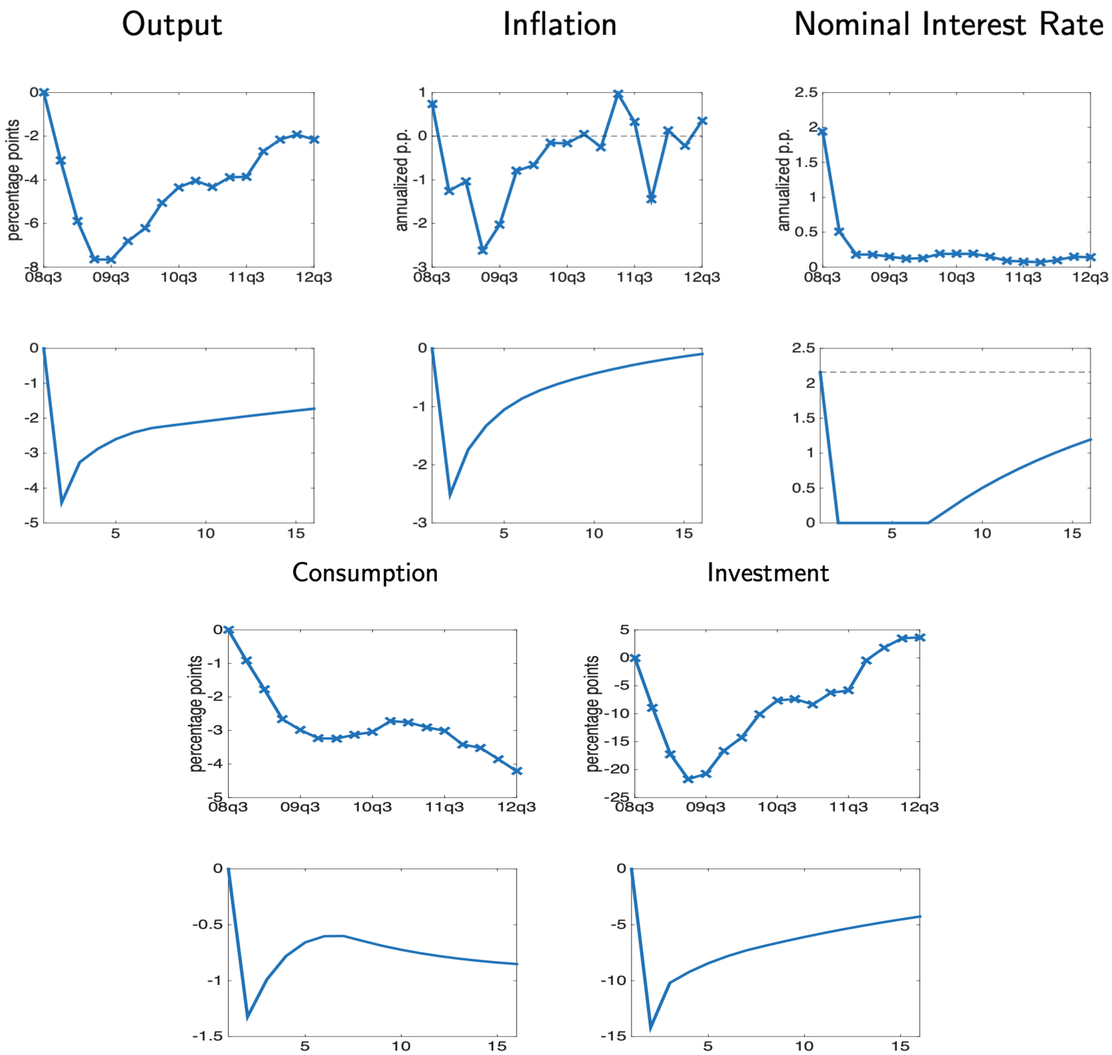


Figure 165: Response of Financial Variables

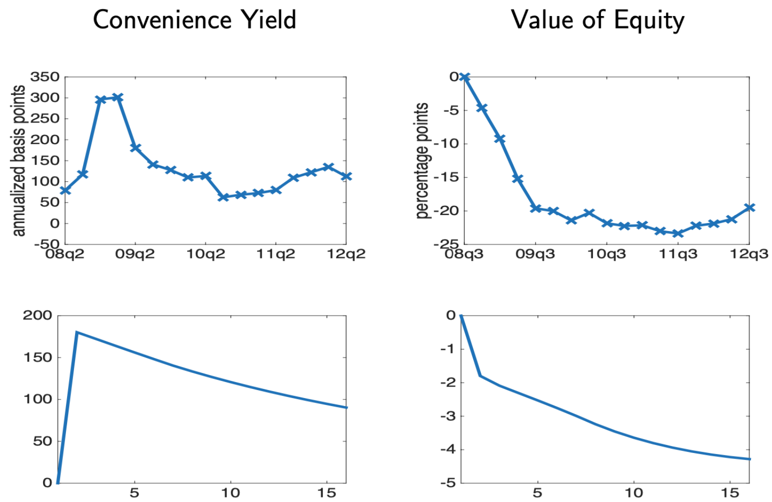
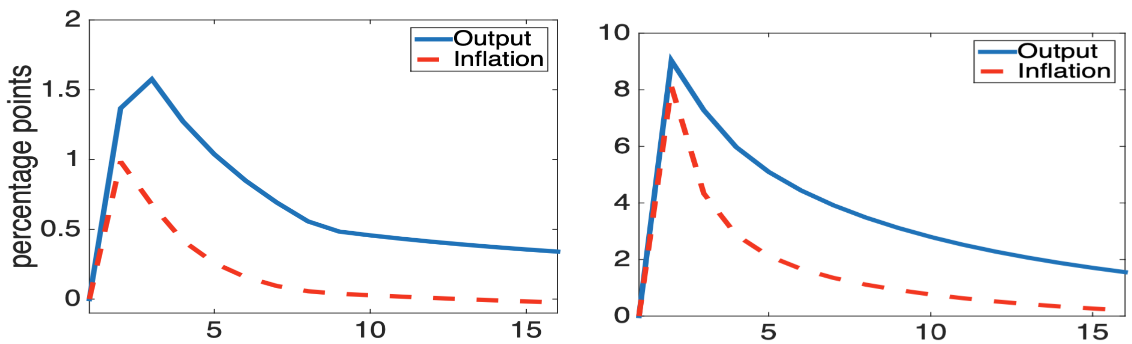


