Macroeconomics

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Preface

These notes are an introduction to postgraduate macroeconomics at a PhD level, focusing on business cycle theory, the analysis of aggregate fluctuations, and monetary economics. I guess you could say that it is an ode to DSGE modelling and the New Keynesian framework. Selecting what topics to include in these notes was a challenge. Macroeconomics is, arguably, one of the most exciting fields in economics; it is rapidly evolving with large strides being made in new discoveries and known unknowns. For example, this introduction in Tobin (1972) is just as true today as it was back then:

> The world economy today is vastly different from the 1930's, when Seymour Harris, the chairman of this meeting, infected me with his boundless enthusiasm for economics and his steadfast confidence in its capacity for good works. Economics is very different, too. Both the science and its subject have changed, and for the better, since World War II. But there are some notable constants. Unemployment and inflation still preoccupy and perplex economists, statesmen, journalists, housewives, and everyone else. The connection between them is the principal domestic economic burden of presidents and prime ministers, and the major area of controversy and ignorance in macroeconomics.

Despite all that we have learned, there is still much that we do not know about the relationship between unemployment and inflation! So, I have decided to cover topics that I feel are most relevant for the budding macroeconomist in the early 21st century. Additionally, and perhaps more importantly – as of this writing – these notes will not be covering topics in international macroeconomics, model estimation, IO and firm heterogeneity, incomplete markets, or inequality. So while topics such as heterogeneous agent New Keynesian (HANK) models and macro-labour economics are on the research frontier, I will not cover them here. Those will have to be tackled in the future, or in a separate piece.

Also, I must make some acknowledgements. In writing these notes, I based them on the lectures I took at the University of Oxford in 2019 for the MPhil Economics first-year macroeconomics course delivered by Guido Ascari, Martin Ellison, and Francesco Zanetti. In addition, I have supplemented the original lecture material with publicly available notes from Ambrogio Cesa-Bianchi, Peter Howitt, Eric Sims, and Karl Whelan, and with notes and teaching material from Luigi Bocola, Toni Braun, Max Croce, Sergio de Ferra, Andrea Ferrero, Ippei Fujiwara, Fumio Hayashi, Yasuo Hirose, Sagiri Kitao, Michael McMahon, Tommaso Monacelli, Taisuke Nakata, Federica Romei, Petr Sedláček, and Luca Sala. If anything, these are a culmination of all that I have learned during my studies at Keio University, Oxford, and Bocconi University.

Finally, I thank Olivia Flett, Ivan Shchapov, and Lovisa Reich for their constructive feedback and spotting my typos. All other typos and errors are mine.

Contents

Part I

Growth and Business Cycles

1 Introduction and Simple General Equilibrium Models

At the heart of modern macroeconomic models is the belief that growth and "business cycles" should be explained by making explicit assumptions regarding the "deep" structural parameters of the economy, namely:

- tastes and preferences of agents;
- production technology; and
- market structure.

In this section we will focus on how to represent agents in a simple economy, define the business cycle, and talk about stylised facts of economic growth. These observations will guide and discipline our choice of modelling, giving us a basis to assess a given model's performance. Rounding out the section will be a focus on the consumption Euler equation – a key equation which we will revisit time and time again.

1.1 Economic growth verses the business cycle

Macroeconomists conduct "business cycle analysis" by breaking down a data series, such as GDP, into a "non-stationary" long-run trend and a "stationary" cyclical component. Consider the plot of US real GDP in Figure 1.1.

Let's use the simplest tool, a log-linear trend, to try and break down the cyclical components of the real GDP time series to estimate the following regression

$$\ln Y_t = y_t = \alpha + gt + \varepsilon_t, \tag{1.1}$$

where Y_t is real GDP, the trend component is $\alpha + gt$, and ε_t is a zero-mean stationary cyclical component. We can define the log difference in real GDP, Δy_t, as having two components: constant trend growth g and the change in the cyclical component $\Delta\varepsilon_t$. We thus have:

$$\begin{aligned}
\Delta y_t &= y_t - y_{t-1} \\
&= \alpha + gt + \varepsilon_t - \alpha - g(t-1) - \varepsilon_{t-1} \\
&= g + \varepsilon_t - \varepsilon_{t-1} \\
&= g - \Delta\varepsilon_t.
\end{aligned}$$

Plotting this log-linear fit gives us the plots that we see in Figures 1.2a and 1.2b.

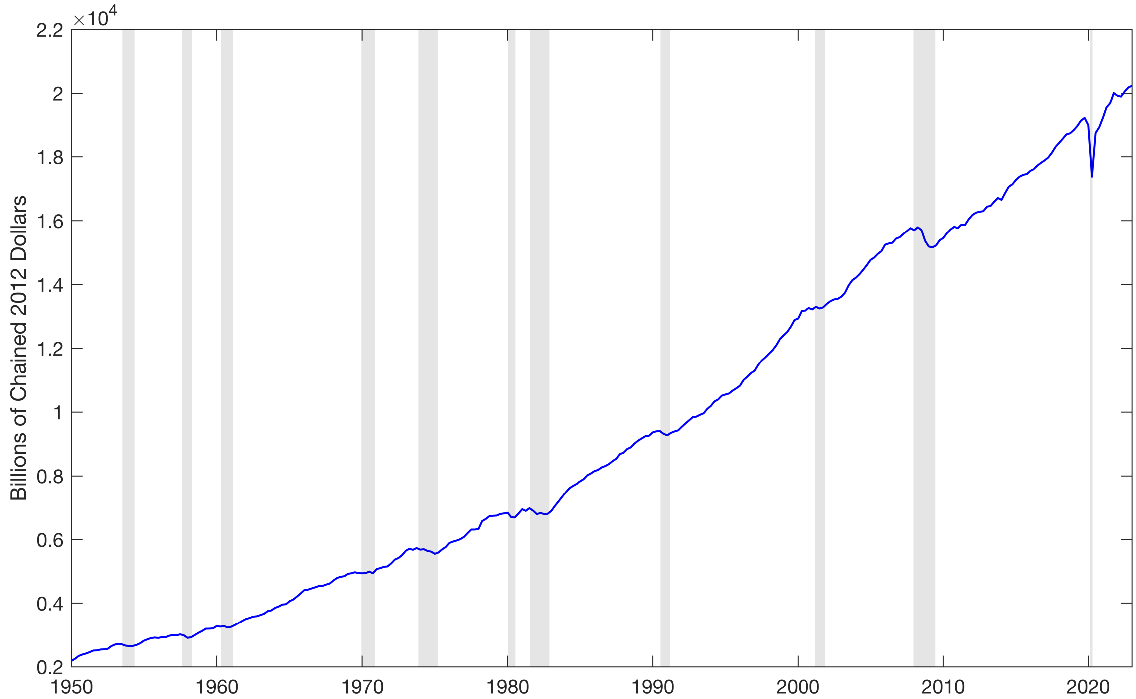


Figure 1.1: US Real GDP
Source: FRED. Shaded regions denote NBER recession periods.

But drawing these straight lines to detrend a series can provide misleading results. For example, suppose that the correct model is

$$y_t = g + y_{t-1} + \varepsilon_t, \tag{1.2}$$

where growth has the constant component g and the random component ε_t.[1] Then, the cycles here are just an accumulation of all the random shocks that have affected Δy_t over time. There is no tendency to revert to the trend, as the expected growth rate is always g no matter what happened in the past. In such a case, Δy_t is stationary: first differencing gets rid of the unit-root (non-stationary stochastic trend component) of the series. In this simple example, if we fit a model like (1.1) to a series like (1.2), there might appear to be mean-reverting cyclical component when there actually is not. The simple takeaway is that detrending a time series – to understand the underlying trend, business cycle component, seasonality, and any other purely random fluctuations – is not as simple as fitting in a straight line.[2]

For instance, one very obvious problem of a simple log-linear trend as in Figure 1.2a is that the slope of the trend clearly depends on the sample period. If we were to only look at a sub-sample, say, post the year 2000 in the US, then the trend line would be flatter. A flatter trend line would

1. In other words, the data is generated by a random walk with drift.
2. Johannes Pfeifer – the Dynare extraordinaire – has a fantastic set of notes, Pfeifer (2013), that discusses these topics in great detail.

then imply a lower level of "underperformance" of the US economy post global financial crisis (GFC) compared to what we see in Figure 1.2b.

So what can we do? Well, we can use a filter, such as the Hodrick-Prescott (HP) filter,[3] to try and break down a series into its various components. The idea behind the HP filter is that the trend must be a smooth time series, rather than a typical zero-mean white noise process. In other words, we may think of the trend component of GDP as being something that is quite "low frequency"; that it evolves as the economy itself is structurally changing and transforming. Anything that is residual of this underlying structural transformation could then be considered as a fluctuation. Hodrick and Prescott suggested choosing the time-varying trend Y_t^* so as to minimise the following

$$\min_{Y_t^*} \sum_{t=1}^{N} \left[(Y_t - Y_t^*)^2 + \lambda \left(\Delta Y_t^* - \Delta Y_{t-1}^* \right) \right]. \tag{1.3}$$

This method tries to minimise the sum of squared deviations between output and its trend $(Y_t - Y_t^*)^2$, but it also contains a term that emphasises minimising the change in the trend growth rate, $\lambda(\Delta Y_t^* - \Delta Y_{t-1}^*)$. Here λ is a parameter that we have to set, and typically this is set to 1600 for quarterly data. The larger the value of λ, the smoother the changes in the growth of the trend. Figure 1.3 shows HP-filtered US real GDP cycles, consumption, investment, and NBER-defined recessions. While Figure 1.4 shows US real GDP with various values set for the HP filters.

As you can see, the HP-filter does seem to fit the quarterly data quite well, and that is probably one of the reasons why it has become an industry standard technique. However, there is also widespread concern about its use, mainly:

1. Business cycle facts are not invariant to the detrending filter used.

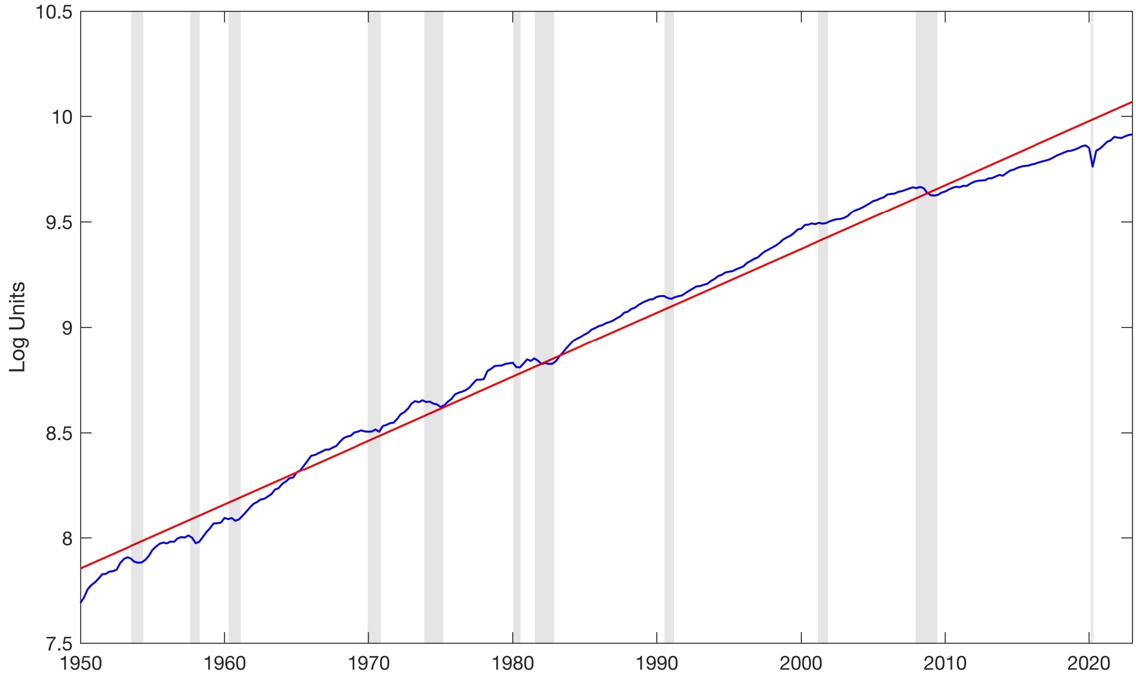
2. Other filters may be more optimal. A little bit of thought will reveal that if variables have different stochastic properties then a different detrending filter should be applied.

3. The HP filter may produce spurious cycles.

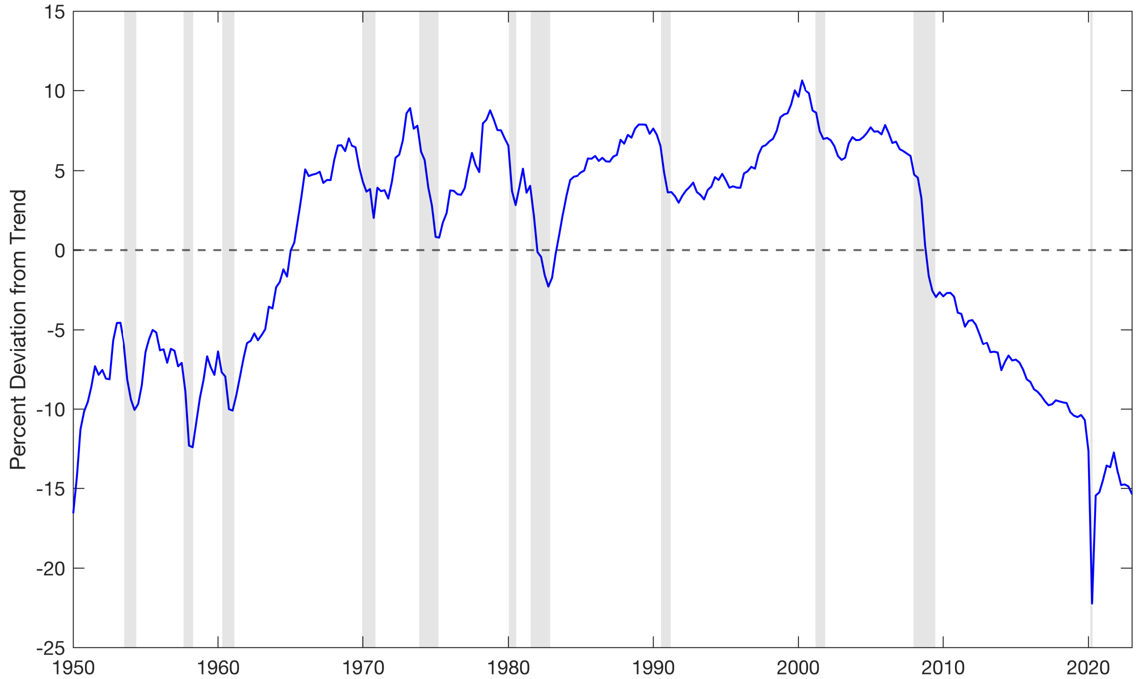
A well known result in the econometrics literature is by Nelson and Kang (1981), who show that if a linear time trend is fitted to a series which follows a random walk then the detrended data will display spurious cycles. In other words, if a researcher mistakenly thinks the trend is deterministic, then the cycles derived will be misspecified. Incorrect assumptions about the stochastic behaviour of a variable similarly may imply that the HP filter will exaggerate the pattern of long term growth cycles at cyclical frequencies and depress the influence of cycles at other frequencies. The result is that the HP filter may overstate the importance of business cycles.

3. For more info see "Postwar U.S. Business Cycles: An Empirical Investigation" by Hodrick and Prescott (1997). This paper, and the HP filter, was actually first drafted in 1981. But it wasn't published until 1997.

Figure 1.2: Log-Linear Trend and Fluctuations of US Real GDP



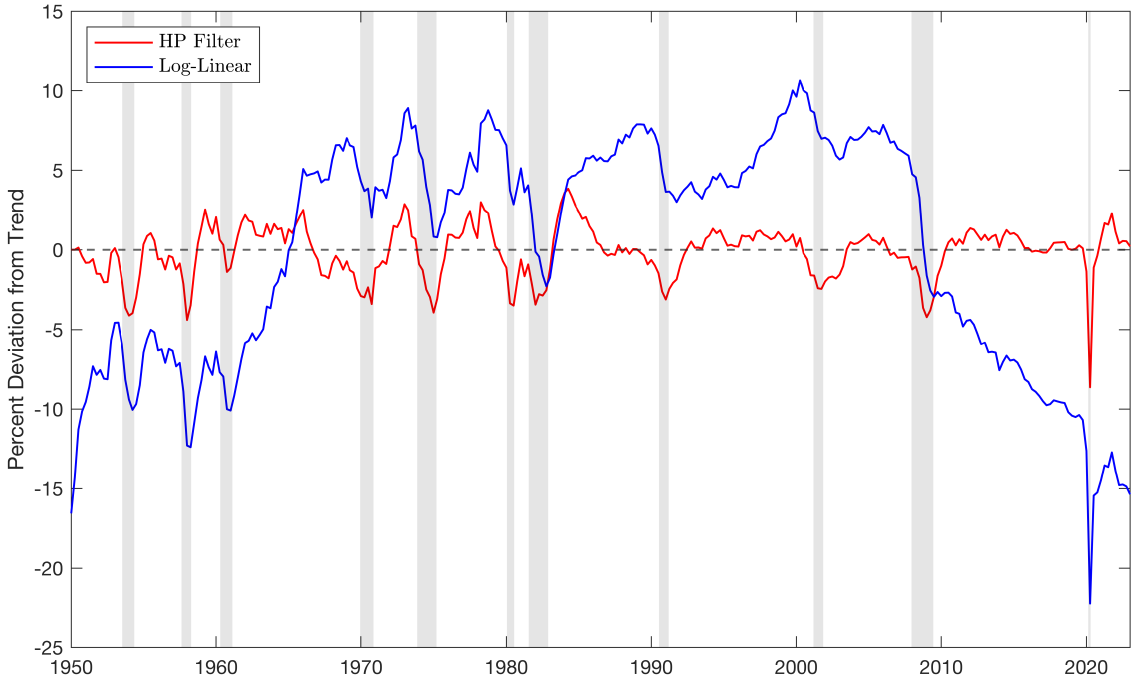
(a) Series and Trend



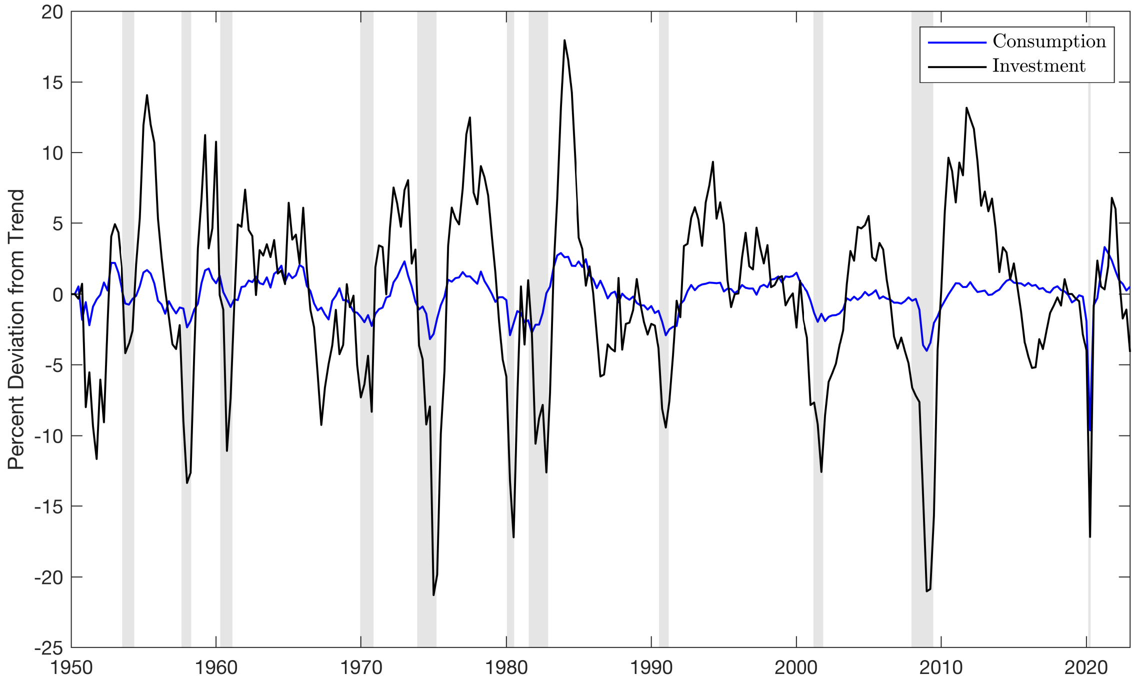
(b) Deviations from Trend

Source: FRED. Shaded regions denote NBER recession periods.

Figure 1.3: HP-Filtered Cycles and NBER Recessions



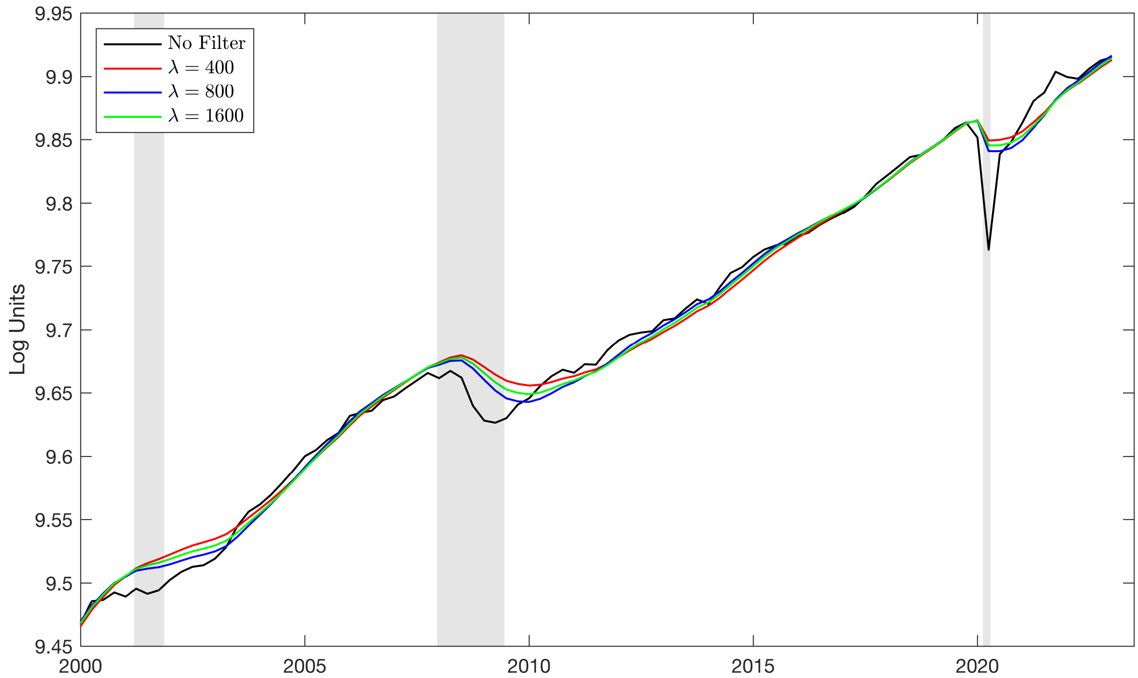
(a) US Real GDP Fluctuations



(b) HP Filtered Consumption and Investment Fluctuations

Source: FRED.

Figure 1.4: US GDP and HP Trends



Even more strikingly, in the context of the Frisch-Slutsky paradigm, the HP filter can be dramatically misleading. Observed stylised facts about the business cycle reflect three factors: (i) an impulse; (ii) a propagation mechanism; and (iii) the data being detrended by the HP filter and the certain statistics reported. It can be shown that for a typical macroeconomic model (ii) is unnecessary – merely assuming a process for the shock and applying the HP filter will be enough to generate business cycle patterns even if they are not there in the model. In other words, so called "stylised facts" are nothing more than artefacts. This is why some call the HP filter the "Hocus Pocus" filter – it can create business cycles from nothing.[4]

1.1.1 The Lucas calculation

But should we care about business cycles? How important are fluctuations away from trend growth compared to the importance of the actual growth rate g? After all, if fluctuations are of minor importance compared to growth, then dedicating complex statistical and mathematical techniques to the explanation of shocks is a waste of time. Lucas considered a simple formulation to try and answer this question by looking at the "welfare cost" of business cycles. Suppose there are three

4. Hamilton (2018) provides a lengthy explanation of the HP filter's flaws, and provides an alternative filtering technique in his piece "Why You Should Never Use the Hodrick-Prescott Filter". In summary, Hamilton's reasoning is: (i) the HP filter produces spurious cycles; (ii) filtered values at the end of the sample are very different from those in the middle; (iii) industry standard values for the smoothing parameter λ are statistically inaccurate; and (iv) there's a better alternative: Regress the variable at date $t + h$ on the four most recent values as of date t. Hamilton shows that his method achieves all the objectives sought by the HP filter but with none of its drawbacks.

economies: A, B, and C. Economy A grows at rate g but has business cycles, economy B grows at rate g too but does not have business cycles, and lastly, economy C grows at rate $g' > g$ but has business cycle fluctuations. So to summarise,

$$\begin{aligned} c_t^A &= \begin{cases} c_0(1+g)^t(1+f) & \text{w.p. } 0.5, \\ c_0(1+g)^t(1-f) & \text{w.p. } 0.5, \end{cases} \\ c_t^B &= c_0(1+g)^t, \\ c_t^C &= \begin{cases} c_0(1+g')^t(1+f) & \text{w.p. } 0.5, \\ c_0(1+g')^t(1-f) & \text{w.p. } 0.5. \end{cases} \end{aligned}$$

Clearly, economy B and C are better off than economy A, but the question is by how much? Suppose that the representative agent household in these economies has the following utility function

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma},$$

where lifetime utility in economy A depends on three things: the initial level of consumption c_0 which affects every period thereafter in the same proportion, the rate of economic growth g, and the size of fluctuations f. We can compute the welfare for economy A as follows:

$$\begin{aligned} \mathbb{W}^A(c_0, g, f) &= \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \beta^t u\left(c_t^A\right) \right] \\ &= \frac{1}{2} \sum_{t=0}^{\infty} \left(\beta^t \frac{1}{1-\sigma} \left[c_0(1+g)^t(1+f) \right]^{1-\sigma} \right) + \frac{1}{2} \sum_{t=0}^{\infty} \left(\beta^t \frac{1}{1-\sigma} \left[c_0(1+g)^t(1-f) \right]^{1-\sigma} \right) \\ &= \frac{1}{2} \sum_{t=0}^{\infty} \beta^t \frac{1}{1-\sigma} \left(c_0(1+g)^t \right)^{1-\sigma} \left[(1+f) + (1-f) \right]^{1-\sigma} \\ &= \frac{1}{2} \sum_{t=0}^{\infty} \beta^t \frac{1}{1-\sigma} c_0^{1-\sigma} \left((1+g)^t \right)^{1-\sigma} \left[(1+f) + (1-f) \right]^{1-\sigma} \\ &= \frac{1}{2} \frac{c_0^{1-\sigma}}{1-\sigma} \left[\frac{(1+f)^{1-\sigma} + (1-f)^{1-\sigma}}{1-\beta(1+g)^{1-\sigma}} \right]. \end{aligned} \tag{1.4}$$

Now, how do we compare this welfare to welfare in economies B and C? Rather, what fraction of their consumption every year would the households in economy A be prepared to give up in order to have the features of economies B or C? For economy B, this would mean we solve for some proportion λ^B in the following equation:

$$\mathbb{W}^A(c_0, g, f) = \mathbb{W}^B(\lambda^B c_0, g, f).$$

So, we have from (1.4):

$$\frac{1}{2}\frac{c_0^{1-\sigma}}{1-\sigma}\left[\frac{(1+f)^{1-\sigma}+(1-f)^{1-\sigma}}{1-\beta(1+g)^{1-\sigma}}\right] = \frac{\left(\lambda^B c_0\right)^{1-\sigma}}{1-\sigma}\frac{1}{1-\beta(1+g)^{1-\sigma}}$$

$$\implies \lambda^B = \left(\frac{1}{2}\left[(1+f)^{1-\sigma}+(1-f)^{1-\sigma}\right]\right)^{\frac{1}{1-\sigma}}. \tag{1.5}$$

If we parameterise $\beta = 0.97$, $\sigma = 2$, $g = 0.015$, and $f = 0.02$, we get $\lambda^B = 0.9996$. What does this mean? Households in economy A would be willing to give up just 0.04% of initial consumption to eliminate fluctuations. What about when we compare A to C? We get

$$\frac{c_0^{1-\sigma}}{1-\sigma}\frac{(1+f)^{1-\sigma}+(1-f)^{1-\sigma}}{1-\beta(1+g)^{1-\sigma}} = \frac{(\lambda^C c_0)^{1-\sigma}}{1-\sigma}\frac{(1+f)^{1-\sigma}+(1-f)^{1-\sigma}}{1-\beta(1+g')^{1-\sigma}}$$

$$\implies \lambda^C = 0.826,$$

when $g' = 0.025$. With this parameterisation $\lambda^C = 0.826$, which means that households in A would be willing to give up 17.4% of initial consumption to raise the rate of economic growth from 1.5% to 2.5% per year while also keeping fluctuations!

So what does this simple exercise show? It seems to suggest that growth matters a lot more than business cycle fluctuations, which could probably be one reason why Lucas chose to focus on long-term growth rather than business cycle research. But there are some things that this hasn't addressed: distributional consequences of business cycles, other values of risk aversion, unemployment, and other social consequences of recessions (e.g. political instability and crime).

So while Lucas' "napkin math" seems to suggest that business cycles aren't as relevant compared to growth, we could say that there are factors associated with business cycles which we want to minimise, and which are not captured by this simple exercise.

With this rather tepid motivation as to the importance of business cycles, let us continue documenting some of their key facts.

1.2 Stylised facts of the business cycle

We've gone on for a bit without formally defining what a "business cycle" is, although I suspect many have got a decent understanding of it by now. A business cycle is made of an expansion (boom) and a contraction (recession). During the expansion all good things (GDP, employment, productivity, and so on) tend to go up, or grow faster than "normal"; and bad things (e.g. unemployment) tend to fall. During the contraction good things go down and bad things go up.

Using some of the techniques previously mentioned (while carefully noting caveats of the HP filter), we can extract the cyclical component (the business cycle) from a raw macroeconomic time-series. Figure 1.5 plots detrended (HP filtered) real US GDP alongside consumption – that is we're plotting their cyclical fluctuations from their trend components. We can see a strong positive relationship between the two variables, with consumption leading GDP by a quarter or two.

Figure 1.5: HP Filtered Cycles of GDP and Consumption

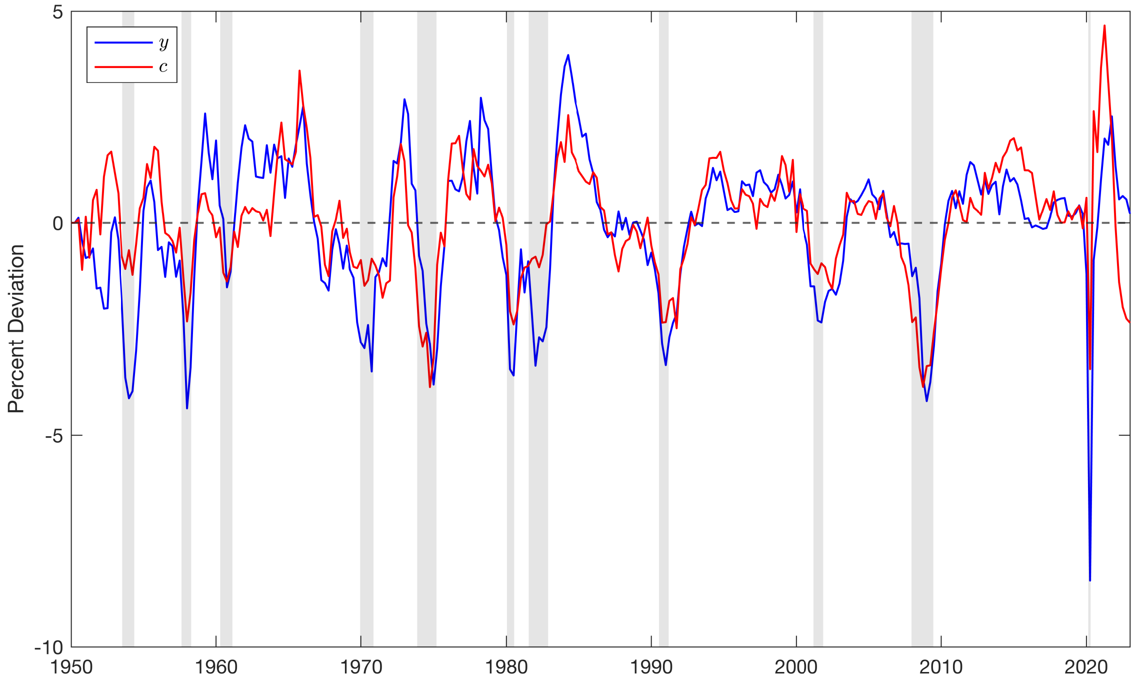


Table 1.1 gives a more complete description of the volatilities and cross correlations of consumption and labour market variables. We quote results from the US because most of the theoretical models we shall examine have been constructed with this data in mind. However, surprisingly, the UK exhibits very similar properties as the US (with a bit more even split between hours and unemployment). There are six main stylised facts which emerge from Table 1.1:

1. Consumption is smoother than output.
2. Volatility in GNP is similar in magnitude to volatility in total hours.
3. Volatility in employment is greater than volatility in average hours. Therefore most labour market adjustments operate on the extensive rather than intensive margin.
4. Productivity is slightly pro-cyclical.
5. Wages are less variable than productivity.
6. There is no correlation between wages and output (nor with employment for that matter).

In terms of the neoclassical model's performance we will show that the model is relatively successful at explaining why consumption is smoother than output (at least for the US). Fact 2 shows how important labour market fluctuations are to the business cycle. The sections on unemployment later in the notes examine a number of models which try to account for fact 3, but this represents

Table 1.1: Cyclical Behaviour of the US Economy (1954Q1-1991Q2)

Variable	SD%	Cross-correlation of output with:								
		$t-4$	$t-3$	$t-2$	$t-1$	t	$t+1$	$t+2$	$t+3$	$t+4$
GNP	1.72	0.16	0.38	0.63	0.85	1.00	0.85	0.63	0.38	0.16
CND	0.86	0.40	0.55	0.68	0.78	0.77	0.64	0.47	0.27	0.06
CD	4.96	0.37	0.49	0.65	0.75	0.78	0.61	0.38	0.11	-0.13
H	1.59	0.09	0.30	0.53	0.74	0.86	0.82	0.69	0.52	0.32
$AveH$	0.63	0.16	0.34	0.48	0.63	0.62	0.52	0.37	0.23	0.09
L	1.14	0.04	0.23	0.46	0.69	0.85	0.86	0.76	0.59	0.40
GNP/L	0.90	0.14	0.20	0.30	0.33	0.41	0.19	0.00	-0.18	-0.25
$AveW$	0.55	0.25	0.21	0.14	0.09	0.03	-0.07	-0.09	-0.09	-0.09

Note: SD% denotes standard deviations, $t-j$ denotes the correlation between GNP at time t and the variable denoted by the first column at time $t-j$. CND stands for non-durable consumption, CD for durable consumption, H for total hours worked, $AveH$ is average hours worked per employee, L is employment, GNP/L is productivity, $AveW$ is average hourly wage based on national accounts. All unemployment data is based on household surveys. Source: "Frontiers of Business Cycle Research" (Cooley and Prescott, 1995).

a significant problem for the basic neoclassical model. Facts 4-6 are also very problematic for the neoclassical model. The findings by Prescott and Cooley are also verified by the findings of King and Rebelo (1999), which are summarised in Table 1.2.

Table 1.2: Business Cycle Statistics for the US Economy

Variable	SD	Relative SD	ρ	corr$(\cdot, Y)$
Y	1.81	1.00	0.84	1.00
C	1.35	0.74	0.80	0.88
I	5.30	2.93	0.87	0.80
N	1.79	0.99	0.88	0.88
Y/N	1.02	0.56	0.74	0.55
w	0.68	0.38	0.66	0.12
r	0.30	0.16	0.60	-0.35
A	0.98	0.54	0.74	0.78

Note: All variables are in logarithms (with the exception of the real interest rate) and have been detrended with the HP filter. SD is standard deviation, ρ denotes a variable's first-order autocorrelation, and corr$(\cdot, Y)$ is a variable's contemporaneous correlation with output. Data sources are described in Stock and Watson (1999), who also created the real rate series. Y is per capita output, C is per capita consumption, I is per capita investment, N is per capita hours, w is the real wage (compensation per hour), r is the real interest rate, and A is total factor productivity. Source: "Resuscitating Real Business Cycles" (King and Rebelo, 1999).

Some facts that emerge from the King and Rebelo (1999) study are:

1. Consumption of non-durables is less volatile than output.
2. Consumer durables are more volatile than output.

3. Investment is three times more volatile than output.

4. Government expenditures are less volatile than output.

5. Total hours worked are about the same volatility as output.

6. Capital is much less volatile than output.

7. Employment is as volatile as output, while hours per worker are much less volatile than output.

8. Labour productivity is less volatile than output

9. The real wage is much less volatile than output.

Clearly, most macroeconomic series are pro-cyclical, exhibiting a positive contemporaneous correlation with output, and are very persistent with an autocorrelation order of roughly 0.8 to 0.9. There are three acyclical series: wages, government expenditures, and the capital stock. So, any model that we build will have to account and explain these facts, which we will soon find is quite a challenge.

1.2.1 Technical aside: The AR(1) model and impulse responses

Cyclical components are positively autocorrelated (i.e., positively correlated with their own lagged values) and also exhibit random-looking fluctuations. One simple model that captures these features is the autoregressive of order 1 (AR(1)) model:

$$y_t = \rho y_{t-1} + \varepsilon_t. \tag{1.6}$$

Suppose an AR(1) series starts out at zero. Then there is a unit shock, $\varepsilon_t = 1$, then all shocks are zero afterwards. In period t we have $y_t = 1$, in period $t+1$ we have $y_{t+1} = \rho$, in period $t+n$ we have $y_{t+n} = \rho^n$, and so on. The shock fades away gradually. How fast it does so depends on the size of ρ. The time path of y after this hypothetical shock is known as the impulse response function (IRF).

We can think of the IRF as the path followed from t onwards when shocks are $(\varepsilon_t+1, \varepsilon_{t+1}, \varepsilon_{t+2}, ...)$ instead of $(\varepsilon_t, \varepsilon_{t+1}, \varepsilon_{t+2}, ...)$, i.e., the incremental effect in all future periods of a unit shock today. IRF graphs are commonly used to illustrate dynamic properties of macro data.

Now consider the series in (1.6), and suppose that the variance of ε_t is σ^2. The long-run variance of y_t is the same as the long-run variance of y_{t-1}, and (remembering that ε_t is independent of y_{t-1}) this is given by

$$\sigma_y^2 = \rho^2 \sigma_y^2 + \sigma_\varepsilon^2,$$

and this simplifies to

$$\sigma_y^2 = \frac{\sigma_\varepsilon^2}{1-\rho^2},$$

which says that the variance of output depends positively on both shock variance, σ_ε^2, and also on the persistence parameter, ρ. So, the volatility of the series is partly due to the size of shocks but also due to the strength of the propagation mechanism.