Figure 166: Effect of Policy Intervention (left: baseline; right: crisis lasts 20 quarters)



Plots show gains over no intervention.

...and that's the money shot right there. The right plot in Figure 166 shows the gains over no intervention in a Great Depression-like scenario. It basically shows that liquidity injections are far more powerful in a Great Depression scenario.

Finally, we can show the role of nominal rigidities and the ZLB:

Figure 167: Role of Nominal Rigidities

Flexible Prices	Nominal Rigidities
$Y_t = C_t \uparrow + \left[1+S\left(\frac{I_t}{I}\right)\right] I_t \downarrow$	$Y_t \downarrow = C_t \downarrow + \left[1+S\left(\frac{I_t}{I}\right)\right] I_t \downarrow$

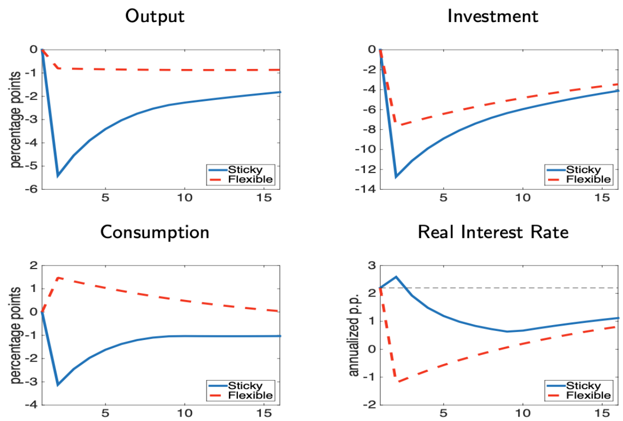
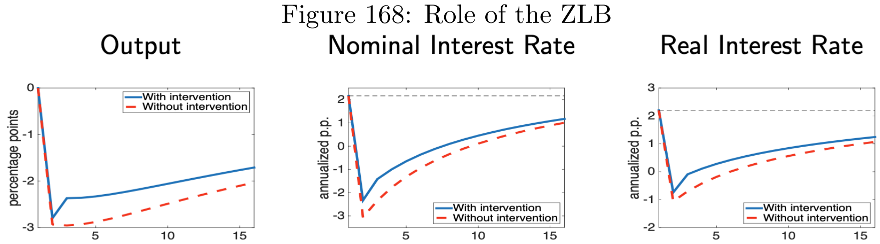


Figure 168: Role of the ZLB



Nominal rigidities and the ZLB play a crucial role in the analysis. Under flexible prices, the KM financial frictions can only account for a drop in investment. In this case, aggregate output is almost unchanged because consumption makes up for the fall in investment. The consumption boom requires the real interest rate to fall in order to induce people to spend more. Thus the real rate of interest on liquid paper absent nominal frictions—the so-called natural rate of interest—needs to fall substantially. Furthermore, the loss of liquidity of private paper drives up the premium people are willing to pay for holding liquid government paper. This additional channel leading to a decline in the natural rate of interest during financial stress is absent in standard DSGEs. But the real interest rate can hardly fall if the nominal interest rate cannot turn negative and prices are sluggish. As a consequence the freeze in the private paper market triggers a drop not only in investment, but also in consumption and aggregate output.

Unconventional policy can alleviate the crisis by targeting directly the source of the problem, which is the loss of liquidity of private paper. By swapping partially illiquid private paper for government liquidity, thus making the aggregate portfolio holdings of the private sector more liquid, the intervention lubricates financial markets, reducing the fall in investment and consumption. Importantly, DEFK are not assuming that the policy intervention violates the private sector resaleability constraint. Instead, the intervention only increases the supply of government paper by purchasing private paper in the open market.

Liquidity shocks can generate large movements in real and financial variables as observed during the GFC. The swap of liquid for illiquid assets (unconventional monetary policy) is effective in reducing impact on real variables and spreads. The effectiveness of liquidity policies depend on the expected duration of a crisis – very large effects if a crisis is expected to last as long as the Great Depression. One caveat to this analysis is that it is NOT intended as a normative analysis.

18.6 Unconventional monetary policy: Quantitative easing

The final topic we will be looking at in this section is an overview of modelling quantitative easing (QE) – a method of unconventional monetary policy (UMP), and arguably the most infamous. Other forms of UMP include, but are not limited to: qualitative easing, ETF acquisitions, forward guidance, and negative interest rates. We will be focusing our attention to the Fed's conduct of QE, though you can find plenty of examples and readings based on the experiences of the Bank of England, Bank of Japan, and European Central Bank.

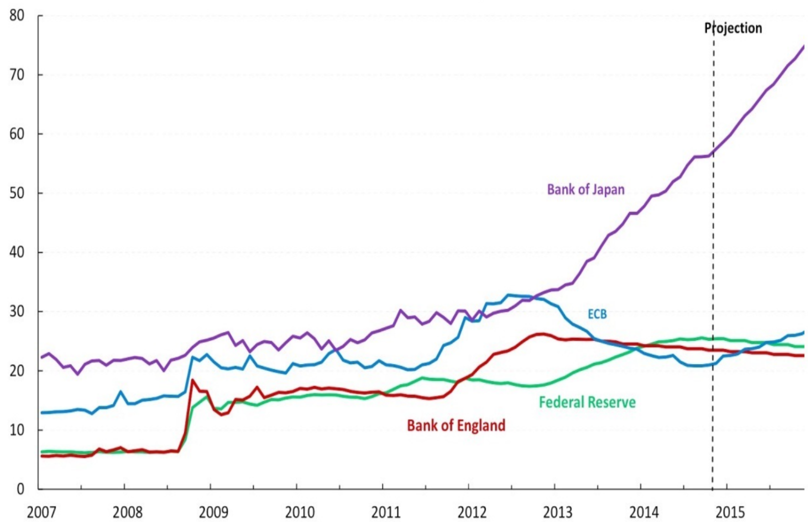
Recall that the FFR was lowered zero by 2009q1. Once the FFR was at the ZLB, the Fed undertook its large-scale asset purchase programs:

- QE1 (2008-2009): Long-term T-Bills and MBS's worth $1.75 trillion;
- QE2 (November 2010): Long-term T-Bills worth $600 billion;
- "Operation Twist-Again" (September 2011): Long-term T-Bills and MBS's worth $400 billion; and

- Continued large-scale asset purchases (September 2012): Long-term T-Bills and MBS's worth $85 billion per-month.

The objective of the various QE programmes is to support aggregate economic activity in periods when the traditional instrument of monetary policy is not available due to the ZLB. The general idea is that asset purchases operate directly on different segments of the yield curve, reducing rates at different maturities while the short-term is at zero.

Figure 169: QE Around the World (Central Bank assets in % of GDP)



Several papers find evidence that QE programmes have indeed been effective in reducing long-term rates:

Table 13: Effects of QE on 10yr Treasury Yield

	Total Impact	Impact (/$100bn)
Hamilton and Wu	-13 bps	-3 bps
Doh	-39 bps	-4 bps
D'Amico and King	-45 bps	-15 bps
Bomfim and Meyer	-60 bps	-3 bps
Gagnon et al.	-75 bps	-4 bps
Neely	-107 bps	-6 bps
Krishnamurthy and Vissing-Jorgensen	-33 bps (QE2)	-5 bps
Swanson	-15 bps (Twist)	

Table 14: Early Estimates of Real Effects of QE

	Shock*	Unemployment**	Inflation**
MacroAdvisers	-50	-0.5 in 2012Q4	+0.1 in 2012Q4
Chung et al.	-20	-0.3 in 2012Q4	+0.1 in 2012Q4
Curdia and Ferrero	-50	-0.3 in 2013Q1	+0.5 in 2014Q4
Baumeister and Benati	-60***	-0.6 in 2009Q4	+1 in 2009Q1

* Reduction in 10-year yield (in bps)
** Peak effect relative to no-QE2 baseline (in %)
*** Shock to term spread that leaves policy rate unchanged