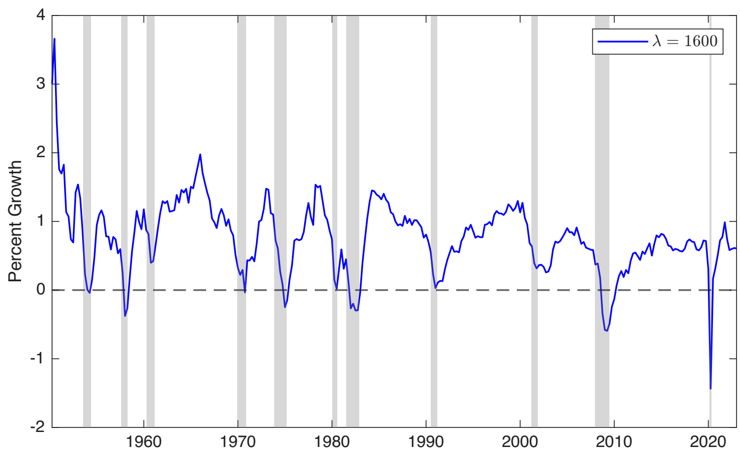
1.3 Stylised facts of economic growth

Statistical properties of long-term economic growth were first summarised by Kaldor (1957). These "remarkable historical constancies revealed by recent empirical investigations" quickly become known as the "Kaldor stylised facts". While initially derived from US and UK data, the Kaldor stylised facts were later found to hold for many other countries too. These stylised facts can be summarised as follows:

1. Output per worker grows at a roughly constant rate that does not diminish over time. $\left(\frac{Y}{L}\right)\uparrow$
2. Capital per worker grows over time. $\left(\frac{K}{L}\right)\uparrow$
3. The capital/output ratio is roughly constant. $\overline{K/Y}$
4. The rate of return to capital is constant. $\bar{r}^K$
5. The share of capital and labour in net income are nearly constant. $\bar{\alpha}$
6. Real wages grow over time. $w\uparrow$
7. Constant ratios of consumption to GDP and investment to GDP. $\overline{C/Y}, \overline{I/Y}$

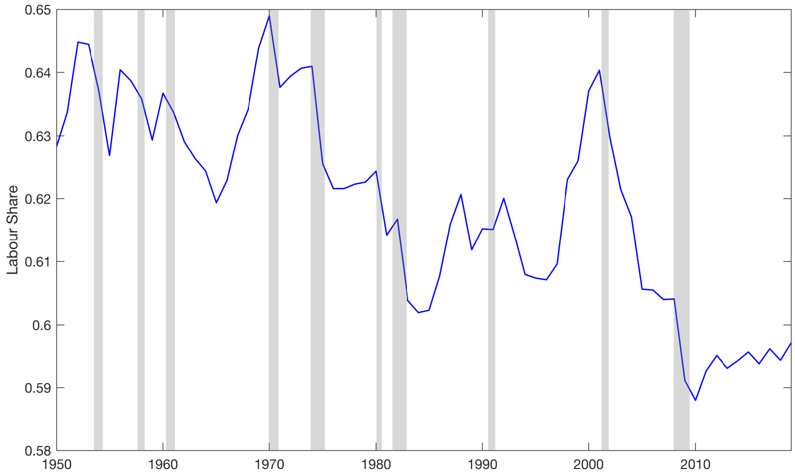
The idea of Kaldor's stylised facts is not that these hold every period, rather that they hold when averaging data over long periods of time. This is exactly what the HP trend is designed to do, so if Kaldor is right we would expect to see fairly constant trend output per worker growth, and so on. Let's see how these stylised facts stack up:

Figure 1.6: GDP Trend Growth

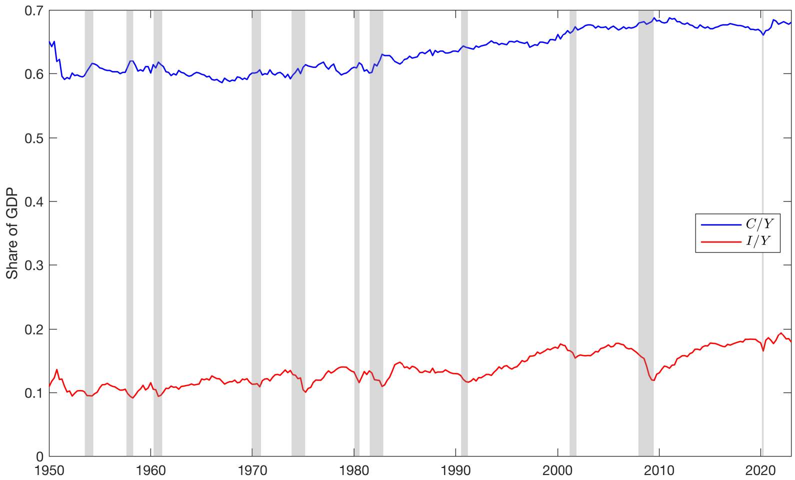


Source: FRED.

Figure 1.7: Labour Share and the Great Ratios in the US



(a) Labour Share of Output



(b) Consumption and Investment Ratios

Source: FRED.

Despite some recent declines in the labour share of output, the Kaldor stylised facts look pretty good. Our business cycle stylised facts and long-term growth stylised facts are looking pretty reliable. The only thing left to do now is to build models that can explain and replicate these facts.

1.3.1 A note on "Capital in the Twenty-First Century"

Though not covered in most macroeconomic courses, it is worth talking a bit about Piketty's 2014 piece, *Capital in the Twenty-First Century,* to see how it relates to the Kaldor stylised facts . Piketty states there are "Two Fundamental Laws of Capitalism":

1. $\alpha = r\beta$, where r is the net rate of profit, and β is defined by the 2nd Law of Capitalism.

2. $\beta = s/g$, where β is the "ratio of wealth to income" and s is the savings rate.

It should be noted that the "2nd Law of Capitalism" is basically from conventional macroeconomic growth research: $sY = gK \implies K/Y = s/g = \beta$. Piketty argues that s has been broadly fixed throughout history, however g has decreased, leading to an increase in the ratio of wealth to income, β. This has lead to the profit share of capital rising in most developed, capitalist economies (which also means that labour's share $(1 - \alpha)$ is decreasing. Certainly, observing the data in Figure 1.7a, we do see a decline in the labour share of income in the twenty-first century.

Some macroeconomists have stated that Piketty should have considered depreciation in his equation, $sY = (g + \delta)K$, so as to attain $\beta = s/(g + \delta)$. It should be noted that adjusting for depreciation does not invalidate the findings of Piketty. Ton Van Schaik has a good piece in VOXEU-CEPR on this, and it's a good read for those interested.[5] I also highly recommend Robert Solow's review of Piketty's book.[6]

1.4 The consumption Euler equation and a general equilibrium model

It's now time to begin building some models that can explain the stylised facts we've observed. We will begin with very simple neoclassical models which feature only households and firms. These include the Solow-Swan model, the Ramsey model, the overlapping generations (OLG) model, and models of endogenous growth such as the AK model. These models are all rudimentary, but provide some key insights, particularly when it comes to explaining long term growth.

As a preview to where we're heading, after neoclassical growth models we will discuss vector autoregression (VAR) models and stochastic difference equations, and then move onto the real business cycle (RBC) model. The RBC model takes the Ramsey model as its foundation, and then builds in mechanisms to account for business cycle fluctuations. We will see that even the baseline RBC model can go a long way in explaining a lot of the business cycle moments we found. The baseline RBC model can be tweaked and enhanced to improve its performance, but ultimately those efforts will lead to a dead end. There are simply too many factors that the RBC model cannot account for without loosening some of the strict assumptions that keep the model together. While the RBC model will struggle to explain some dynamics in the data, it will serve as a "best case scenario" benchmark. We will then add money to the RBC model, a monetary authority/government, imperfect competition, and sticky prices to then get the New Keynesian (NK) model. The NK model will serve as our workhorse model[7] to explain macroeconomic shocks and optimal policy. But, again, that's all way ahead; for now, let's build some neoclassical models. We'll start with a model that is quite similar to a Robinson Crusoe economy with fixed labour.[8]

5. https://voxeu.org/article/piketty-s-two-laws
6. https://newrepublic.com/article/117429/capital-twenty-first-century-thomas-piketty-reviewed
7. With some additional tweaks and modifications, of course. This is macroeconomics after all.
8. See McCandless (2008) for a good treatment of neoclassical growth models (as well as RBC models).

We assume the existence of a utility function $u(c_t^i)$ where c_t^i is consumption of household i. Notice that utility depends only on current consumption – that is, preferences are intertemporally separable. This implies that previous consumption choices do not influence marginal utility in this period. Clearly, previous values of c_t^i will influence the current choice of consumption through budget constraint effects but they do not directly influence the utility function. A number of recent studies have stressed the importance of not having intertemporally separable preferences, but for the sake of our simple model we won't pay attention to that. Households have to make two decisions: (i) how much to spend, and (ii) how much to save. Households receive a gross interest rate, $R_t = (1 + r_t)$, on any savings, and receive an endowment y_t^i each period. Both R_t and y_t^i are treated as beyond the household's control and are known with certainty into the infinite future. Assume that the household wishes to maximise the present value of the discounted stream of utility. That is

$$\max_{\{c_{t+s}^i, a_{t+s}^i\}} \sum_{s=0}^{\infty} \beta^s u(c_{t+s}^i), \tag{1.7}$$

subject to the following constraints:

$$c_{t+s}^i + a_{t+s}^i = y_{t+s}^i + R_{t+s-1} a_{t+s-1}^i, \tag{1.8}$$

$$\lim_{T\to\infty} \frac{a_T^i}{\prod_{s=t+1}^{T-1} R_s} = 0, \tag{1.9}$$

where a_t^i denotes the household's asset holdings and $\beta \in (0, 1)$ is the household's discount factor. The second constraint is the no-Ponzi condition that rules out consumption plans based on ever-increasing levels of debt. It serves as the transversality condition to uniquely pin down the optimal path for consumption. Note that if we define $\tilde{R}_{t+1} = R_0 R_1 R_2 ... R_{t+1}$ for $t > 0$ then we can solve the period budget constraint (1.8) forward to get a present value budget constraint:

$$\sum_{t=0}^{\infty} \frac{c_t^i}{\tilde{R}_t} = a_0^i + \sum_{t=0}^{\infty} \frac{y_t^i}{\tilde{R}_t}, \tag{1.10}$$

which states that the present discounted value of consumption must equal initial assets plus the present discounted value of the endowment stream. To see this, begin by writing (1.8) as

$$y_t^i + R_{t-1} a_{t-1}^i - c_t^i - a_t^i = 0,$$

and then roll the budget constraint forward one period and then substitute the result for a_t^i back into the period t budget constraint:

$$\begin{aligned} 0 &= y_{t+1}^i + R_t a_t^i - c_{t+1}^i - a_{t+1}^i \\ \implies a_t^i &= \frac{c_{t+1}^i + a_{t+1}^i - y_{t+1}^i}{R_t}, \end{aligned}$$

put back into (1.8):

$$y_t^i + R_{t-1}a_{t-1}^i - c_t^i - \left(\frac{c_{t-1}^i + a_{t-1}^i - y_{t+1}^i}{R_t}\right) = 0.$$

Do this again for a_{t+1}^i to get

$$y_t^i + \frac{y_{t+1}^i}{R_t} + R_{t-1}a_{t-1}^i - c_t^i - \frac{c_{t+1}^i}{R_t} - \frac{1}{R_t}\left(\frac{c_{t+2}^i + a_{t+2}^i - y_{t+2}^i}{R_{t+1}}\right) = 0$$

$$\Leftrightarrow y_t^i + \frac{y_{t+1}^i}{R_t} + \frac{y_{t+2}^i}{R_t R_{t+1}} + R_{t-1}a_{t-1}^i - c_t^i - \frac{c_{t+1}^i}{R_t} - \frac{c_{t+2}^i}{R_t R_{t+1}} - \frac{1}{R_t}\frac{1}{R_{t+1}}a_{t+2}^i = 0,$$

and eventually we have

$$\sum_{s=0}^{\infty} \frac{y_{t+s}^i R_t}{\prod_{j=0}^{s} R_{t+j}} + R_{t-1}a_{t-1}^i - \sum_{s=0}^{\infty} \frac{c_{t+s}^i R_t}{\prod_{j=0}^{s} R_{t+j}} - \frac{a_{t+\infty}^i}{R_t R_{t+1} \ldots R_{t+\infty-1}} = 0.$$

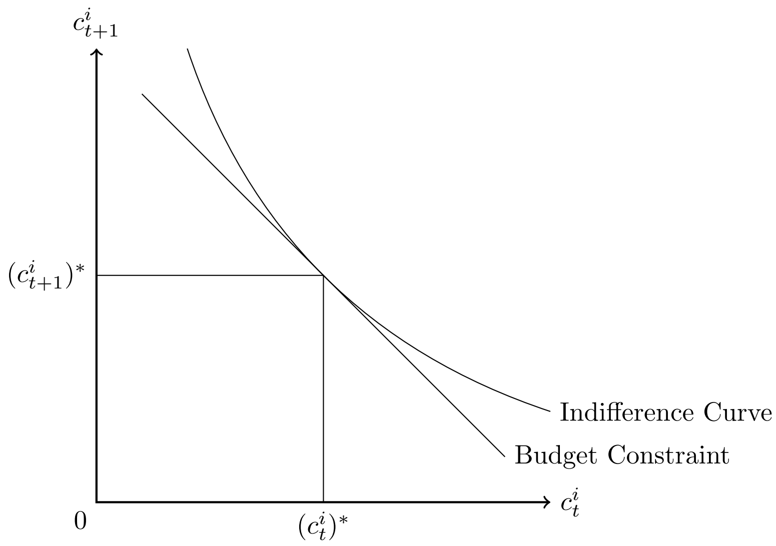
Rearrange and assume that $t-1$ is period 0 to get (1.10).

Now, how are we going to solve the household maximisation problem? We have four method available to us here.

1.4.1 Graphical approach

This method should be familiar to those who took undergraduate macroeconomics courses. Consider two consecutive periods, t and $t+1$, in the maximisation problem (1.7). From the utility function we can draw the indifference curves in $(c_t^i, c_{t+1}^i) \in \mathbb{R}^2$ space.

Figure 1.8: Graphical Solution to Household Problem



The utility function is

$$u(c_t^i) + \beta u(c_{t+1}^i) + \sum_{s=2}^{\infty} \beta^s u(c_{t+s}^i),$$

and the slope of an indifference curve can be calculated by total differentiation of the utility function and is given by

$$\begin{aligned} 0 &= u_{c,t} dc_t^i + \beta u_{c,t+1} dc_{t+1}^i \\ \implies \frac{dc_{t+1}^i}{dc_t^i} &= -\frac{1}{\beta} \frac{u_{c,t}}{u_{c,t+1}}, \end{aligned}$$

and this is what we call the marginal rate of substitution (MRS). We then add the budget constraint with a slope given by iterating the budget constraint forward:

$$\begin{aligned} a_t^i &= R_{t-1} a_{t-1}^i + y_t^i - c_t^i, \\ a_{t+1}^i &= R_t a_t^i + y_{t+1}^i - c_{t+1}^i, \end{aligned}$$

to then get

$$\frac{a_{t+1}^i - y_{t+1}^i + c_{t+1}^i}{R_t} = R_{t-1} a_{t-1}^i + y_t^i - c_t^i,$$

where it's clear that

$$\begin{aligned} -\frac{1}{R_t} dc_{t+1}^i &= dc_t^i \\ \implies \frac{dc_{t+1}^i}{dc_t^i} &= -R_t, \end{aligned}$$

which is the marginal rate of transformation (MRT). Use basic microeconomic theory to justify that the solution to the household's problem is where MRS = MRT:

$$\begin{aligned} \frac{1}{\beta} \frac{u_{c,t}}{u_{c,t+1}} &= R_t \\ \Leftrightarrow u_{c,t} &= \beta R_t u_{c,t+1}, \end{aligned} \tag{1.11}$$

which is the consumption Euler equation.

1.4.2 Direct substitution/sledgehammer approach

Next is the most brute-force method of solving the household's problem. Simply rearrange the budget constraint (1.8) to get a_t^i in terms of the other variables, and then substitute into the objective function (1.7):

$$\max_{\{a_{t+s}^i\}} \sum_{s=0}^{\infty} \beta^s u(R_{t+s-1} a_{t+s-1}^i + y_{t+s}^i - a_{t+s}^i).$$

Differentiating the above summation term with respect to a_t, and setting the derivative equal to zero gives

$$-u_{c,t} + u_{c,t+1}\beta R_t = 0,$$

and after rearranging we get

$$u_{c,t} = \beta R_t u_{c,t+1},$$

which is simply (1.11).

1.4.3 Value function approach

This is the dynamic programming approach, which has a large range of uses in macroeconomics.[9] Write the value function as

$$V(a^i_{t-1}) = \max_{a^i_t} \left[u(R_{t-1}a^i_{t-1} + y^i_t - a^i_t) + \beta V(a^i_t) \right], \tag{1.12}$$

noting that a^i_{t-1} is the state variable. The first order condition (FOC) with respect to assets a^i_t is

$$\begin{aligned} 0 &= -u_{c,t} + \beta V'(a^i_t) \\ \implies u_{c,t} &= \beta V'(a^i_t). \end{aligned} \tag{1.13}$$

As is usual in dynamic programming, we do not know the form of the value function $V(a^i_{t-1})$, but we do know its first derivative $V'(a^i_{t-1})$. Differentiating the value function (1.12) yields

$$V'(a^i_{t-1}) = u_{c,t}R_{t-1},$$

and if we roll one period ahead

$$V'(a^i_t) = u_{c,t+1}R_t,$$

then substitute into (1.13), we get

$$u_{c,t} = \beta R_t u_{c,t+1},$$

which is nothing but the consumption Euler equation.

1.4.4 The Lagrangian approach

This should also be very familiar from undergraduate macroeconomics. Begin by setting up the Lagrangian:

$$\mathcal{L}^i = \sum_{s=0}^{\infty} \beta^s u(c^i_{t+s}) + \sum_{s=0}^{\infty} \lambda^i_{t+s}\beta^s \left(R_{t+s-1}a_{t+s-1} + y^i_{t+s} - c^i_{t+s} - a^i_{t+s} \right).$$

9. See *Recursive Macroeconomic Theory* by Ljungqvist and Sargent, or *Recursive Methods in Economic Dynamics* by Stokey, Lucas, and Prescott.

This is the present value formulation of the Lagrangian as the Lagrangian multiplier, λ_{t+s}^i, is discounted by β^s back to its present value. It is equally valid to work with the current value Lagrangian and write the second term without discounting, i.e. $\tilde{\lambda}_{t+s}^i = \lambda_{t+s}^i \beta^s$. They are mathematically equivalent but sometimes it is more convenient to work with one than the other. The FOCs with respect to c_t^i, c_{t+1}^i, and a_t^i are

$$
\begin{aligned}
u_{c,t} &= \lambda_t^i, \\
u_{c,t+1} &= \lambda_{t+1}^i, \\
\lambda_{t+1}^i \beta R_t - \lambda_t^i &= 0.
\end{aligned}
$$

Do some substitution and rearranging and then we get

$$
u_{c,t} = \beta R_t u_{c,t+1},
$$

which is the consumption Euler equation.

1.4.5 Implications of the consumption Euler equation

We will soon assign particular functional forms to the utility function, but from (1.11) we can already notice some of the major implications of the neoclassical model for consumption. We can see that what determines the growth in the marginal utility of consumption (which as we shall see is closely linked to consumption) is the interest rate, R_t. In our model we have assumed that the consumer can only invest in one asset, a_t^i. However, Equation (1.11) holds for any asset the consumer invests in so we should think of R_t more widely as the return on any asset.

To see this more clearly, assume that $R_t = \bar{R}$ and that $\beta \bar{R} = 1$. This then implies that $\mathbb{E}_t [u_{c,t+1}/u_{c,t}] = 1$ so that agents do not expect their marginal utility to change between time periods. As a consequence, they are not expecting their consumption to change either. Similar reasoning suggests that if $\beta R_t > 1$ then the expectation of the ratio of marginal utilities of consumption is less than one, which given that marginal utility is declining in consumption (i.e. $u'' < 0$) implies that agents must be expecting consumption to increase. Similarly, if $\beta R_t < 1$ then consumption is expected to fall.

In all cases, the only thing which determines consumption growth is the rate of return/interest rate and not income. The rationale for the interest rate effect is as follows. If consumers know that savings this period are going to earn a high rate of return, there is an incentive for them to save more by having lower consumption. For a given end of period consumption level, the lower the level of initial consumption the faster is the growth rate of consumption.

1.4.6 Econometric evidence on the consumption Euler equation

The first paper to examine the consumption Euler equation was Hall (1978). He focused on utility functions which were well approximated by quadratic functions and assumed a constant interest rate which satisfies $\beta R = 1$. The result of this model is that consumption changes should be unpredictable. This paper sparked one of the largest literature fields in applied econometrics. Hall found that consumption growth was not predicted by income growth, but could be forecast by stock market prices. He interpreted this as a mild victory for the model. Subsequent work has been less kind to the model and has found that consumption growth does display a small but significant dependence on past income growth. However, the overall prediction that agents try and smooth their consumption over the business cycle is partly correct (see Table 1.1).

However, what if we let interest rates to be time varying and assume a standard form for preferences: $u(c^i) = (c^i)^{1-\sigma}/(1-\sigma)$. With some other assumptions, the Euler equation for consumption is then:

$$\Delta \ln c^i_{t+1} = \alpha + \frac{1}{\sigma} \ln R_t + \varepsilon^i_t. \tag{1.14}$$

The above equation should be familiar – and we will be returning to it many times throughout these notes. Here, σ^{-1} is the intertemporal elasticity of substitution,[10] and it governs the change in consumption growth to a change in [real] interest rates. Take a closer look at what (1.14) is saying though: first, it's saying that consumption growth is positively correlated with changes in interest rates; and secondly if we calibrate σ to something in the range of 2 to 5, then σ^{-1} is small – somewhere in the range of 0.2 to 0.5. The problem is these implications are strongly rejected by the data.[11] Interest rates tend to be counter-cyclical – they tend to be high in recessions and low in growth periods – and consumption growth is strongly procyclical. More obviously: consumption growth should be affected by other factors besides interest rates, suggesting that α and ε^i_t in (1.14) are misspecified. So, while this very simple approach has some good features – which we will continue to use – such as consumption smoothing and intertemporal substitution, it has too many shortcomings.

1.4.7 Taking the model to general equilibrium

The consumption Euler equation explains the dynamics of consumption of an individual i. To make further progress we make the simplifying assumption that utility is logarithmic in consumption, i.e., $u(c^i_t) = \log c^i_t$. In this case, the marginal utility of consumption is given by $u'(c^i_t) = 1/c^i_t$ and the consumption Euler equation is therefore

$$c^i_{t+1} = \beta R_t c^i_t.$$

General equilibrium requires that all individuals satisfy their Euler equations for consumption and that markets clear (Walras' Law), which is achieved here by the interest rate adjusting to clear the market. Since there is no aggregate savings device, market clearing requires that individual

10. Remember that if σ is the coefficient of relative risk aversion, then with constant relative risk aversion (CRRA) preferences σ^{-1} is the elasticity of intertemporal substitution.

11. A very good paper on this, which also provides a good review of the literature, is Ascari, Magnusson, and Mavroeidis (2021). But it's best to read this paper once you have become accustomed to the New Keynesian framework.

net claims must sum to zero and $\sum_i a_t^i = 0,\ \forall t$. In this case, all the endowment is consumed each period and $\sum_i y_t^i = \sum_i c_t^i,\ \forall t$. When we aggregate the individual consumption Euler equations with logarithmic utility, we find that $\sum_i c_{t+1}^i = \beta R_t \sum_i c_t^i$, and hence:

$$\sum_i y_{t+1}^i = \beta R_t \sum_i y_t.$$

Defining $\bar{y}_t$ and $\bar{y}_{t+1}$ as the average endowments in periods t and $t+1$, we see that the rate of interest is determined by the ratio of endowments in the two periods

$$\beta R_t = \frac{\bar{y}_{t+1}}{\bar{y}_t} = \frac{\bar{c}_{t+1}}{\bar{c}_t}.$$

1.5 Comments and key readings

The simple example illustrates a lot of what will become familiar in macroeconomics. The dominant approach is to view outcomes as the result of purposeful interaction of many agents in many markets. The two main elements are optimisation and equilibrium: (i) taking some sets X and Z as given, agents optimise, and (ii) there is a consistency requirement on X and Z such as Walrasian market clearing consistency or Nash equilibrium. Modern macroeconomics stresses the importance of dynamic optimisation, motivated in part by the Lucas critique (absence of stable behavioural equations) and the notion of intertemporal substitution.

Key readings for this section are Barro and King (1984), Hall (1978), Hall (1988), Lucas (1978), Mankiw (1990), Mankiw, Rotemberg, and Summers (1985), and Romer (2012) (chapters 4 and 5).

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2 The Overlapping Generations Model

2.1 Introduction

The endowment economy we examined in the previous section had no firms and only financial capital. This isn't a crazy idea if labour is supplied inelastically and is the only input in production. However, physical capital is clearly important in the economy. We now allow agents in the economy to accumulate physical capital. To keep things simple, however, we will continue to assume that labour supply is exogenous. There are two important classes of models with capital accumulation: the overlapping generation (OLG) model and the representative agent model. We will start with the OLG model in this section.

Peter Diamond produced a version of the OLG model introduced by Samuelson in which the savings rate is endogenous and can change with other parameters of the economy – addressing one of the biggest weaknesses of the Solow-Swan model.

Neoclassical growth models such as the OLG model typically have two entities (firms and households[1]) and three markets: goods, labour, and capital. We can generally disregard financial markets as they will be redundant. While our analysis of the endowment economy focused mainly on the determination of interest rates, these models focus extensively on the determination of consumption and the capital stock. In the baseline case we will assume no population growth and no technological progress. We will later consider these as extensions. Agents in the OLG model live for two periods and must make decisions in the first period of their lives about their consumption in both periods of life. Sounds quite morbid, but it's a simplification we make for analytical tractability. Individuals who have substantial income in the first period of life may save some of this in the form of capital or lending, and are able to consume more than they otherwise might in the second period of life. People live for two periods in these economies because this is the smallest number of periods that permits a savings decision.[2]

2.2 The basic two-period OLG model

The basic OLG model is described as follows. Let there be an infinite sequence of time, $t = 0, 1, 2, ..., \infty$. The generation born in period t is referred to as generation t. There are $N(t)$ members of generation t, and people live for 2 periods, and generation t is young in t and old in $t + 1$. Generation t does not exist in period $t + 2$.

A member h of generation t has utility

$$u_t^h(c_t^h(t), c_t^h(t+1)), \tag{2.1}$$

where, to clarify the notation, $c_t^h(t + 1)$ denotes the consumption of the aggregate consumption good by individual h of generation t in period $t + 1$. Production takes place in competitive firms

1. We can add government into the model too.
2. Many OLG models can feature three-periods for example.

with homogenous of degree 1 (HOD1) production technology with constant returns to scale (CRS), implying that they do not produce economic profits.[3] Production in period t is given by

$$Y(t) = F(K(t), L(t)),$$

where $L(t)$ is the total labour used in production and $K(t)$ is the total capital.

Individuals are endowed with lifetime endowment of labour given by

$$l_t^h = [l_t^h(t), l_t^h(t+1)].$$

Total labour is given as

$$L(t) = \sum_{h=1}^{N(t)} l_t^h(t) + \sum_{h=1}^{N(t-1)} l_{t-1}^h(t).$$

Aggregate labour of the young at time t is the first component of the RHS, and the aggregate labour of the old is the second component of the RHS. We also assume that $K(t)$ depreciates fully. This assumption removes the complication of a capital market between members of different generations.

The economy has the following resource constraint:

$$Y(t) = F(K(t), L(t)) \geq \sum_{h=1}^{N(t)} c_t^h(t) + \sum_{h=1}^{N(t-1)} c_{t-1}^h(t) + K(t+1).$$

The production of period t goes either to consumption of the young or the old or to capital for use in period $t+1$.

We assume that the economic organisation of the economy is one of perfectly competitive markets where individuals are owners of their own labour. Members of generation t earn income in period t by offering all their labour endowment to firms at market wage, w_t, and use income to fuel consumption in period t, to fund borrowing and lending to other members of generation t, and

3. It's worth expanding on how and why we can assume this. Suppose firms have the production technology,

$$f(x_t) = x,$$

and firms want to maximise profits, $\Pi = px - wx$. Clearly, if $p > w$, then no profitable production plan exists for this firm. So $p \leq w$ is a necessary condition, in which case max profits will be zero, and will occur for any CRS technology. Why? Suppose we have (p, w) where $\Pi > 0$, then

$$\Pi^* = pf(x^*) - wx^* > 0,$$

and suppose we scale up production by $\lambda > 1$, so our profits will be

$$pf(\lambda x^*) - w\lambda x^* = \lambda(pf(x^*) - wx^*) = \lambda\Pi^* > \Pi^*.$$

This means that if profits are ever positive, they can always be scaled up, and are unbounded and no maximal production plan will exist. So, the only nontrivial profit maximising position for a CRS firm is one involving zero profits.

for accumulation of private capital. The budget constraint for individual h when they're young is

$$w_t l_t^h(t) = c_t^h(t) + a^h(t) + k^h(t+1), \tag{2.2}$$

where $a^h(t)$ are net asset holdings of individual h. $a^h(t) < 0$ implies net borrowing from other members of generation t. Because of the overlapping nature of the generations, borrowing and lending can only occur among members of the same generation.

Suppose that a young person of generation t lends some goods to an old member of generation $t-1$ in period t with the expectation of being paid back in periods $t+1$. In period $t+1$, this rather naive young person hunts for the member of generation $t-1$ so that he/she can be paid back. Unfortunately, members of generation $t-1$ are now all dead and the dead cannot be forced to pay back their debts. Individuals know this and will not make loans to members of other generations. Individuals cannot borrow or lend across generations so

$$\sum_{h=1}^{N(t)} a^h(t) = 0.$$

In period $t+1$, a member of generation t has income from the labour supplied in period $t+1$, from interest earned on any loans that were made in period t, and from the rent on capital that they accumulated in period t. Since this is the last period of life, all income will be consumed. Therefore, the budget constraint for generation t individual in period $t+1$ is:

$$c_t^h(t+1) = w_{t+1} l_t^h(t+1) + R_t a^h(t) + R_{t+1} k^h(t+1), \tag{2.3}$$

where R_t is the interest paid on loans between period t and $t+1$.

Individuals are assumed to have perfect foresight in the sense that they know, when young, what wages and rents will be when they are old. In addition, no fraud is permitted so that all loans are paid back with the agreed upon interest.

Factor prices are determined by their marginal products due to competitive equilibrium:

$$w_t = F_L(K(t), L(t)), \tag{2.4}$$

$$R_t = F_K(K(t), L(t)), \tag{2.5}$$

where $F_i(\cdot, \cdot)$ is the partial derivative of the production function with respect to its i-th component.

We can combine the budget constraints of the young and old. From (2.2):

$$a^h(t) = w_t l_t^h(t) - c_t^h(t) - k^h(t+1),$$

and substitute this expression into (2.3) to get:

$$c_t^h(t+1) = w_{t+1} l_t^h(t+1) + R_t w_t l_t^h(t) - R_t c_t^h(t) - R_t k^h(t+1) + R_{t+1} k^h(t+1),$$

collecting terms, we can yield an expression for $c_t^h(t)$,

$$c_t^h(t) = \frac{w_{t+1} l_t^h(t+1) - c_t^h(t+1)}{R_t} + w_t l_t^h(t) - k^h(t+1) \left[1 - \frac{R_{t+1}}{R_t}\right].$$

Since we assume that there are no arbitrage opportunities, the return on capital should equal the return on loans amongst members of a particular cohort, $R_t = R_{t+1}$. Thus the budget constraint becomes:

$$c_t^h(t) + \frac{c_t^h(t+1)}{R_t} = w_t l_t^h(t) + \frac{w_{t+1} l_t^h(t+1)}{R_t}. \tag{2.6}$$

In words: The present value of lifetime consumption must equal the present value of lifetime wage income.

A competitive equilibrium consists of a sequence of prices

$$\{w_t, R_t\}_{t=0}^{\infty},$$

and quantities

$$\left\{ \{c_t^h(t)\}_{h=1}^{N(t)}, \{c_{t-1}^h(t)\}_{h=1}^{N(t-1)}, K(t+1) \right\}_{t=0}^{\infty},$$

such that each member h of each generation $t > 0$ maximises utility (2.1) subject to their lifetime budget constraint given by (2.6), and so that the equilibrium conditions

$$\begin{aligned} R_{t+1} &= R_t, \\ w_t &= F_L(K(t), L(t)), \\ R_t &= F_K(K(t), L(t)), \\ L(t) &= \sum_{h=1}^{N(t)} l_t^h(t) + \sum_{h=1}^{N(t-1)} l_{t-1}^h(t), \end{aligned}$$

hold each period.

Note that in the above definition we did not define the individual holdings of either lending or of capital. This is because they offer exactly the same return and there are an infinite number of distributions of lending and capital holdings among members of a generation that would meet the equilibrium conditions. Two example distributions for an economy where all members of a generation are identical are i) person $h = 1$ borrows from everyone else and holds all the capital; and ii) no one borrows and each person holds $K(t+1)/N(t)$ units of capital. These two distributions would result in the same total capital stock and the same equilibrium as the above definition.

Now, substitute the lifetime budget constraint (2.6) into the utility function, to set up household h of generation t's problem

$$\max_{c_t^h(t)} u\left(c_t^h(t), R_t w_t l_t^h(t) - w_{t+1} l_t^h(t+1) - R_t c_t^h(t)\right),$$

where, for individual h, the assumption of perfect foresight means that the values of all the other parameters are known. The FOC is:

$$\begin{aligned} &u_1\left(c_t^h(t), R_t w_t l_t^h(t) + w_{t+1} l_t^h(t+1) - R_t c_t^h(t)\right) \\ &\quad = R_t u_2\left(c_t^h(t), R_t w_t l_t^h(t) + w_{t+1} l_t^h(t+1) - R_t c_t^h(t)\right), \end{aligned} \tag{2.7}$$

where $u_i(\cdot,\cdot)$ is the partial derivative of the utility function with respect to its i-th element. Using the budget constraint when young (2.2), we can find a savings function for individual h of generation t, $s_t^h(\cdot)$, where

$$s_t^h(w_t, w_{t+1}, R_t) = a^h(t) + k^h(t+1).$$

Summing the savings of all members of generation t, we define an aggregate savings function $S(\cdot)$, as equal to

$$S_t(\cdot) = \sum_{h=1}^{N(t)} s_t^h(\cdot) = \sum_{h=1}^{N(t)} a^h(t) + \sum_{h=1}^{N(t)} k^h(t+1).$$

Given that, in equilibrium,

$$\sum_{h=1}^{N(t)} a^h(t) = 0,$$

and

$$K(t+1) = \sum_{j=1}^{N(t)} k^h(t+1),$$

the aggregate savings equation can be written as

$$S_t(w_t, w_{t+1}, R_t) = K(t+1).$$

Substituting R_{t+1} for R_t, and using the equilibrium conditions for factor prices ((2.4) and (2.5)) in periods t and $t+1$, we can write aggregate savings as:

$$S_t\left(\underbrace{F_L(K(t), L(t))}_{w_t}, \underbrace{F_L(K(t+1), L(t+1))}_{w_{t+1}}, \underbrace{F_K(K(t+1), L(t+1))}_{R_t}\right) = K(t+1).$$

The above expression gives $K(t+1)$ as an implicit functions of the labour supplies in each periods, $L_t(t), L_{t-1}(t), L_t(t+1)$, the parameters of the utility functions and the production function, and $K(t)$. Since, as the model is constructed, all of these except $K(t)$ are constants through time, one can find the capital stock in $t+1$ as a function of the capital stock in time t:

$$K(t+1) = G(K(t)). \tag{2.8}$$

This is a first-order difference equation/law of motion that describes the growth path of the economy.