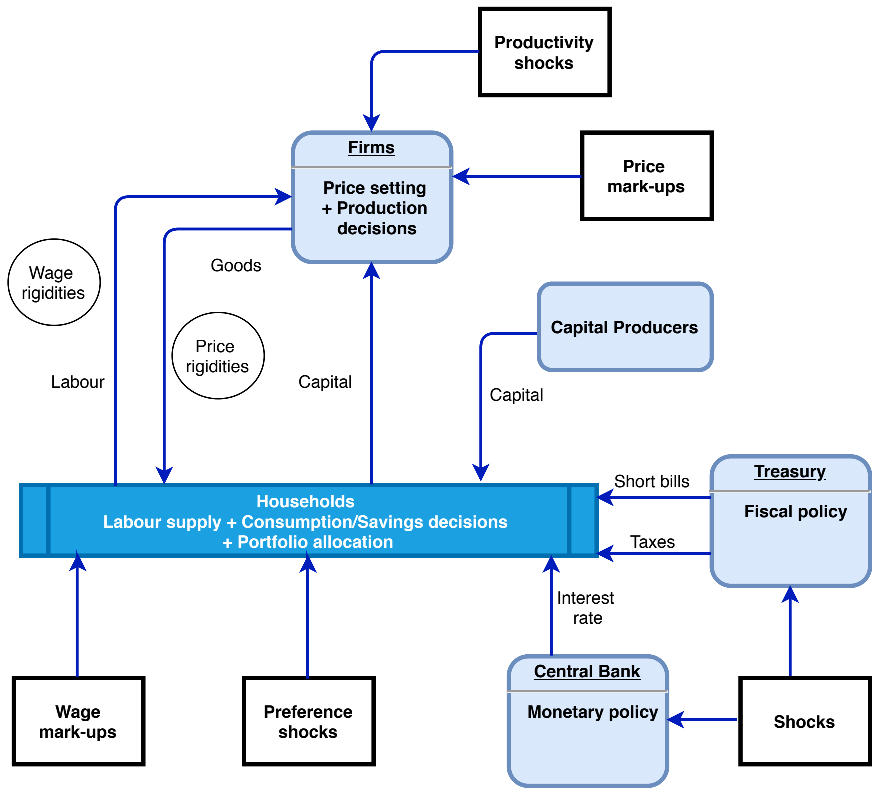
Yet, agreement on the effectiveness of QE programmes in supporting the macroeconomy is far from universal. From a theoretical perspective, QE programmes were criticised before their implementation, based on some version of the irrelevance result in Wallace (1981). QE also completely ineffective in the baseline New Keynesian model of Eggertson and Woodford (2003). In that framework, injecting reserves in exchange for longer term securities is a neutral operation. To the extent that market participants take full advantage of arbitrage opportunities, QE programmes should have no effect on real economic outcomes. Chen et al. (2012) (CCF) extend this result to a New Keynesian model with credit frictions. If households perceive the assets purchased (such as short-term government bonds) as equivalent to reserves, again QE programmes have no effect on the macroeconomy. Ex post, the criticism has continued due to the difficulty of identifying empirically the effects of asset purchases from other macroeconomic forces.

18.6.1 Breaking the classical neutrality result

General equilibrium effects are at the heart of Wallace's irrelevance theorem. By going beyond the effects of asset purchases on interest rates, we can evaluate the extent of the criticisms against QE. At the same time, we want to give QE programmes a chance. CCF introduce limits to arbitrage and market segmentation in a simple form that encompasses frictionless financial markets. Their strategy is to identify the degree of segmentation – and ultimately the effectiveness of asset purchases on macroeconomic activity – directly from the data, without assuming a priori that QE programmes are bound to fail.

To do this, they augment a standard DSGE model with nominal and real rigidities, along the lines of Christiano et al. (2005) and Smets and Wouters (2003; 2007) with segmented bond markets.

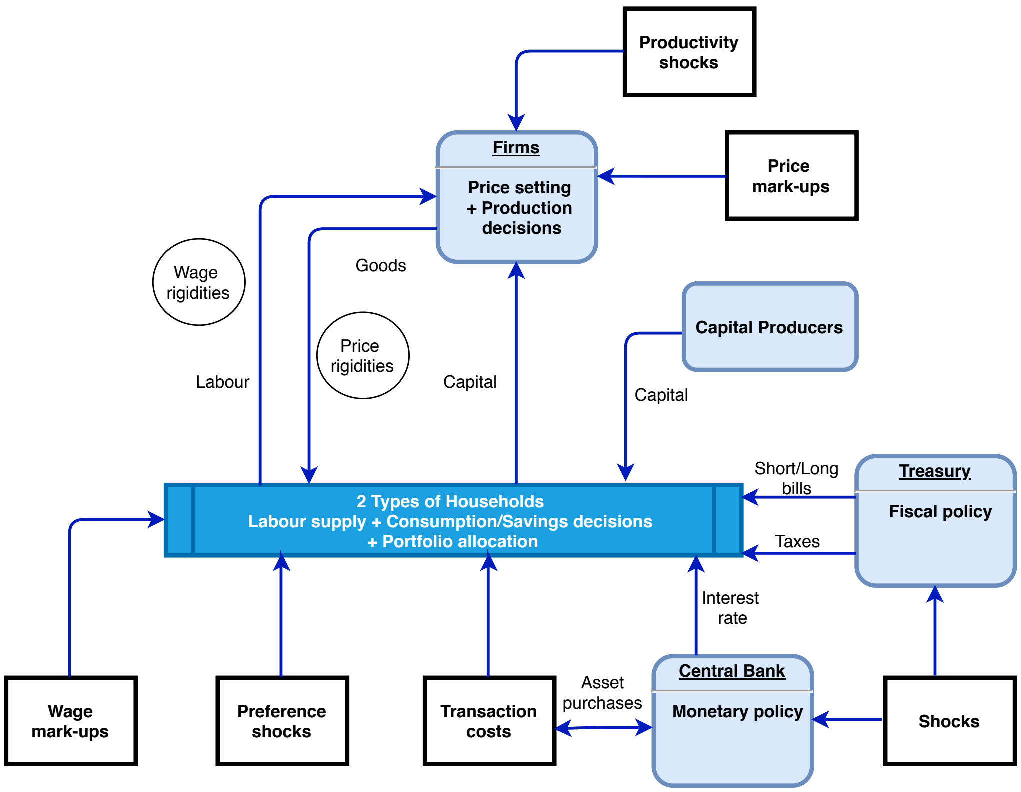
Figure 170: Block Diagram of a Standard DSGE Model



The empirical/policymaker view is that asset purchases reduce long-term rates via "portfolio balance" effect, and that they affect the real economy via segmented bond markets due to a "preferred habitat" motive. So CCF assume that investors have heterogeneous preferences for assets of different maturities (in line with the preferred habitat motive), and that even if the short-term rate is constrained by the ZLB, monetary policy can still be effective by directly influencing current long-term rates (in line with portfolio balancing effects).

So the key ingredients to the CCF model are, once again, modified assets and frictions. We now have short- and long-term bonds, transaction costs for long-term bonds, and market segmentation which limits arbitrage. In addition, there are two types of households: unrestricted and restricted households.[152] Unrestricted households save in short- and long-term bonds (albeit they pay a transaction cost to trade in long-term bonds), while restricted households can only save in long bonds, but pay no transaction costs. The rest of the model is otherwise fairly standard – the government sector operates an interest rate policy rule, a tax rule, and a bond supply rule.

Figure 171: CCF Model Overview



18.6.2 Households

The utility function for individual i of type j:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta_j^s b_{t+s}^j \left[\frac{\left(\frac{C_{t+s}^j(i)}{Z_{t+s}} - h \frac{C_{t+s-1}^j(i)}{Z_{t+s-1}} \right)^{1-\sigma_j}}{1-\sigma_j} - \frac{\varphi_{t+s}^j L_{t+s}^j(i)^{1+\upsilon}}{1+\upsilon} \right],$$

[152] This is a modelling trick to get what we want.

where b_t^j is a preference shock, $h \in (0,1)$ is a habit parameter, υ is the inverse Frisch inverse elasticity of labour supply, and φ_t^j is a labour supply shock. The period t budget constraint for unrestricted households is:

$$P_t C_t^u(i) + B_t^u(i) + (1+\zeta_t) P_{L,t} B_t^{L,u}(i) = R_{t-1} B_{t-1}^u(i) + P_{L,t} R_{L,t} B_{t-1}^{L,u}(i) + W_t^u(i) L_t^u(i) + \mathcal{P}_t - T_t,$$

and for restricted households is:

$$P_t C_t^r(i) + P_{L,t} B_t^{L,t}(i) = P_{L,t} R_{L,t} B_{t-1}^{L,r}(i) + W_t^r(i) L_t^r(i) + \mathcal{P}_t - T_t,$$

where superscripted variables with u and r denote variables of unrestricted and restricted households, respectively, variables with an L superscript or subscript denote long-bond variables, $\mathcal{P}_t$ are distributed profits from ownership of intermediate goods firms, and T_t denote lump-sum taxes. Note that ζ_t, which appears in the budget constraint for the unrestricted household is a transaction cost per unit of long-term bond purchased.

18.6.3 Dealing with long-bonds

As mentioned, two types of bonds exist. Short-term bonds B_t are one-period securities purchased at time t that pay nominal return R_t at time $t+1$. Long-term bonds are lot harder to deal with, however. Why? Suppose you have a 10-year bond and you issued it today, then next year it becomes a nine year bond. Heck, if your model is quarterly, then in the next period the 10-year bond will become a nine year and three-quarters bond, and so on. So a 10-year bond in a quarterly model would require you to track 40 maturities.

But there's a trick to get rid of this. Following Woodford (2001), long term bonds can be thought of as perpetuities that cost $P_{L,t}$ at time t and pay an exponentially decaying coupon κ^s at time $t+s+1$ for $\kappa \in (0,1)$. So, for example, if a bond was issued in period t, in period $t+1$ it would pay a perpetuity payment of κ^0, in $t+2$ is pays κ^1, in $t+3$ it pays κ^2, and so on. This allows the ratio of the payment profiles of different bonds to be constant.

OK, another example: Consider two long-term portfolios of government bonds, B_{t-1} and B_t, issued in periods $t-1$ and t, respectively. The B_{t-1} portfolio pays 1 in period t, κ in period $t+1$, κ^2 in period $t+2$, and so on. The B_t portfolio pays 1 in period $t+1$, κ in periods $t+2$, κ^2 in period $t+3$, and so on. Let $q_{0,t}$ and $q_{1,t}$ denote the period t prices of B_{t-1} and B_t, respectively. Taking into account the structure of payoffs, these two prices have to satisfy the following asset pricing equations:

$$q_{0,t} = \sum_{j=1}^{\infty} \prod_{i=1}^{j} \frac{\kappa^j}{R_{t+i-1}},$$
$$q_{1,t} = \sum_{j=1}^{\infty} \prod_{i=1}^{j} \frac{\kappa^{j-1}}{R_{t+i-1}},$$

where the one-period nominal interest rate is used to for discounting the payoffs. It is then straightforward to see that market prices of the two portfolios satisfy:

$$q_{0,t} = \kappa q_t.$$

Thus, in CCF model we have that:

$$P_{L,t}(s) = \kappa^s P_{L,t},$$

where $P_{L,t}(s)$ is the price of a perpetuity issued s periods ago and $P_{L,t}$ is the price of a perpetuity issued today. Further, the value of perpetuities purchased at $t-1$ must equal the value of the portfolio of outstanding perpetuities issued since $t-s$:

$$\begin{aligned} P_{L,t-1}B^L_{t-1} &= \sum_{s=1}^{\infty} P_{L,t-1}(s) B^L_{t-s} \\ &= \sum_{s=1}^{\infty} \kappa^{s-1} P_{L,t-1} B^L_{t-s} \\ \implies B^L_{t-1} &= \sum_{s=1}^{\infty} \kappa^{s-1} B^L_{t-s}. \end{aligned}$$