2.2.1 An example OLG economy

Suppose that the agents in our model possessed log-utility:

$$u(c_t) = \ln c_t,$$

and the production technology is of Cobb-Douglas form:

$$F(K(t), L(t)) = K(t)^{\alpha} L(t)^{1-\alpha}, \ \alpha \in (0,1).$$

Our problem would be

$$\max_{c_t^h(t)} \ \ln c_t^h(t) + \beta \ln c_t^h(t+1),$$

and using our lifetime budget constraint (2.6) we can write this as

$$\max_{c_t^h(t)} \ \ln c_t^h(t) + \beta \ln \left(R_t w_t l_t^h(t) - w_{t+1} l_t^h(t+1) - R_t c_t^h(t) \right),$$

and with the following FOC:

$$\begin{aligned} 0 &= \frac{1}{c_t^h(t)} - \frac{\beta R_t}{\underbrace{R_t w_t l_t^h(t) - w_{t+1} l_t^h(t+1) - R_t c_t^h(t)}_{c_t^h(t+1)}} \\ \implies 1 &= \beta \frac{R_t c_t^h(t)}{c_t^h(t+1)}. \end{aligned}$$

The above equation is nothing but the consumption Euler equation. Now, substitute the optimal consumption given by the Euler equation back into the budget constraint:

$$\begin{aligned} c_t^h(t) + \frac{c_t^h(t+1)}{R_t} &= w_t l_t^h(t) + \frac{w_{t+1} l_t^h(t+1)}{R_t} \\ \implies c_t^h(t) + \frac{1}{R_t}\left[\beta R_t c_t^h(t)\right] &= w_t l_t^h(t) + \frac{w_{t+1} l_t^h(t+1)}{R_t} \\ c_t^h(t)(1+\beta) &= w_t l_t^h(t), \end{aligned}$$

where we also assume that the agent does not work when they're old, so we have

$$c_t^h(t) = \frac{w_t l_t^h(t)}{1+\beta}. \tag{2.9}$$

Now that have consumption per period for an individual h of generation t in period t, we want to pin down aggregate savings, which help us get the law of motion of capital in this model. But first, we need our factor prices:

$$\frac{\partial Y(t)}{\partial K(t)} = R_t = \alpha \left[\frac{K(t)}{L(t)}\right]^{\alpha-1} = \alpha k(t)^{\alpha-1},$$
$$\frac{\partial Y(t)}{\partial L(t)} = w_t = (1-\alpha) \left[\frac{K(t)}{L(t)}\right]^{\alpha} = (1-\alpha) k(t)^{\alpha},$$

and from our household FOC, we have

$$c_t^h(t) = \frac{w_t l_t^h(t)}{1+\beta} = \frac{l_t^h(t)}{1+\beta}(1-\alpha)k(t)^\alpha,$$

and aggregating across the cohort yields

$$\begin{aligned} C_t(t) &= \frac{1}{1+\beta}(1-\alpha)K(t)^\alpha L(t)^{1-\alpha} \\ &= \left(\frac{1-\alpha}{1+\beta}\right) Y(t). \end{aligned}$$

So savings is given by

$$\begin{aligned} S(t) &= Y(t) - C_t(t) \\ &= Y(t) - \left(\frac{1-\alpha}{1+\beta}\right) Y(t) \\ &= \left(\frac{\alpha+\beta}{1+\beta}\right) Y(t), \end{aligned}$$

and since $S(t) = K(t+1)$,

$$K(t+1) = \left(\frac{\alpha+\beta}{1+\beta}\right) Y(t).$$

If we assume that labour is supplied inelastically by the young, then the law of motion of capital can be written as

$$K(t+1) = \left(\frac{\alpha+\beta}{1+\beta}\right) K(t)^\alpha. \tag{2.10}$$

The steady-state capital stock, $\bar{K}$, is given by

$$\begin{aligned} \bar{K} &= \left(\frac{\alpha+\beta}{1+\beta}\right) \bar{K}^\alpha \\ \bar{K}^{1-\alpha} &= \left(\frac{\alpha+\beta}{1+\beta}\right) \\ \therefore \bar{K} &= \left(\frac{\alpha+\beta}{1+\beta}\right)^{\frac{1}{1-\alpha}}. \end{aligned} \tag{2.11}$$

In other words, $\bar{K}$ satisfies the condition $\Delta K(t+1) = 0$:

$$\Delta K(t+1) = 0 = K(t+1) - K(t),$$

and this satisfies these conditions for two values of $\bar{K}$: $\bar{K} = 0$ and the value for $\bar{K}$ in (2.11). Actually, also, we could log-linearise the law of motion of capital (2.10):

$$\begin{aligned} \ln K(t+1) &= \ln\left(\frac{\alpha+\beta}{1+\beta}\right) + \alpha \ln K(t) \\ \ln \bar{K} + \frac{1}{\bar{K}}(K(t+1) - \bar{K}) &\approx \ln\left(\frac{\alpha+\beta}{1+\beta}\right) + \alpha \ln \bar{K} + \frac{\alpha}{\bar{K}}(K(t) - \bar{K}), \end{aligned}$$

and we know from (2.11) that in the steady-state $\ln \bar{K} = \ln\left(\frac{\alpha+\beta}{1+\beta}\right) + \alpha \ln \bar{K}$, so we have:

$$\begin{aligned} \frac{1}{\bar{K}}(K(t+1) - \bar{K}) &= \frac{\alpha}{\bar{K}}(K(t) - \bar{K}) \\ \therefore \hat{K}(t+1) &= \alpha \hat{K}(t). \end{aligned}$$

2.2.2 Convergent dynamics in the OLG model

The behaviour of this model out of a steady-state is similar to that of the Solow-Swan model. If the initial capital stock is between the two steady-states, $0 < K(0) < \left(\frac{\alpha+\beta}{1+\beta}\right)^{\frac{1}{1-\alpha}}$, the capital stock will grow, converging on the positive steady-state. This can be seen by simply looking for the range of initial capital stocks for which $K(t+1) > K(t)$, or where

$$\Delta K(t+1) = \left(\frac{\alpha+\beta}{1+\beta}\right) K(t)^\alpha - K(t) > 0.$$

This condition holds for positive $K(t)$ when $K(t) < \left(\frac{\alpha+\beta}{1+\beta}\right)^{\frac{1}{1-\alpha}}$. In addition, the rate of growth of the capital stock declines as it grows. Define the gross rate of growth of capital as $\Delta K(t) = K(t+1)/K(t)$. This can be written as

$$\Delta K(t) = \left(\frac{\alpha+\beta}{1+\beta}\right) \frac{K(t)^\alpha}{K(t)} = \left(\frac{\alpha+\beta}{1+\beta}\right) K(t)^{\alpha-1}.$$

Taking the derivative of the growth rate with respect to the capital stock yields:

$$\frac{d\Delta K(t)}{dK(t)} = (\alpha - 1)\left(\frac{\alpha+\beta}{1+\beta}\right) K(t)^{\alpha-2} < 0.$$

As in the Solow-Swan model, the larger the initial capital stock, the slower the growth rate of capital. In addition, since output is defined by a Cobb-Douglas technology, the gross growth rate of output, $\Delta Y(t) = Y(t+1)/Y(t)$ is equal to

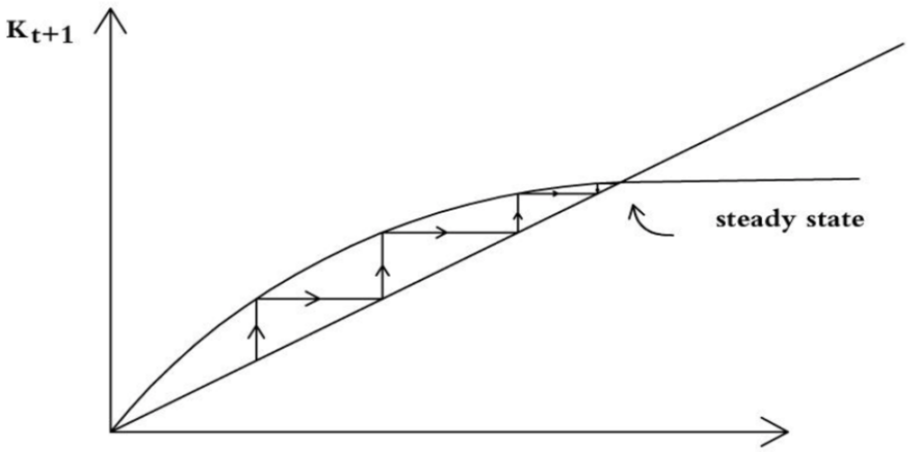
$$\Delta Y(t) = \frac{K(t+1)^\alpha L(t+1)^{1-\alpha}}{K(t)^\alpha L(t)^{1-\alpha}} = \frac{K(t+1)^\alpha}{K(t)^\alpha} = \Delta K(t)^\alpha,$$

where the second equality is given because we have inelastic labour supply equal to unity. The derivative of the gross growth rate of output with respect to the capital stock is:

$$\frac{d\Delta Y(t)}{dK(t)} = \frac{d\left[\left(\frac{\alpha+\beta}{1+\beta}\right)K(t)^{\alpha-1}\right]^\alpha}{dK(t)} = \alpha\left[\left(\frac{\alpha+\beta}{1+\beta}\right)K(t)^{\alpha-1}\right]^{\alpha-1}(\alpha-1)\left(\frac{\alpha+\beta}{1+\beta}\right)K(t)^{\alpha-2} < 0,$$

so output growth slows as the capital stock increases.

Figure 2.1: Convergence in a Simple OLG Model



As Figure 2.1 shows, we have monotonic convergent dynamics, although there is a possibility of dynamic inefficiency as it may be possible to generate Pareto improvements by transferring resources from each young generation to the current old generation.[4]

2.3 Fiscal policy and non-Ricardian equivalence

With the possibility of efficiency gains by reallocating resources from the old to the young, we now amend the basic OLG model slightly to see the effects of government spending. To make things simple, I will use a change of notation here.[5] Let fiscal policy in the two period OLG model be given by $\{G_t, T_t^0, T_t^1, B_t\}$ where G_t is government spending which directly benefits the young (such as schooling), T_t^0 is taxes on the young, T_t^1 is taxes on the old, B_t is government debt. Interest on

4. See the discussion in Section 3.8.
5. Here, I also use the start of period notation for government debt.

government debt is given by $R_t^b = R_t$, so the government budget constraint is:

$$B_{t+1} = G_t - T_t^0 - T_t^1 + R_t^b B_t.$$

The household problem is now:

$$\max_{C_t^0, C_{t+1}^1} \quad u(C_t^0 + G_t) + \beta u(C_{t+1}^1),$$

subject to

$$C_t^0 + B_{t+1}^1 + K_{t+1}^1 = w_t - T_t^0, \tag{2.12}$$
$$C_{t+1}^1 = (B_{t+1}^1 + K_{t+1}^1)R_{t+1} - T_t^1, \tag{2.13}$$

where B_{t+1}^1 is saving by the household in the form of government debt. Assuming a well-behaved utility function and $T_t^1 = 0$ so there are no taxes on the old, the FOC with respect to K_{t+1}^1 yields the consumption Euler equation:

$$C_{t+1}^1 = \beta \underbrace{R_{t+1}}_{F_K(K_{t+1}^1)} (C_t^0 + G_t), \tag{2.14}$$

and with the household budget constraints and market clearing factor prices, we can find the law of motion of capital. From (2.13) and (2.14), we have

$$(B_{t+1}^1 + K_{t+1}^1)R_{t+1} = \beta R_{t+1}(C_t^0 + G_t),$$

and then substitute in the value for C_t^0 from (2.12),

$$\begin{aligned}
(B_{t+1}^1 + K_{t+1}^1)R_{t+1} &= \beta R_{t+1}(w_t - T_t^0 - B_{t+1}^1 - K_{t+1}^1 + G_t) \\
(B_{t+1}^1 + K_{t+1}^1)R_{t+1} + \beta R_{t+1} K_{t+1}^1 &= \beta R_{t+1}(w_t - T_t^0 - B_{t+1}^1 + G_t) \\
B_{t+1}^1 + K_{t+1}^1 + \beta K_{t+1}^1 &= \beta(w_t - T_t^0 - B_{t+1}^1 + G_t) \\
K_{t+1}^1 &= \frac{\beta}{1+\beta}(w_t - T_t^0 - B_{t+1}^1 + G_t) - \frac{B_{t+1}^1}{1+\beta},
\end{aligned}$$

which gives:

$$K_{t+1}^1 = \frac{\beta}{1+\beta}\left(F_L(K_t^1) - T_t^0 + G_t\right) - B_{t+1}^1.$$

Ricardian equivalence states that it does not matter whether a given sequence of government spending is funded through taxes or debt. To see where this fails in OLG models, we fix the sequence of government spending to $G_t = \bar{G}$ at time t, and $G_{t+i} = 0$, $\forall i > 0$. Now consider two financing schemes. The first funds the one-off government spending by a tax on the young so that $T_t^0 = G_t$. The second places no taxes in period t and instead borrows $B_{t+1} = G_t$ and taxes the young $R^b_{t+1}G_t$ in period $t+1$ to repay the debt. If Ricardian equivalence holds then these two financing schemes will have equivalent aggregate effects. In the first case, government policy does nothing:

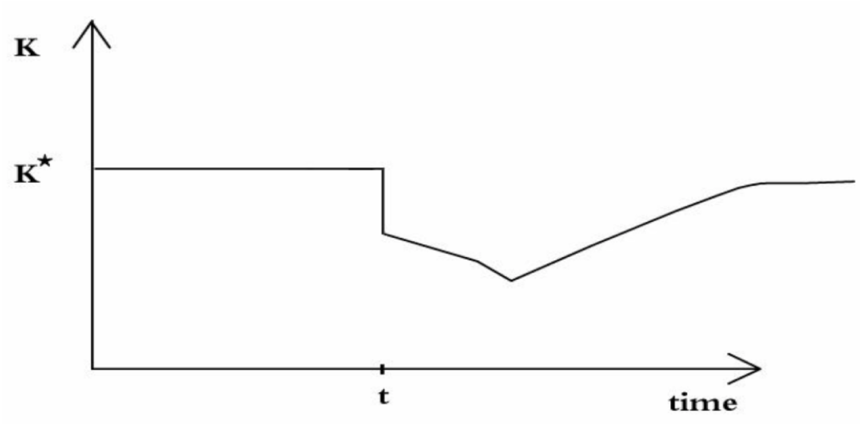
$$K^1_{t+1} = \frac{\beta}{1+\beta}\left(F_L(K^1_t) - T^0_t + G_t\right) = \frac{\beta}{1+\beta}F_L(K^1_t),$$

so if the economy is in steady-state at time t it will stay there. In contrast, with the second policy we have that:

$$\begin{aligned} K^1_{t+1} &= \frac{\beta}{1+\beta}F_L(K^1_t) - \frac{1}{1+\beta}G_t \\ K^1_{t+2} &= \frac{\beta}{1+\beta}\left(F_L(K^1_t) - T^0_{t+1}\right) \\ K^1_{t+i} &= \frac{\beta}{1+\beta}F_L(K^1_{t+i-1}),\ \forall i > 3. \end{aligned}$$

So financing matters and Ricardian equivalence fails to hold in the OLG model. If the economy starts in steady-state then it will deviate from steady-state for several periods. Figure 2.2 illustrates this.

Figure 2.2: Path of Capital Stock After a Bond Financed Fiscal Expansion



2.4 Adding technological change

The model so far has abstracted from technological change. In general, we can think of technological change as entering the production function $Y_t = F(K_t, L_t, \theta_t)$, where θ_t is a technology term that is given exogenously. Fortunately, there are cases where this allows for a simple characterisation of a balanced growth path that satisfies the Kaldor facts. Example production functions which account for technological changed are the Hicks-neutral production function $Y_t = \theta_t F(K_t, L_t)$, the capital-augmenting production function $Y_t = F(\theta_t K_t, L_t)$, and the labour-augmenting production function $Y_t(K_t, \theta_t L_t)$. The latter of these technologies allows a characterisation consistent with the Kaldor stylised facts!

Consider the labour augmenting production function. Also, let's assume that households in that model possess a constant-relative risk aversion (CRRA) utility function of the form

$$U(c) = \frac{c^{1-\sigma}}{1-\sigma},$$

and let's assume that the law of motion for technological change is:

$$\theta_{t+1} = (1+g)\theta_t,$$

where g is the growth rate of the economy. Now, in a model with exogenous labour, we have the following equilibrium condition (from the consumption Euler equation):

$$\begin{aligned} u'(C_t) &= \beta R_{t+1} u'(C_{t+1}) \\ u'(w_t - K^1_{t+1}) &= \beta R_{t+1} u'(K^1_{t+1} R_{t+1}) \\ u'(\theta_t F_L(K^1_t, \theta_t L_t) - K^1_{t+1}) &= \beta F_K(K^1_{t+1}, \theta_{t+1} L_{t+1}) u'(K_{t+1} F_K(K^1_{t+1}, \theta_{t+1} L_{t+1})) \\ u'\left(\theta_t F_L\left(\frac{K^1_t}{\theta_t}, 1\right) - K^1_{t+1}\right) &= \beta F_K\left(\frac{K^1_{t+1}}{\theta_{t+1}}, 1\right) u'\left(K^1_{t+1} F_K\left(\frac{K^1_{t+1}}{\theta_{t+1}}, 1\right)\right), \end{aligned} \tag{2.15}$$

since labour is exogenous at $L_t = 1, \ \forall t$. To see why we can write this, consider the case where Y_t is a Cobb-Douglas technology:

$$\begin{aligned} F(K_t, \theta_t L_t) &= Y_t = K_t^\alpha (\theta_t L_t)^{1-\alpha}, \\ \implies F_L(K_t, \theta_t L_t) &= (1-\alpha) K_t^\alpha \theta_t^{1-\alpha} L_t^{-\alpha} \\ &= (1-\alpha)\left(\frac{K_t}{L_t}\right)^\alpha \theta_t^{1-\alpha} \\ \implies F_L(K_t, \theta_t) &= (1-\alpha) K_t^\alpha \theta_t^{1-\alpha} \\ &= (1-\alpha)\frac{K_t^\alpha}{\theta_t^{\alpha-1}}, \end{aligned}$$

and divide through by θ_t to get:

$$\begin{aligned} (1-\alpha)\frac{K_t^\alpha}{\theta_t^{\alpha-1}}\theta_t^{-1} &= (1-\alpha)\left(\frac{K_t}{\theta_t}\right)^\alpha \\ &= F_L\left(\frac{K_t}{\theta_t}, 1\right). \end{aligned}$$

We could also easily show this for marginal product of capital. It's also worth noting that the model will have a balanced growth path property if it can be written as a dynamic equation in K_t/θ_t. When the production function has constant returns to scale then derivatives are HOD0, which means that a doubling of all inputs leads to a doubling of output, and doubling of factor prices and resources has no effect in input demand.[6]

Now, assuming we have CRRA utility, (2.15) yields:

$$\left(\theta_t F_L\left(\frac{K_t^1}{\theta_t},1\right)-K_{t+1}^1\right)^{-\sigma}=\beta F_K\left(\frac{K_{t+1}^1}{\theta_{t+1}},1\right)\left[K_{t+1}^1 F_K\left(\frac{K_{t+1}^1}{\theta_{t+1}},1\right)\right]^{-\sigma},$$

divide through by $\theta_{t+1}^{-\sigma}$ from our law of motion of technological growth:

$$\left(\frac{\theta_t}{\theta_{t+1}}F_L\left(\frac{K_t^1}{\theta_t},1\right)-\frac{K_{t+1}^1}{\theta_{t+1}}\right)^{-\sigma}=\beta F_K\left(\frac{K_{t+1}^1}{\theta_{t+1}},1\right)\left[\frac{K_{t+1}^1}{\theta_{t+1}} F_K\left(\frac{K_{t+1}^1}{\theta_{t+1}},1\right)\right]^{-\sigma},$$
$$\Leftrightarrow\left(\frac{1}{1+g}F_L\left(\frac{K_t^1}{\theta_t},1\right)-\frac{K_{t+1}^1}{\theta_{t+1}}\right)^{-\sigma}=\beta F_K\left(\frac{K_{t+1}^1}{\theta_{t+1}},1\right)\left[\frac{K_{t+1}^1}{\theta_{t+1}} F_K\left(\frac{K_{t+1}^1}{\theta_{t+1}},1\right)\right]^{-\sigma},$$

which is now a first order difference equation in K_t/θ_t with only a minor change compared to what we saw in the baseline OLG model without technological change. If we assumed log utility, where $\sigma = 1$ and a Cobb-Douglas production technology, then the law of motion for capital relative to technology becomes:

$$\frac{K_{t+1}^1}{\theta_{t+1}}=\frac{\beta(1-\alpha)}{(1+\beta)(1+g)}\left(\frac{K_t^1}{\theta_t}\right)^{\alpha}.$$

Not much has changed, however. $g > 0$ changes the speed of convergence but nothing else. The model with exogenous technological change has the same qualitative properties as the model without it.

2.5 Comments and key readings

An OLG model allows us to make the savings decision endogenous to the model in a relatively simple way. Since agents live only two periods, their optimisation problem involves only those two periods. The overlapping nature of the economy and the fact that the old are holding all the capital that they saved from the previous periods gives the model some persistence. In the version shown here, we did not make the labour supply decision endogenous, but this can be done relatively easily, since it adds only two more variables to the decision problem of each agent: the labour to supply when young and that when old.

Key readings for this section are far and wide. See Acemoglu (2009), Blanchard and Fischer (1989), Ljungqvist and Sargent (2018), McCandless (2008), and Romer (2012). For a more rigorous treatment of OLG models see McCandless and Wallace (1992) and Sargent (1987).

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3 The Ramsey-Cass-Koopmans Model

3.1 Introduction

The Ramsey-Cass-Koopmans model (henceforth referred to as just the Ramsey model) is the basic model of a competitive capitalist economy. Competitive firms rent capital and hire labour to produce and sell output, and a fixed number of infinitely lived households supply labour, hold capital, consume, and save. This model, which was developed by Ramsey (1928), Cass (1965), and Koopmans (1965), avoids all market imperfections and all issues raised by heterogeneous households and links among generations. It provides us with an excellent benchmark model to build on from. Thus, there will be no exogenous dynamics for now, so all the dynamics will be induced by the mechanism of capital accumulation, which will feedback on interest rates and savings decisions. Our ultimate aim is to understand the dynamic properties of the Walrasian equilibrium: Is it stable, and does it allow for growth? To what extent does growth generated in the model have properties similar to observed growth?

3.2 Households

To keep our analysis simple, for now, we assume a constant large number of households (i.e., the population of households does not grow[1]) all bundled in a single representative agent. We have already stated the behaviour of households, so we can go ahead and write the problem of the representative agent household as:

$$\max_{\{c(t)\}} \int_0^\infty u(c(t)) \exp(-\rho t) dt, \tag{3.1}$$

subject to

$$\begin{aligned} c(t) + I(t) &= w(t)l + r(t)k(t) + \Pi(t), \\ \dot{k}(t) &= I(t) - \delta k(t), \end{aligned}$$

where the objective is the present discounted value of utility in continuous time, $u(\cdot)$ is the instantaneous utility function, and ρ is the discount factor. The first constraint is the standard budget constraint requiring consumption and investment to be equal to labour income, capital income, plus any profits from firms owned by the household. The latter of which is treated as exogenous by the household, and in equilibrium will be zero because firms with constant returns to scale in competitive markets don't make profits. The second constraint defines the law of motion of capital with $\dot{k}(t) = \frac{dk(t)}{dt}$, and states the change in capital with respect to time is a function of investment

1. You could easily relax this assumption. Romer (2012) does a full setup of the Ramsey model with population growth.

less depreciation. The two constraints can be combined by substitution:

$$\dot{k}(t) = w(t)l + (r(t) - \delta)k(t) + \Pi(t) - c(t). \tag{3.2}$$

We will impose regularity conditions on the utility function: $u' > 0$ and $u'' < 0$. Our now-familiar forms of utility, CRRA and log utility, will abide by these conditions which will help us achieve nice analytical results. Finally, note that labour l is supplied inelastically by the household.

3.2.1 Technical aside: CRRA utility

It's worth pointing out some basic characteristics of the CRRA utility functional form:

$$u(c(t)) = \frac{c(t)^{1-\sigma}}{1-\sigma},\ \sigma > 0.$$

In the context of the Ramsey model, this functional form is needed for the economy to converge to a balanced growth path. It is known as constant-relative-risk-aversion because the coefficient of relative risk aversion,

$$-c\frac{u''(c)}{u'(c)}, \tag{3.3}$$

is σ, and thus is independent of c.

Since there is no uncertainty in this model, the household's attitude toward risk is not directly relevant. But σ also determines the household's willingness to shift consumption between different periods. When σ is smaller, marginal utility falls more slowly as consumption rises, and so the household is more willing to allow its consumption to vary over time. If σ is close to zero, for example, utility is almost linear in c, and so the household is willing to accept large swings in consumption to take advantage of small differences between the discount rate and the rate of return on savings. It is for this reason that σ is also known as the inverse of the elasticity of substitution.

Two additional features of the CRRA utility function are worth mentioning. First, $c^{1-\sigma}$ is increasing in c if $\sigma < 1$ but decreasing if $\sigma > 1$; dividing $c^{1-\sigma}$ by $1-\sigma$ thus ensures that the marginal utility of consumption is positive regardless of the value of σ. Secondly, in the special case of $\sigma \to 1$, the CRRA utility function simplifies to log utility, $\ln c$; this is often a useful case to consider.

3.3 Firms

Firms' behaviour is relatively simple in the Ramsey model. At each point in time they employ the stocks of labour and capital, pay them their marginal products, and sell the resulting output. Because the production function has constant returns to scale (CRS) and the economy is competitive, firms earn zero profits – as previously stated. Thus, the problem of the firm is:

$$\max_{k(t),l(t),y(t)} \Pi(t) = y(t) - w(t)l(t) - r(t)k(t), \tag{3.4}$$

subject to

$$y(t) = F(k(t), l(t)), \tag{3.5}$$

$$\lim_{k(t)\to\infty} F_k(k(t), l(t)) = 0, \tag{3.6}$$

where (3.6) is basically to ensure that the firms' production technology satisfies the Inada conditions: assumptions about the shape of the production functions which guarantee the stability of a growth path.[2]

3.4 Equilibrium

Equilibrium is a sequence of wages and rental rates $\{w(t), r(t)\}$ and allocations $\{k(t), c(t), I(t)\}$ which satisfy optimality and market clearing. The firm's first order conditions are as usual, and thus the real interest rate and what is paid to capital at time t is:

$$r(t) = F_k(k(t), l(t)),$$

and labour's marginal product is:

$$w(t) = F_l(k(t), l(t)).$$

To solve the household problem, we set up the Hamiltonian[3]:

$$\mathcal{H} = u(c(t)) \exp(-\rho t) + \lambda(t)(w(t)l + (r(t) - \delta)k(t) + \Pi(t) - c(t)), \tag{3.7}$$

and taking the derivative of the Hamiltonian with respect to the control variable $c(t)$ and setting it equal to zero gives the consumption FOC:

$$\mathcal{H}_{c(t)} = u'(c(t)) \exp(-\rho t) - \lambda(t) = 0, \tag{3.8}$$

2. Formally, we want $F(0,0) = 0$, $F(\cdot)$ is concave, $\lim_{k(t)\to 0} F_k(k(t), l(t)) = \infty$, and the condition in (3.6). Thankfully, Cobb-Douglas production technology satisfies these conditions.

3. We could have equivalently set up a current-value Hamiltonian:

$$\bar{\mathcal{H}} = u(c(t)) + \lambda(t)(w(t)l + (r(t) - \delta)k(t) + \Pi(t) - c(t)),$$

which would have given us the following FOCs:

$$\bar{\mathcal{H}}_{c(t)} = u'(c(t)) - \lambda(t) = 0,$$

$$\bar{\mathcal{H}}_{k(t)} = \lambda(t)(r(t) - \delta) = \rho\lambda(t) - \dot{\lambda}(t),$$

and the transversality/"no-Ponzi" condition:

$$\lim_{t\to\infty} \exp(-\rho t)\lambda(t)k(t) = 0$$

and differentiating the Hamiltonian with respect to the state variable $k(t)$, using our rules for differentiating Hamiltonian functions, gives:

$$\mathcal{H}_{k(t)} = \lambda(t)(r(t) - \delta) = -\dot{\lambda}(t), \tag{3.9}$$

and where we have the "no-Ponzi"/transversality condition:

$$\lim_{t\to\infty} \exp\left(\int_0^t (\delta - r_\tau) d\tau\right) k(t) = 0. \tag{3.10}$$

There are now seven equations in seven unknowns $\{r(t), w(t), k^f(t), k^h(t), c(t), l(t), \lambda(t)\}$ where $k^f(t)$ is the capital stock that solves the firm problem and $k^h(t)$ is the capital stock that solves the household problem. We summarise the equations which constitute equilibrium below:

$$F_K(k^f(t), l(t)) = r(t), \tag{3.11}$$
$$F_L(k^f(t), l(t)) = w(t), \tag{3.12}$$
$$u'(c(t)) \exp(-\rho t) = \lambda(t), \tag{3.13}$$
$$\lambda(t)(r(t) - \delta) = -\dot{\lambda}(t), \tag{3.14}$$
$$w(t)l + (r(t) - \delta)k(t) + \Pi(t) - c(t) = \dot{k}^h(t), \tag{3.15}$$
$$l(t) = l, \tag{3.16}$$
$$k^f(t) = k^h(t). \tag{3.17}$$

Let's to try to solve for equilibrium by first starting with the household's problem. From (3.8) we know:

$$\lambda(t) = u'(c(t)) \exp(-\rho t),$$

and if we differentiate (remember to use product rule here) this with respect to t, we get

$$\frac{d\lambda(t)}{dt} = \dot{\lambda}(t) = u''(c(t))\dot{c}(t) \exp(-\rho t) - \rho u'(c(t)) \exp(-\rho t).$$
$$\implies \dot{c}(t) = \frac{\dot{\lambda}(t) + \rho u'(c(t)) \exp(-\rho t)}{u''(c(t)) \exp(-\rho t)}$$

Now, substitute our value for $\dot{\lambda}(t)$ from (3.9) (be careful of the minus sign):

$$\begin{aligned}
\dot{c}(t) &= \frac{\rho u'(c(t)) \exp(-\rho t) - u'(c(t)) \exp(-\rho t)(r(t) - \delta)}{u''(c(t)) \exp(-\rho t)} \\
&= \frac{\rho u'(c(t)) - u'(c(t))(r(t) - \delta)}{u''(c(t))} \\
&= \frac{u'(c(t)) [\rho - r(t) + \delta]}{u''(c(t))},
\end{aligned}$$

and then divide both the LHS and RHS by $c(t)$ to get:

$$\frac{\dot{c}(t)}{c(t)} = \frac{u'(c(t))}{u''(c(t))c(t)} \left[\rho - r(t) + \delta\right] = \frac{1}{\sigma(c(t))} \left[r(t) - \rho - \delta\right], \tag{3.18}$$

where

$$\sigma(c(t)) = -c(t)\frac{u''(c(t)}{u'(c(t))}.$$

(3.18) is nothing but the familiar Euler equation, simply written in continuous time, and $\sigma(c(t))$ is the definition of the coefficient of relative risk aversion given in (3.3)! It states that the rate of consumption growth is higher if the rate of interest is high, if depreciation is low, or if the discount factor is low. The intuition for these effects is the same as we saw in the simple discrete time general equilibrium/Robinson Crusoe-like model in the first section. Note that $\sigma(c(t))$ depends only on the first and second derivatives of the utility function. The CRRA form of utility is particular neat here as $\sigma(c(t))$ is constant and independent of consumption.

Under CRRA preferences, the equilibrium conditions become:

$$\dot{c}(t) = \frac{F_K(k(t), l) - \rho - \delta}{\sigma} c(t), \tag{3.19}$$

$$\dot{k}(t) = F(k(t), l) - c(t) - \delta k(t), \tag{3.20}$$

$$\lim_{t\to\infty} \exp\left(\int_0^\infty (\delta - F_K(k(\tau), l)d\tau\right) k(t) = 0. \tag{3.21}$$

3.5 The dynamics of the economy

To make further progress with the Ramsey model, we will need to consider the steady-state. The steady-state is the point at which $\dot{c}(t) = \dot{k}(t) = 0$. Looking at our equilibrium conditions at the steady-state we have:

$$0 = \frac{F_K(k, l) - \rho - \delta}{\sigma} c, \tag{3.22}$$

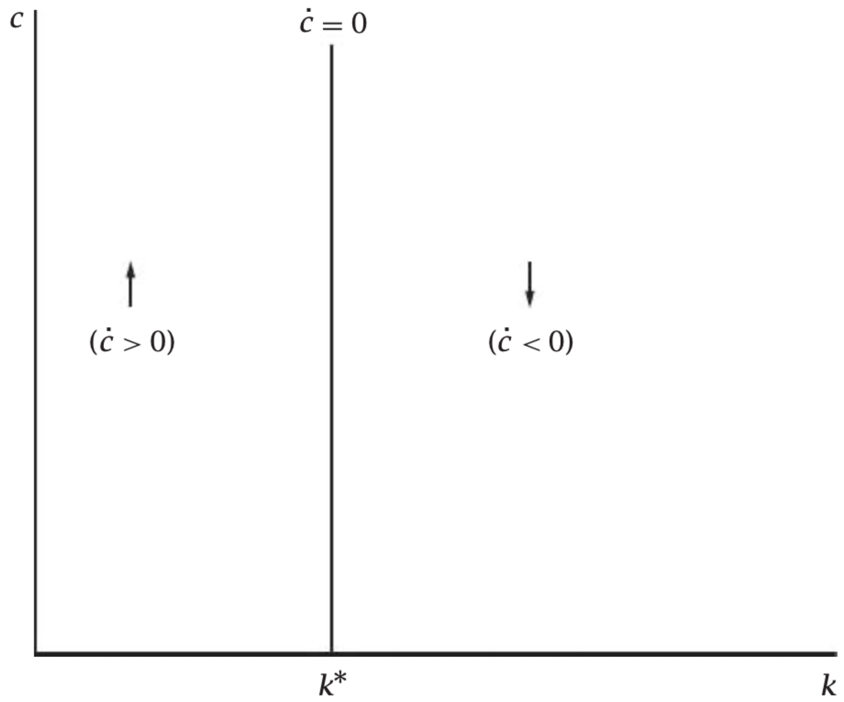
$$0 = F(k, l) - c - \delta k, \tag{3.23}$$

and note that we have gotten rid of all time-subscripts. This is common notation in macroeconomics, because the non-time denoted variables are referred to as steady-state equilibrium values.[4] Now, we know that in order for (3.22) to hold,[5] it must be that $F_K(k, l) = r = \rho + \delta$. Let k^* denote this level of k. When k exceeds k^*, $F_K(k, l)$ is less than $\rho + \delta$, and so $\dot{c}$ is negative; when k is lower than k^*, $F_K(k, l)$ is more than $\rho + \delta$, and so $\dot{c}$ is positive. Figure 3.1 illustrates these dynamics.

4. Or barred variables, such as $\bar{x}$ being the steady-state value of x_t.

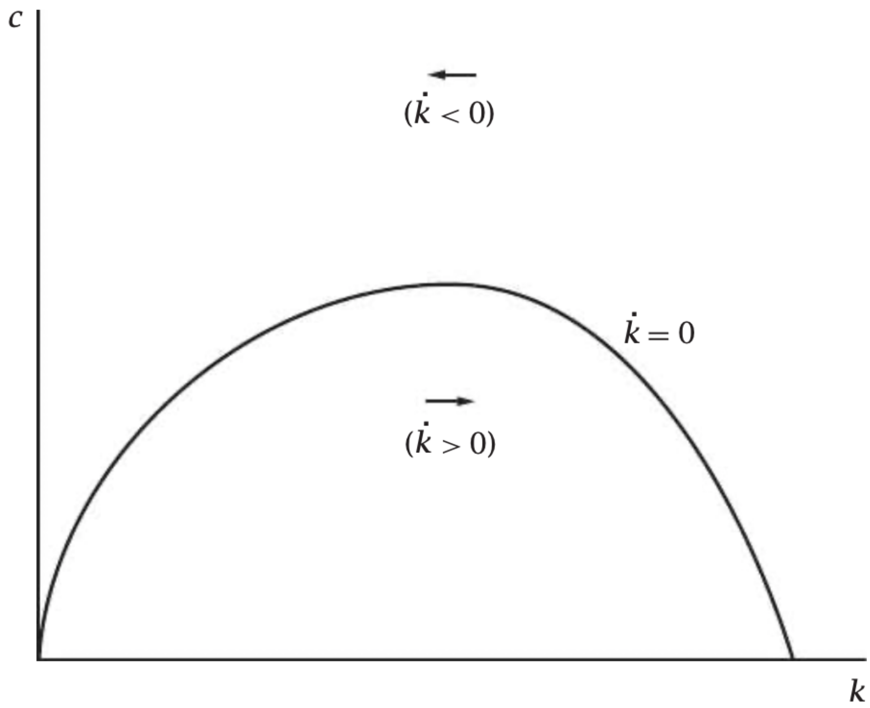
5. Implicitly we are assuming that $c \neq 0$. Well, $c = 0$ is mathematically feasible, but economically of no interest to us as our economy would be in a situation with no consumption and no capital.

Figure 3.1: Dynamics of c



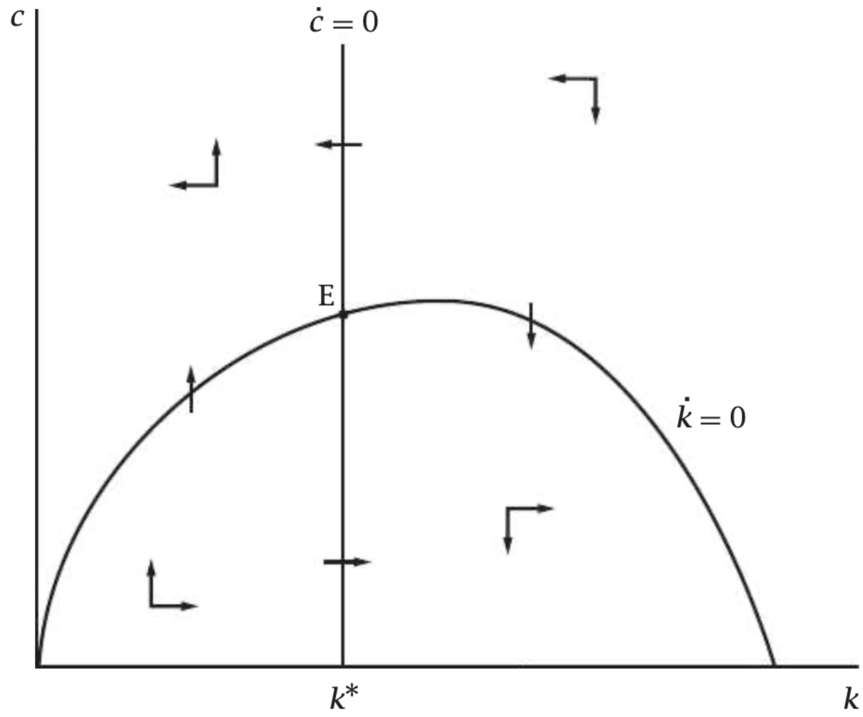
As in the Solow-Swan model, $\dot{k}$ equals actual investment minus break-even investment. Here, without population growth, actual investment is output minus consumption, and break-even investment is simply δk. From (3.23), $\dot{k}$ is zero when consumption equals the difference between the actual output and break-even investment lines when you plot a simple Solow-Swan diagram like as in Figure 2.1. This value of c is increasing in k until $F_K(k, l) = \delta$ (the golden-rule level of k) and is then decreasing. When c exceeds the level that yields $\dot{k} = 0$, k is falling; when c is lower than this level, k is rising. For k sufficiently large, break-even investment exceeds total output, and so $\dot{k}$ is negative for all positive values of c. Figure 3.2 plots these dynamics.

Figure 3.2: Dynamics of k



Now if we combine the information from Figures 3.1 and 3.2, we can illustrate a phase diagram which describe the convergence-to-equilibrium dynamics of the Ramsey model. This is done in Figure 3.3.

Figure 3.3: Dynamics of c and k



Notice that Figure 3.3 is drawn with k^* (the level of k that implies $\dot{c} = 0$) less than the golden-rule level of k. To see that this must be the case, recall that k^* is defined by $F_K(k^*, l) = \rho + \delta$, and that the golden-rule k is defined by $F_K(k^{GR}, l) = \delta$. Since $F_{KK}(k, l) < 0$, k^* is less than k^{GR} if $\rho + \delta$ is greater than δ, which it obviously is. This is equivalent to arguing that $\rho > 0$, which is a key assumption we make to ensure that lifetime utility does not diverge.[6]

3.5.1 The saddle path

Figure 3.4 shows how c and k evolve over time to satisfy households' intertemporal Euler equation and the law of motion of capital for various given initial values of c and k. We can trace out the paths for points such as A, B, C, and D using the phase diagram, where we see the economy trail off to the upper-left and down-right quadrants. But we notice there is some critical point between C and D – point F in the diagram – such that at that level of initial c, the economy converges to the stable point, point E. We can suppose that for any positive initial level of k, there is a unique initial level of c that is consistent with the Euler equation, the law of motion of capital, households' budget constraint, and the requirement that k not be negative. The function giving this initial c as a function of k is known as the saddle path; and is shown in Figure 3.5.

6. In a model with population growth and no depreciation, this condition is $\rho - n - (1 - \sigma)g > 0$, where n is the population growth rate.