Thus, recall the budget constraint for the restricted household:

$$P_t C_t^r(i) + P_{L,t} B_t^{L,t}(i) = P_{L,t} R_{L,t} B_{t-1}^{L,r}(i) + W_t^r(i) L_t^r(i) + \mathcal{P}_t - T_t,$$

we can rewrite this as:

$$P_t C_t^r(i) + P_{L,t} B_t^{L,t}(i) = P_{L,t} R_{L,t} \sum_{s=1}^{\infty} \kappa^{s-1} B_{t-s}^{L,r}(i) + W_t^r(i) L_t^r(i) + \mathcal{P}_t - T_t.$$

You may be wondering how to select κ. Well, you can pick κ to match the average duration of US government debt from the following relationship between price and the period yield to maturity (YTM):

$$R_{L,t} = \frac{1}{P_{L,t}} + \kappa.$$

18.6.4 Risk/Term premium and segmentation

Earlier we hand-waved the difference between restricted households and unrestricted households as simply being a trick to get the model dynamics we wanted. This wasn't exactly true, though. We can think of restricted households as institutional investors (e.g. pension funds), who face legal restrictions on the type of assets in their portfolio. Unrestricted households, then, could be thought of as everyone else. We can even do some microfoundation for the transaction costs ζ_t:

$$\zeta_t = \zeta\left(\frac{P_{L,t} B_t^L}{B_t}, \epsilon_{\zeta,t}\right),$$

where the transaction costs are a function of the relative market value of long-term debt and a shock process.[153]

[153]This also motivated by empirical evidence such as in Gagnon et al. (2011).

Moving onto the Euler equations for the unrestricted household:

$$\begin{aligned}\text{Short-term bond: } 1 &= \mathbb{E}_t\left[M^u_{t,t+1}R_t\right],\\ \text{Long-term bond: } 1 &= \mathbb{E}_t\left[M^u_{t,t+1}\frac{P_{L,t+1}R_{L,t+1}}{P_{L,t}}\frac{1}{1+\zeta_t}\right],\end{aligned}$$

this implies that arbitrage between the two bonds is subject to a transaction cost term, and that the Euler equation for long-term bonds absent these transaction costs is:

$$1 = \mathbb{E}_t\left[M^u_{t,t+1}\frac{P^{EH}_{L,t+1}R^{EH}_{L,t+1}}{P^{EH}_{L,t}}\right].$$

The risk/term premium up to a first order approximation is thus:

$$\begin{aligned}\widehat{RP}_t &= \hat{R}_{L,t} - \hat{R}^{EH}_{L,t}\\ &= \frac{1}{D_L}\sum_{s=0}^{\infty}\left(\frac{D_L-1}{D_L}\right)^s \mathbb{E}_t\zeta_{t+s},\end{aligned}$$

where D_L is the steady-state duration of the two securities.

The Euler equation for restricted households is:

$$1 = \mathbb{E}_t\left[M^r_{t,t+1}\frac{P_{L,t+1}R_{L,t+1}}{P_{L,t}}\right],$$

since the restricted household cannot arbitrage between the two bonds.

18.6.5 Fiscal and monetary policy

The supply side of the model is fairly standard, but here we will quickly describe the government sector. The government budget constraint is:

$$B_t + P_{L,t}B^L_t = R_{t-1,t}B_{t-1} + P_{L,t}R_{L,t}B^L_{t-1} + P_tG_t - T_t.$$

The long-term bond supply rule is:

$$P_{L,t}B_{z,t}^{L} = \bar{B}\left(P_{L,t-1}B_{z,t-1}^{L}\right)^{\rho_B}\exp(\epsilon_{B,t}).$$

The tax rule is:

$$T_{z,t} - G_{z,t} = \bar{T}\left(P_{L,t-1}B_{z,t-1}^{L} + B_{z,t-1}\right)^{\phi_t}\exp(\epsilon_{T,t}).$$

The central bank operates a fairly standard Taylor Rule with smoothing:

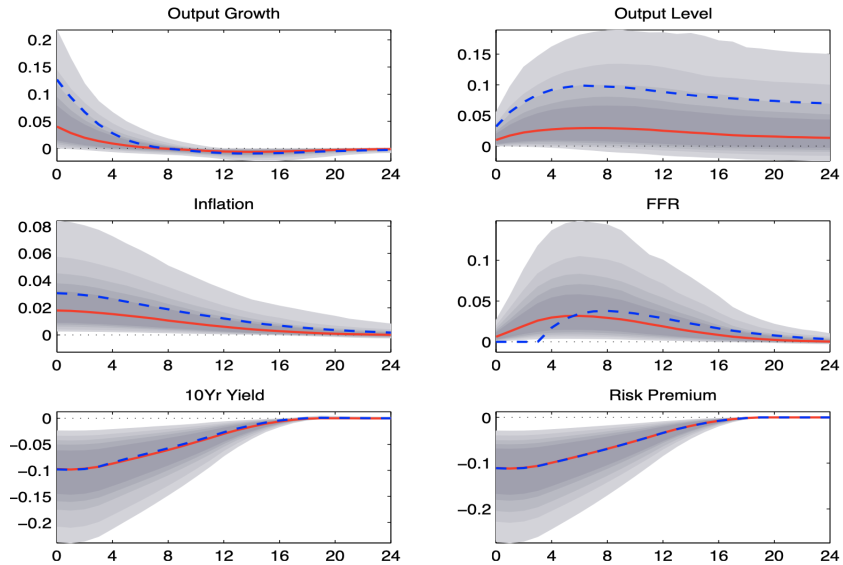
$$\frac{R_t}{\bar{R}} = \left(\frac{R_{t-1}}{\bar{R}}\right)^{\rho_m}\left[\left(\frac{\Pi_t}{\bar{\Pi}}\right)^{\phi_\pi}\left(\frac{Y_t/Y_{t-4}}{\exp(4\gamma)}\right)^{\phi_y}\right]^{1-\rho_m}\exp(\epsilon_{m,t}).$$