Figure 3.4: The Behaviour of c and k for Various Initial Values of c

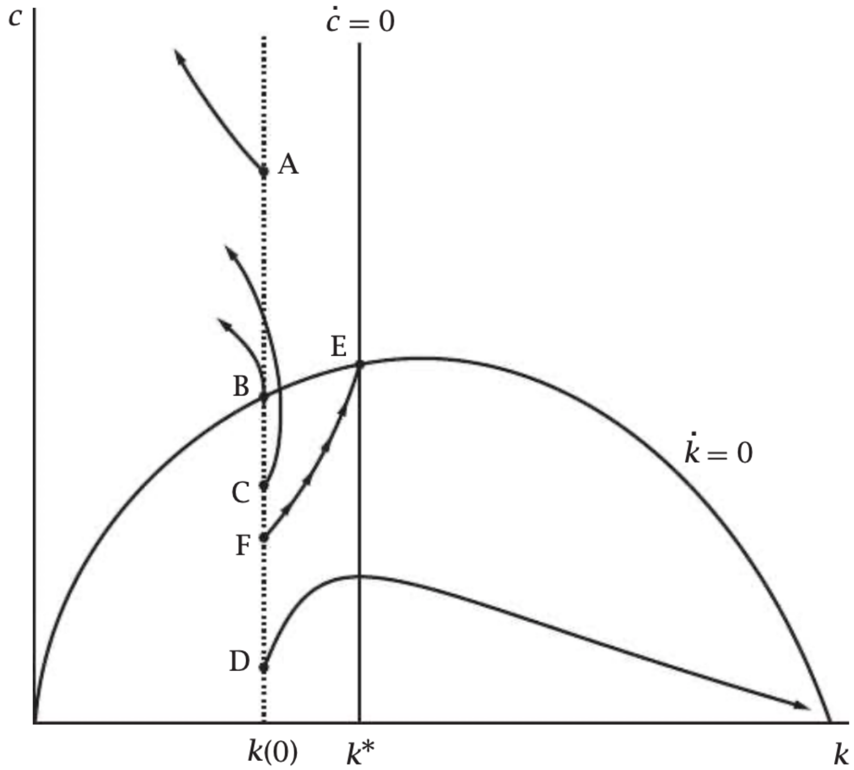
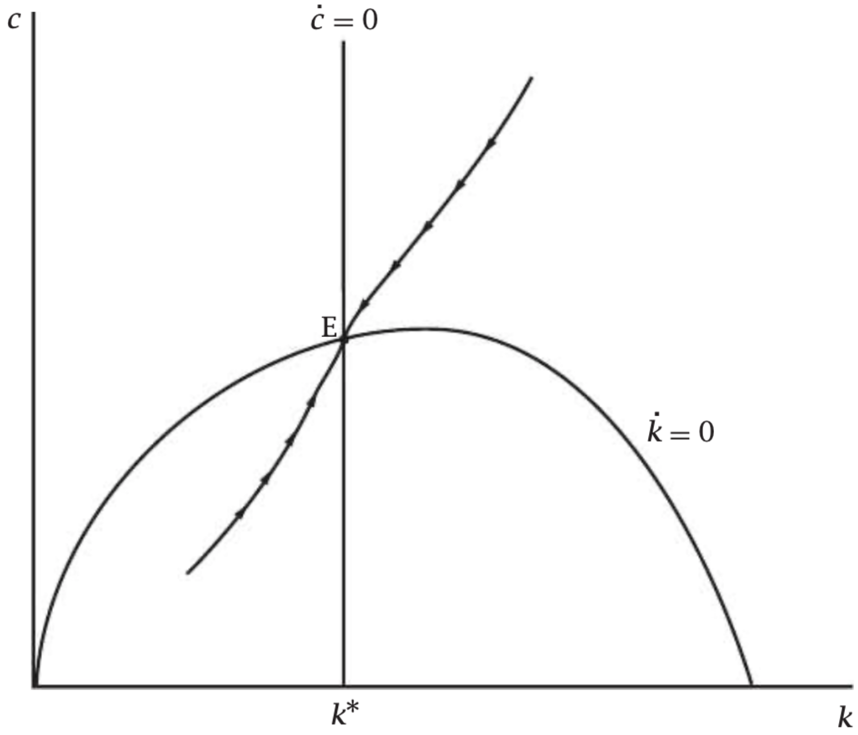


Figure 3.5: The Saddle Path



3.6 Adding government to the model

Thus far, we have left government out of our model. Yet, modern economies devote their resource not just to investment and private consumption but also to public uses. It is thus natural to extend our model to include a government sector.

Assume that the government buys output at rate $G(t)$, and that government purchases are assumed not to affect utility from private consumption; this can occur if the government devotes the goods to some activity that does not affect utility at all, or if utility equals the sum of utility from private consumption and utility from government provided goods. The purchases are financed by lump-sum taxes $T(t)$ which directly equal $G(t)$; thus the government always runs a balanced budget. Assume that the representative that the representative agent has preferences given by

$$\int_0^\infty \exp(-\rho t)\left[\ln C(t) + \ln G(t)\right] dt,$$

where ρ is the discount factor, $C(t)$ is private consumption, and $G(t)$ is government spending. Production in this economy is given by a constant returns to scale production technology satisfying the Inada conditions. Households supply an exogenous amount of labour in every period. There is no population growth, no technological change, and capital depreciates at the rate $\delta > 0$. Suppose the economy is initially at the steady-state and subsequently the government increases spending to a permanently higher level.

How will this permanent increase in government spending affect the new steady-state level of output? Begin by assuming competitive markets, which allows to write the representative household's problem as

$$\max_{\{C(t)\}} \int_0^\infty \exp(-\rho t)[\ln C(t) + \ln G(t)]dt,$$

subject to

$$\dot{K}(t) = w(t)L(t) + r(t)K(t) + \Pi(t) - C(t) - \delta K(t) - T(t),$$

and so our present-value Hamiltonian is

$$\mathcal{H} = \exp(-\rho t)\left\{\ln C(t) + \ln G(t)\right\} + \lambda(t)\left[w(t)L(t) + r(t)K(t) + \Pi(t) - C(t) - \delta K(t) - T(t)\right],$$

where labour is standardised to unity so $L(t) = 1, \forall t$, $r(t)$ is the return on capital, $\Pi(t)$ is firms' profits (and are zero due to perfect competition). As stated, government expenditure does not affect utility from private consumption; utility is additively separable between private consumption and government expenditure. Therefore, we can take the derivative of the Hamiltonian with respect to the control variable, $C(t)$, to yield our first FOC:

$$\mathcal{H}_{C(t)} = \frac{\exp(-\rho t)}{C(t)} = \lambda(t), \tag{3.24}$$

and taking the derivative with respect to the state variable $K(t)$ yields our second FOC:

$$\mathcal{H}_{K(t)} = \lambda(t)(r(t) - \delta) = -\dot{\lambda}_t \tag{3.25}$$

Differentiating (3.24) with respect to time yields

$$\dot{\lambda}(t) = \frac{-\rho\exp(-\rho t)C(t) - \dot{C}(t)\exp(-\rho t)}{C(t)^2},$$

and we can combine this with (3.25) to yield

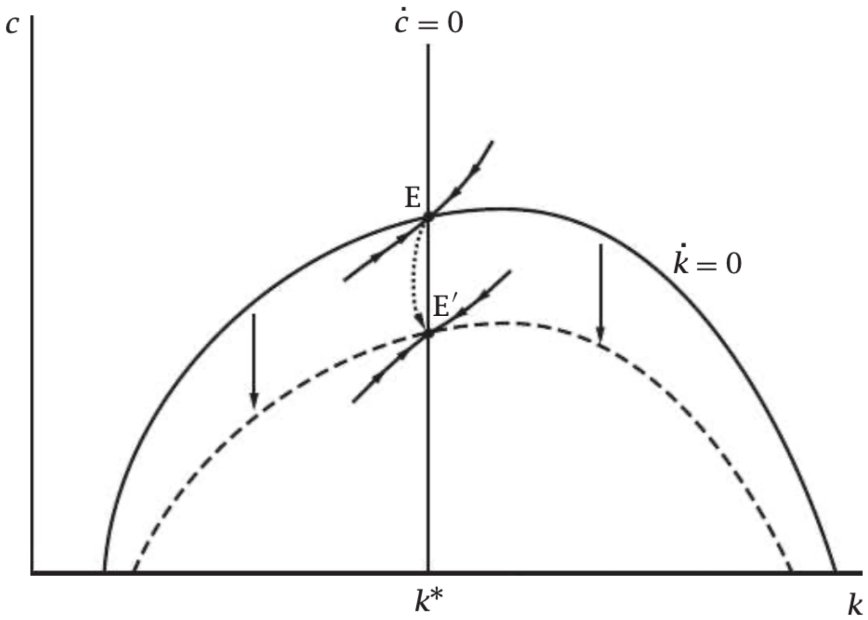
$$\begin{aligned}
\lambda(t)(\delta - r(t)) &= \frac{-\rho\exp(-\rho t)C(t) - \dot{C}(t)\exp(-\rho t)}{C(t)^2} \\
\Leftrightarrow \frac{\exp(-\rho t)}{C(t)}(\delta - r(t)) &= \frac{-\rho\exp(-\rho t)C(t) - \dot{C}(t)\exp(-\rho t)}{C(t)^2} \\
(\delta - r(t)) &= \frac{-\rho C(t)^2}{C(t)^2} - \frac{C(t)\dot{C}(t)}{C(t)^2} \\
\delta - r(t) + \rho &= -\frac{\dot{C}(t)}{C(t)} \\
\implies \dot{C}(t) &= C(t)(r(t) - \rho - \delta).
\end{aligned}$$

Set $\dot{C}(t) = 0$ and our law of motion equation $\dot{K}(t) = 0$, and we have two equations describing equilibrium dynamics[7]:

$$\dot{C}(t) = C(t)(F_K(K(t), L) - \delta - \rho) = 0,$$
$$\dot{K}(t) = F(K(t), L) - C(t) - \delta K(t) - T(t) = 0.$$

These dynamics are identical to an economy where government expenditures do not enter the household's utility at all. Thus, any increases in $G(t)$ will simply be funded by an increase in $T(t)$ and a downward shift in the locus of equilibria dictated by $\dot{K}(t) = 0$. That is, the economy will undergo a crowding out affect and household private consumption will fall instantly by the same amount as $G(t)$ increases. The dynamics are illustrated in Figure 3.6.

Figure 3.6: The Effects of a Permanent Increase in Government Purchases



In words: we know that in response to a permanent and surprise increase in government spending, C must jump so that the economy is on its new saddle path. If not, then as before, either capital would become negative at some point or households would accumulate infinite wealth. In this case, the adjust takes a simple form: C falls by the amount of the increase in G, and the economy is immediately on its new balanced growth path. Households do not have an opportunity to smooth their consumption moving from the initial equilibrium E to E' due to the unanticipated increase in G. Intuitively, the permanent increases in government purchases and taxes reduce households' lifetime wealth (there's a crowding out effect). And because the increases in purchases

7. Here I simply rewrite the law of motion of capital as being total output minus consumption (which is actual investment), and minus break-even investment $\delta K(t)$ minus taxes $T(t)$.

and taxes are permanent, there is no scope for households to raise their utility by adjusting the time pattern of their consumption. Thus the size of the immediate fall in consumption is equal to the full amount of the increase in government purchases, and the capital stock and the real interest rate are unaffected.

3.7 Considering technological progress in the Ramsey model

The baseline Ramsey model we developed had a time invariant steady-state level. In order to match the Kaldor stylised facts, we need the steady-state to constantly shift and grow. How would we do this? We could assume that the production technology features a labour augmenting technology term such as:

$$Y(t) = F(K(t), A(t)L),$$

where A grows exogenously by the rate γ:

$$\frac{\dot{A}(t)}{A(t)} = \gamma.$$

How would our new steady-state convergence equations look like? Under CRRA preferences, our equilibrium conditions are:

$$\begin{aligned}
\frac{\dot{C}(t)}{C(t)} &= \frac{F_K(K(t), A(t)L) - \delta - \rho}{\sigma}, \\
\dot{K}(t) &= F(K(t), A(t)L) - C(t) - \delta K(t) - T(t), \\
\lim_{t\to\infty} &= \exp\left(\int_0^\infty (\delta - F_K(K(\tau), A(t)L))d\tau\right) K(t) = 0,
\end{aligned}$$

and consider the following transformation of our variables:

$$\begin{aligned}
\tilde{K}(t) &= \frac{K(t)}{A(t)}, \\
\implies \dot{K}(t) &= \dot{A}(t)\tilde{K}(t) + A(t)\dot{\tilde{K}}(t), \\
\implies F_L(K(t), A(t)L) &= A(t)F_L(\tilde{K}(t), L),
\end{aligned}$$

and

$$\begin{aligned}
\tilde{C}(t) &= \frac{C(t)}{A(t)}, \\
\implies \dot{C}(t) &= \dot{A}(t)\tilde{C}(t) + A(t)\dot{\tilde{C}}(t), \\
\implies F_K(K(t), A(t)L) &= F_K(\tilde{K}(t), L).
\end{aligned}$$

Using our transformed variables, equilibrium is now

$$\frac{\dot{\tilde{C}}(t)}{\tilde{C}(t)} = \frac{F_K(\tilde{K}(t), L) - \sigma\gamma - \delta\rho}{\sigma}, \tag{3.26}$$

$$\dot{\tilde{K}}(t) = F(\tilde{K}(t), L) - \tilde{C}(t) - (\delta + \gamma)\tilde{K}(t). \tag{3.27}$$

With this tweak, our Ramsey model now manages to reproduce the Kaldor stylised facts of persistent growth. Our model exhibits transitional dynamics by converging to a balanced growth path which is constantly growing at rate γ. Success!

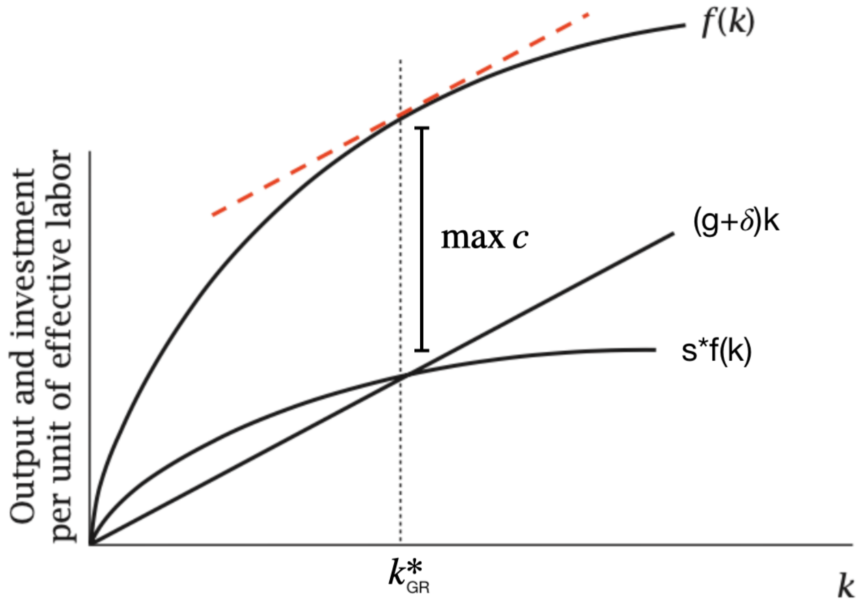
But this was done in a pretty ad-hoc way... So maybe it's too early to celebrate.

3.8 The social optimum and the golden-rule level of capital

One thing worth mentioning about the Ramsey model is a note on dynamic efficiency. Throughout this section, we've been comparing the Ramsey model to the Solow-Swan model – and for obvious reasons; the two models are very similar. The primary difference between the two models is that in the Solow-Swan model saving in each period is some fixed proportion of total output, s, whereas in the Ramsey model savings/investment fluctuates depending on the capital stock. Recall your undergraduate macroeconomics and look at Figure 3.7, which illustrates a savings rate, s^*, which coincides with the "golden rule" of investment: the level of investment which permits the highest level of consumption and golden rule level of capital, k^*_{GR} in the Solow-Swan model. Savings levels above or below s^* are of course possible in the Solow-Swan model, and the model does provide for a balanced growth path which will converge to a steady-state k that is either above or below k^*_{GR}. That is to say that the Solow-Swan model may be dynamically inefficient.[8]

8. An economy is said to be dynamically inefficient if it is saving too much.

Figure 3.7: STEADY-STATE (GOLDEN RULE) IN THE SOLOW-SWAN MODEL



Now consider the plot of the convergent dynamics of the Ramsey model in Figure 3.5. In the Ramsey model, it is not possible to be on the balanced growth path with a capital stock above k^*. This is because, as we now know, savings in the Ramsey model is derived from the behaviour of households whose utility depends on their consumption, and there are no externalities. As a result, it cannot be an equilibrium for the economy to follow a path where higher consumption can be attained in every period; if the economy were on such a path, households would reduce their saving and take advantage of this opportunity. Also, remember that k^* in the Ramsey model is lower than the golden rule level of capital corresponding to k^*_{GR} in the Solow-Swan model.

If the initial capital stock exceeds the golden rule of level of capital (if $k(0)$ is greater than the k associated with the peak of the $\dot{k} = 0$ locus), initial consumption is above the level needed to keep k constant; thus $\dot{k}$ is negative, and k gradually approaches k^*, which is below the golden-rule level. Finally the fact that k^* is less than the golden-rule capital stock implies that the economy does not converge to the balanced growth path that yields the maximum sustainable level of c.

We saw the intuition for this result earlier, but just to repeat: Consider the baseline Ramsey model with no technological growth. In this case, k^* is defined by $F_K(k^*, l) = \rho + \delta$ and k^{GR} is defined by $F_K(k^{GR}, l) = \delta$. A condition for convergence is that $\rho + \delta < \delta \Leftrightarrow \rho < 0$. Since $k^* < k^{GR}$, an increase in saving starting at $k = k^*$ would cause consumption per worker to eventually rise above its previous level and remain there. But because households value present consumption more than future consumption, the benefit of the eventual permanent increase in consumption is bounded. At some point – specifically, when k exceeds k^* – the tradeoff between the temporary short-term sacrifice and the permanent long-term gain is sufficiently unfavourable that accepting it reduces rather than raises lifetime utility. Thus k converges to a value below the golden-rule level. Because k^* is the optimal level of k for the economy to converge to, it is known as the "modified golden-rule capital stock".

3.9 Comments and key readings

The Ramsey model obviously isn't very realistic. For one, the assumption that agents have perfect foresight is pretty hard to reconcile. But it offers a very important foundation for when we move onto models in a stochastic setting, where agents have to form rational expectations about the future based on information known today (the real business cycle model).

Key readings for the Ramsey model are Blanchard and Fischer (1989), McCandless (2008), and Romer (2012).

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4 Endogenous Growth

4.1 Introduction

The neoclassical models presented in the previous sections takes the rate of technological change as being determined exogenously. There is good reason, however, to believe that technological change depends on economic decisions, because it comes from industrial innovations made by profit-seeking firms, and depends on the funding of science, the accumulation of human capital, and other such activities. What we want to do is endogenise growth in our macroeconomic model (a concept we will frequently revisit), so that it is determined within the model and not just exogenously assumed.

Incorporating endogenous technology into growth theory forces us to deal with the difficult phenomenon of increasing returns to scale. More specifically, people must be given an incentive to improve technology. But because the aggregate production function F exhibits constant returns to scale (CRS) in K and L alone, Euler's Theorem tells us that it will take all of the economy's output to pay capital and labour their marginal products in producing final output, leaving nothing over to pay for the resources used in improving technology.[1] Thus, a theory of endogenous technology cannot be based on the usual theory of competitive equilibrium, which requires that all factors be paid their marginal products.

We will develop two approaches to tackle this challenge: the AK model and the varieties model.

4.2 The AK model based on Arrow (1962) and Frankel (1962)

Arrow's (1962) approach to endogenous growth was to propose that technological progress is an unintended consequence of producing new capital goods, a phenomenon dubbed "learning by doing". Learning by doing was assumed to be purely external to the firms responsible for it. That is, if technological progress depends on the aggregate production of capital, and firms are all very small, then they can be assumed all to take the rate of technological progress as being given independently of their own production of capital goods. So each firm maximises profit by paying K and L their marginal products, without offering any additional payment for their contribution to technological progress.

Learning by doing formed the basis of the first model of endogenous growth theory, which is known as the AK model. The AK model assumes that when people accumulate capital, learning

1. Euler's Theorem states that if F is homogeneous of degree 1 (HOD1) in K and L (the definition of CRS) then:

$$F_K(K, L)K + F_L(K, L)L = F(K, L). \tag{4.1}$$

The marginal products of K and L are F_K and F_L, respectively. So if K and L are paid their marginal products then the LHS is the total payment to capital plus the total payment to labour, and the equation states that these payments add up to total output. To verify Euler's Theorem take the equation

$$F(\lambda K, \lambda L) = \lambda F(K, L), \tag{4.2}$$

which defines HOD1, and differentiate both sides with respect to λ at the point $\lambda = 1$. Since the (4.2) must hold for all $\lambda > 0$, the two derivatives must be equal, which implies (4.1).

by doing generates technological progress that tends to raise the marginal product of capital, thus offsetting the tendency for the marginal product to diminish when technology is unchanged. The model results in a production function of the form $Y = AK$, in which the marginal product of capital is equal to the constant A.

The AK model predicts that a country's long-run growth rate will depend on economic factors such as thrift and the efficiency of resource allocation. In subsequent sections we will develop alternative models of endogenous growth that emphasise not thrift and efficiency, but creativity and innovation, which we see as the main driving forces behind economic growth. But given its historical place as the first endogenous growth model, the AK paradigm is an important part of any economist's toolkit. Accordingly, we devote this section to developing the AK model.

We begin by assuming that firm output is given by

$$F(K_t, A) = AK_t. \tag{4.3}$$

Fundamental to the AK model is that K embodies both physical and human capital, and thus firms' production technology is a special case of the Cobb-Douglas production function $Y_t = AK_t^\alpha L_t^{1-\alpha}$ where $\alpha = 1$, and hence the model's name. Since $\alpha = 1$, the model relies on constant returns to scale production. The fact that the return on capital is now a constant, A, eliminates any potential for there being transition dynamics.

Households in the AK model behave as they do in the Ramsey model or Solow-Swan model, choosing consumption to maximise the present discounted valued of utility subject to a budget constraint a transversality condition. Households maximise their present discounted value of utility

$$\max_{\{C_t\}} \int_0^\infty \exp(-\rho t) u(C_t) dt,$$

subject to a budget constraint and a transversality condition. The optimality condition is from the now familiar consumption Euler equation. With CRRA preferences this is:

$$\frac{\dot{C}_t}{C_t} = \frac{r_t - \rho - \delta}{\sigma}, \tag{4.4}$$

Since the firms' production technology is given by (4.3), their profit maximisation problem can be written as:

$$\arg\max_{K_t} \; AK_t - rK_t,$$

where the rental rate of capital is given by:

$$r_t = F_K(A_t, A) = A.$$

Thus, the law of motion for capital is

$$\dot{K}_t = AK_t - C_t - \delta K_t. \tag{4.5}$$

The model does not have a steady-state but it has a balanced growth path:

$$\frac{\dot{C}_t}{C_t} = \frac{A - \rho - \delta}{\sigma} = g, \tag{4.6}$$

$$\frac{\dot{K}_t}{K_t} = \underbrace{A - \frac{C_t}{K_t}}_{sA} - \delta = g \tag{4.7}$$

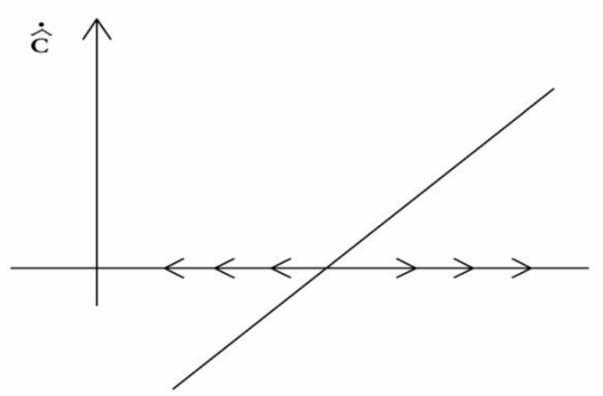
$$\implies \frac{C_t}{K_t} = A - \delta - g. \tag{4.8}$$

Putting things together, assuming log utility for simplicity, and defining $\tilde{C}_t = \frac{C_t}{K_t}$, we then have:

$$\begin{aligned} \frac{\dot{\tilde{C}}_t}{\tilde{C}_t} &= \frac{K_t\dot{C}_t - C_t\dot{K}_t}{K_t^2}\frac{K_t}{C_t} \\ &= \frac{\dot{C}_t}{C_t} - \frac{\dot{K}_t}{K_t} \\ &= A - \rho - \delta\left(A - \frac{C_t}{K_t} - \delta\right) \\ &= \tilde{C}_t - \rho. \end{aligned} \tag{4.9}$$

When we plot $\dot{\tilde{C}}_t/\tilde{C}_t$ as a function of $\tilde{C}_t$, we see that the relationship is upward sloping. Hence there are no transitional dynamics and economy jumps directly to the balanced growth path, as shown in Figure 4.1.

Figure 4.1: Dynamics in the AK Model



4.2.1 Microfoundations of the AK model

The most straightforward microfoundation for the AK model is the idea is that knowledge accumulates through accumulating capital:

$$\begin{aligned} Y_t &= F(K_t, A_t L_t), \\ A_t &= \psi \frac{K_{t-1}}{L_{t-1}}. \end{aligned}$$

Implicit in this is an assumption there is an externality in capital accumulation. Knowledge advances at the aggregate level as the capital stock increases, but the effect for an individual firm is so small that they do no internalise it in their own (firm-specific) investment decisions. With a CRS production function:

$$\begin{aligned} \frac{Y_t}{L_t} &= \frac{1}{L_t} F(K_t, A_t L_t) \\ &= \frac{K_t}{L_t} F\left(1, \frac{A_t L_t}{K_t}\right) \\ &= \frac{K_t}{L_t} F\left(1, \psi \frac{K_{t-1} L_t}{L_{t-1} K_t}\right) \\ &\approx \frac{K_t}{L_t} F\left(1, \psi \frac{1}{1+g}\right), \end{aligned} \tag{4.10}$$

which is of the desired AK form.

A different argument is based on human capital, assuming that it accumulates over time and plays a part in production. Production depends on physical and human capital, the latter of which can be increased through investment. The structural equations of this one-sector model are therefore:

$$Y_t = F(K_t, H_t), \tag{4.11}$$

$$Y_t = C_t + I_t^K + I_t^H, \tag{4.12}$$

$$\dot{K}_t = I_t^K - \delta K_t, \tag{4.13}$$

$$\dot{H}_t = I_t^H - \delta H_t. \tag{4.14}$$

None of that should be too controversial, so let's go ahead and set up the present value Hamiltonian of the household problem:

$$\mathcal{H} = u(C_t)\exp(-\rho t) + \lambda_{1,t}(F(K_t, H_t) - C_t - I_t^K - I_t^H) + \lambda_{2,t}(I_t^K - \delta K_t) + \lambda_{3,t}(I_t^H - \delta H_t),$$

and this problem has the following FOCs:

$$u'(C_t)\exp(-\rho t) = \lambda_{1,t}, \tag{4.15}$$

$$\lambda_{1,t} = \lambda_{2,t}, \tag{4.16}$$

$$\lambda_{1,t} = \lambda_{3,t}, \tag{4.17}$$

$$\lambda_{1,t} F_K(K_t, H_t) - \lambda_{2,t}\delta = -\dot{\lambda}_{2,t}, \tag{4.18}$$

$$\lambda_{1,t} F_H(K_t, H_t) - \lambda_{3,t}\delta = -\dot{\lambda}_{3,t}. \tag{4.19}$$

These conditions imply that $\lambda_{2,t} = \lambda_{3,t}$ so the marginal products of physical and human capital are equalised and $F_k(K_t, H_t) = F_H(K_t, H_t)$. It follows that K_t/H_t is constant because of the HOD1 of $F(\cdot)$. With a constant K_t/H_t the production function works like in the AK model. More interesting would be a two-sector model in which the first sector products output (for consumption o investment) and the second sector produces human capital (e.g. schools). Then K_t/H_t becomes a state variable, and there can be transition dynamics.

4.2.2 Neoclassical models vs the AK model

In this section, we briefly reflect on a now closed debate between advocates of the neoclassical approach and those of the AK model.

The AK model can first account for persistent and positive growth rates of output (we could also show positive growth rates for output per capita) – something that we observe empirically, and a feature which neoclassical models cannot address. Where the AK model struggles is with its inability to explain convergence, which is another important empirical observation. Neoclassical models such as the Solow-Swan and Ramsey model predict that economies with lower GDP and capital per-capita levels undergo rapid growth initially, before plateauing out as the economy reaches its steady-state. The main mechanism for this convergence dynamic is the fact that production in neoclassical model had constant returns to scale in capital and labour together. In other words, the further an economy is away from its steady-state, the faster it grows as it has a relatively high marginal product of capital.

Now consider the case of two geographically proximate economies with similar characteristics (e.g. states in the US), such as similar population growth, savings, and depreciation rates. Suppose that one economy had lower GDP and capital per-capita than the other. Under the AK model and constant returns to scale, the two economies would never converge to one another as they grow at exogenous rates. Whereas under a neoclassical model, the two economies would tend toward their steady-states – which is something that we observe in the data.

So, to conclude: AK models can explain long-run growth, but can't say much for convergence. However, neoclassical models can explain convergence but do not have a convincing story for long-run technological growth. As I previously mentioned, the fact that the AK model lumps physical and human capital together under a catch-all capital term is the model's primary weakness.

4.3 The fixed varieties model

The objective here is to build a tractable model where changes in the production possibility set are endogenously determined as a response to economic incentives. The difficulty is that we need some sort of imperfection in the goods market if technological change is to be remunerated. Otherwise, if factors are paid their marginal product and there are constant returns to scale, this exhausts the output. We therefore use a monopolistic-competitive setup where: (i) final goods production is competitive but needs intermediate goods, and (ii) intermediate goods are produced by a monopolist.

4.3.1 Firms

We first setup the varieties model where the number of goods is fixed. Some of the setup we establish here will make a comeback later on when we look at the New Keynesian model – where we have final good products and intermediate goods. For now, let's assume that the production of final goods are:

$$Y_t = L_t^{1-\alpha} \sum_{i=1}^{N_i} X_{i,t}^{\alpha}, \tag{4.20}$$

where N_i is the number of intermediate goods. The production functions shows increasing returns to specialisation, as seen for the first time by Adam Smith in the pin factory. Firms that produce final goods maximise their profits:

$$\max_{L_t, \{X_{i,t}\}_{i=1}^{N_i}} Y_t - w_t L_t - \sum_{i=1}^{N_i} P_{i,t} X_{i,t},$$

subject to the production function (4.20). The Lagrangian for this problem is

$$\mathcal{L} = Y_t - w_t L_t - \sum_{i=1}^{N_i} P_{i,t} X_{i,t} + \lambda_t \left(Y_t - L_t^{1-\alpha} \sum_{i=1}^{N_i} X_{i,t}^{\alpha} \right),$$

giving the FOCs:

$$\begin{aligned}
\mathcal{L}_{L_t} &= -w_t - (1-\alpha) L_t^{-\alpha} \lambda_t \sum_{i=1}^{N_i} X_{i,t}^{\alpha} = 0, \\
\mathcal{L}_{X_{i,t}} &= -P_{i,t} - \alpha \lambda_t L_t^{1-\alpha} X_{i,t}^{\alpha-1} = 0, \\
\mathcal{L}_{Y_t} &= 1 + \lambda_t = 0, \\
\implies w_t &= (1-\alpha) L_t^{-\alpha} \sum_{i=1}^{N_i} X_{i,t}^{\alpha}, && (4.21) \\
\implies P_{i,t} &= \alpha L_t^{1-\alpha} X_{i,t}^{\alpha-1}. && (4.22)
\end{aligned}$$

Intermediate goods, which are slightly differentiated from one another, are produced using a simply technology that transforms one unit of the final good into one unit of $X_{i,t}$. The profit maximisation problem of the intermediate good monopolistically competitive firm is therefore:

$$\max_{P_{i,t}, X_{i,t}} \Pi_t = P_{i,t} X_{i,t} - y_{i,t},$$

subject to

$$\begin{aligned} y_{i,t} &= X_{i,t}, \\ P_{i,t} &= \alpha L_t^{1-\alpha} X_{i,t}^{\alpha-1}. \end{aligned}$$

The Lagrangian for intermediate firm i is:

$$\mathcal{L} = P_{i,t} X_{i,t} - X_{i,t} + \lambda_t \left(P_{i,t} - \alpha L_t^{1-\alpha} X_{i,t}^{\alpha-1} \right),$$

and the FOCs are:

$$\mathcal{L}_{P_{i,t}} = X_{i,t} + \lambda_t = 0, \tag{4.23}$$

$$\mathcal{L}_{X_{i,t}} = P_{i,t} - 1 - \lambda_t \alpha(\alpha - 1) L_t^{1-\alpha} X_{i,t}^{\alpha-2} = 0. \tag{4.24}$$

Combining the FOCs we have

$$\begin{aligned} 1 &= P_{i,t} + X_{i,t} \alpha(\alpha - 1) L_t^{1-\alpha} X_{i,t}^{\alpha-2} \\ &= P_{i,t} + \alpha(\alpha = 1) L_t^{1-\alpha} X_{i,t}^{\alpha-1} \\ &= P_{i,t} + \alpha^2 L_t^{1-\alpha} X_{i,t}^{\alpha-1} - \alpha L_t^{1-\alpha} X_{i,t}^{\alpha-1}, \end{aligned}$$

and since $P_{i,t} = \alpha L_t^{1-\alpha} X_{i,t}^{\alpha-1}$, we have:

$$\begin{aligned} 1 &= \alpha P_{i,t} \\ \implies P_{i,t} &= \frac{1}{\alpha}. \end{aligned} \tag{4.25}$$

Plugging this value for $P_{i,t}$ into one second constraint gives:

$$\begin{aligned} \frac{1}{\alpha} &= \alpha L_t^{1-\alpha} X_{i,t}^{\alpha-1} \\ \frac{1}{\alpha^2} &= L_t^{1-\alpha} X_{i,t}^{\alpha-1} \\ \frac{L_t^{\alpha-1}}{\alpha^2} &= X_{i,t}^{\alpha-1} \\ \implies X_{i,t} &= L_t \alpha^{\frac{2}{1-\alpha}}. \end{aligned} \tag{4.26}$$

4.3.2 Households

The household maximises the present discounted value of its utility from consumption in the standard way:

$$\max_{\{C_t, L_t, A_t\}} \int_0^\infty u(C_t)\exp(-\rho t)dt,$$

subject to

$$\begin{aligned} C_t + \dot{A}_t &= w_t L_t + r_t A_t + \Pi_t, \\ L_t &= \bar{L}, \end{aligned}$$

noting that profits $\Pi_t > 0$ in equilibrium because the intermediate producers are monopolistically competitive. The consumption Euler equation is standard:

$$\frac{\dot{C}_t}{C_t} = r_t - \rho.$$

4.3.3 Equilibrium

Equilibrium is a sequence of prices $\{P_{i,t}, w_t, r_t\}$ and a set of quantities $\{Y_t, \{X_{i,t}\}_{i=1}^{N_i}, C_t, A_t, L_t\}$ such that: Given prices, $Y_t, \{X_{i,t}\}_{i=1}^{N_i}, L_t$ solves the problem of final good firms; given perceived demand, $\{X_{i,t}\}_{i=1}^{N_i}$ solves the intermediate goods firms' problem; given prices, C_t, A_t solves the households problem; and, markets clear.

Putting everything together defines the equilibrium:

$$X_{i,t} = \bar{L}\alpha^{\frac{2}{1-\alpha}}, \tag{4.27}$$

$$P_{i,t} = \frac{1}{\alpha}, \tag{4.28}$$

$$Y_t = \bar{L}N_i\alpha^{\frac{2\alpha}{1-\alpha}}, \tag{4.29}$$

$$C_t = Y_t - N_i X_{i,t} = \bar{L}N_i\alpha^{\frac{2\alpha}{1-\alpha}} - N_i\bar{L}\alpha^{\frac{2}{1-\alpha}}, \tag{4.30}$$

$$A_t = 0, \tag{4.31}$$

$$r_t = \rho, \tag{4.32}$$

$$w_t = (1-\alpha)N_i\alpha^{\frac{2\alpha}{1-\alpha}}, \tag{4.33}$$

which is static and exhibits no growth.

4.4 Endogenous number of varieties

To get growth up and running we endogenise N_i, the number of varieties or intermediate goods. We allow monopolistically competitive firms to create a new variety a cost of η, where η is sufficiently small to ensure existence of equilibrium. The per-period profit of a new variety to the intermediate firm is:

$$\begin{aligned}
\Pi_{i,t} &= P_{i,t}X_{i,t} - X_{i,t} \\
&= \frac{1}{\alpha}\bar{L}\alpha^{\frac{2}{1-\alpha}} - \bar{L}\alpha^{\frac{2}{1-\alpha}} \\
&= \bar{L}\alpha^{\frac{2}{1-\alpha}-1} - \bar{L}\alpha^{\frac{2}{1-\alpha}} \\
\therefore \Pi_i &= \bar{L}\alpha^{\frac{2}{1-\alpha}}\left(\frac{1-\alpha}{\alpha}\right),
\end{aligned} \tag{4.34}$$

which in present value terms is

$$\int_0^\infty \Pi_i \exp(-rt)dt = \frac{\Pi_i}{r}.$$

The intermediate firm will bring new varieties of intermediate goods to the market as long as the present value of profits exceeds the entry cost, i.e., equilibrium requires $\Pi_i/r = \eta$ in which case consumption has to grow according to:

$$\frac{\dot{C}_t}{C_t} = \frac{\Pi_i}{\eta} - \rho \equiv g. \tag{4.35}$$

Can we build an equilibrium in which consumption, output, and the number of varieties grows at rate g? The resource constraint in the growing economy is:

$$\begin{aligned}
C_t &= Y_t - N_t X_{i,t} \\
&= \bar{L}N_t\left(\alpha^{\frac{2\alpha}{1-\alpha}} - \alpha^{\frac{2}{1-\alpha}}\right) - \underbrace{\dot{N}_t\eta}_{\text{Cost of new varieties}} \\
&= N_t\left\{\bar{L}\left(\alpha^{\frac{2\alpha}{1-\alpha}} - \alpha^{\frac{2}{1-\alpha}}\right) - g\eta\right\}.
\end{aligned} \tag{4.36}$$

It's clear that N_t will grow at the same rate as C_t. Since $Y_t = \bar{L}N_t\alpha^{\frac{2\alpha}{1-\alpha}}$ we also have Y_t growing at the same rate and we have succeeded in finding an equilibrium where everything grows. In this equilibrium we have profits being given by (4.34) so the economy grows at the rate:

$$g \equiv \frac{\Pi_i}{\eta} - \rho \Leftrightarrow \bar{L}\alpha^{\frac{2}{1-\alpha}}\left(\frac{1-\alpha}{\alpha\eta}\right) - \rho. \tag{4.37}$$

4.4.1 Efficiency concerns in the varieties model

We now examine whether these equilibrium are efficient (they will not be). The first calculation is a static optimum for the model with a fixed number of varieties. For a given N_i is the household getting the maximal amount of consumption for its labour input? Maximising consumption involves solving the following problem:

$$\underset{Y_t, L_t, \{X_{i,t}\}_{i=1}^{N_i}}{\text{argmax}} \; Y_t - \sum_{i=1}^{N_i} X_{i,t},$$

subject to

$$
\begin{aligned}
Y_t &= L_t^{1-\alpha} \sum_{i=1}^{N_i} X_{i,t}^{\alpha}, \\
L_t &= \bar{L}.
\end{aligned}
$$

We know from our previous problem that from the FOCs we have:

$$
\begin{aligned}
X_{i,t}^* &= \bar{L}\alpha^{\frac{1}{1-\alpha}}, \\
Y_t^* &= \bar{L}N_i\alpha^{\frac{\alpha}{1-\alpha}}, \\
C_t^* &= \bar{L}N_i\left(\alpha^{\frac{\alpha}{1-\alpha}} - \alpha^{\frac{1}{1-\alpha}}\right),
\end{aligned}
$$

and from the expanding varieties model with monopolistic competition we had:

$$
X_{i,t} = \bar{L}\alpha^{\frac{2}{1-\alpha}},
$$

and since $\alpha^{\frac{2}{1-\alpha}} < \alpha^{\frac{1}{1-\alpha}}$ we have that $X_{i,t} < X_{i,t}^*$. In other words, the economy with imperfect supply of intermediate goods produces too few intermediate goods relative to the efficient level. This should not be too surprising – the intermediate firms artificially restricts supply of the intermediate goods to maximise profits. Consumption is similarly restricted because $C_t < C_t^*$, which is easiest to check numerically for all $\alpha \in (0, 1)$.