To stimulate the Fed's QE2 program and its purchase of \$600 billion worth of long-term bonds from the private sector, CCF assume that the central bank will keep the interest rate constant at the ZLB for four quarters.

18.6.6 Simulation results

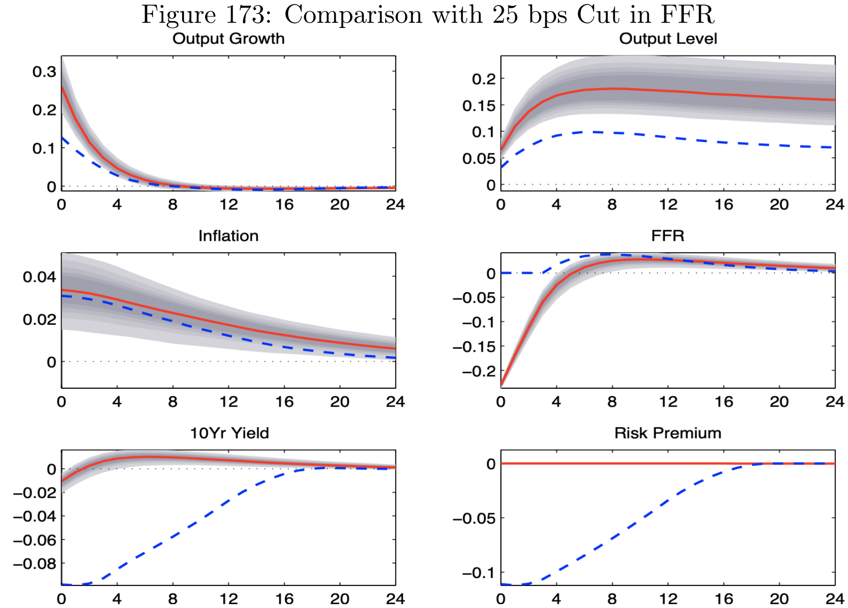
We won't cover the calibration strategy and Bayesian estimation strategy of the CCF paper. Instead, we'll skip straight to the simulation results as they contain the key intuition of the paper.

Figure 172: The Role of the ZLB



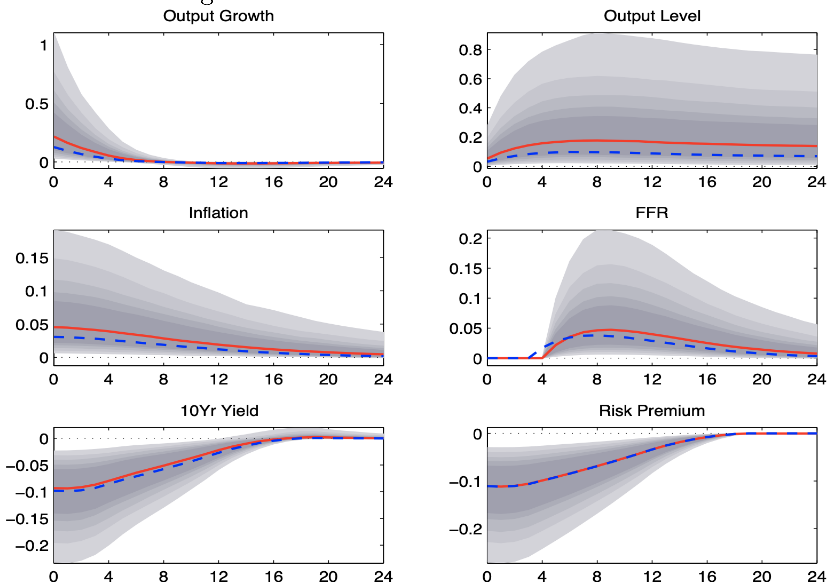
The blue line represents the posterior simulation of QE2 – essentially what we observed in the aftermath of the GFC. The red line represents a "what if" counter-factual scenario where we are not bound by the ZLB.

Figure 173: Comparison with 25 bps Cut in FFR



The blue line is the posterior simulation of QE2. The red line represents a "what if" scenario where the FFR was able to be cut by 25 bps.

Figure 174: Extended ZLB Commitment



The blue line is the posterior simulation of QE2. The red line represents a "what if" scenario where QE is implemented by the Fed keeps at the ZLB for an extra period.

So, what are the conclusions? It's safe to say that QE2 had modest effects on macroeconomic variables – not as strong as conventional monetary policy, as shown figure showing a 25 bps cut to the FFR, but the effect was not insignificant either. Besides, there's little worth squabbling over the virtues of conventional monetary policy and cutting the FFR when you've already exhausted them and are sitting at the ZLB. In conclusion, we can say that the various QE programmes and extended ZLB periods likely lead to expansionary effects in macro variables.

There are caveats to this analysis, however – similar to the caveats we noted when we covered the DEFK paper previously. The model features a very stylised financial sector (no banks or corporate bonds, for example), it features no trade in firm ownership, it uses log-linear approximations (for risk, for example), it features no time variation in market segmentation, and preferred habits are the only channel of QE transmissions.

18.7 Comments and key readings

So, prior to the 2008 GFC, what did we macroeconomists think we knew?

1. **One target: Stable inflation**: This was the result of a coincidence between the reputational need of central bankers to focus on inflation rather than activity and the intellectual support for inflation targeting provided by the New Keynesian model.

2. **Low inflation**: The danger of a low inflation rate was thought, however, to be small. In a world of small shocks, 2 percent inflation seemed to provide a sufficient cushion to make the zero lower bound unimportant.

3. **One instrument: The policy rate**: Real effects of monetary policy took place through interest rates and asset prices, not through any direct effect of money – all interest rates and asset prices were linked through arbitrage.

4. **A limited role for fiscal policy**: Wide skepticism about the effects of fiscal policy, itself largely based on Ricardian equivalence arguments – lags in the design and the implementation of fiscal policy, together with the short length of recessions, implied that fiscal measures were likely to come too late.

5. **Financial regulation**: Not a macro policy tool and largely ignored.

6. **The Great Moderation**: Better macroeconomic policy could deliver, and had indeed delivered, higher economic stability.

What have we learned?

1. **Stable inflation may be necessary, but is not sufficient**: Both inflation and the output gap may be stable, but the behaviour of some asset prices and credit aggregates, or the composition of output, may be undesirable – and potentially trigger major macroeconomic adjustments later on.

2. **Low inflation limits the scope of monetary policy in deflationary recessions**: The zero nominal interest rate bound has proven costly. Higher average inflation would have made it possible to cut interest rates more.

3. **Financial intermediation matters**: During crises rates are no longer linked through arbitrage, and the policy rate is no longer a sufficient instrument for policy. Interventions, either through the acceptance of assets as collateral, or through their straight purchase by the central bank, can affect the rates on different classes of assets, for a given policy rate.

4. **Countercyclical fiscal policy is an important tool**: To the extent that monetary policy had largely reached its limits, policymakers had little choice but to rely on fiscal policy. From its early stages, the recession was expected to be long lasting, so that it was clear that fiscal stimulus would have ample time to yield a beneficial impact despite implementation lags.

5. **Regulation is not macroeconomically neutral**: Financial regulation contributed to the amplification effects that transformed the decrease in U.S. housing prices into a major world economic crisis. Mark-to-market rules, when coupled with constant regulatory capital ratios, forced financial institutions to take dramatic measures to reduce their balance sheets, exacerbating fire sales and deleveraging.

6. **Reinterpreting the Great Moderation**: It may even be that success in responding to standard demand and supply shocks, and in moderating fluctuations, was in part responsible for the larger effects of the financial shocks in this crisis. The Great Moderation led too many (including policymakers and regulators) to understate macroeconomic risk, ignore, in particular, tail risks, and take positions (and relax rules) from leverage to foreign currency exposure, which turned out to be much riskier after the fact.

As we wrap things up to a close, there is another quote from Tom Sargent worth sharing:

> "One of my favourite quotes is by George Stigler who said, "A war can ravage half a continent, but raise no new issues in economic theory." And in some sense these crises are like that. As I look at this crisis, there are two wise things from the past that keep coming

back and haunting me. The first is the passages in Friedman and Schwartz where they raise the possibility that the establishment of the Federal Reserve System, which was designed to arrest a crisis, might actually make them worse. They sketch a mechanism, basically only in footnotes, that by making people think they didn't have to worry about crises; they actually caused people to make decisions that made it more likely that crises would happen. That was one thing. The other thing was that the establishment of a Federal Reserve arrested some mechanisms in earlier crises that the market had worked out to halt crises. They occurred both in England and the United States and took the form of temporary suspensions of convertibility by banks. They were a stopper of runs. And they noted that the Fed stopped that. So, that is the other thing that haunts me a little bit.

Fast forward to the 1970s and in the Journal of Business and the late 70s, I think '78; there is a pair of articles, one by Merton and one by Kareken and Wallace. These articles were written before the massive deregulation in the United States and at a time when many people in macro, like the leaders like Friedman and Tobin, uniformly celebrated deposit insurance. Kareken and Wallace wrote down a model in which if you didn't have deposit insurance there would be equilibrium where depositors would be vigilant. And they worked out equilibrium where if depositors wanted deposits that were safe they would choose banks that had safe portfolios. Then they described an equilibrium where you had banks of indeterminate size with safe portfolios. Then they did the experiment. They put on deposit insurance, which is not correctly priced. And they re-analysed the problem of depositors and the banks. Well the depositors didn't care. They had no reason to be vigilant. The banks did care because they had the incentives to become as large as possible and as risky as possible. This is 1978 so what Kareken and Wallace concluded is if you are going to have deposit insurance then you are going to have to regulate bank portfolios. Or else you are going to have to price it right. So this was a warning against deregulation. Kareken and Wallace's message was ignored for various reasons although I do not think it is really the fault of economists that it was."

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