The second calculation asks whether the growing economy is dynamically efficient, i.e. does it grow at the optimal rate? To answer this we solve for the growth rate that maximises the present discounted value of utility. Under logarithmic utility for simplicity we have:

$$
\max_{\{C_t, N_t\}} \int_0^{\infty} (\log C_t)\exp(-\rho t)dt,
$$

subject to

$$
Y_t = C_t + \eta\dot{N}_t \Leftrightarrow \dot{N}_t = \frac{Y_t - C_t}{\eta},
$$

where Y_t is the statically efficient output level. The Hamiltonian is:

$$
\mathcal{H} = (\log C_t)\exp(-\rho t) + \lambda_t\left(\frac{\bar{L}N_t\alpha^{\frac{\alpha}{1-\alpha}} - C_t}{\eta}\right),
$$

and its FOCs are:

$$
\begin{aligned}
\mathcal{H}_{C_t} &= \frac{\exp(-\rho t)}{C_t} - \frac{\lambda_t}{\eta} = 0 \\
\implies \frac{\dot{C}_t}{C_t} &= -\frac{\dot{\lambda}_t}{\lambda_t} - \rho, \qquad (4.38)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{H}_{N_t} &= \lambda_t \frac{\bar{L}\alpha^{\frac{\alpha}{1-\alpha}}}{\eta} = -\dot{\lambda}_t \\
\implies -\frac{\dot{\lambda}_t}{\lambda_t} &= \frac{\bar{L}\alpha^{\frac{\alpha}{1-\alpha}}}{\eta},
\end{aligned} \tag{4.39}
$$

and therefore, combining the two FOCs, the optimal growth rate is:

$$
\frac{\dot{C}_t^*}{C_t^*} = \frac{\bar{L}\alpha^{\frac{\alpha}{1-\alpha}}}{\eta} - \rho. \tag{4.40}
$$

Compared to what happens in the monopoly model, dynamic efficiency requires the economy to grow faster. This is because

$$
\frac{\dot{C}_t^*}{C_t^*} = \frac{\bar{L}\alpha^{\frac{\alpha}{1-\alpha}}}{\eta} - \rho > \frac{\dot{C}_t}{C_t} = \frac{\bar{L}\alpha^{\frac{2}{1-\alpha}}}{\eta} - \rho.
$$

We see that monopoly power in the intermediate goods market creates two distinct distortions. Not only is an efficiently low quantity of intermediate goods produced, the growth in variety of those goods is also inefficiently restricted.

4.5 Comments and key readings

There was quite a lot of digest in this section. We looked at the strengths and weaknesses of the AK model against the neoclassical Solow-Swan and Ramsey models. There is a large literature for further study and research including: *Introduction to Modern Economic Growth* by Acemoglu (2009), *Endogenous Growth Theory* by Aghion and Howitt (1997), *Economic Growth* by Barro and Sala-i-Martin (2003), Rebelo (1991), and *Advanced Macroeconomics* by Romer (2012).

For the varieties model, the above textbooks are also very good. In addition, see Aghion and Howitt (1992) and Romer (1990). The key message to takeaway from the varieties model (with endogenous growth) is that we had a loss of efficiency as soon as we introduced monopolistic competition. Keep this in mind when we introduce monopolistic competition to the Real Business Cycle model.

Additionally, more recently there's been a bit of a revival in the macroeconomics literature when it comes to research on long term growth. While we won't cover it in these notes, those interested in this area of research should take a look at papers such as those by Charles Jones and the quantitative macroeconomics field.

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5 Primer to DSGE Models

Before we tackle our first dynamic stochastic general equilibrium (DSGE) model, the Real Business Cycle model, it's worth going over a few important mathematical concepts. Some of the concepts are essential to understand now, but some of the other concepts, such as solution methods for DSGE models can be revisited later. But it's good to be aware of them now, and keep them in mind as we move on in the course.

5.1 Vector autoregressions

As we saw in the first section, AR models are useful tools for understanding the dynamics of individual variables such as output or consumption, but they ignore the interrelationships between variables. A vector autoregression (VAR) model captures the dynamics of n different variables allowing each variable to depend on lagged values of all variables. More specifically, with VAR models we can examine the impulse responses of all n variables to all n shocks. Consider the following simple VAR(1) model with two variables and one lag:

$$y_{1,t} = a_{11} y_{1,t-1} + a_{12} y_{2,t-1} + e_{1,t},$$
$$y_{2,t} = a_{21} y_{1,t-1} + a_{22} y_{2,t-1} + e_{2,t},$$

where $e_{1,t}$ and $e_{2,t}$ are shocks to the system. What are these shocks? They could be shocks which macroeconomists are interested in such as: exogenous policy changes;[1] changes in preferences such as attitudes to consumption, saving, work, or leisure; technology shocks – random increases or decreases in the efficiency with which firms produce goods and services; or, shocks to various frictions, such as increases or decreases in the efficiency with which various markets operate.

The time series perspective – that business cycle are being determined by various random shocks which are propagated throughout the economy over time[2] – is central to how modern macroeconomists now view economic fluctuations. VARs are a very common framework for modelling macroeconomic dynamics and the effects of shocks. But while VARs can describe how things work, they cannot explain why things work – hence why we need models based on economic theory (e.g., DSGE models!).

These VAR models were introduced to the economics discipline by Christopher Sims in 1980. Sims was telling macroeconomists to "get real." He criticised the widespread use of highly specialised macro-models that made very strong identifying restrictions (in the sense that each equation in the model usually excluded most of the model's other variables from the right-hand side) as well as very strong assumptions about the dynamic nature of these relationships. VARs were an alternative that allowed one to model macroeconomic data accurately, without having to impose lots of incredible restrictions. In the phrase used in an earlier paper by Sargent and Sims (who

1. More on this soon.
2. In other words, the Frisch-Slutsky paradigm: shocks to the economy causes impulses which lead to a propagation mechanism based on the structure of the economy, which in turn results in fluctuations.

shared the Nobel prize) it was "macro modelling without pretending to have too much a priori theory". We will see that VARs are not theory free. But they do make the role of theoretical identifying assumptions far clearer than was the case for the types of models Sims was criticising.

5.1.1 Matrix representation of VARs and the vector moving average representation

Let's consider our simple VAR(1) model:

$$y_{1,t} = a_{11}y_{1,t-1} + a_{12}y_{2,t-1} + e_{1,t},\ e_{1,t} \overset{\text{IID}}{\sim} \mathcal{N}(0,\sigma_1^2),$$
$$y_{2,t} = a_{21}y_{1,t-1} + a_{22}y_{2,t-1} + e_{2,t},\ e_{2,t} \overset{\text{IID}}{\sim} \mathcal{N}(0,\sigma_2^2),$$

which we can express more compactly using matrices. Let

$$\boldsymbol{Y}_t = \begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix},\ \boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},\ \boldsymbol{e}_t = \begin{bmatrix} e_{1,t} \\ e_{2,t} \end{bmatrix},$$

and so we can write the simple VAR(1) model as:

$$\boldsymbol{Y}_t = \boldsymbol{A}\boldsymbol{Y}_{t-1} + \boldsymbol{e}_t. \tag{5.1}$$

VARs express variables as a function of what happened yesterday and today's shocks. But what happened yesterday depended on yesterday's shocks and on what happened the day before, and so on. So with a bit of recursion, and like we do with AR(1) models, we can express the VAR(1) model as a vector moving average (VMA) model:

$$\begin{aligned}
\boldsymbol{Y}_t &= \boldsymbol{e}_t + \boldsymbol{A}\boldsymbol{Y}_{t-1} \\
&= \boldsymbol{e}_t + \boldsymbol{A}\left[\boldsymbol{e}_{t-1} + \boldsymbol{A}\boldsymbol{Y}_{t-2}\right] \\
&= \boldsymbol{e}_t + \boldsymbol{A}\boldsymbol{e}_{t-1} + \boldsymbol{A}^2\left[\boldsymbol{e}_{t-2} + \boldsymbol{A}\boldsymbol{Y}_{t-3}\right] \\
&\vdots \\
\boldsymbol{Y}_t &= \boldsymbol{e}_t + \boldsymbol{A}\boldsymbol{e}_{t-1} + \boldsymbol{A}^2\boldsymbol{e}_{t-2} + \boldsymbol{A}^3\boldsymbol{e}_{t-3} + ... + \boldsymbol{A}^t\boldsymbol{e}_0.
\end{aligned}$$

This makes it clear how today's values for the series are the cumulation of all the shocks from the past. It is also useful for deriving predictions about the properties of VARs.

5.1.2 Impulse response functions

Suppose there is an initial shock identified as:

$$\boldsymbol{e}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

and then all shock terms are zero afterwards, i.e., $\boldsymbol{e}_t = \boldsymbol{0},\ \forall t > 0$. Using our VMA representation we see that the response in $\boldsymbol{Y}_t$ after n periods is

$$\boldsymbol{A}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

So the impulse response function (IRF) for VARs are directly analogous to the IRFs for AR(1) models that we looked at before.

VARs are often used for forecasting. Suppose we observe our vector of variables $\boldsymbol{Y}_t$. What is our forecast for $\boldsymbol{Y}_{t+1}$? Using forward iteration, we could write the following for the next period:

$$\boldsymbol{Y}_{t+1} = \boldsymbol{A}\boldsymbol{Y}_t + \boldsymbol{e}_{t+1}.$$

But because $\mathbb{E}_t \boldsymbol{e}_{t+1} = 0$, an unbiased forecast at time t is $\boldsymbol{A}\boldsymbol{Y}_t$. In other words, $\mathbb{E}_t \boldsymbol{Y}_{t+1} = \boldsymbol{A}\boldsymbol{Y}_t$. The same reasoning tells us that $\boldsymbol{A}^2\boldsymbol{Y}_t$ is an unbiased forecast of $\boldsymbol{Y}_{t+2}$, and $\boldsymbol{A}^3\boldsymbol{Y}_t$ is an unbiased forecast of $\boldsymbol{Y}_{t+3}$, and so on. So once a VAR is estimated and organised in this form, it is very easy to construct forecasts.

The model (5.1) we've been looking at may seem like a small subset of all possible VARs because it doesn't have a constant term and only has lagged values from one period ago. However, we can easily add a third variable here which takes the constant value 1 each period. The equation for the constant term will just state that it equals its own lagged values. So this formulation actually incorporates models with constant terms. What about more than one lagged term? It turns out the first-order matrix representation can represent VARs with longer lags. Consider the two-lag system:

$$y_{1,t} = a_{11}y_{1,t-1} + a_{12}y_{1,t-2} + a_{13}y_{2,t-1} + a_{14}y_{2,t-2} + e_{1,t}$$
$$y_{2,t} = a_{21}y_{1,t-1} + a_{22}y_{1,t-2} + a_{23}y_{2,t-1} + a_{24}y_{2,t-2} + e_{2,t},$$

and define the vector

$$\boldsymbol{Z}_t = \begin{bmatrix} y_{1,t} \\ y_{1,t-1} \\ y_{2,t} \\ y_{2,t-1} \end{bmatrix}.$$

This system can be represented in matrix form as

$$\boldsymbol{Z}_t = \boldsymbol{A}\boldsymbol{Z}_{t-1} + \boldsymbol{e}_t, \tag{5.2}$$

where

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \boldsymbol{e}_t = \begin{bmatrix} e_{1,t} \\ 0 \\ e_{2,t} \\ 0 \end{bmatrix}.$$

The representation (5.2) is called the "companion form" matrix representation.

5.1.3 Interpreting shocks

The system we've been looking at is usually called a "reduced-form" VAR model. It is a purely econometric model, without any theoretical element, and fluctuations in the system are driven by the shocks $\boldsymbol{e}_t$. But how should we interpret these shocks?

Suppose that $e_{1,t}$ is a shock that affects only $y_{1,t}$ on impact and $e_{2,t}$ is a shock that affects only $y_{2,t}$ on impact. For instance, one can use the IRFs generated from an inflation-output VAR to calculate the dynamic effects of "a shock to inflation" and "a shock to output".

But we may imagine that the shocks are an "aggregate supply" shock and an "aggregate demand" shock and that both of these shocks have a direct effect on both inflation and output. How can we breakdown which "part" of $e_{1,t}$, say, affects only output? If we knew this or had an "identification strategy" to find this, then we could identify the component(s) of $\boldsymbol{e}_t$ that only affect inflation and that only affect output. We could then interpret $\boldsymbol{e}_t$ as being the reduced-form shocks which are comprised of "structural shocks", $\boldsymbol{\varepsilon}_t$.[3]

Suppose reduced-form and structural shocks are related by

$$e_{1,t} = c_{11}\varepsilon_{1,t} + c_{12}\varepsilon_{2,t},$$
$$e_{2,t} = c_{21}\varepsilon_{1,t} + c_{22}\varepsilon_{2,t},$$

and in matrix form we can write this as

$$\boldsymbol{e}_t = \boldsymbol{C}\boldsymbol{\varepsilon}_t.$$

These two VMA representations describe the data equally well:

$$\boldsymbol{Y}_t = \boldsymbol{e}_t + \boldsymbol{A}\boldsymbol{e}_{t-1} + \boldsymbol{A}^2\boldsymbol{e}_{t-2} + \boldsymbol{A}^3\boldsymbol{e}_{t-3} + \ldots + \boldsymbol{A}^t\boldsymbol{e}_0,$$
$$\Leftrightarrow \boldsymbol{Y}_t = \boldsymbol{C}\boldsymbol{\varepsilon}_t + \boldsymbol{A}\boldsymbol{C}\boldsymbol{\varepsilon}_{t-1} + \boldsymbol{A}^2\boldsymbol{C}\boldsymbol{\varepsilon}_{t-2} + \boldsymbol{A}^3\boldsymbol{C}\boldsymbol{\varepsilon}_{t-3} + \ldots + \boldsymbol{A}^t\boldsymbol{C}\boldsymbol{\varepsilon}_0.$$

We can interpret the model as one with reduced form shocks, $\boldsymbol{e}_t$, and IRFs given by $\boldsymbol{A}^n$; or as a model with structural shocks, $\boldsymbol{\varepsilon}_t$, and IRFs are given by $\boldsymbol{A}^n\boldsymbol{C}$. We could do this for any $\boldsymbol{C}$ if we knew the structural shocks.

3. There is a vast literature on the debate between structural and reduced-form modelling, especially in the empirical macro and micro literature. We will do our best to not getting bogged down by this here.

Another way to see how reduced-form shocks can be different from structural shocks is if there are contemporaneous interactions between variables – which is likely in macroeconomics. Consider the following model:

$$y_{1,t} = a_{12}y_{2,t} + b_{11}y_{1,t-1} + b_{12}y_{2,t-1} + \varepsilon_{1,t},$$
$$y_{2,t} = a_{21}y_{1,t} + b_{21}y_{1,t-1} + b_{22}y_{2,t-1} + \varepsilon_{2,t},$$

which can be written in matrix form as:

$$\boldsymbol{A}\boldsymbol{Y}_t = \boldsymbol{B}\boldsymbol{Y}_{t-1} + \boldsymbol{\varepsilon}_t,$$

where

$$\boldsymbol{A} = \begin{bmatrix} 1 & -a_{12} \\ -a_{21} & 1 \end{bmatrix}, \ \boldsymbol{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Now, if we estimate the "reduced-form" VAR model,

$$\boldsymbol{Y}_t = \boldsymbol{D}\boldsymbol{Y}_{t-1} + \boldsymbol{e}_t,$$

then the reduced-form shocks and coefficients are:

$$\boldsymbol{D} = \boldsymbol{A}^{-1}\boldsymbol{B},$$
$$\boldsymbol{e}_t = \boldsymbol{A}^{-1}\boldsymbol{\varepsilon}_t.$$

Again, the following two decompositions both describe the data equally well:

$$\boldsymbol{Y}_t = \boldsymbol{e}_t + \boldsymbol{D}\boldsymbol{e}_{t-1} + \boldsymbol{D}^2\boldsymbol{e}_{t-2} + \boldsymbol{D}^3\boldsymbol{e}_{t-3} + ...,$$
$$\Leftrightarrow \boldsymbol{Y}_t = \boldsymbol{A}^{-1}\boldsymbol{\varepsilon}_t + \boldsymbol{D}\boldsymbol{A}^{-1}\boldsymbol{\varepsilon}_{t-1} + \boldsymbol{D}^2\boldsymbol{A}^{-1}\boldsymbol{\varepsilon}_{t-2} + ... + \boldsymbol{D}^t\boldsymbol{A}^{-1}\boldsymbol{\varepsilon}_0.$$

For the structural model, the impulse responses to the structural shocks from n periods are given by $\boldsymbol{D}^n\boldsymbol{A}^{-1}$. This is true for any matrix $\boldsymbol{A}$.

So why should we care about this? There seems to be no problem with forecasting with reduced-form VARs: Once you know the reduced-form shocks and how they affected today's value of the variables, you can use the reduced-form coefficients to forecast, right? The problem comes when you start asking "what if" questions/counterfactuals. For example, "what happens if there is a shock to the first variable in the VAR?" In practice, the error series in a reduced-form VAR are usually correlated with each other. So are you asking "What happens when there is a shock to the first variable only?" or, are you asking "What usually happens when there is a shock to the first variable given that this is usually associated with a corresponding shock to the second variable?"

Most interesting questions about the structure of the economy relate to the impact of different types of shocks that are uncorrelated with each other. A structural identification that explains how the reduced-form shocks are actually combinations of uncorrelated structural shocks is far more likely to give clear and interesting answers.

5.1.4 Structural VARs: A general formulation

In its general formulation, the structural VAR (SVAR) is:

$$\underset{n\times n}{\boldsymbol{A}}\underset{n\times 1}{\boldsymbol{Y}_t} = \underset{n\times n}{\boldsymbol{B}}\underset{n\times 1}{\boldsymbol{Y}_{t-1}} + \underset{n\times n}{\boldsymbol{C}}\underset{n\times 1}{\boldsymbol{\varepsilon}_t}, \quad \boldsymbol{\varepsilon}_t \overset{\text{IID}}{\sim} \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma}). \tag{5.3}$$

The model is fully described by the following parameters: n^2 parameters in $\boldsymbol{A}$, n^2 parameters in $\boldsymbol{B}$, n^2 parameters in $\boldsymbol{C}$, and $\frac{n^2+n}{2}$ parameters in $\boldsymbol{\Sigma}$ which describes the patterns of covariances of the underlying shock terms. Adding all these together, we see that the most general form of the SVAR is a model with $3n^2 + \frac{n^2+n}{n}$ parameters. But estimating the reduced-form VAR,

$$\boldsymbol{Y}_t = \boldsymbol{D}\boldsymbol{Y}_{t-1} + \boldsymbol{e}_t,$$

only gives us information on $n^2 + \frac{n^2+n}{2}$ parameters: the coefficients in $\boldsymbol{D}$ and the estimated variance-covariance matrix of the reduced form errors.

To obtain information about structural shocks, we thus need to impose $2n^2$ a priori theoretical restrictions on our SVAR. This will leave us with $n^2 + \frac{n^2+n}{2}$ known reduced-form parameters and $n^2 + \frac{n^2+n}{2}$ structural parameters that we want to know. This can be expressed as $n^2 + \frac{n^2+n}{2}$ equations in $n^2 + \frac{n^2+n}{2}$ unknowns, so we can get a unique solution. For example, asserting that the reduced-form VAR is in fact the structural model is the same as imposing the $2n^2$ a priori restrictions so that $\boldsymbol{A} = \boldsymbol{C} = \boldsymbol{I}_n$.

SVARs generally identify their shocks as coming from distinct independent sources, and thus assume that they are uncorrelated. The error series in reduced-form VARs are usually correlated with each other. One way to view these correlations is that the reduced-form errors are combinations of a set of statistically independent structural errors. The most popular SVAR method is the recursive identification method. This method (used in the Sims (1980) paper) uses simple regression techniques to construct a set of uncorrelated structural shocks directly from the reduced-form shocks. This method sets $\boldsymbol{A} = \boldsymbol{I}_n$ and constructs the matrix $\boldsymbol{C}$ so that the structural shocks will be uncorrelated.

5.1.5 The Cholesky decomposition

It's probably best to go through an example. Start with a reduced-form VAR with three variables and the errors, $e_{1,t}, e_{2,t}$, and $e_{3,t}$:

$$\begin{aligned} \boldsymbol{Y}_t &= \boldsymbol{D}\boldsymbol{Y}_{t-1} + \boldsymbol{e}_t, \\ \Leftrightarrow \begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} e_{1,t} \\ e_{2,t} \\ e_{3,t} \end{bmatrix}, \end{aligned} \tag{5.4}$$

where the joint distribution of $\boldsymbol{e}_t$ is:

$$\begin{bmatrix} e_{1,t} \\ e_{2,t} \\ e_{3,t} \end{bmatrix} \overset{\text{IID}}{\sim} \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma}),\ \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_y^2 & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_z^2 \end{bmatrix}.$$

Now, we want to express the shocks $\boldsymbol{e}_t$ as a function of structural shocks. Apply the following restriction:

$$\begin{aligned} e_{1,t} &= c_{11}\varepsilon_{1,t} \\ e_{2,t} &= c_{21}\varepsilon_{1,t} + c_{22}\varepsilon_{2,t} \\ e_{3,t} &= c_{31}\varepsilon_{1,t} + c_{32}\varepsilon_{2t} + c_{33}\varepsilon_{3,t}, \end{aligned}$$

or, in matrix form:

$$\begin{aligned} \begin{bmatrix} e_{1,t} \\ e_{2,t} \\ e_{3,t} \end{bmatrix} &= \begin{bmatrix} c_{11} & 0 & 0 \\ c_{21} & c_{22} & 0 \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{bmatrix}, \\ \Leftrightarrow \boldsymbol{e}_t &= \boldsymbol{C}\boldsymbol{\varepsilon}_t, \end{aligned} \tag{5.5}$$

where the joint distribution of $\boldsymbol{\varepsilon}_t$ is:

$$\begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{bmatrix} \overset{\text{IID}}{\sim} \mathcal{N}\left(\boldsymbol{0}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right).$$

How do we get $\boldsymbol{C}$? Estimation is one option, but the easier way is to use the Choleski decomposition for the variance-covariance matrix $\boldsymbol{\Sigma}$:

$$\begin{aligned} \boldsymbol{\Sigma} &= \boldsymbol{C}\boldsymbol{C}^\top, \\ \implies \boldsymbol{C}^{-1}\boldsymbol{\Sigma}(\boldsymbol{C}^\top)^{-1} &= \boldsymbol{I}_n. \end{aligned}$$

To see how this works, first note that the transpose of equation (5.5) is:

$$\begin{aligned}\boldsymbol{e}_t^\top &= (\boldsymbol{C}\boldsymbol{\varepsilon}_t)^\top \\ &= \boldsymbol{\varepsilon}_t^\top \boldsymbol{C}^\top,\end{aligned}$$

so post multiply (5.5) with $\boldsymbol{e}_t^\top$ to get:

$$\boldsymbol{e}_t\boldsymbol{e}_t^\top = \boldsymbol{C}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t^\top \boldsymbol{C}^\top, \tag{5.6}$$

or equivalently:

$$\begin{bmatrix} e_{1,t} \\ e_{2,t} \\ e_{3,t} \end{bmatrix} \begin{bmatrix} e_{1,t} & e_{2,t} & e_{3,t} \end{bmatrix} = \begin{bmatrix} c_{11} & 0 & 0 \\ c_{21} & c_{22} & 0 \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t} & \varepsilon_{2,t} & \varepsilon_{3,t} \end{bmatrix} \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ 0 & c_{22} & c_{32} \\ 0 & 0 & c_{33} \end{bmatrix},$$

and if we take expectations of (5.6), we get:

$$\begin{aligned}\mathbb{E}_t\left[\boldsymbol{e}_t\boldsymbol{e}_t^\top\right] &= \mathbb{E}_t\left[\boldsymbol{C}\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t^\top \boldsymbol{C}^\top\right] \\ \Leftrightarrow \boldsymbol{\Sigma} &= \boldsymbol{C}\boldsymbol{I}_n\boldsymbol{C}^\top = \boldsymbol{C}\boldsymbol{C}^\top. \end{aligned} \tag{5.7}$$

Identification done! We have shown that we can get $\boldsymbol{C}$ by a Choleski decomposition for the variance-covariance matrix.

So, from (5.4), if we substitute in equation (5.5), we have

$$\begin{aligned}\begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} c_{11} & 0 & 0 \\ c_{21} & c_{22} & 0 \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{bmatrix} \\ \Leftrightarrow \boldsymbol{Y}_t &= \boldsymbol{D}\boldsymbol{Y}_{t-1} + \boldsymbol{C}\boldsymbol{\varepsilon}_t,\end{aligned}$$

which can be transformed into:

$$\boldsymbol{C}^{-1}\boldsymbol{Y}_t = \boldsymbol{C}^{-1}\boldsymbol{D}\boldsymbol{Y}_{t-1} + \boldsymbol{\varepsilon}_t, \tag{5.8}$$

which is nothing but a SVAR with what macro-econometricians like to call a "short-run restriction."

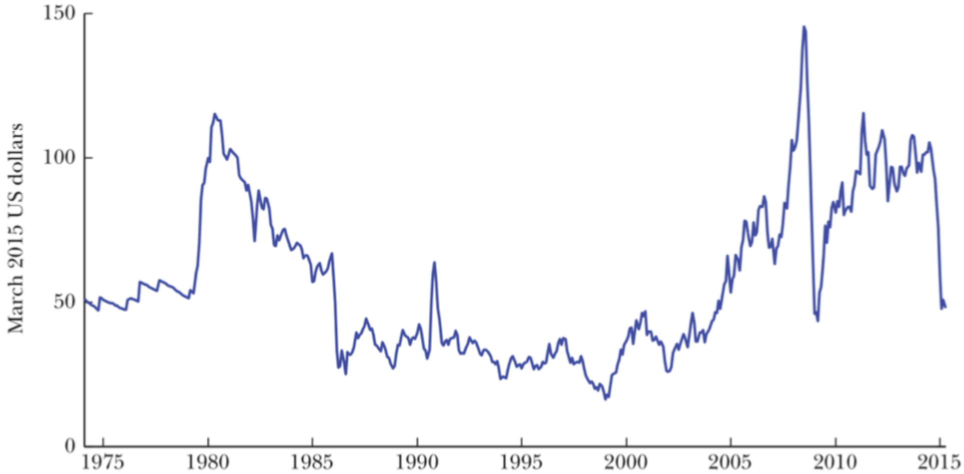
Note now that, by construction, the $\boldsymbol{\varepsilon}_t$ shocks constructed in this way are uncorrelated with each other. This method posits a sort of "causal chain" of shocks. The first shock affects all of the variables at time t. The second only affects two of them at time t, and the last shock only affects the last variable at time t.

There is a serious drawback to this however: The causal ordering is not unique. Any one of the VAR variables can be listed first, and any one can be listed lasted. This means there are $n! = 1 \times 2 \times 3 \times ... \times n$ possible recursive orderings. We need to think very carefully about our own prior thinking about causation!

5.1.6 Example: Kilian (2009) and Baumeister and Kilian (2016)

Oil shocks – large run-ups and subsequent declines in the price of crude oil – regularly receive attention. Many recessions have been preceded by an increase in the price of oil. Why exactly this has occurred is not obvious: oil usage is actually a relatively small input compared to GDP.

Figure 5.1: Inflation-Adjusted WTI Price of Crude Oil (1974.1-2015.3)



Source: US Energy Information Administration. Note: The West Texas Intermediate (WTI) oil price series has been deflated with the seasonally adjusted US consumer price index for all urban consumers.

Empirical studies, prior to Kilian's, generally asked the question "what are the effects of an oil price shock?" Kilian (2009) and Baumeister and Kilian (2016) asked "what is an oil price shock and are there different kinds of oil price shocks?" He uses VAR analysis to distinguish between shocks to oil prices due to global demand, shocks due to oil supply, and shocks due to speculation in the oil market.

The three variable, monthly VAR model of Killian is based on the following:

$$\boldsymbol{z}_t = \begin{bmatrix} \Delta prod_t \\ rea_t \\ rpo_t \end{bmatrix},$$

where $\Delta prod_t$ is the growth rate of oil production, rea_t is real global economic activity, and rpo_t is the real price of oil.

The VAR structure is

$$\boldsymbol{A}_0 \boldsymbol{z}_t = \boldsymbol{\alpha} + \sum_{i=1}^{24} \boldsymbol{A}_i \boldsymbol{z}_{t-i} + \boldsymbol{\varepsilon}_t,$$

where $\boldsymbol{\varepsilon}_t$ are the structural shocks, and $\boldsymbol{A}_0$ is a lower-triangular matrix,

$$\boldsymbol{A}_0 = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}.$$

Kilian's identifying assumptions are:

- Oil production does not respond within the month to world demand and oil prices.
- World demand is affected within the month by oil production, but not by oil prices.
- Oil prices responded immediately to oil production and world demand.

It follows that if $\boldsymbol{A}_0$ is lower-triangular, then so is its inverse, $\boldsymbol{A}_0^{-1}$. Thus, the reduced form model is

$$\boldsymbol{z}_t = \boldsymbol{A}_0^{-1}\boldsymbol{\alpha} + \sum_{i=1}^{24} \boldsymbol{A}_0^{-1}\boldsymbol{A}_i \boldsymbol{z}_{t-i} + \boldsymbol{A}_0^{-1}\varepsilon_t.$$

Reduced-form shocks, $\boldsymbol{e}_t$, are related to the structural shocks, $\boldsymbol{\varepsilon}_t$, by

$$\begin{aligned} \boldsymbol{e} &= \boldsymbol{A}_0^{-1}\boldsymbol{\varepsilon}_t \\ \Leftrightarrow \begin{bmatrix} e_t^{\Delta prod} \\ e_t^{rea} \\ e_t^{rpo} \end{bmatrix} &= \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_t^{\Delta prod} \\ \varepsilon_t^{rea} \\ \varepsilon_t^{rpo} \end{bmatrix}. \end{aligned}$$

As you can see, the oil production reduced-form shock, $e_t^{\Delta prod}$, is considered a structural shock ($e_t^{\Delta prod} = \varepsilon_t^{\Delta prod}$); the reduced form economic activity shock, e_t^{rea}, combines the structural oil shock and the structural activity shock, ε_t^{rea}; and the reduced form oil price shock, e_t^{rpo}, is a combination of all three structural shocks.

So, relative to the general model of an SVAR,

$$\boldsymbol{A}\boldsymbol{Y}_t = \boldsymbol{B}\boldsymbol{Y}_{t-1} + \boldsymbol{C}\boldsymbol{\varepsilon}_t,$$

where are our $2n^2 = 18$ identifying restrictions? Well, we set $\boldsymbol{C} = \boldsymbol{I}$, assuming contemporaneous interactions between the variables (9 restrictions); we assumed that $\boldsymbol{A}_0$ is a lower triangle matrix (3 restrictions); we assume that the diagonal of $\boldsymbol{A}_0$ are unit coefficients (3 restrictions); and we assume that the structural shocks are orthogonal, i.e., 3 off-diagonal elements of $\boldsymbol{\Sigma}$ are zero (3 restrictions. Thus we get our 18 restrictions.

In addition to the standard IRFs, Kilian shows how the real price oil can be decomposed into components related to these three shocks. How did he do this? Recall the VMA representation:

$$\boldsymbol{Y}_t = \boldsymbol{\varepsilon}_t + \boldsymbol{A}\boldsymbol{\varepsilon}_{t-1} + \boldsymbol{A}^2\boldsymbol{\varepsilon}_{t-2} + \ldots + \boldsymbol{A}^t\boldsymbol{\varepsilon}_0.$$

One can do this calculation three times, each time with only one type of shock "turned on" and the others set to zero. Adding these up, one will get the realised values of $\boldsymbol{Y}_t$. Alternatively, one can do a dynamic simulation of the model

$$\boldsymbol{Y}_t = \boldsymbol{A}\boldsymbol{Y}_{t-1} + \boldsymbol{\varepsilon}_t,$$

in each case letting $\boldsymbol{\varepsilon}_t$ represent one of the realised historical shocks with the others set to zero.

The main results from Kilian's findings were:

- Shocks to oil supply have limited effects on oil prices and have been of negligible importance in driving oil prices over time.
- Both global demand and speculative oil price shocks can have significant effects on oil prices, but speculative oil price shocks have limited effects on global economic activity.
- Speculative oil-market shocks have accounted for most of the month-to-month movements in oil prices.
- Steady increase in oil prices from 2000 onwards was mostly due to strong global demand.
- How the economy reacts to an "oil price shock" will depend on the origins of that shock.

The last point helps to explain why the world economy survived a long period of increasing oil prices in the 2000s without going into recession (due to oil shocks – the 2008 GFC had little to do with oil prices).

5.1.7 Another VAR example: Stock and Watson (2001)

Stock and Watson in their *Journal of Economic Perspectives* 2001 piece, "Vector Autoregressions", examine the effect of monetary policy shocks. The paper is a useful introduction to VAR methods. You can think of these VARs as useful in two ways. First, as an exercise in positive analysis: monetary policy co-moves with lots of other macro variables, by only identifying the structural or exogenous shocks to policy can we discover its true effects. Second, there's the normative analysis perspective: It may help a policy maker to answer the question "If I choose to raise interest rates by 25 basis points today, what is likely to happen over the next year to inflation and output relative to the case where I keep rates unchanged? Should I do this or not?" Essentially, this is a question about impulse responses.

Stock and Watson's VAR features monthly data on inflation, π_t, the unemployment rate, u_t, and the Federal Funds Rate (FFR), i_t. They posit a lower-triangle causal chain of the form:

$$\boldsymbol{A}\boldsymbol{Z}_t = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \pi_t \\ u_t \\ i_t \end{bmatrix} = \boldsymbol{B}\boldsymbol{Z}_{t-1} + \boldsymbol{\varepsilon}_t.$$

Their identification assumptions are: (i) inflation depends only on lagged values of the other variables (perhaps motivated by the idea of sticky prices); (ii) unemployment depends on contemporaneous inflation but not the FFR; and (iii) the FFR depends on both contemporaneous inflation and unemployment (Fed using its knowledge about the current state of the economy when it is setting rates). The IRFs from the Stock and Watson paper are reproduced in Figure 5.2.

Figure 5.2: IRFs from Recursive VAR, First Identification

Results reproduced by Whelan (2016).

Looking at the IRFs, we see that most of the results seem to make sense. Let's focus on the third row (responses of a shock to the FFR): An increase in the interest rate leads to a rise in unemployment and a delayed decline in inflation. However, the short-run response of the inflation rate is a bit puzzling: the interest rate increase seems to raise the inflation rate for a few periods before it falls.

This "price puzzle" result has been obtained in a number of other VAR studies.[4] It provides a good illustration of the potential limitations of VAR analysis. Some think the explanation is

4. See, for example, Christiano, Eichenbaum, and Evans (1999).

that the Fed is acting on information not captured in the VAR (e.g., information about commodity prices) and that this information may provide signals of future inflationary pressures. Thus, interest rate increases could occur just before an increase in inflation. The VAR may be capturing this pattern and confusing causation and correlation. Indeed, subsequent research has managed to solve the puzzle – through a variety of measures – one example is by factoring in commodity prices into inflation measures.

Secondly, as eluded to earlier, the ordering of a VAR is very important. Consider the IRFs in Figure 5.3.

Figure 5.3: IRFs from Recursive VAR, Second Identification

Results reproduced by Whelan (2016).

Here the ordering has been changed to: 1) interest rates, 2) unemployment, and 3) inflation. A researcher could rationalise this ordering on the grounds that the Fed can only respond to the economy with a lag because it takes time to collate data on the economy, but that inflation should be able to respond immediately to economic events. This sounds reasonable enough, but the results from this identification don't make much sense: the interest rate shock raises inflation now for almost four years and unemployment drops for a while after the increase in interest rates!

5.1.8 Long-run restrictions and the Blanchard-Quah method

The identifying assumptions in the recursive VAR approach require knowledge of how certain variables react in an instantaneous way to certain shocks. Sometimes, because certain variables are sluggish or because information about some variables is only available with a lag, we can be pretty confident about these restrictions. But often they are pure guesswork. Economic theory gives us little guidance – in fact, economic theory usually tells us about how variables react in the long-run rather than what will happen contemporaneously. For example, shocks in the IS-LM model or aggregate demand shocks have no effect on output and a positive effect on prices in the long run. This suggests an alternative approach: use these theoretically-inspired long-run restrictions to identify shocks and impulse responses.

Consider the VAR model:

$$\boldsymbol{Z}_t = \boldsymbol{B}\boldsymbol{Z}_{t-1} + \boldsymbol{C}\boldsymbol{\varepsilon}_t, \tag{5.9}$$

where the variance-covariance matrix of the structural shocks is:

$$\mathbb{E}_t\left[\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t^\top\right] = \begin{bmatrix} \mathbb{E}_t[\varepsilon_1^2] & \mathbb{E}_t[\varepsilon_1\varepsilon_2] \\ \mathbb{E}_t[\varepsilon_2\varepsilon_1] & \mathbb{E}_t[\varepsilon_2^2] \end{bmatrix} = \boldsymbol{I}_2,$$

so the structural shocks are uncorrelated and have unit variance. Note that the variance-covariance matrix of the observed reduced-form errors is:

$$\boldsymbol{\Sigma} = \mathbb{E}_t[\boldsymbol{e}_t\boldsymbol{e}_t^\top] = \mathbb{E}_t\left[(\boldsymbol{C}\boldsymbol{\varepsilon}_t)(\boldsymbol{C}\boldsymbol{\varepsilon}_t)^\top\right] = \boldsymbol{C}\mathbb{E}_t[\boldsymbol{\varepsilon}_t\boldsymbol{\varepsilon}_t^\top]\boldsymbol{C}^\top = \boldsymbol{C}\boldsymbol{C}^\top,$$

and as we saw before, the observed variance-covariance structure of the reduced-form shocks tells us something about how they are related to the uncorrelated, unit variance, structural shocks.

Now, suppose

$$\boldsymbol{Z}_t = \begin{bmatrix} \Delta y_t \\ \Delta x_t \end{bmatrix},$$

then the long-effect of the shock on y_t is the sum of its effects of Δy_t, $\Delta y_{t+1}, \Delta y_{t+2}$, and so on. The long-run effect is the sum of the impulse responses, and the impulse responses for the model (5.9) are: $\boldsymbol{C}$ in the impact period, $\boldsymbol{B}\boldsymbol{C}$ after one period, $\boldsymbol{B}^2\boldsymbol{C}$ after two periods, and so on. This implies the IRFs are given by $\boldsymbol{B}^n\boldsymbol{C}$ after n periods. Thus, the long-run level effects are:

$$\boldsymbol{D} = (\boldsymbol{I} + \boldsymbol{B} + \boldsymbol{B}^2 + \boldsymbol{B}^3 + ... + \boldsymbol{B}^n)\boldsymbol{C}.$$

If the eigenvalues of $\boldsymbol{B}$ are inside unit circle, then

$$\boldsymbol{I} + \boldsymbol{B} + \boldsymbol{B}^2 + \boldsymbol{B}^3 + ... + \boldsymbol{B}^n = (\boldsymbol{I} - \boldsymbol{B})^{-1},$$

which is the matrix equivalent to the scalar case,

$$1 + a + a^2 + a^3 + ... + a^n = \frac{1}{1-a}.$$

Thus, the long-run responses are

$$\boldsymbol{D} = (\boldsymbol{I} - \boldsymbol{B})^{-1}\boldsymbol{C}.$$

Now, note that

$$\boldsymbol{D}\boldsymbol{D}^\top = (\boldsymbol{I} - \boldsymbol{B})^{-1}\boldsymbol{C}\boldsymbol{C}^\top \left((\boldsymbol{I} - \boldsymbol{B})^{-1}\right)^\top.$$

But, we know that $\boldsymbol{C}\boldsymbol{C}^\top = \boldsymbol{\Sigma}$ is the variance-covariance matrix of the reduced-form shocks – which can be estimated – so we have:

$$\boldsymbol{D}\boldsymbol{D}^\top = (\boldsymbol{I} - \boldsymbol{B})^{-1}\boldsymbol{\Sigma} \left((\boldsymbol{I} - \boldsymbol{B})^{-1}\right)^\top. \tag{5.10}$$

Now, make a restriction about the long-run effects described in $\boldsymbol{D}$: Assume that $\boldsymbol{D}$ is lower triangular so only the first shock has a long-run effect on the first variable, and only the first and second shocks have long-run effects on the second variable, and so on. In the two variable case, this is just:

$$\boldsymbol{D} = \begin{bmatrix} d_{11} & 0 \\ d_{21} & d_{22} \end{bmatrix}.$$

Since we assume that $\boldsymbol{D}$ is lower triangular, a unique lower-triangle matrix $\boldsymbol{D}$, when post-multiplied by its transpose, will equal symmetric matrix, $\boldsymbol{D}\boldsymbol{D}^\top$. As we saw before, this is known as the Cholesky factor of the symmetric matrix. Typically, in most software packages, $\boldsymbol{D}$ can be calculated as the Cholesky factor of the known matrix $(\boldsymbol{I} - \boldsymbol{B})^{-1}\boldsymbol{\Sigma}\left((\boldsymbol{I} - \boldsymbol{B})^{-1}\right)^\top$.

Now, recall that $\boldsymbol{D} = (\boldsymbol{I} - \boldsymbol{B})^{-1}\boldsymbol{C}$, so the crucial matrix, $\boldsymbol{C}$, defining the structural shocks can be calculated as

$$\boldsymbol{C} = (\boldsymbol{I} - \boldsymbol{B})\boldsymbol{D}.$$

Now, let's look at some applications. Blanchard and Quah (1989) (BQ) used a two-variable VAR in the log-difference of GDP, Δy_t, and the unemployment rate, U_t. Because the VAR is estimated to be stationary (eigenvalues inside unit circle) both structural shocks have zero long-run effect on the unemployment rate. The lower diagonal assumption thus implies that of the two structural shocks, only one of them could have a long-run effect on the level of output. BQ labelled this the "supply shock" while the shock that has no effect on long-run output was labelled the "demand shock":

$$\begin{bmatrix} \Delta y_t \\ U_t \end{bmatrix} = \begin{bmatrix} d_{11} & 0 \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_{s,t} \\ \varepsilon_{d,t} \end{bmatrix}, \tag{5.11}$$

where $\varepsilon_{s,t}$ is the supply shock and $\varepsilon_{d,t}$ is the demand shock. The relative importance of supply verses demand shocks in determining output is a long-running theme in macroeconomics. Keynesians emphasise the importance of demand shocks while more classically-orientated economists, such as advocates for the RBC approach, see supply shocks as being more important. BQ's results implied that demand shocks were responsible for the vast majority of short-run fluctuations.

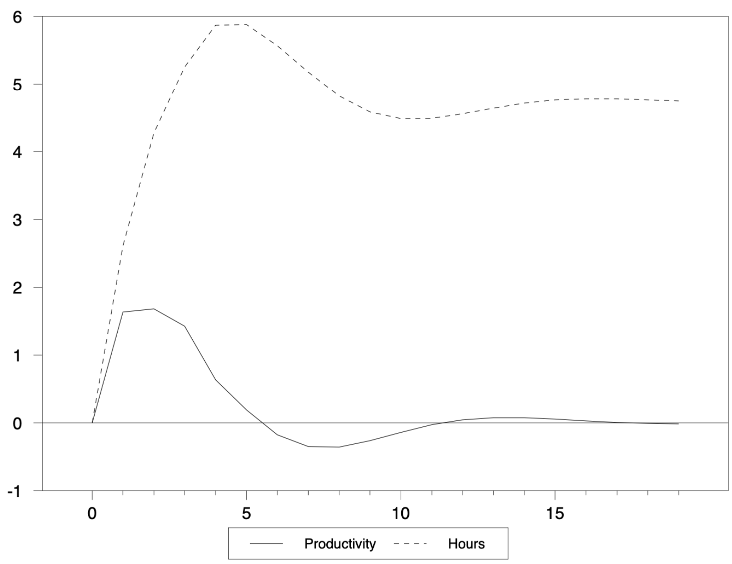
Now we look at Galí (1999), who suggested that BQ's formulation was a little bit restrictive. The assumption that neither supply nor demand shocks can change unemployment rates in the long-run may not be correct. Galí's paper applied a similar analysis to BQ, but for a formulation that moved a bit closer to the debate about RBC models and their predictions for the labour market. RBC models assume technology shocks drive the business cycle, and explain why hours worked are higher in booms than in recessions (i.e., make hay while the sun shines).

Galí's long-run restriction was as follows:

$$\begin{bmatrix} \Delta \ln\left(\frac{y}{h}\right) \\ \Delta \ln(h) \end{bmatrix} = \begin{bmatrix} d_{11} & 0 \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} \varepsilon_s \\ \varepsilon_d \end{bmatrix}. \tag{5.12}$$

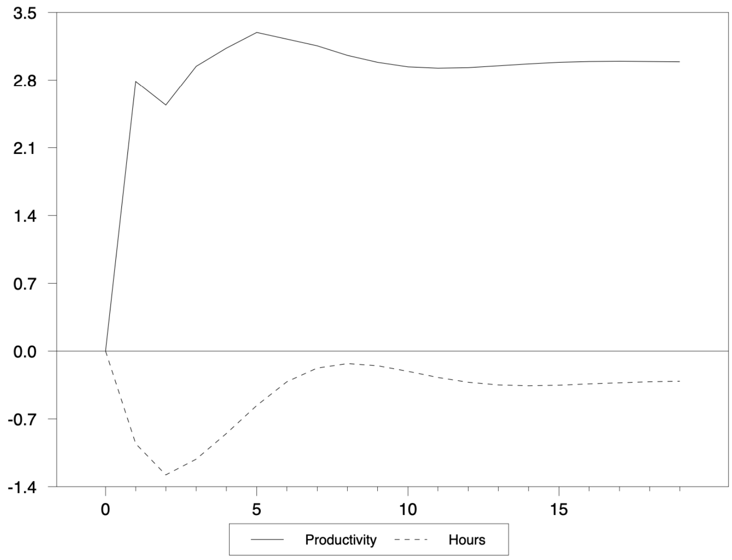
The lower-diagonal assumption about long-run responses now means that the supply shock (now called the "technology shock") can affect productivity in the long-run, while the non-technology shock cannot. The model lets the data dictate the long-run effects of technology and non-technology shocks on hours worked. If the technology shock increases hours worked then that's essentially a score for the neoclassical camp. If hours worked falls, then that's a win for the Keynesians.

Figure 5.4: Response to Non-Technology Shock



Source: Galí (1999).

Figure 5.5: Response to Technology Shock



Source: Galí (1999).

Looks like that's a win for the Keynesians. A positive technology shock actually decreases hours worked. Short-run output seems to be demand-driven not supply-driven. More efficiency means that demanded output can be supplied with less labour. Furthermore, non-technology shocks seem to cause both output and productivity to rise in the short-run.

5.2 Moving beyond VARs: Solving Models with Rational Expectations

Having described econometric methods for measuring the shocks that hit the macroeconomy and their dynamic effects, we now turn to developing theoretical models that explain these patterns. This requires models with explicit dynamics and with stochastic shocks. Obviously, VARs are dynamic stochastic models, however they are econometric models, not theoretical models, and they have their limitations (as we previously saw). They do not explicitly characterise the underlying decisions rules adopted by firms and households – i.e., they don't tell us how or why things happen. This "why" element is crucial if the stories underlying our forecasts or analysis of policy effects are to be believed.

The goal of the modern DSGE approach is to develop models that can explain macroeconomic dynamics as well as the VAR approach, but that are based upon the fundamental idea of optimising firms and households.

5.2.1 Introducing expectations

A key sense in which DSGE models differ from VARs is that while VARs just have backward-looking dynamics, DSGE models have both backward-looking and forward-looking dynamics. The backward-looking dynamics stem, for instance, from identities linking today's capital stock with last period's capital stock and this period's investment. For example:

$$K_t = (1 - \delta)K_{t-1} + I_t.$$

The forward-looking dynamics stem from optimising behaviour: What agents expect to happen tomorrow is very important for what they decide to do today – think about our consumption Euler equation. Modelling this idea requires an assumption about how people formulate their expectations.

Almost all economic transactions rely crucially on the fact that the economy is not a "one-period game". Economic decisions have an intertemporal element to them. A key issue in macroeconomics is how people formulate expectations about them in the presence of uncertainty. Prior to the 1970s, this aspect of macroeconomic theory was largely ad hoc. Generally, it was assumed that agents used some simple extrapolative rule whereby the expected future value of a variable was close to some weighted average of its recent past values – e.g., recall how agents formed inflationary expectations in the AD-AS model.

This approach was criticised in the 1970s by economists such as Robert Lucas[5] and Thomas Sargent. Lucas and Sargent instead promoted the use of an alternative approach which they called "Rational Expectations". In economics, rational expectations usually means two things: (i) Agents use publicly available information in an efficient manner. Thus, they do not make systematic mistakes when formulating expectations; and, (ii) That agents understand the structure of the model economy and base their expectations of variables on this knowledge.

Rational Expectations is a strong assumption. No one truly understands the structure of an economy – not even macroeconomists. But one reason for using Rational Expectations as a baseline assumption is that once one has specified a particular model of the economy, any other assumption about expectations means that people are making systematic errors, which seems inconsistent with rationality. In other words, we think it's entirely reasonable to presume that agents are optimising to get what's best for them. We can easily disagree on what "the best" is for them, but I think we can agree that they will *try to* act optimally.

5. See also the "Lucas Critique". In a nutshell, the Lucas' Critique states that it is fraught with hazard to try to predict the effects of a policy change based on correlations (or regression coefficients) based on historical data. We say that a parameter is "structural" if it is invariant to the rest of the economic environment, and, in particular, the policy environment. A parameter is "reduced form" if it is not invariant to the environment, or, more generally, if that parameter cannot be mapped back into some economic primitive.

5.2.2 Rational Expectations: Lucas' island model

To perhaps clarify or further motivate the utility of Rational Expectations and the Lucas critique, let's take a refresher of Lucas' island model (Lucas, 1972, 1973, 1975). I will follow the derivation in Campante, Sturzenegger, and Velasco (2021), but the treatment in Blanchard and Fischer (1989) is good too.

Suppose that there are many agents, indexed by $i \in (0, 1)$, whereby each agent is on their own island – picture Robinson Crusoe on an island; only that there are many of them occupying their own island. Thus, each agent produces and consumes output each period. The only catch is that, unlike in a Robinson Crusoe economy, agents observe demand for output across the islands each period. But they need to discern if the demand increase (or decrease) is real or nominal. If the demand change is real, then production must be changed. Conversely, if the change in demand is only nominal, then agents need only adjust prices. Of course, how much information each agent has is the main source of tension in the model and is where all the action is. Let's start with the perfect (or full) information case before moving onto the case of imperfect (or asymmetric) information.

Full information. Every agent i maximises his or her utility function,

$$u_i = C_i - \frac{L_i^{1+\varphi}}{1+\varphi}, \quad \varphi > 0, \tag{5.13}$$

where C_i is real consumption, L_i is labour supply, and φ is the inverse-Frisch elasticity of labour supply. They each have access to a linear production technology,

$$Q_i = L_i, \tag{5.14}$$

where Q_i is the quantity of output. This means that their potential consumption is

$$C_i = \frac{P_i Q_i}{P}, \tag{5.15}$$

where P_i is the price of each agent i's output, P is the aggregate (average) market price for the output of the islands, and hence P_i/P is the relative price of output i.

Optimisation is straightforward. Substitute (5.14) into (5.15) then substitute into (5.13) to get

$$u_i = \frac{P_i L_i}{P} - \frac{L_i^{1+\varphi}}{1+\varphi}.$$

This will then give us a first order condition (FOC) for labour,

$$\frac{P_i}{P} - L_i^{\varphi} = 0,$$

which can be rearranged to give the labour supply curve:

$$L_i = \left(\frac{P_i}{P}\right)^{\frac{1}{\varphi}}.$$

If we take logs (denote lower case letters as log levels), then we can write this as:

$$l_i = \frac{p_i - p}{\varphi}. \tag{5.16}$$

So intuitively, we have that labour supply (production) increases in the relative price of the agent i's output.

Next, let us conjecture an expression for good i. Let's say that demand for good i depends on average income, y, relative prices, and a good specific preference shock, z_i,

$$q_i = y - \eta(p_i - p) + z_i, \quad \eta > 0. \tag{5.17}$$

In words, the above expression says that demand for good i increases in average incomes – which is the same for all agents – and good specific taste shocks but is decreasing in the relative price of good i. Seems intuitive enough. One more thing to note about (5.17) is that we assume that $\sum_i z_i = 0$ with $z_i \overset{\text{IID}}{\sim} \mathcal{N}(0, \nu_z)$, i.e., the preference shock is a relative taste shock and, in aggregate, all the shocks wash out to be zero across all the agents and islands.

As for aggregate demand, let us suppose the following:

$$y = m - p. \tag{5.18}$$

Don't think too hard about this – just think about your old AD-AS undergrad macro lectures: average incomes are a function of a policy shifter, m (can just think of this as monetary policy), and the average price of goods. We just need to note that m is normally distributed with mean $\mathbb{E}[m]$ and variance ν_m.

Equilibrium occurs when the goods market clears – when the demand for good i is equal to its supply. So set (5.16) equal to (5.17):

$$\frac{p_i - p}{\varphi} = y - \eta(p_i - p) + z_i.$$

Doing some algebra we can then write the optimal price for good i as

$$p_i = \frac{\varphi}{1 + \varphi\eta}(y + z_i) + p. \tag{5.19}$$

Crucially, remember that here p is known – thus all variables are known by all the agents. So this will pin down p_i, and so by averaging across all prices we get (somewhat mechanically):

$$p = \frac{\varphi}{1+\varphi\eta}(y + z_i) + p,$$

which implies that

$$y = 0.$$

Substitute the equilibrium average income level back into our expression for aggregate demand (5.18) to get

$$p = m. \tag{5.20}$$

This is the classical monetary neutrality result – when agents have access to full information, prices respond to the policy variable. In other words, as agents observe the monetary policy shocks and average price level, and where prices are fully flexible, real variables are completely unaffected as it is only nominal variables that react to shocks.

Imperfect information. Now consider the case that agents do not observe the average price level; they of course observe their own price. If p_i changes, agent i needs to know this is just a nominal price change or of this is a relative price change for their output. If it is a relative price change, then the agent must change the amount of labour to supply and thus the quantity of output to produce as in (5.16). In other words, there is now some uncertainty in the economy, and so agents will supply labour based on rational expectations, which will be determined by the mathematical expectation that is consistent with the model.

Now denote $s_i = p_i - p$ as the relative price and write the labour supply curve (5.16) as

$$l_i = \frac{1}{\varphi}\mathbb{E}[s_i|p_i].$$

Remember, however, that we assumed that both sources of uncertainty in the model, z_i and m, were normally distributed. Thus, it must be the case that if z_i and m are jointly normally distributed, then s_i, p_i, and p will also be jointly normal. The conditional expectation $\mathbb{E}[s_i|p_i]$ can be written as the following linear function

$$\mathbb{E}[s_i|p_i] = a + bp_i.$$

That is, we have a very simple signal extraction problem. We want to observe s_i but the signal is contaminated with some noise – in this case from p. With the assumption of log normality, we can write $\mathbb{E}[s_i|p_i]$ as

$$\mathbb{E}[s_i|p_i] = \frac{\nu_s}{\nu_s + \nu_p}(p_i - \mathbb{E}[p]), \tag{5.21}$$

where ν_s and ν_p are the variances of the relative price and the aggregate price level, respectively. This equation is key. If the signal that each agent observes is very noisy (high ν_p), then the change in demand for good i (the change in p_i) mostly indicates a nominal change in demand.

Substitute (5.21) into (5.16) to then get

$$l_i = \frac{\nu_s}{\varphi(\nu_s + \nu_p)}(p_i - \mathbb{E}[p]),$$

where for simplicity we can define $\beta = \frac{\nu_s}{\varphi(\nu_s+\nu_p)}$. Averaging this over all agents gets:

$$\begin{aligned} \mathbb{E}[l_i] &= \beta \mathbb{E}\left[p_i - \mathbb{E}[p]\right] \\ y &= \beta(p - \mathbb{E}[p]), \end{aligned} \tag{5.22}$$

which was called the "Lucas supply curve" – or the Phillips curve you saw in undergrad AD-AS macroeconomics courses. Think back to your macro lectures and take a look at what (5.22) is saying. It's an upward sloping supply which is positive in prices being above what they were expected to be. The intuition here is that when prices are above their expected level, agents attribute a portion of this to an increase in the relative price (i.e. a real increase) of goods on their island. Furthermore, the higher that β is (i.e., the "cleaner" the signal agents receive about relative price changes), the higher output (and hence labour) will increase when prices are above their expected level. Simply genius.

Tying this back to aggregate demand, set aggregate supply (5.22) equal to aggregate demand (5.18) to get:

$$\beta(p - \mathbb{E}[p]) = m - p.$$

We can then get the aggregate price level:

$$p = \frac{m}{1+\beta} + \frac{\beta}{1+\beta}\mathbb{E}[p]. \tag{5.23}$$

Put this back into aggregate demand to then get the equilibrium level of income:

$$y = \frac{\beta m}{1+\beta} - \frac{\beta}{1+\beta}\mathbb{E}[p]. \tag{5.24}$$

With rational expectations, agents will form their own expectations of the aggregate price level, and so if we take expectations of the aggregate price level we have:

$$\begin{aligned} \mathbb{E}[p] &= \frac{1}{1+\beta}\mathbb{E}[m] + \frac{\beta}{1+\beta}\mathbb{E}[p] \\ \implies \mathbb{E}[p] &= \mathbb{E}[m]. \end{aligned}$$

Note this is eerily similar to money neutrality result we got in the perfect information case (5.20).

Now use the above equation, and to a little add and subtraction, to rewrite (5.23):

$$\begin{aligned} p &= \frac{m}{1+\beta} + \frac{\beta}{1+\beta}\mathbb{E}[p] \\ &= \frac{m + \mathbb{E}[m] - \mathbb{E}[m]}{1+\beta} + \frac{\beta}{1+\beta}\mathbb{E}[m] \\ &= \frac{(1+\beta)}{1+\beta}\mathbb{E}[m] + \frac{m - \mathbb{E}[m]}{1+\beta} \\ p &= \mathbb{E}[m] + \frac{m - \mathbb{E}[m]}{1+\beta}, \end{aligned}$$

then substitute this into (5.18) to get

$$y = \frac{\beta}{1+\beta}(m - \mathbb{E}[m]).$$

This fabulous result is basically the Lucas critique in a nutshell. Changes in aggregate demand (due to monetary policy in this instance) will only have an effect on output so long as shocks to demand are unexpected. Not only that, but the changes in output are proportional to the signal extraction problem that agents face.

5.2.3 Technical aside: First-order stochastic difference equations

A lot of models in economics take the form:

$$y_t = x_t + a\mathbb{E}_t y_{t+1}, \tag{5.25}$$

which just says that y today is determined by x and by tomorrow's expected value of y given the information we have today. But what determines this expected value? Rational Expectations implies a very specific answer. Under Rational Expectations, the agents in the economy understand the equation and formulate their expectation in a way that is consistent with it:

$$\mathbb{E}_t y_{t+1} = \mathbb{E}_t x_{t+1} + a\mathbb{E}_t\mathbb{E}_{t+1} y_{t+2},$$

where we can simplify the second expression on the RHS by the law of iterated expectations (LIE):[6]

$$\mathbb{E}_t y_{t+1} = \mathbb{E}_t x_{t+1} + a\mathbb{E}_t y_{t+2}.$$

Substituting our expression for $\mathbb{E}_t y_{t+1}$ into our expression for y_t yields:

$$y_t = x_t + a\mathbb{E}_t x_{t+1} + a^2\mathbb{E}_t y_{t+2},$$

6. LIE in a nutshell: It is not rational for me expect to have a different expectation next period for y_{t+2} than the one that I have today.

and if we kept repeating this by substituting for $\mathbb{E}_t y_{t+2}$, then $\mathbb{E}_t y_{t+3}$, and so on, we would get:

$$y_t = x_t + a\mathbb{E}_t x_{t+1} + a^2\mathbb{E}_t x_{t+2} + ... + a^{N-1}\mathbb{E}_t x_{t+N-1} + a^N\mathbb{E}_t y_{t+N},$$
$$\Leftrightarrow y_t = \sum_{j=0}^{N-1} a^j\mathbb{E}_t x_{t+j} + a^N\mathbb{E}_t y_{t+N},$$

where usually we assume that

$$\lim_{N\to\infty} a^N\mathbb{E}_t y_{t+N} = 0.$$

So, the solution is:

$$y_t = \sum_{k=0}^{\infty} a^k\mathbb{E}_t x_{t+k}. \tag{5.26}$$

This solution underlies the logic of a very large amount of modern macroeconomics.

5.2.4 Example of a first-order difference equation

Consider an asset[7] that can be purchased today for price P_t and which yields a dividend D_t. Suppose there is a close alternative to this asset that will yield a guaranteed rate of return of r. Then, a risk neutral investor will only invest in the asset if it yields the same rate of return, i.e., if

$$\frac{D_t + \mathbb{E}_t P_{t+1}}{P_t} = 1 + r. \tag{5.27}$$

We can rearrange this to get:

$$P_t = \frac{D_t}{1+r} + \frac{\mathbb{E}_t P_{t+1}}{1+r},$$

and then iterating forward we get:

$$P_t = \sum_{j=0}^{\infty} \left(\frac{1}{1+r}\right)^{j+1} \mathbb{E}_t D_{t+j}. \tag{5.28}$$

This equation, which states that asset prices should equal a discounted present-value sum of expected future dividends is usually known as the dividend-discount model.

5.2.5 Forward and backward solutions

The model

$$y_t = x_t + a\mathbb{E}_t y_{t+1} \tag{5.29}$$

can also be written as:

$$y_t = x_t + ay_{t+1} + a\varepsilon_{t+1},$$

7. This is a simple "Lucas tree" type of asset.

where ε_{t+1} is a forecast error that cannot be predicted at date t. Moving the time subscripts back one period and rearranging this yields:

$$y_t = a^{-1} y_{t-1} - a^{-1} x_{t-1} - \varepsilon_t.$$

This backward-looking equation which can also be solved via recursive substitution to give:

$$y_t = -\sum_{j=0}^{\infty} a^{-j} \varepsilon_{t-j} - \sum_{j=1}^{\infty} a^{-j} x_{t-j}. \tag{5.30}$$

The forward and backward solutions are both correct solutions to the first-order stochastic difference equation (as are all linear combinations of them). Which solution we choose to work with depends on the value of the parameter a. If $|a| > 1$, then the weights on future values of x_t in the forward solution (5.26) will explode. In this case, it is most likely that the forward solution will not converge to a finite sum. Even if it does, the idea that today's value of y_t depends more on values of x_t far in the distant future than it does on today's values is not one that we would be comfortable with. In this case, practical applications should focus on the backwards solutions.

However, the equation holds for any set of shocks ε_t such that $\mathbb{E}_{t-1}\varepsilon_t = 0$. So the solution is indeterminate: We can't actually predict what will happen with y_t even if we knew the full path for x_t.

But if $|a| < 1$, then the weights in the backwards solution are explosive and the forward solution is the one to focus on. Also, this solution is determinate. Knowing the path of x_t will tell you the path of y_t. In most cases, it is assumed that $|a| < 1$, and we can assume that

$$\lim_{n\to\infty} a^n \mathbb{E}_t y_{t+n} = 0,$$

amounts to a statement that y_t can't grow too fast.

What if it doesn't hold? Then the solution can have other elements. Let

$$y_t^* = \sum_{j=0}^{\infty} a^j \mathbb{E}_t x_{t+j},$$

and let $y_t = y_t^* + b_t$ be any other solution. The solution must satisfy

$$y_t^* + b_t = x_t + a\mathbb{E}_t y_{t+1}^* + a\mathbb{E}_t b_{t+1}.$$

By construction, one can show that $y_t^* = x_t + a\mathbb{E}_t y_{t+1}^*$. Now, the above equation means that the additional component satisfies

$$b_t = a\mathbb{E}_t b_{t+1},$$

and because $|a| < 1$, this means that b is always expected to get bigger in absolute value, going to infinity in expectation. This is a bubble. Note that the term bubble is usually associated with irrational behaviour by investors. But in this simple model, the agents have rational expectations. This is a rational bubble.

There may be restrictions in the real economy that stop b from growing forever. But constant growth is not the only way to satisfy $b_t = a\mathbb{E}_t b_{t+1}$. The following process also works:

$$b_{t+1} = \begin{cases} (aq)^{-1}b_t + e_{t+1}, & \text{w.p. } q, \\ e_{t+1}, & \text{w.p. } 1-q, \end{cases}$$

where $\mathbb{E}_t e_{t+1} = 0$. This is a bubble that everyone knows is going to crash eventually. And even then, a new bubble can get going. Imposing $\lim_{n\to\infty} a^n \mathbb{E}_t y_{t+n} = 0$ rules out bubbles of this (or any other) form.

5.3 The DSGE recipe

The forward solution to (5.29),

$$y_t = \sum_{j=0}^{\infty} a^j \mathbb{E}_t x_{t+j},$$

provides useful insights into how the variable y_t is determined. However, without some assumptions about how x_t evolves over time, it cannot be used to give precise predictions about the dynamics of y_t (and ideally, we want to be able to simulate the behaviour of y_t).

One reason why there is a strong linkage between DSGE modelling and VARs is because we assume that the exogenous "driving variables" such as x_t are generated by backward-looking time series models like in VARs. Consider for instance the case where the process driving x_t is AR(1),

$$x_t = \rho x_{t-1} + \varepsilon_t, \quad |\rho| < 1.$$

In this case, we have

$$\mathbb{E}_t x_{t+j} = \rho^j x_t.$$

Now the model's solution can be written as

$$y_t = \left[\sum_{j=0}^{\infty} (a\rho)^j\right] x_t,$$

and because $|a\rho| < 1$, the infinite sum converges to

$$\sum_{j=0}^{\infty} (a\rho)^j = \frac{1}{1-a\rho}.$$

Which should look familiar if you did undergrad macro – it's how we derived the Keynesian multiplier formula. So, in this case, the model solution is

$$y_t = \frac{1}{1 - a\rho} x_t.$$

Macroeconomists call this a reduced-form solution for the model. Together with the equation describing the evolution for x_t, it can be easily simulated on a computer (e.g., Dynare will do this for you automatically).

While this example is obviously very simple, it illustrates the general principle for getting predictions from DSGE models:

1. Obtain structural equations involving expectations of future driving variables (in this case, the $\mathbb{E}_t x_{t+j}$ terms).
2. Make assumptions about the time series process for the driving variables (in this case, x_t).
3. Solve for a reduced-form solution that can be simulated on the computer along with the driving variables.

Finally, note that the reduced-form of this model also has a VAR-like representation, which can be shown as follows

$$\begin{aligned} y_t &= \frac{1}{1 - a\rho} (\rho x_{t-1} + \varepsilon_t) \\ &= \rho y_{t-1} + \frac{1}{1 - a\rho} \varepsilon_t. \end{aligned}$$

So both the x_t and y_t series have purely backward-looking representations. Even this simple model helps to explain how theoretical models tend to predict that the data can be described well using a VAR.

5.3.1 Second order stochastic difference equations

First, let's define the term "jump variable," as it's a concept that will pop up a lot when solving DSGE models. Variables that are characterised by

$$y_t = \sum_{j=0}^{\infty} a^j \mathbb{E}_t x_{t+j}, \tag{5.31}$$

are jump variables. They only depend on what's happening today what's expected to happen tomorrow. If expectations about the future change, they will jump. Nothing that happened in the past will restrict their movement. This may be an okay characterisation of financial variables like stock prices but it's harder to argue with it as a description of variables in the real economy like employment, consumption, or investment.

Many models in macroeconomics feature variables which depend on both the expected future value and their past values. They are characterised by second-order difference equations of the form

$$y_t = ay_{t-1} + b\mathbb{E}_t y_{t+1} + x_t \tag{5.32}$$

Here's one way of solving second order stochastic difference equations. Suppose there was a value, λ, such that the expression,

$$v_t = y_t - \lambda y_{t-1},$$

followed a first order stochastic difference equation of the form:

$$v_t = \alpha \mathbb{E}_t v_{t+1} + \beta x_t.$$

If such a value of λ existed, we would know how to solve for v_t, and then back out the values for y_t. From the fact that $y_t = v_t + \lambda y_{t-1}$, we can rewrite the original equation as:

$$\begin{aligned} v_t + \lambda y_{t-1} &= ay_{t-1} + b(\mathbb{E}_t v_{t+1} + \lambda y_t) + x_t \\ &= ay_{t-1} + b\mathbb{E}_t v_{t+1} + b\lambda(v_t + \lambda y_{t-1}) + x_t, \end{aligned}$$

which after rearranging yields:

$$(1 - b\lambda)v_t = b\mathbb{E}_t v_{t+1} + x_t + (b\lambda^2 - \lambda + a)y_{t-1}, \tag{5.33}$$

which is now a first order stochastic difference equation in v_t! So just to recap, we postulated that there existed a λ such that the variable, v_t, it defined followed a first order stochastic difference equation, and whereby it satisfies the condition:

$$b\lambda^2 - \lambda + a = 0.$$

This is a quadratic equation, so there are two values of λ that satisfy it. For either of these values, we can characterise v_t by

$$\begin{aligned} v_t &= \frac{b}{1 - b\lambda}\mathbb{E}_t v_{t+1} + \frac{1}{1 - b\lambda} x_t \\ &= \frac{b}{1 - b\lambda}\left[\frac{b}{1 - b\lambda}\mathbb{E}_t v_{t+2} + \frac{1}{1 - b\lambda} x_{t+1}\right] + \frac{1}{1 - b\lambda} x_t \\ &= \frac{b}{1 - b\lambda}\left[\frac{b}{1 - b\lambda}\left[\frac{b}{1 - b\lambda}\mathbb{E}_t v_{t+3} + \frac{1}{1 - b\lambda} x_{t+2}\right] + \frac{1}{1 - b\lambda} x_{t+1}\right] + \frac{1}{1 - b\lambda} x_t \\ &\vdots \\ \implies v_t &= \frac{1}{1 - b\lambda}\sum_{j=0}^{\infty}\left(\frac{b}{1 - b\lambda}\right)^j \mathbb{E}_t x_{t+j}, \end{aligned}$$

which as you can see, is a jump variable, and y_t obeys:

$$y_t = \lambda y_{t-1} + \frac{1}{1-b\lambda} \sum_{j=0}^{\infty} \left(\frac{b}{1-b\lambda} \right)^j \mathbb{E}_t x_{t+j}.$$

Usually, only one of the potential values of λ is less than one in absolute value, so this delivers the unique stable solution.[8]

5.4 Systems of stochastic difference equations

5.4.1 Introduction

Thus far, we have only looked at a single equation linking two variables. However, it turns out that the logic of the first-order stochastic difference equation underlies the solution methodology for just about all rational expectations models. Suppose one has a vector of variables:

$$\boldsymbol{Z}_t = \begin{bmatrix} z_{1,t} \\ z_{2,t} \\ \vdots \\ z_{n,t} \end{bmatrix}.$$

It turns out that a lot of macroeconomic models can be represented by an equation of the form

$$\boldsymbol{Z}_t = \boldsymbol{B} \mathbb{E}_t \boldsymbol{Z}_{t+1} + \boldsymbol{X}_t, \tag{5.34}$$

where $\boldsymbol{B}$ is an $n \times n$ matrix. The logic of recursive or iterated substitution can also be applied to this model to give a solution of the form:

$$\boldsymbol{Z}_t = \sum_{j=0}^{\infty} \boldsymbol{B}^j \mathbb{E}_t \boldsymbol{X}_{t+j}. \tag{5.35}$$

5.4.2 Eigenvalues and eigenvectors

As with the single-equation model, this will only give you a stable non-explosive solution under certain conditions. A value, λ_i, is an eigenvalue of the matrix $\boldsymbol{B}$ if there exists an "eigenvector" $\boldsymbol{e}_i$ such that:

$$\boldsymbol{B}\boldsymbol{e}_i = \lambda_i \boldsymbol{e}_i.$$

Many $n \times n$ matrices have n distinct eigenvalues. Denote by $\boldsymbol{P}$ the matrix that has as its columns n eigenvectors corresponding to these eigenvalues. In this case:

$$\boldsymbol{B}\boldsymbol{P} = \boldsymbol{P}\boldsymbol{\Lambda},$$

8. If this all seems a bit abstract now, don't worry. We will go more in-depth in the next section.

where

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & & & \boldsymbol{O} \\ & \lambda_2 & & \\ & & \ddots & \\ \boldsymbol{O} & & & \lambda_n \end{bmatrix}$$

is a diagonal matrix of eigenvalues, and $\boldsymbol{O}$ denotes triangular matrices.[9] Now assume that we can write:

$$\boldsymbol{B} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1}. \tag{5.36}$$

This tells us something about the relationship between eigenvalues and higher powers of $\boldsymbol{B}$, because:

$$\boldsymbol{B}^n = \boldsymbol{P}\boldsymbol{\Lambda}^n\boldsymbol{P}^{-1} = \boldsymbol{P} \begin{bmatrix} \lambda_1^n & & & \boldsymbol{O} \\ & \lambda_2^n & & \\ & & \ddots & \\ \boldsymbol{O} & & & \lambda_n^n \end{bmatrix} \boldsymbol{P}^{-1}.$$

So, the difference between lower and higher powers of $\boldsymbol{B}$ is that the higher powers depend on the eigenvalues taken to the power of n. If all of the eigenvalues are inside the unit circle (i.e., less than one in absolute value) then all of the entries in $\boldsymbol{B}^n$ will tend towards zero as $n \to \infty$. So, a condition that ensures that model of the form (5.34) has unique stable forward-looking solution is that the eigenvalues of $\boldsymbol{B}$ are all inside the unity circle.

How are eigenvalues calculated? Consider a simple 2×2 matrix,

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},$$

and suppose that $\boldsymbol{A}$ has two eigenvalues, λ_1 and λ_2, and define $\boldsymbol{\lambda}$ as the vector:

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}.$$

The fact that there are eigenvectors which when multiplied by $\boldsymbol{A} - \boldsymbol{\lambda I}$ equal a vector of zeros (i.e., we have $(\boldsymbol{A} - \boldsymbol{\lambda I})\boldsymbol{e}_i = \boldsymbol{0}$) means that the determinant of the matrix,

$$\boldsymbol{A} - \boldsymbol{\lambda I} = \begin{bmatrix} a_{11} - \lambda_1 & a_{12} \\ a_{21} & a_{22} - \lambda_2 \end{bmatrix},$$

equals zero, i.e.,

$$\det(\boldsymbol{A} - \boldsymbol{\lambda I}) = 0.$$

9. Just be aware that sometimes I will use the matrix $\boldsymbol{O}$ to signify a square null matrix; other times, such as this, I use it in substitution for a void of null values. These two cases should be clear from context. For more information on this notation see Turkington (2013).

So, solving the quadratic formula:

$$(a_{11} - \lambda_1)(a_{22} - \lambda_2) - a_{12}a_{21} = 0,$$

gives the two eigenvalues of $\boldsymbol{A}$.

5.4.3 The Binder-Pesaran method

Consider a vector $\boldsymbol{Z}_t$ characterised by

$$\boldsymbol{Z}_t = \boldsymbol{A}\boldsymbol{Z}_{t-1} + \boldsymbol{B}\mathbb{E}_t\boldsymbol{Z}_{t+1} + \boldsymbol{H}\boldsymbol{X}_t. \tag{5.37}$$

The restriction to one-lag and one-lead form is apparent, and the companion matrix trick can be used to allow this model to represent models with n leads and lags. In this sense, this equation summarises all possible linear rational expectations models.

Binder and Pesaran (1996) solved this model in a manner exactly analogous to the second-order difference equation discussed earlier: find a matrix $\boldsymbol{C}$ such that $\boldsymbol{W}_t = \boldsymbol{Z}_t - \boldsymbol{C}\boldsymbol{Z}_{t-1}$ obeys a first-order matrix equation of the form

$$\boldsymbol{W}_t = \boldsymbol{F}\mathbb{E}_t\boldsymbol{W}_{t+1} + \boldsymbol{G}\boldsymbol{X}_t.$$

In other words, we transform the problem of solving the "second-order" system in equation (5.37) into a simpler first-order system.

What must matrix $\boldsymbol{C}$ be? Using the fact that $\boldsymbol{Z}_t = \boldsymbol{W}_t + \boldsymbol{C}\boldsymbol{Z}_{t-1}$, the model can be rewritten as:

$$\begin{aligned}\boldsymbol{W}_t + \boldsymbol{C}\boldsymbol{Z}_{t-1} &= \boldsymbol{A}\boldsymbol{Z}_{t-1} + \boldsymbol{B}(\mathbb{E}_t\boldsymbol{W}_{t+1} + \boldsymbol{C}\boldsymbol{Z}_t) + \boldsymbol{H}\boldsymbol{X}_t \\ &= \boldsymbol{A}\boldsymbol{Z}_{t-1} + \boldsymbol{B}(\mathbb{E}_t\boldsymbol{W}_{t+1} + \boldsymbol{C}(\boldsymbol{W}_t + \boldsymbol{C}\boldsymbol{Z}_{t-1})) + \boldsymbol{H}\boldsymbol{X}_t.\end{aligned}$$

This rearranges to:

$$(\boldsymbol{I} - \boldsymbol{B}\boldsymbol{C})\boldsymbol{W}_t = \boldsymbol{B}\mathbb{E}_t\boldsymbol{W}_{t+1} + (\boldsymbol{B}\boldsymbol{C}^2 - \boldsymbol{C} + \boldsymbol{A})\boldsymbol{Z}_{t-1} + \boldsymbol{H}\boldsymbol{X}_t.$$

Because $\boldsymbol{C}$ is the matrix such that $\boldsymbol{W}_t$ follows a first-order forward-looking matrix equation, it follows that

$$\boldsymbol{B}\boldsymbol{C}^2 - \boldsymbol{C} + \boldsymbol{A} = \boldsymbol{O}.$$

This "matrix quadratic equation" can be solved to give $\boldsymbol{C}$. Solving these equations is non-trivial, however. One method uses the fact that $\boldsymbol{C} = \boldsymbol{B}\boldsymbol{C}^2 + \boldsymbol{A}$ to solve iteratively is as follows. Provide an initial guess, say $\boldsymbol{C}_0 = \boldsymbol{I}$, and then iterate on $\boldsymbol{C}_n = \boldsymbol{B}\boldsymbol{C}_{n-1}^2 + \boldsymbol{A}$ until all the entries in $\boldsymbol{C}_n$

converge. Once we know $\boldsymbol{C}$, we have:

$$\boldsymbol{W}_t = \boldsymbol{F}\mathbb{E}_t\boldsymbol{W}_{t+1} + \boldsymbol{G}\boldsymbol{X}_t,$$

where

$$\boldsymbol{F} = (\boldsymbol{I} - \boldsymbol{B}\boldsymbol{C})^{-1}\boldsymbol{B},$$
$$\boldsymbol{G} = (\boldsymbol{I} - \boldsymbol{B}\boldsymbol{C})^{-1}\boldsymbol{H}.$$

Assuming that all the eigenvalues of $\boldsymbol{F}$ are inside the unit circle, this has a stable forward-looking solution:

$$\boldsymbol{W}_t = \sum_{j=0}^{\infty} \boldsymbol{F}^j \mathbb{E}_t[\boldsymbol{G}\boldsymbol{X}_{t+j}],$$

which can be written in terms of the original equation as:

$$\boldsymbol{Z}_t = \boldsymbol{C}\boldsymbol{Z}_{t-1} + \sum_{j=0}^{\infty} \boldsymbol{F}^j \mathbb{E}_t[\boldsymbol{G}\boldsymbol{X}_{t+j}].$$

Finally, consider the case in which the driving variables $\boldsymbol{X}_t$ follow a VAR representation of the form:

$$\boldsymbol{X}_t = \boldsymbol{D}\boldsymbol{X}_{t-1} + \boldsymbol{\varepsilon}_t,$$

where $\boldsymbol{D}$ has eigenvalues inside the unit circle. This implies $\mathbb{E}_t\boldsymbol{X}_{t+j} = \boldsymbol{D}^j\boldsymbol{X}_t$, so the model solution is:

$$\boldsymbol{Z}_t = \boldsymbol{C}\boldsymbol{Z}_{t-1} + \left[\sum_{j=0}^{\infty} \boldsymbol{F}^j\boldsymbol{G}\boldsymbol{D}^j\right]\boldsymbol{X}_t.$$

The infinite sum in this equation will converge to a matrix $\boldsymbol{P}$, so the model has a reduced-form representation:

$$\boldsymbol{Z}_t = \boldsymbol{C}\boldsymbol{Z}_{t-1} + \boldsymbol{P}\boldsymbol{X}_t,$$

which can be simulated along with the VAR process for the driving variables. This provides a relatively simple recipe for simulating DSGE models: Specify the $\boldsymbol{A}, \boldsymbol{B}$, and $\boldsymbol{H}$ matrices; solve for $\boldsymbol{C}, \boldsymbol{F}$, and $\boldsymbol{G}$; specify a VAR process for the driving variables; and then obtain the reduced form representations.

5.5 Comments and key readings

In addition to the examples shown in this section, other classical references to gain a better understanding of how VAR models work are Eichenbaum and Evans (1995) and Christiano, Eichenbaum, and Evans (1999, 2005). These seminal papers talk a lot about the effects of monetary policy, which I've wanted to keep to a minimum here because we have yet to set up the New Keynesian model.

But for those interested, you can also take a look at some extensions to basic VAR modelling such as sign restrictions (Uhlig, 2005) and proxy VARs and instruments (Romer and Romer, 2004; Gertler and Karadi, 2015; Jarociński and Karadi, 2020). There is, of course, a vast amount of literature on Bayesian VARs – which is natural when one thinks about the utility of forming priors based on economic theory.

In more recent years, there has also been a big discussion in the macroeconometrics literature about an alternative to the VAR model: estimation based on local projection (LP) (Jordà, 2005; Miranda-Agrippino and Ricco, 2021). The discussion is beyond the scope of this course and these notes. But those interested on this debate in empirical macroeconomics should thoroughly read Li, Plagborg-Møller, and Wolf (2022) and follow the discussion by Fabio Canova.

Speaking of which, I would recommend the toolkits of Canova[10] and Ambrogio Cesa-Bianchi[11] for those looking for easy-to-use software packages to do VAR estimation (and LP and BVAR for the case of Canova's toolkit). While those toolkits are MATLAB based, each statistical language has their ecosystem of toolkits for empirical work. So if you're a Python or R user, there should be plenty of options available.

All these mathematical "prerequisites" were certainly a lot to digest. If some of it has gone over your head then not to worry – we'll revisit some of these concepts later on with some more examples. Without further ado, let's move onto the RBC model...

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6 The Real Business Cycle Model

6.1 Introduction

As we saw in the first section, modern economies undergo significant short-run fluctuations in aggregate output and employment. What's more is that these fluctuations don't really follow a pattern that we can heuristically predict or forecast easily. We do, however, know that these fluctuations have some intriguing characteristics. Understanding the causes and characteristics of these aggregate fluctuations is a central goal of macroeconomics. Critically, by understanding these factors, we can build models which can replicate business cycle moments and to hopefully consider optimal policy responses to these fluctuations.

In this section (and the ones that follow), we develop the leading theories concerning the causes and nature of macroeconomic fluctuations. We have so far worked with rudimentary general equilibrium/Walrasian models, and have slowly been increasing our proficiency by building evermore sophisticated models. Now, we are ready to take on the challenge of building a Walrasian model in order to explain business cycles.

First, it's important to state the assumptions we'll be making. Our Walrasian model will feature perfectly competitive markets without externalities, asymmetric information, missing markets, or other imperfections. All the neoclassical models we've looked at so far have featured these assumptions, so can we pick a familiar to model to build upon? The Ramsey model seems like a very good candidate to start with. We know that absent of any shocks, the Ramsey model with converge to a balanced growth path, and then grows smoothly. It then seems sensible to incorporate business cycle fluctuations and shocks into the Ramsey model. From there, we can also look at things like worker productivity and government purchases. Because the Ramsey model features no money or prices,[1] the shocks we will introduce to the Ramsey model will all be in real terms. These shocks will thus change the actual productive capacity of the economy. Hence, the modified Ramsey model is known as the Real Business Cycle (RBC) model.

The RBC model also features one other significant departure from the Ramsey model: labour will be endogenous and will be allowed to vary. Instead of just optimising over consumption in each period, households in the RBC model will be able to choose how much time they would like to allocate between working and leisure. The motivation for this is, again, wanting to build a model which can explain the business cycle facts we looked at in the opening section. We saw that despite investment and output fluctuating quite wildly throughout the business cycle, working hours were quite invariant. We hope that our RBC model can capture this phenomenon.

As we will soon find out, however, the RBC model does a pretty poor job of explaining actual fluctuations. Thus, we will have to move beyond the baseline RBC model to far more sophisticated models – as is standard in the discipline of macroeconomics. At the same time, however, what these models are trying to accomplish remains the ultimate goal of business cycle research: building a

1. More accurately, the Ramsey model does not feature any nominal variables. Output is treated as a numeraire good. Households consume and save it, and get paid in units of it. Think of it like something like rice or corn.

general equilibrium model from microeconomic foundations and a specification of the underlying shocks that explains, both qualitatively and quantitatively, the main features of macroeconomic fluctuations. Despite its empirical failings, the RBC model established a research agenda which remains as the central orthodoxy in macroeconomics to this day: the research of dynamic stochastic general equilibrium (DSGE) models. The RBC model represented such a significant departure from the models that came before it, that many economists see it as the progenitor of the current DSGE paradigm.

6.2 The social planner's (centralised) problem

There are a few ways to set up an RBC model: either from the perspective of a benevolent social planner[2] which is able to allocate all resources in the economy, or by setting up competitive markets and finding market equilibria. The Ramsey social planner seeks to maximise social welfare subject to the economy's resource constraints; whereas in competitive markets agents optimise their utility or profit given their endowments. In the RBC model, both approaches yield the same outcome – an important point that we will later come back to.[3] For now, to keep things simple, we will set up the RBC model from the perspective of the Ramsey social planner.

The Ramsey planner solves the following problem:

$$\underset{\{C_t,N_t\}}{\operatorname{argmax}}\ \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s U(C_{t+s}, N_{t+s}), \tag{6.1}$$

where C_t is aggregate consumption, N_t is hours worked or aggregate labour supply, and β is the representative household's rate of time preference (their discount factor) – importantly, let's note that the household experiences disutility from supplying labour.[4] In words, the Ramsey planner wishes to maximise households' welfare by assigning the optimal amounts of consumption and labour supply each period. Furthermore, the Ramsey planner wishes to maximise (6.1) subject to the following economy-wide resource constraints:

$$Y_t = C_t + I_t, \tag{6.2}$$

$$Y_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha}, \tag{6.3}$$

$$K_t = I_t + (1-\delta)K_{t-1}, \tag{6.4}$$

2. Often referred to as the "Ramsey social planner".

3. In macroeconomic models, solving the Ramsey planner's problem yields the social welfare maximising, Pareto-efficient solution. This is because in other models we will look at, markets are not fully competitive or efficient, so the competitive equilibrium will be unable to achieve a first-best outcome. Here, in the RBC model where all markets are complete and efficient, the Ramsey policy coincides with the competitive market solution.

4. Alternatively we could write:

$$U(C_t, N_t) = u(C_t) - v(N_t),$$

where $u(\cdot)$ and $v(\cdot)$ are subutility functions.

and a process for the technology shock term A_t:[5]

$$\ln A_t = \rho \ln A_{t-1} + \varepsilon_t,\ \varepsilon_t \overset{\text{IID}}{\sim} \mathcal{N}(0, \sigma_a^2), \tag{6.5}$$

where Y_t is output, I_t is investment into new capital, A_t is total factor productivity term, K_t is productive capital, and δ is the depreciation rate.

The question that arises now is: how do we go about maximising (6.1)? The main issue is that we have a stream of future consumption and labour decisions to make, constrained to the fact that we don't know what A_t will be in the future. Technically, the best way to solve this problem is using stochastic dynamic programming,[6] but we don't have time for that. Instead, we will use a trick and simplification: we treat the Ramsey problem as a deterministic problem and then substitute $\mathbb{E}_t X_{t+i}$ for X_{t+i}. Can we do this? Sure.

Suppose

$$G(x) = \sum_{j=1}^{N} p_j F(a_j, x),$$

which is maximised by setting

$$G'(x) = \sum_{j=1}^{N} p_j F'(a_j, x) = \mathbb{E}_t F'(x) = 0,$$

so, the FOCs for maximising $\mathbb{E}_t F(x)$ are just $\mathbb{E}_t F'(x) = 0$.

Now, we can combine our constraints to simply get:

$$A_t K_{t-1}^{\alpha} N_t^{1-\alpha} = C_t + K_t - (1-\delta) K_{t-1}. \tag{6.6}$$

Then, we can set up the Ramsey planner's problem as a Lagrangian:

$$\begin{aligned} \mathcal{L} = {} & \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left[U(C_{t+s}, N_{t+s}) \right] \\ & + \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \lambda_{t+s} \left[A_{t+s} K_{t+s-1}^{\alpha} N_{t+s}^{1-\alpha} + (1-\delta) K_{t+s-1} - C_{t+s} - K_{t+s} \right]. \end{aligned} \tag{6.7}$$

But this is still a hideous equation to work with. It involves two infinite sums, so technically there is an infinite number of FOCs for current and future expected values of consumption, capital, and labour. So, what can we do? This is macroeconomics, so we will use another trick/simplification.

We want to take a snapshot of how the variables behave in the period in which we are optimising in, t. Most of our variables are denoted in period t with the subscript t, so they're fine. But we

5. Technically we have:

$$\ln A_t = (1-\rho) \ln \bar{A} + \rho \ln A_{t-1} + \varepsilon_t.$$

But $\bar{A} = 1$, so $\ln \bar{A} = 0$.

6. Try reading Stokey, Lucas, and Prescott (1989) – but good luck. It's a tough read.

have K_{t-1} and A_{t-1} in the law of motion equations for capital and technology, respectively. So, what we can do is set up the Lagrange with the objective function based in period t, a single constraint dated in period t, and then we can add in a second constraint from period $t+1$. Then, the period t variables appear as:

$$\begin{aligned}\mathcal{L} &= U(C_t, N_t) + \lambda_t(A_t K_{t-1}^{\alpha} N_t^{1-\alpha} + (1-\delta)K_{t-1} - C_t - K_t) \\ &\quad + \beta \mathbb{E}_t \lambda_{t+1} \left(A_{t+1} K_t^{\alpha} N_{t+1}^{1-\alpha} + (1-\delta)K_t - C_{t+1} - K_{t+1} \right).\end{aligned}$$

After that, the period t variables don't ever appear again. So, the FOCs for the period t variables consist of differentiating this equation with respect to these variables and setting the derivatives equal to zero. Then, the period $t+i$ variables appear exactly as the period t variables do, except that they are in expectation form and they are multiplied by the discount rate β^i. But this means that the FOCs for the period $t+i$ variables will be identical to those for period t variables. So differentiating this equation gives us the equations for the optimal dynamics at all times.

Thus, we yield the following FOCs:

$$\mathcal{L}_{C_t} = U_C(C_t) - \lambda_t = 0, \tag{6.8}$$

$$\mathcal{L}_{K_t} = -\lambda_t + \beta \mathbb{E}_t \lambda_{t+1} \left(\alpha \frac{Y_{t+1}}{K_t} + 1 - \delta \right) = 0, \tag{6.9}$$

$$\mathcal{L}_{N_t} = U_N(N_t) + \lambda_t (1-\alpha) \frac{Y_t}{N_t} = 0, \tag{6.10}$$

$$\mathcal{L}_{\lambda_t} = A_t K_{t-1}^{\alpha} N_t^{1-\alpha} + (1-\delta)K_{t-1} - C_t - K_t = 0. \tag{6.11}$$

Easy!

6.3 The Keynes-Ramsey condition (consumption Euler equation)

Define the marginal value of an additional unit of capital next year as

$$R_{t+1}^k = \mathbb{E}_t \frac{\left(\alpha \frac{Y_{t+1}}{K_t} + 1 - \delta \right) Q_{t+1}}{Q_t}, \tag{6.12}$$

where Q_t is the real price of capital. So Q_{t+1}/Q_t can be considered as the "capital gain".

Now, we're going to make an assumption which may seem quite strange or abstract – but it will make sense soon (especially when we solve the RBC model for the decentralised equilibrium). Recall that I mentioned that the RBC model is nested in a environment of perfectly competitive markets, with full information and complete asset markets. We're going to make use of that last point here – because of complete financial markets, we're going to declare the following "no-arbitrage condition":

$$R_t = R_t^k. \tag{6.13}$$

For reasons that we will explore later, this condition just says that the gross return on capital,

R_t^k, is equal to the risk-free gross real interest rate, R_t. Our assumption of perfectly competitive markets also means that the price of capital is constant, $Q_t = Q_{t+1} = ... = Q_{t+s}, \forall s > 1$.

Then, with those abstractions done, the FOC for capital (6.9) can be written as:

$$\lambda_t = \beta \mathbb{E}_t \left[\lambda_{t+1} R_{t+1}\right],$$

and this can be combined with the FOC for consumption (6.8) to yield:

$$U_C(C_t) = \beta \mathbb{E}_t \left[U_C(C_{t+1}) R_{t+1}\right], \tag{6.14}$$

which is nothing but the consumption Euler equation – sometimes referred to as the Keynes-Ramsey condition. As a quick refresher, we can interpret the Keynes-Ramsey condition as: decreasing consumption by Δ today at the cost of $U_C(C_t)\Delta$ in utility; invest to get $R_{t+1}\Delta$ tomorrow; that investment is worth $\beta \mathbb{E}_t \left[U_C(C_{t+1}) R_{t+1} \Delta\right]$ in terms of utility today; and, along the optimal path, an agent must be indifferent between these options.

If we assume CRRA utility and a simple linear technology for the disutility from supplying labour, we can write the utility function as:

$$U(C_t, N_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \eta N_t,$$

then the Keynes-Ramsey condition (6.14) becomes:

$$C_t^{-\sigma} = \beta \mathbb{E}_t C_{t+1}^{-\sigma} R_{t+1},$$

and the intratemporal Euler equation for labour and leisure, derived from (6.10), becomes:

$$-\eta + C_t^{-\sigma}(1-\alpha)\frac{Y_t}{N_t} = 0.$$

6.4 Equilibrium and log-linearisation

The RBC model can be defined by the following seven equations:

$$Y_t = C_t + I_t, \tag{6.15}$$

$$Y_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha}, \tag{6.16}$$

$$K_t = I_t + (1-\delta) K_{t-1}, \tag{6.17}$$

$$R_t = \alpha \frac{Y_t}{K_{t-1}} + 1 - \delta, \tag{6.18}$$

$$C_t^{-\sigma} = \beta \mathbb{E}_t \left[C_{t+1}^{-\sigma} R_{t+1}\right], \tag{6.19}$$

$$\frac{Y_t}{N_t} = \frac{\eta}{1-\alpha} C_t^{\sigma}, \tag{6.20}$$

$$\ln A_t = \rho \ln A_{t-1} + \varepsilon_t, \tag{6.21}$$

so we have seven equations in seven unknown variables. Notice that a lot of the RBC model equations are non-linear – and we haven't discussed any strategies of solving systems of stochastic non-linear equations. So what can we do? Again, this is macroeconomics, so there's a trick: we linearise the model equations via log-linearisation, from which we can then solve the model.

The idea is to use Taylor series approximations. In general, any non-linear function $F(x_t, y_t)$ can be approximated around any point $F(x_t^*, y_t^*)$ using the formula:

$$\begin{aligned} F(x_t, y_t) &= F(x_t^*, y_t^*) + F_x(x_t^*, y_t^*)(x_t - x_t^*) + F_y(x_t^*, y_t^*)(y_t - y_t^*) \\ &\quad + F_{xx}(x_t^*, y_t^*)(x_t - x_t^*)^2 + F_{yy}(x_t^*, y_t^*)(y_t - y_t^*)^2 + F_{xy}(x_t^*, y_t^*)(x_t - x_t^*)(y_t - y_t^*) + ... \end{aligned}$$

If the gap between (x_t, y_t) and (x_t^*, y_t^*) is small, then terms in second and higher powers and cross-terms will all be very small and can be ignored (i.e. a first-order Taylor series approximation will suffice), leaving something like:

$$F(x_t, y_t) \approx \alpha + \beta_1 x_t + \beta_2 y_t.$$

But if we linearise about a point that (x_t, y_t) is far away from (x_t^*, y_t^*) (e.g. this could be because F is very non-linear), then this approximation will not be accurate.[7]

DSGE models use a particular version of this technique. They take logs and then linearise the logs of variables about a simple "steady-state" path in which all real variables are growing at the same rate. The steady-state path is relevant because the stochastic economy will, on average, tend to fluctuate around the values given by this path, making the approximation an accurate one. This will give us a set of linear equations in terms of deviations of the logs of these variables from their steady-state values.

Remember that log-differences are approximately percentage deviations:

$$\log X - \log Y \approx \frac{X - Y}{Y},$$

so this approach gives us a system that expresses variables in terms of their percentage deviations from the steady-state paths. In other words, it can be thought of as giving a system of variables that represents the business-cycle component of the model! Coefficients are elasticities and IRFs are easy to interpret. Also, believe it or not, log-linearisation is easy – we won't have to take a lot of derivatives.

From here, it's important to note down some notation. Let "hatted" variables (e.g. $\hat{X}_t$) denote log-deviations of variables from their steady-state values, denoted by a "bar" (e.g. $\bar{X}$):

$$\hat{X}_t = \log X_t - \log \bar{X}.$$

7. In such a case we would be better off solving the model with numerical nonlinear methods (which we won't cover here).

The key to the log-linearisation method is that every variable can be written as:

$$X_t = \bar{X}\frac{X_t}{\bar{X}} = \bar{X}e^{\hat{X}_t},$$

and the big trick is that a first-order Taylor approximation of $e^{\hat{X}_t}$ is given by:

$$e^{\hat{X}_t} \approx 1 + \hat{X}_t.$$

So, we can write variables as:

$$X_t \approx \bar{X}(1 + \hat{X}_t).$$

The next trick is for variables multiplying each other such as:

$$X_t Y_t \approx \bar{X}\bar{Y}(1 + \hat{X}_t)(1 + \hat{Y}_t) \approx \bar{X}\bar{Y}(1 + \hat{X}_t + \hat{Y}_t),$$

because you set terms like $\hat{X}_t\hat{Y}_t = 0$ since we're looking at small deviations from steady-state and multiplying these small deviations together gives a term close to zero.

Anything else? Nope, that's it. It's also worth noting, however, that there are a few ways to do log-linearisation. The above gives a short-cut, broad picture approach to log-linearisation.[8] It's probably best that we go through a few examples (and that you practice) in order to nail down how log-linearisation works.

6.4.1 The Taylor expansion (standard) method

Consider a nonlinear model that can represented by a set of equations of the general form

$$F(\boldsymbol{X}_t) = \frac{G(\boldsymbol{X}_t)}{H(\boldsymbol{X}_t)}, \tag{6.22}$$

where $\boldsymbol{X}_t$ is a vector of the variables of the model that can include forward-looking variables and lagged variables (jump and state variables, respectively), in addition to contemporaneous variables. The process of log-linearisation is to first take the logs of the functions $F(\cdot)$, $G(\cdot)$, and $H(\cdot)$, and then take a first-order Taylor series approximation. Taking logs of (6.22) gives:

$$\ln F(\boldsymbol{X}_t) = \ln G(\boldsymbol{X}_t) - \ln H(\boldsymbol{X}_t),$$

and taking the first-order Taylor series expansion around the steady state, $\bar{\boldsymbol{X}}$, gives

$$\ln F(\bar{\boldsymbol{X}}) + \frac{F'(\bar{\boldsymbol{X}})}{F(\bar{\boldsymbol{X}})}(\boldsymbol{X}_t - \bar{\boldsymbol{X}}) \approx \ln G(\bar{\boldsymbol{X}}) + \frac{G'(\bar{\boldsymbol{X}})}{G(\bar{\boldsymbol{X}})}(\boldsymbol{X}_t - \bar{\boldsymbol{X}}) - \ln H(\bar{\boldsymbol{X}}) - \frac{H'(\bar{\boldsymbol{X}})}{H(\bar{\boldsymbol{X}})}(\boldsymbol{X}_t - \bar{\boldsymbol{X}}),$$

8. "A Toolkit for Analyzing Nonlinear Dynamic Stochastic Models Easily" Uhlig (1998) gives a very rigorous treatment of log-linearisation.

where the notation $X'(\bar{\boldsymbol{X}})$ is used to indicate the gradient at the steady state. Notice that the model is now linear in $\boldsymbol{X}_t$, since $F'(\bar{\boldsymbol{X}})/F(\bar{\boldsymbol{X}})$, $G'(\bar{\boldsymbol{X}})/G(\bar{\boldsymbol{X}})$, $H'(\bar{\boldsymbol{X}})/H(\bar{\boldsymbol{X}})$, $\ln F(\bar{\boldsymbol{X}})$, $\ln G(\bar{\boldsymbol{X}})$, and $\ln H(\bar{\boldsymbol{X}})$ are constants. Since the following holds:

$$\ln F(\bar{\boldsymbol{X}}) = \ln G(\bar{\boldsymbol{X}}) - \ln H(\bar{\boldsymbol{X}}),$$

we can eliminate the three log components and the previous expression simplifies to

$$\frac{F'(\bar{\boldsymbol{X}})}{F(\bar{\boldsymbol{X}})}(\boldsymbol{X}_t - \bar{\boldsymbol{X}}) \approx \frac{G'(\bar{\boldsymbol{X}})}{G(\bar{\boldsymbol{X}})}(\boldsymbol{X}_t - \bar{\boldsymbol{X}}) - \frac{H'(\bar{\boldsymbol{X}})}{H(\bar{\boldsymbol{X}})}(\boldsymbol{X}_t - \bar{\boldsymbol{X}}).$$

The implicit assumption here is that if we stay close enough to the steady state, $\bar{\boldsymbol{X}}$, we can ignore the second-order or higher terms of the Taylor expansion – i.e., a first-order approximation is sufficient to capture the dynamics of the model. Let's work through some examples.

Example: The production function. This method works particularly well when you have multiplicative terms. So let's start with our production technology:

$$Y_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha}, \tag{6.23}$$

and then take logs:

$$\log Y_t = \log A_t + \alpha \log K_{t-1} + (1-\alpha)\log N_t,$$

where we know that

$$\ln X_t = \ln \bar{X} + \frac{X_t - \bar{X}}{\bar{X}},$$

and so we have:

$$\ln \bar{Y} + \frac{Y_t - \bar{Y}}{\bar{Y}} \approx \ln \bar{A} + \frac{A_t - \bar{A}}{\bar{A}} + \alpha\left[\ln \bar{K} + \frac{K_{t-1} - \bar{K}}{\bar{K}}\right] + (1-\alpha)\left[\ln \bar{N} + \frac{N_t - \bar{N}}{\bar{N}}\right],$$

and we know that in the steady-state we have $\ln \bar{Y} = \ln \bar{A} + \alpha \ln \bar{K} + (1-\alpha)\ln \bar{N}$, so

$$\begin{aligned} \frac{Y_t - \bar{Y}}{\bar{Y}} &\approx \frac{A_t - \bar{A}}{\bar{A}} + \alpha\frac{K_{t-1} - \bar{K}}{\bar{K}} + (1-\alpha)\frac{N_t - \bar{N}}{\bar{N}} \\ \Leftrightarrow \hat{Y}_t &= \hat{A}_t + \alpha\hat{K}_{t-1} + (1-\alpha)\hat{N}_t. \end{aligned} \tag{6.24}$$

Example: The growth model. Consider a simple growth model where the representative agent maximises

$$\sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi}\right),$$

subject to the budget constraint,

$$C_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha} + (1-\delta)K_{t-1} - K_t. \tag{6.25}$$

The first-order conditions are:

$$C_t^{-\sigma} = \beta \mathbb{E}_t \left[\alpha A_{t+1} K_t^{\alpha-1} N_{t+1}^{1-\alpha} + (1-\delta) \right] C_{t+1}^{-\sigma}, \tag{6.26}$$

$$N_t^{\varphi} = C_t^{-\sigma} \left[(1-\alpha) A_t K_{t-1}^{\alpha} N_t^{-\alpha} \right]. \tag{6.27}$$

Taking logs of the budget constraint and first order conditions gives:

$$\begin{aligned} \ln C_t &= \ln \left[A_t K_{t-1}^{\alpha} N_t^{1-\alpha} + (1-\delta) K_{t-1} - K_t \right], \\ -\sigma \ln C_t &= \ln \beta + \ln \left[\alpha A_{t+1} K_t^{\alpha-1} N_{t+1}^{1-\alpha} + (1-\delta) \right] - \sigma \ln C_{t+1}, \\ \varphi \ln N_t &= -\sigma \ln C_t + \ln(1-\alpha) + \ln A_t + \alpha \ln K_{t-1} - \alpha \ln N_t. \end{aligned}$$

Then, take the first order Taylor expansions about the steady state (this is always an algebraic nightmare):

$$\begin{aligned} \ln \bar{C} + \frac{1}{\bar{C}}(C_t - \bar{C}) &\approx \ln \left[\bar{A}\bar{K}^{\alpha}\bar{N}^{1-\alpha} + (1-\delta)\bar{K} - \bar{K} \right] + \frac{\bar{K}^{\alpha}\bar{N}^{1-\alpha}}{\bar{A}\bar{K}^{\alpha}\bar{N}^{1-\alpha} + (1-\delta)\bar{K} - \bar{K}}(A_t - \bar{A}) \\ &+ \frac{\alpha\bar{A}\bar{K}^{\alpha-1}\bar{N}^{1-\alpha} + 1 - \delta}{\bar{A}\bar{K}^{\alpha}\bar{N}^{1-\alpha} + (1-\delta)\bar{K} - \bar{K}}(K_{t-1} - \bar{K}) + \frac{(1-\alpha)\bar{A}\bar{K}^{\alpha}\bar{N}^{-\alpha}}{\bar{A}\bar{K}^{\alpha}\bar{N}^{1-\alpha} + (1-\delta)\bar{K} - \bar{K}}(N_t - \bar{N}) \\ &+ \frac{-1}{\bar{A}\bar{K}^{\alpha}\bar{N}^{1-\alpha} + (1-\delta)\bar{K} - \bar{K}}(K_t - \bar{K}), \\ -\sigma \ln \bar{C} - \sigma\frac{1}{\bar{C}}(C_t - \bar{C}) &\approx \ln \beta + \frac{0}{\beta} + \ln\left(\alpha\bar{A}\bar{K}^{\alpha-1}\bar{N}^{1-\alpha} + 1 - \delta \right) + \frac{\alpha\bar{K}^{\alpha-1}\bar{N}^{1-\alpha}}{\alpha\bar{A}\bar{K}^{\alpha-1}\bar{N}^{1-\alpha} + 1 - \delta}(A_{t+1} - \bar{A}) \\ &\frac{(\alpha-1)\alpha\bar{A}\bar{K}^{\alpha-2}\bar{N}^{1-\alpha}}{\alpha\bar{A}\bar{K}^{\alpha-1}\bar{N}^{1-\alpha} + 1 - \delta}(K_t - \bar{K}) + \frac{(1-\alpha)\alpha\bar{A}\bar{K}^{\alpha-1}\bar{N}^{-\alpha}}{\alpha\bar{A}\bar{K}^{\alpha-1}\bar{N}^{1-\alpha} + 1 - \delta}(N_{t+1} - \bar{N}) \\ &- \sigma \ln C - \sigma\frac{1}{\bar{C}}(C_{t+1} - \bar{C}), \\ \varphi \ln \bar{N} + \varphi\frac{1}{\bar{N}}(N_t - \bar{N}) &\approx -\sigma \ln \bar{C} - \sigma\frac{1}{\bar{C}}(C_t - \bar{C}) + \ln(1-\alpha) + \frac{0}{1-\alpha} + \ln \bar{A} + \frac{1}{\bar{A}}(A_t - \bar{A}) \\ &+ \alpha \ln \bar{K} + \alpha\frac{1}{\bar{K}}(K_{t-1} - \bar{K}) - \alpha \ln \bar{N} - \alpha\frac{1}{\bar{N}}(N_t - \bar{N}). \end{aligned}$$

Use steady state identities to get rid of the logs, and do some more algebra (e.g. multiple and divide some of the fractions), and use some steady state facts (like $\bar{A} = 1$) to get the final log-linearised expressions:

$$\begin{aligned} \hat{C}_t = &\frac{1}{1 - \delta\left(\bar{K}/\bar{N}\right)^{1-\alpha}}\hat{A}_t + \frac{\alpha\bar{K}^{\alpha-1}\bar{N}^{1-\alpha} + 1 - \delta}{\bar{K}^{\alpha-1}\bar{N}^{1-\alpha} - \delta}\hat{K}_{t-1} + \frac{1-\alpha}{1 - \delta\left(\bar{K}/\bar{N}\right)^{1-\alpha}}\hat{N}_t \\ &- \frac{1}{\left(\bar{K}/\bar{N}\right)^{\alpha-1} - \delta}\hat{K}_t, \end{aligned} \tag{6.28}$$

$$\begin{aligned} -\sigma\hat{C}_t = &\frac{\alpha\bar{K}^{\alpha-1}\bar{N}^{1-\alpha}}{\alpha\bar{K}^{\alpha-1}\bar{N}^{1-\alpha} + 1 - \delta}\mathbb{E}_t\hat{A}_{t+1} + \frac{(\alpha-1)\alpha\bar{K}^{\alpha-1}\bar{N}^{1-\alpha}}{\alpha\bar{K}^{\alpha-1}\bar{N}^{1-\alpha} + 1 - \delta}\hat{K}_t \\ &+ \frac{(1-\alpha)\alpha\bar{K}^{\alpha-1}\bar{N}^{1-\alpha}}{\alpha\bar{K}^{\alpha-1}\bar{N}^{1-\alpha} + 1 - \delta}\mathbb{E}_t\hat{N}_{t+1} - \sigma\mathbb{E}_t\hat{C}_{t+1}, \end{aligned} \tag{6.29}$$

$$\varphi\hat{N}_t = -\sigma\hat{C}_t + \hat{A}_t + \alpha\hat{K}_{t-1} - \alpha\hat{N}_t \tag{6.30}$$

6.4.2 The Uhlig method

As you can see, the standard Taylor expansion method is extremely cumbersome. Uhlig (1998) recommends using a simpler method for finding log-linear approximations of functions. His method does not require taking derivatives and gives the same results as the above method.

We actually covered this briefly before. But just for clarification, consider an equation of a set of variables, X_t. Define $\hat{X}_t = \ln X_t - \ln\bar{X}$. One can write the original variable as

$$X_t = \bar{X}\exp\hat{X}_t,$$

since

$$\bar{X}\exp\hat{X}_t = \bar{X}\exp\big(\ln X_t - \ln\bar{X}\big) = \bar{X}\exp\left(\ln\left(\frac{X_t}{\bar{X}}\right)\right) = \bar{X}X_t/\bar{X} = X_t.$$

Let's look at an example:

$$\frac{A_t B_t^\alpha}{C_t^\delta} = \frac{\bar{A}\exp\left(\hat{A}_t\right)\bar{B}^\alpha\exp\left(\alpha\hat{B}_t\right)}{\hat{C}^\delta\exp\left(\delta\hat{C}_t\right)},$$

and this becomes

$$\frac{\bar{A}\bar{B}^\alpha}{\bar{C}^\delta}\exp\left(\hat{A}_t + \alpha\hat{B}_t - \delta\hat{C}_t\right).$$

Now we take a Taylor expansion of the exponential term around the steady state – this time it's $\hat{X} = \ln\bar{X} - \ln\bar{X} = 0$!

$$\begin{aligned}\exp\left(\hat{A}_t + \alpha\hat{B}_t - \delta\hat{C}_t\right) &\approx \exp\left(\hat{A} + \alpha\hat{B} - \delta\hat{C}\right) + \exp\left(\hat{A} + \alpha\hat{B} - \delta\hat{C}\right)(\hat{A}_t - \hat{A}) \\ &\quad + \alpha\exp\left(\hat{A} + \alpha\hat{B} - \delta\hat{C}\right)(\hat{B}_t - \hat{B}) - \delta\exp\left(\hat{A} + \alpha\hat{B} - \delta\hat{C}\right)(\hat{C}_t - \hat{C}) \\ &= 1 + \hat{A}_t + \alpha\hat{B}_t - \delta\hat{C}_t.\end{aligned}$$

Thus

$$\frac{\bar{A}\bar{B}^\alpha}{\bar{C}^\delta}\exp\left(\hat{A}_t + \alpha\hat{B}_t - \delta\hat{C}_t\right) = \frac{\bar{A}\bar{B}^\alpha}{\bar{C}^\delta}\left(1 + \hat{A}_t + \alpha\hat{B}_t - \delta\hat{C}_t\right).$$

Example: The production function. Suppose we have:

$$Y_t = A_t K_{t-1}^\alpha N_t^{1-\alpha}.$$

Substitute $X_t = \bar{X}\exp\hat{X}_t$ for each variable:

$$\bar{Y}\exp\hat{Y}_t = \bar{A}\bar{K}^\alpha\bar{N}^{1-\alpha}\exp\left(\hat{A}_t + \alpha\hat{K}_{t-1} + (1-\alpha)\hat{N}_t\right)$$

This is approximated by a Taylor expansion (using the "Uhlig method"):

$$\bar{Y}(1+\hat{Y}_t) = \bar{A}\bar{K}^{\alpha}\bar{N}^{1-\alpha}\left(1+\hat{A}_t+\alpha\hat{K}_{t-1}+(1-\alpha)\hat{N}_t\right),$$

Since in steady state we have $\bar{Y} \equiv \bar{A}\bar{K}^{\alpha}\bar{N}^{1-\alpha}$, we can then write:

$$\hat{Y}_t = \hat{A}_t + \alpha\hat{K}_{t-1} + (1-\alpha)\hat{N}_t.$$

Example: The resource constraint. Start with

$$Y_t = C_t + I_t,$$

now, we could take logs and then do some total derivatives to log-linearise, but it's far easier to use the methodology explained above (often referred to as the Uhlig method). Rewrite our equation as:

$$\begin{aligned} \bar{Y}e^{\hat{Y}_t} &= \bar{C}e^{\hat{C}_t} + \bar{I}e^{\hat{I}_t} \\ \Leftrightarrow \bar{Y}(1+\hat{Y}_t) &= \bar{C}(1+\hat{C}_t) + \bar{I}(1+\hat{I}_t), \end{aligned}$$

and we know that in the steady-state $\bar{Y} \equiv \bar{C} + \bar{I}$, so terms cancel out, so

$$\begin{aligned} \bar{Y}\hat{Y}_t &= \bar{C}\hat{C}_t + \bar{I}\hat{I}_t \\ \therefore \hat{Y}_t &= \frac{\bar{C}}{\bar{Y}}\hat{C}_t + \frac{\bar{I}}{\bar{Y}}\hat{I}_t. \end{aligned}$$

So, in summary, his rules are:

$$\exp\left(\hat{X}_t + a\hat{Y}_t\right) \approx 1 + \hat{X}_t + a\hat{Y}_t, \tag{6.31}$$

$$\hat{X}_t\hat{Y}_t \approx 0, \tag{6.32}$$

$$a\mathbb{E}_t \exp\left(\hat{X}_{t+1}\right) \approx a + a\mathbb{E}_t\hat{X}_{t+1}, \tag{6.33}$$

$$\mathbb{E}_t X_{t+1} = \bar{X}\left(1 + \mathbb{E}_t\hat{X}_{t+1}\right). \tag{6.34}$$

6.4.3 The total derivative method

This method is a bit of a headache, but it does come in handy when we have to deal with messy expressions. It essentially uses the fact that the differential of a variable, say X_t, about its steady-state can be written as $\frac{1}{\bar{X}}dX_t$, where $dX_t = X_t - \bar{X}$. Again, it's better to demonstrate this, so let's take:

$$R_t = \alpha\frac{Y_t}{K_{t-1}} + 1 - \delta,$$

and don't bother taking logs (since we don't have to deal with any power terms); just take the total derivative:

$$\begin{aligned} dR_t &= \alpha \frac{1}{\bar{K}} dY_t - \alpha \frac{\bar{Y}}{\bar{K}^2} dK_{t-1} \\ \Leftrightarrow R_t - \bar{R} &= \alpha \frac{1}{\bar{K}} (Y_t - \bar{Y}) - \alpha \frac{\bar{Y}}{\bar{K}^2} (K_{t-1} - \bar{K}), \end{aligned}$$

and then divide the LHS and RHS by $\bar{R}$, and then do some manipulation to the terms on the RHS:

$$\begin{aligned} \frac{R_t - \bar{R}}{\bar{R}} &= \frac{1}{\bar{R}} \left[\alpha \frac{1}{\bar{K}} (Y_t - \bar{Y}) \frac{\bar{Y}}{\bar{Y}} - \alpha \frac{\bar{Y}}{\bar{K}^2} (K_{t-1} - \bar{K}) \right] \\ \Leftrightarrow \hat{R}_t &= \frac{1}{\bar{R}} \left[\alpha \frac{\bar{Y}}{\bar{K}} \hat{Y}_t - \alpha \frac{\bar{Y}}{\bar{K}} \hat{K}_{t-1} \right], \end{aligned}$$

and then clean up a bit to get:

$$\hat{R}_t = \frac{\alpha}{\bar{R}} \frac{\bar{Y}}{\bar{K}} \left(\hat{Y}_t - \hat{K}_{t-1} \right). \tag{6.35}$$

Now let's look at the Keynes-Ramsey condition since it has an exponent term:

$$C_t^{-\sigma} = \beta \mathbb{E}_t \left[C_{t+1}^{-\sigma} R_{t+1} \right],$$

and then take logs:[9]

$$-\sigma \ln C_t = \ln \beta - \sigma \mathbb{E}_t \ln C_{t+1} + \mathbb{E}_t \ln R_{t+1},$$

then take total derivatives:

$$\begin{aligned} \frac{-\sigma}{\bar{C}} dC_t &= \frac{-\sigma}{\bar{C}} \mathbb{E}_t dC_{t+1} + \frac{1}{\bar{R}} \mathbb{E}_t dR_{t+1} \\ \Leftrightarrow -\sigma \frac{C_t - \bar{C}}{\bar{C}} &= -\sigma \frac{\mathbb{E}_t C_{t+1} - \bar{C}}{\bar{C}} + \frac{\mathbb{E}_t R_{t+1} - \bar{R}}{\bar{R}} \\ -\sigma \hat{C}_t &= -\sigma \mathbb{E}_t \hat{C}_{t+1} + \mathbb{E}_t \hat{R}_{t+1} \\ \hat{C}_t &= \mathbb{E}_t \hat{C}_{t+1} - \frac{1}{\sigma} \mathbb{E}_t \hat{R}_{t+1}. \end{aligned} \tag{6.36}$$

Again, not too difficult since the terms were multiplicative.

Those with eagle eyes are probably outraged that I've seemingly ignored Jensen's Inequality[10] by taking logs of the expected future value of consumption and the interest rate. In general, you would be correct: the log of an expectation is not equal to the expectation of a log term. So let's log-linearise the Keynes-Ramsey condition without applying logs. To start, replace the variables

9. I've taken natural logs here to show that it doesn't matter whether you take logs with base 10 or e.
10. Recall that:

$$\ln \mathbb{E}[x] \geq \mathbb{E}[\ln x].$$

C_t, C_{t+1}, and R_{t+1} using the Uhlig/substitution method and write

$$\begin{aligned}
\left(\bar{C}\exp\left\{\hat{C}_t\right\}\right)^{-\sigma} &= \beta\mathbb{E}_t\left(\bar{C}\exp\left\{\hat{C}_{t+1}\right\}\right)^{-\sigma}\bar{R}\exp\left\{\hat{R}_{t+1}\right\} \\
\left(\exp\left\{\hat{C}_t\right\}\right)^{-\sigma} &= \mathbb{E}_t\left(\exp\left\{\hat{C}_{t+1}\right\}\right)^{-\sigma}\exp\left\{\hat{R}_{t+1}\right\} \\
\mathbb{E}_t\exp\left\{\hat{R}_{t+1}\right\} &= \left(\frac{\mathbb{E}_t\exp\left\{\hat{C}_{t+1}\right\}}{\exp\left\{\hat{C}_t\right\}}\right)^{\sigma} \\
\mathbb{E}_t\exp\left\{\hat{R}_{t+1}\right\} &= \frac{\mathbb{E}_t\exp\left\{\sigma\hat{C}_{t+1}\right\}}{\exp\left\{\sigma\hat{C}_t\right\}}.
\end{aligned}$$

Now apply the approximation by replacing the exponent terms:

$$\begin{aligned}
1+\mathbb{E}_t\hat{R}_{t+1} &= \frac{1+\sigma\mathbb{E}_t\hat{C}_{t+1}}{1+\sigma\hat{C}_t} \\
\left(1+\sigma\hat{C}_t\right)\left(1+\mathbb{E}_t\hat{R}_{t+1}\right) &= 1+\sigma\mathbb{E}_t\hat{C}_{t+1} \\
1+\mathbb{E}_t\hat{R}_{t+1}+\sigma\hat{C}_t+\underbrace{\sigma\hat{C}_t\mathbb{E}_t\hat{R}_{t+1}}_{=0} &= 1+\sigma\mathbb{E}_t\hat{C}_{t+1} \\
\hat{C}_t &= \mathbb{E}_t\hat{C}_{t+1}-\frac{1}{\sigma}\mathbb{E}_t\hat{R}_{t+1},
\end{aligned}$$

which is what we had above.

6.4.4 Taylor approximation method: Single variable case

We won't use this method for the RBC model, but it can come in handy in future applications. Consider the following non-linear first-order difference equation:

$$X_t = f(X_{t-1}),$$

where f is any non-linear functional form you can think of (something not too crazy, though). A first-order Taylor expansion of the RHS about the steady-state gives:

$$X_t \approx f(\bar{X}) + f'(\bar{X})(X_{t-1}-\bar{X}),$$

and in the steady-state if we assume $\bar{X} = f(\bar{X})$,then our Taylor expansion becomes:

$$\begin{aligned}
X_t &\approx \bar{X} + f'(\bar{X})(X_{t-1}-\bar{X}) \\
\Leftrightarrow X_t - \bar{X} &\approx f'(\bar{X})(X_{t-1}-\bar{X})
\end{aligned}$$

then divide this by $\bar{X}$:

$$\frac{X_t - \bar{X}}{\bar{X}} \approx f'(\bar{X}) \frac{\bar{X}_{t-1} - \bar{X}}{\bar{X}},$$

and with a bit cleaning up we have:

$$\hat{X}_t = f'(\bar{X})\hat{X}_{t-1}. \tag{6.37}$$

Consider the following example:

$$K_t = (1-\delta)K_{t-1} + AK_{t-1}^{\alpha},$$

and then apply the formula in (6.37) to get:

$$\hat{K}_t = \left[1 - \delta + \alpha A \bar{K}^{\alpha - 1}\right] \hat{K}_{t-1}.$$

6.4.5 Taylor approximation method: Multivariate case

The Taylor approximation has a vector version as well as a scalar version. Suppose have:

$$X_t = f(X_{t-1}, Y_t),$$

where f is a non-linear function. The vector (bivariate) version of a first-order Taylor expansion about the steady-state is:

$$X_t = f(\bar{X}, \bar{Y}) + f_X(\bar{X}, \bar{Y})(X_{t-1} - \bar{X}) + f_Y(\bar{X}, \bar{Y})(Y_t - \bar{Y}),$$

and again, set the steady-state condition $\bar{X} = f(\bar{X}, \bar{Y})$, and with a bit of rearranging we get:

$$X_t - \bar{X} = f_X(\bar{X}, \bar{Y})(X_{t-1} - \bar{X}) + f_Y(\bar{X}, \bar{Y})(Y_t - \bar{Y}),$$

and then divide through by $\bar{X}$:

$$\frac{X_t - \bar{X}}{\bar{X}} = f_X(\bar{X}, \bar{Y}) \frac{(X_{t-1} - \bar{X})}{\bar{X}} + f_Y(\bar{X}, \bar{Y}) \frac{(Y_t - \bar{Y})}{\bar{X}},$$

use the steady-state trick ("create something out of nothing") on the second term on the RHS:

$$\frac{X_t - \bar{X}}{\bar{X}} = f_X(\bar{X}, \bar{Y}) \frac{(X_{t-1} - \bar{X})}{\bar{X}} + f_Y(\bar{X}, \bar{Y}) \frac{(Y_t - \bar{Y})}{\bar{X}} \frac{\bar{Y}}{\bar{Y}},$$

and then clean up

$$\hat{X}_t \approx f_X(\bar{X}, \bar{Y})\hat{X}_{t-1} + f_Y(\bar{X}, \bar{Y}) \frac{\bar{Y}}{\bar{X}} \hat{Y}_t. \tag{6.38}$$

Consider the following example

$$K_t = (1-\delta)K_{t-1} + sZ_t K_{t-1}^{\alpha},$$

and so taking partial derivatives and following formula in (6.38) gives (you can try and verify it yourself):

$$\hat{K}_t = \left[(1-\delta) + \alpha s \bar{Z} \bar{K}^{\alpha-1}\right] \hat{K}_{t-1} + \left[s\bar{K}^{\alpha}\right] \frac{\bar{Z}}{\bar{Y}} \hat{Z}_t.$$

6.5 Log-linearised system and the steady-state

The full log-linearised system is given by following seven equations:

$$\hat{Y}_t = \frac{\bar{C}}{\bar{Y}}\hat{C}_t + \frac{\bar{I}}{\bar{Y}}\hat{I}_t, \tag{6.39}$$

$$\hat{Y}_t = \hat{A}_t + \alpha\hat{K}_{t-1} + (1-\alpha)\hat{N}_t, \tag{6.40}$$

$$\hat{K}_t = \frac{\bar{I}}{\bar{K}}\hat{I}_t + (1-\delta)\hat{K}_{t-1}, \tag{6.41}$$

$$\hat{R}_t = \frac{\alpha}{\bar{R}}\frac{\bar{Y}}{\bar{K}}\left[\hat{Y}_t - \hat{K}_{t-1}\right], \tag{6.42}$$

$$\hat{C}_t = \mathbb{E}_t\hat{C}_{t+1} - \frac{1}{\sigma}\mathbb{E}_t\hat{R}_{t+1}, \tag{6.43}$$

$$\hat{N}_t = \hat{Y}_t - \sigma\hat{C}_t, \tag{6.44}$$

$$\hat{A}_t = \rho\hat{A}_{t-1} + \varepsilon_t. \tag{6.45}$$

We are almost ready to take this basic RBC model to the computer (e.g., Dynare). We simply need to calibrate the model (macroeconomist speak for assigning values to our structural parameters), and to solve for steady-state values. In other words, we need to obtain numerical values for $\frac{\bar{C}}{\bar{Y}}, \frac{\bar{I}}{\bar{K}}, \frac{\alpha}{\bar{R}}\frac{\bar{Y}}{\bar{K}}$. We can do this by taking the original non-linearised RBC model and figuring out what things look like along a zero-growth path.

Start with the steady-state interest rate. This is linked to consumption behaviour via the consumption Euler equation:

$$1 = \beta\mathbb{E}_t\left(\frac{C_t}{C_{t+1}}\right)^{\sigma} R_{t+1}.$$

Because we have no trend growth in technology in our model, the steady-state features consumption, investment, and output all taking on constant values with no uncertainty. Thus, in steady-state, we have $\bar{C}_t = \bar{C}_{t+1} = \bar{C}$, so

$$\bar{R} = \frac{1}{\beta}. \tag{6.46}$$

In other words, in a no-growth economy, the rate of return on capital is determined by the rate of time preference.

Next, take the equation for the rate of return on capital (in period t):

$$R_t = \alpha \frac{Y_t}{K_{t-1}} + 1 - \delta.$$

In the steady-state we have:

$$\bar{R} = \frac{1}{\beta} = \alpha \frac{\bar{Y}}{\bar{K}} + 1 - \delta,$$

thus, with a bit of rearranging, we get:

$$\frac{\bar{Y}}{\bar{K}} = \frac{\beta^{-1} + \delta - 1}{\alpha}. \tag{6.47}$$

So we have

$$\begin{aligned} \frac{\alpha}{\bar{R}} \frac{\bar{Y}}{\bar{K}} &= [\alpha\beta] \left[\frac{\beta^{-1} + \delta - 1}{\alpha} \right] \\ &= 1 - \beta(1 - \delta), \end{aligned} \tag{6.48}$$

which is one of the steady-state values we needed.

Now, look at the law of motion of capital:

$$K_t = I_t + (1 - \delta) K_{t-1},$$

and use the fact that in the steady-state we have $\bar{K}_t = \bar{K}_{t-1} = \bar{K}$, so:

$$\frac{\bar{I}}{\bar{K}} = \delta, \tag{6.49}$$

which is also what we were looking for.

Putting things together, we have:

$$\frac{\bar{I}}{\bar{Y}} = \frac{\frac{\bar{I}}{\bar{K}}}{\frac{\bar{Y}}{\bar{K}}} = \frac{\delta}{\frac{\beta^{-1} + \delta - 1}{\alpha}} = \frac{\alpha\delta}{\beta^{-1} + \delta - 1}, \tag{6.50}$$

and

$$\frac{\bar{C}}{\bar{Y}} = 1 - \frac{\bar{I}}{\bar{Y}} = 1 - \frac{\alpha\delta}{\beta^{-1} + \delta - 1}. \tag{6.51}$$

So the final, log-linearised RBC model is:

$$\hat{Y}_t = \left[1 - \frac{\alpha\delta}{\beta^{-1} + \delta - 1} \right] \hat{C}_t + \left[\frac{\alpha\delta}{\beta^{-1} + \delta - 1} \right] \hat{I}_t, \tag{6.52}$$

$$\hat{Y}_t = \hat{A}_t + \alpha \hat{K}_{t-1} + (1 - \alpha) \hat{N}_t, \tag{6.53}$$

$$\hat{K}_t = \left[\frac{\alpha\delta}{\beta^{-1} + \delta - 1} \right] \hat{I}_t + (1 - \delta) \hat{K}_{t-1}, \tag{6.54}$$

$$\hat{R}_t = [1 - \beta(1-\delta)]\left[\hat{Y}_t - \hat{K}_{t-1}\right], \tag{6.55}$$

$$\hat{C}_t = \mathbb{E}_t\hat{C}_{t+1} - \frac{1}{\sigma}\mathbb{E}_t\hat{R}_{t+1}, \tag{6.56}$$

$$\hat{N}_t = \hat{Y}_t - \sigma\hat{C}_t, \tag{6.57}$$

$$\hat{A}_t = \rho\hat{A}_{t-1} + \varepsilon_t. \tag{6.58}$$

We will explore the performance of this model via numerical simulation. But first, let's compare this Ramsey social planner equilibrium to the decentralised equilibrium.

6.6 The decentralised equilibrium

As I mentioned previously, there are alternatives in how to set up the RBC model, but these will give us the same outcome. In this section I will now setup the RBC model without the Ramsey social planner. General equilibrium will be achieved via competitive markets as households and firms optimise over their endowments. Furthermore, I will make the assumption that firms own the capital stock in the economy, while the households own the firms – again, whether the firms own the capital stock or households own the capital stock, both will lead to the same outcome.

6.6.1 The household problem

Let there be a continuum of households indexed by $i \in [0,1]$.[11] Each household allocates its time between work and leisure, and it picks a stream of consumption $\{C_t^i\}_{t=0}^{\infty}$ to maximise its present discounted value of lifetime utility. In exchange for supplying labour, households earn a competitive wage, w_t, which they takes as given.

In order to insure themselves against any idiosyncratic risk,[12] each individual household can write and issue state-contingent securities, B_t^i, whereby the counterparty is another household $j \neq i$. Because there is a continuum of households, and because financial markets are complete,[13] the households are fully insured against idiosyncratic risk. The only risk they face is the risk arising from aggregate shocks, A_t. Additionally, in equilibrium, securities are in "zero net supply":

$$\int_0^1 B_t^i di = B_t = 0.$$

All households write and enter into debt contracts with one another; however it's because of this that overall – in aggregate – the sum of all debts must be zero in equilibrium. Again, like most things in macroeconomics, things will be clearer after a bit of derivation.

In addition to taking wages as given, households also take the market clearing real interest rate as given – which we will define soon after obtaining the household's optimality conditions. Additionally, since households own firms, they earns firms' profit in the form of dividend imputations,

11. This means that there are basically a large number or infinite number of individual households.
12. In other words, a shock that adversely (or positively) affects a single individual household.
13. In other words, these securities can be considered as Arrow-Debreu securities.

$\mathcal{D}_t^i$. So, the household problem can be written as:

$$\underset{\{C_t^i, N_t^i, B_t^i\}}{\text{argmax}} \ \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s U(C_{t+s}^i, N_{t+s}^i),$$

subject to

$$C_t^i + B_t^i \leq w_t N_t^i + \mathcal{D}_t^i + R_t B_{t-1}^i. \tag{6.59}$$

Note the timing I have used in the budget constraint for the assets. Here (and throughout these notes) I have assumed "end-of-period" timing. This basically means that households start the period by inheriting assets B_{t-1}, and in the current period t they must pick the asset amount B_t. The assets they picked in period $t-1$ pay out a gross real return of R_t today, and assets picked today, B_t, are expected to pay out a return of R_{t+1} in the next period. Forming a Lagrangian by assuming that an individual's budget constraint (6.59) binds with equality (and using the trick in (6.7)) gives us:

$$\begin{aligned}\mathcal{L}^i = U(C_t^i, N_t^i) + \lambda_t^i \left(w_t N_t^i + \mathcal{D}_t^i + R_t B_{t-1}^i - C_t^i - B_t^i \right) \\ + \beta \mathbb{E}_t \lambda_{t+1}^i \left(w_{t+1} N_{t+1}^i + \mathcal{D}_{t+1}^i + R_t B_t^i - C_{t+1}^i - B_{t+1}^i \right),\end{aligned}$$

and the following FOCs:

$$\mathcal{L}_C^i = U_C(C_t^i) - \lambda_t^i = 0, \tag{6.60}$$

$$\mathcal{L}_N^i = U_N(N_t^i) + \lambda_t^i w_t = 0, \tag{6.61}$$

$$\mathcal{L}_B^i = -\lambda_t^i + \beta \mathbb{E}_t \lambda_{t+1}^i R_{t+1} = 0, \tag{6.62}$$

$$\mathcal{L}_\lambda^i = w_t N_t^i + \mathcal{D}_t^i + R_t B_{t-1}^i - C_t^i - B_t^i = 0. \tag{6.63}$$

These seem very familiar. They're essentially identical to the problem which the Ramsey planner solved. Let's park them here for a bit – we'll return to them later.

6.6.2 The firm problem

There is a representative firm.[14] The firm wants to maximise the present discounted value of real net profits. It discounts future cash flows by a household stochastic discount factor (SDF). The way we'll define the SDF puts cash flows (measured in goods) in terms of current consumption since these firms are ultimately owned by households and the households care about consumption. Define the SDF as:

$$M_{t,t+s}^i = \beta^s \mathbb{E}_t \frac{U_C(C_{t+s}^i)}{U_C(C_t^i)}, \ s > t, \tag{6.64}$$

14. You can either think of a continuum of perfectly competitive firms that employ workers and capital to produce, or you can just aggregate them all together to make one representative firm since they all behave identically. It doesn't matter in the RBC model.

where t is the current period. Why do the firms use this formulation for the stochastic discount factor? Because this is how consumers value future dividend flows. One unit of dividends returned to the household at time $t+s$ generates $U_C(C_{t+s})$ additional units of utility, which must be discounted back to the present period, by β^s. Dividing by $U_C(C_t^i)$ gives the current consumption equivalent value of the future utils.[15]

But, look at (6.60), (6.62), and (6.64). If we combine these equations we can write:

$$\mathbb{E}_t \frac{1}{R_{t+1}} = \beta \mathbb{E}_t \frac{U_C(C_{t+1}^i)}{U_C(C_t^i)} = M_{t,t+1}^i, \quad \forall i.$$

We just did something – well, a few things – pretty great. First, we have written an expression for the household's FOC wrt assets B_t using the definition of the household's SDF. Next, we've managed to define the gross real interest rate as being the inverse of the SDF. Recall back to your undergrad lectures and you should realise that for zero-coupon bonds there is an inverse relationship between yields and prices. For these one-period assets, you can basically think of the SDF as the price and R_t as the yield. Finally, we have – admittedly in an ad-hoc and hand waiving way[16] – managed to drop the index i. This means that we can write the household problem using a representative household setup!

The firm produces output, Y_t, with a CRS production function,

$$Y_t = A_t F(K_{t-1}, N_t),$$

with the usual assumptions that we make. It hires labour, purchases new capital goods, and issues one-period debt promises, D_t. The firm also pays R_t^k on debt issued in the previous period, and the interest paid on debt is equal to the interest paid on assets due to a no-arbitrage condition. The firm's problem can be written as:

$$\mathbb{V}_t = \max_{\{N_t, I_t, D_t, K_t\}} \mathbb{E}_t \sum_{s=0}^{\infty} M_{t,t+s} \left[A_{t+s} F(K_{t-1+s}, N_{t+s}) - w_{t+s} N_{t+s} - I_{t+s} + D_{t+s} - R_{t+s} D_{t-1+s} \right],$$

subject to

$$K_t = I_t + (1-\delta) K_{t-1}.$$

Rearranging the law of motion for capital, and substituting for I_t in the objective function gives us:

$$\mathbb{V}_t = \max_{\{N_t, D_t, K_t\}} \mathbb{E}_t \sum_{s=0}^{\infty} M_{t,t+s} \begin{pmatrix} A_{t+s} F(K_{t-1+s}, N_{t+s}) - K_{t+s} + (1-\delta) K_{t-1+i} \\ -w_{t+s} N_{t+s} + D_{t+s} - R_{t+s}^k D_{t-1+s} \end{pmatrix}$$

15. We will explore this further when look at topics in macro-finance.
16. There is actually a huge literature on modelling heterogeneous agents whereby we cannot just assume a representative agents. We kind of already covered this in OLG models where had young and old households. But modelling a cross-section of agents is beyond the scope of this course.

$$\Leftrightarrow \mathbb{V}_t = \max_{\{N_t, D_t, K_t\}} \left\{ \begin{array}{c} A_t F(K_{t-1}, N_t) - K_t + (1-\delta)K_{t-1} - w_t N_t + D_t - R_t^k D_{t-1} \\ \mathbb{E}_t \left[M_{t,t+1} \left(A_{t+1} F(K_t, N_{t+1}) - K_{t+1} + (1-\delta)K_t - w_{t+1} N_{t+1} + D_{t+1} - R_{t+1}^k D_t \right) \right] \end{array} \right\},$$

which basically says that the firm's revenue each period is equal to output, and that its costs each period are the wage bill, investment in new physical capital, and servicing costs on its debt.

The FOCs from the firm problem are:

$$\frac{\partial \mathbb{V}_t}{\partial N_t} = A_t F_N(K_{t-1}, N_t) - w_t = 0$$

$$\implies w_t = A_t F_N(K_{t-1}, N_t), \tag{6.65}$$

$$\frac{\partial \mathbb{V}_t}{\partial D_t} = 1 - \mathbb{E}_t M_{t,t+1} R_{t+1}^k = 0$$

$$\Leftrightarrow 1 = \mathbb{E}_t \beta \frac{U'(C_{t+1})}{U'(C_t)} R_{t+1}^k$$

$$\implies U_C(C_t) = \beta \mathbb{E}_t U_C(C_{t+1}) R_{t+1}^k, \tag{6.66}$$

$$\frac{\partial \mathbb{V}_t}{\partial K_t} = -1 + \mathbb{E}_t M_{t,t+1} A_{t+1} F_K(K_t, N_{t+1}) + (1-\delta) = 0$$

$$\Leftrightarrow 1 = \mathbb{E}_t \beta \frac{U_C(C_{t+1})}{U_C(C_t)} A_{t+1} F_K(K_t, N_{t+1}) + (1-\delta)$$

$$\implies U_C(C_t) = \beta \mathbb{E}_t U_C(C_{t+1}) A_{t+1} F_K(K_t, N_{t+1}) + (1-\delta). \tag{6.67}$$

Let's interpret these FOCs a bit. (6.65) is pretty intuitive: The wage rate w_t is equal to the marginal productivity of labour. However, look at (6.66) and (6.67) – they're essentially the same, and must therefore hold in equilibrium as long as the household is optimising. In fact they're directly analogous to the consumption Euler equation or Keynes-Ramsey condition in (6.14)! This means that the amount of debt the firm issues is indeterminate, since the condition will hold for any choice of D_t. This is essentially the Modigliani-Miller theorem:[17] it doesn't matter how the firm finances its purchases of new capital – debt or equity – and hence the debt/equity mix is indeterminate.

6.6.3 Technology process

In order to close the model, we need to specify a stochastic process for the exogenous variable(s). The only exogenous variable in this model is A_t – which is the same as before when we solved for the Ramsey social planner equilibrium:

$$\ln A_t = \rho \ln A_{t-1} + \varepsilon_t. \tag{6.68}$$

17. "The Cost of Capital, Corporation Finance and the Theory of Investment" (Modigliani and Miller, 1958).

6.6.4 Competitive equilibrium

A competitive equilibrium is a set of prices $\{R_t, w_t\}$ and allocations $\{C_t, N_t, K_t, D_t, B_t\}$ taking K_{t-1}, D_{t-1}, B_{t-1}, A_{t-1} and the stochastic process for A_t as given; the optimality conditions (6.60)-(6.67); the labour and bond market clearing conditions ($N_t^d = N_t^s$ and $B_t = D_t$, $\forall t$); and both budget constraints holding with equality.

Consolidating the household and firm budget constraints gives:

$$\begin{aligned}
C_t + B_t &= w_t N_t + R_{t-1}B_{t-1} + A_t F(K_{t-1}, N_t) - w_t N_t - I_t + D_t - R_{t-1}D_{t-1} \\
\implies A_t F(K_{t-1}, N_t) &= C_t + I_t,
\end{aligned}$$

in other words, bond market-clearing plus both budget constraints holding just gives the standard accounting identity that output must be consumed or invested.

If you combine the household's FOC for labour supply (6.61) with the firm's FOC, you get:

$$-U_N(N_t) = U_C(C_t)A_t F_N(K_{t-1}, N_t). \tag{6.69}$$

The FOC for bonds/debt (6.67) along with the FOC for the firm's choice of its capital stock (6.66) imply that:

$$\beta \mathbb{E}_t U_C(C_{t+1}) \left[A_{t+1} F_K(K_t, N_{t+1}) + (1-\delta)\right] = \beta \mathbb{E}_t R_{t+1}^k U_C(C_{t+1}),$$

which can be rewritten as:

$$R_{t+1}^k = A_{t+1} F_K(K_t, N_{t+1}) + 1 - \delta, \tag{6.70}$$

which is what we proposed in (6.12).

Putting all the equations together, the equilibrium conditions for the decentralised RBC model are:

$$U_C(C_t) = \beta \mathbb{E}_t U_C(C_{t+1}) \left[A_{t+1} F_K(K_t, N_{t+1}) + 1 - \delta\right], \tag{6.71}$$

$$U_N(N_t) = U_C(C_t) A_t F_N(K_{t-1}, N_t), \tag{6.72}$$

$$K_t = A_t F(K_{t-1}, N_t) - C_t + (1-\delta)K_{t-1}, \tag{6.73}$$

$$\ln A_t = \rho \ln A_{t-1} + \varepsilon_t, \tag{6.74}$$

$$Y_t = A_t F(K_{t-1}, N_t), \tag{6.75}$$

$$Y_t = C_t + I_t, \tag{6.76}$$

$$U_C(C_t) = \beta \mathbb{E}_t U_C(C_{t+1}) R_{t+1}, \tag{6.77}$$

$$w_t = A_t F_N(K_{t-1}, N_t), \tag{6.78}$$

$$R_t = A_t F_K(K_{t-1}, N_t). \tag{6.79}$$

But these are nothing the same as the equilibrium conditions for when we solved for the Ramsey social planner. Why is this the case?

6.7 First and Second Welfare Theorems of Economics

Recall the fundamental welfare theorems of economics from Mas-Colell, Whinston, and Green (1995):

The First Fundamental Welfare Theorem: If the economy is described by complete markets, no externalities or non-convexities then every equilibrium of the competitive market is socially optimal.

The Second Fundamental Welfare Theorem: If household preferences and firm production sets are convex, there is a complete set of markets with publicly known prices, and every agent acts as a price taker, then any Pareto optimal outcome can be achieved as a competitive equilibrium if appropriate lump-sum transfers of wealth are arranged.

The result that the competitive equilibrium of a representative agent economy and that of a perfectly competitive one, that is otherwise identical, is not surprising. The statement of the second fundamental welfare theorem holds for finite dimensional economies. Our economies have an infinite number of periods and, therefore, an infinite number of goods. The conditions for existence of a competitive equilibrium in infinite horizon economies are somewhat more complex than those for finite dimensional ones and some extra assumptions are required.[18] Here, we simply assume that a competitive equilibrium exists and are interested in its relationship to a Ramsey planner economy.

The first fundamental welfare theorem tells us that that any competitive equilibrium is necessarily Pareto optimal, so that the equilibrium found using a decentralised economy with factor and goods markets is also Pareto optimal. The second welfare theorem tells us that, since the production technologies and preferences are the same between the centralised and decentralised economies, then with the right initial wealth conditions, the competitive economy can achieve an equilibrium that is identical to the Ramsey planner economy.

The second fundamental welfare theorem permits us to use a representative agent economy to mimic a competitive economy. Since the second fundamental theorem is carefully worded, it should be clear that using a representative agent economy will not always give the appropriate results. If the economy is not perfectly competitive, if part of the economy has some monopoly power, or if there are some external or internal restrictions that prevent some agents from being perfectly competitive, then the equilibrium found by the decentralised economy will not necessarily be achievable with a representative agent economy.[19]

However, when the conditions are right (like in the RBC model), solving a representative agent economy is often technically much simpler than solving a decentralised economy. In this case, the

18. See Stokey, Lucas, and Prescott (1989) chapter 16 for details.

19. It is partly for this reason that neoclassical models (which invariably satisfy the second welfare theorem) are better understood and articulated than Keynesian models. The latter involve many departures from the welfare theorems and as a result it is far harder to characterise the equilibrium properties of such models.

second fundamental welfare theorem states that, with appropriate initial conditions, the solution of the representative agent economy is one for the decentralised economy.

6.8 Assessing the RBC model

Let's return to our log-linearised RBC model given by (6.52)-(6.58):

$$\begin{aligned}
\hat{Y}_t &= \left(1 - \frac{\alpha\delta}{\beta^{-1}+\delta-1}\right)\hat{C}_t + \left(\frac{\alpha\delta}{\beta^{-1}+\delta-1}\right)\hat{I}_t, \\
\hat{Y}_t &= \hat{A}_t + \alpha\hat{K}_{t-1} + (1-\alpha)\hat{N}_t, \\
\hat{K}_t &= \left(\frac{\alpha\delta}{\beta^{-1}+\delta-1}\right)\hat{I}_t + (1-\delta)\hat{K}_{t-1}, \\
\hat{R}_t &= [1-\beta(1-\delta)]\left(\hat{Y}_t - \hat{K}_{t-1}\right), \\
\hat{C}_t &= \mathbb{E}_t\hat{C}_{t+1} - \frac{1}{\sigma}\mathbb{E}_t\hat{R}_{t+1}, \\
\hat{N}_t &= \hat{Y}_t - \sigma\hat{C}_t, \\
\hat{A}_t &= \rho\hat{A}_{t-1} + \varepsilon_t,
\end{aligned}$$

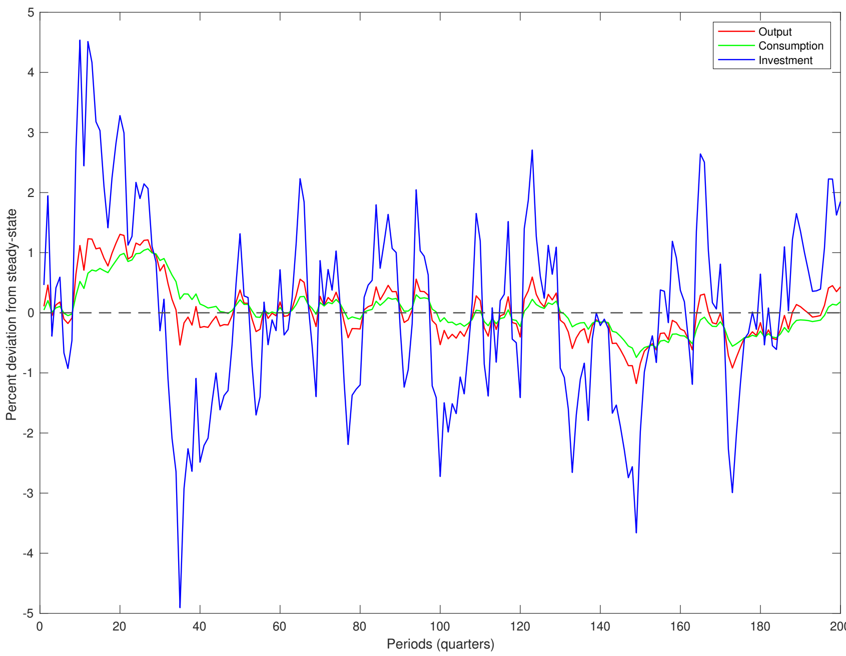
and let's see how the model performs. If we parameterise the model with the following values: $\alpha = 0.3$, $\beta = 0.985$, $\delta = 0.015$, $\rho = 0.85$, $\sigma = 4$, and $\sigma_a^2 = 0.3$ and simulate, what do we get? For those who want to run the model on their own computer, below is the Dynare code for this baseline RBC model:

```
// The RBC Model

// DECLARE VARIABLES
var
Y           $\hat{Y}$                  (long_name='Output')
C           $\hat{C}$                  (long_name='Consumption')
N           $\hat{N}$                  (long_name='Hours Worked')
I           $\hat{I}$                  (long_name='Investment')
K           $\hat{K}$                  (long_name='Capital')
R           $\hat{R}$                  (long_name='Interest Rate')
// TECHNOLOGY
A           $\hat{A}$                  (long_name='TFP')
;
// EXOGENOUS SHOCKS
varexo
EPS_A       $\varepsilon^a$     (long_name='TFP Shock')
;

// PARAMETERS
parameters
betta        $\beta$            (long_name='Discount factor')
alphha       $\alpha$           (long_name='Capital share')
delta        $\delta$           (long_name='Depreciation rate')
sigma        $\sigma$           (long_name='Risk aversion')
rho          $\rho$             (long_name='TFP persistence')
;
// PARAMETERISE
betta = 0.985;
alphha = 0.3;
delta = 0.015;
sigma = 4;
rho = 0.85;

// DECLARE MODEL
model(linear);
[name='Economy resource constraint']
Y = (1 - alphha*delta/(betta^(-1) + delta - 1))*C + (alphha*
```

```
        delta/(betta^(-1) + delta - 1))*I;

    [name='Production']
    Y = A + alphha*K(-1) + (1-alphha)*N;

    [name='Law of motion of capital']
    K = (alphha*delta/(betta^(-1) + delta - 1))*I + (1-delta)*K(-1)
        ;

    [name='Interest rate']
    R = (1 - betta*(1-delta))*(Y - K(-1));

    [name='Consumption Euler equation']
    C = C(+1) - 1/sigma*R(+1);

    [name='Labour supply']
    N = Y - sigma*C;

    [name='TFP process']
    A = rho*A(-1) + 0.3*EPS_A;

    end;

    steady;
    check;
    model_diagnostics;

    // DECLARE SHOCKS
    shocks;
    var EPS_A = 1;
    end;

    stoch_simul(order=1,irf=40,periods=200);
    rplot Y I C;
```

Figure 6.1 shows results from a 200-period simulation of the RBC model. It demonstrates the main successful feature of the RBC model: It generates business cycles and they look very realistic! Reasonable parameterisations of the model can roughly match the magnitude of observed fluctuations in output, and the model can match the fact that investment is far more volatile than consumption. You can see why RBC models were met with high praise when Kydland and Prescott (1982) and Hansen (1985) brought forward the RBC research agenda.

Figure 6.1: RBC Models Generate Cycles with Volatile Investment

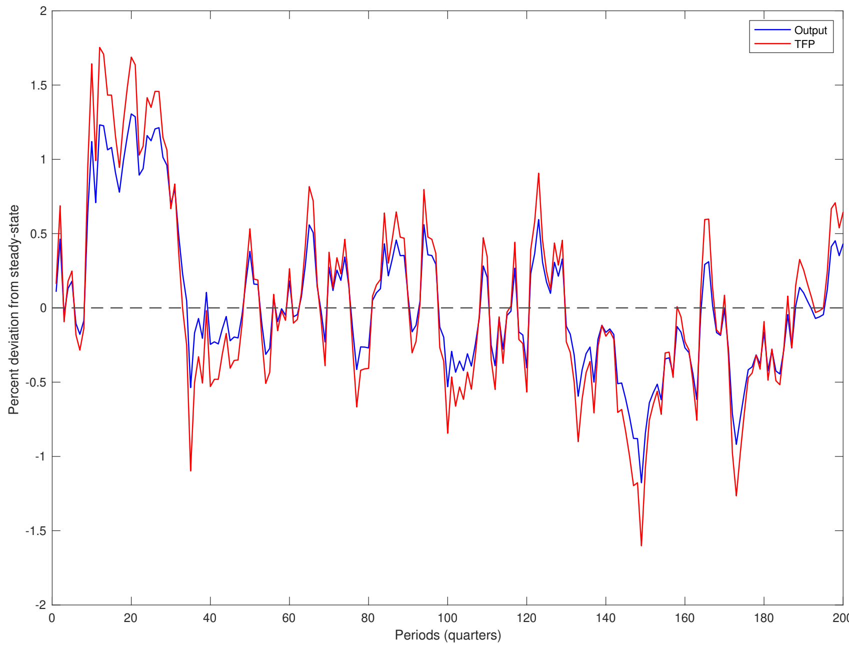


But, despite the successes of the RBC model, they also had some major weaknesses which were heavily criticised by the Keynesians. One reason is that they have not quite lived up to the hype of their early advocates. Part of that hype stemmed from the idea that RBC models contained important propagation mechanisms[20] for turning technology shocks into business cycles. The idea was that increases in technology induced extra output through higher capital accumulation and by incentivising people to work more. In other words, some of the early research suggested that even in a world of IID technology shocks, one would expect RBC models to still generate business cycles.

But, as you can see on Figure 6.2, the RBC model's output fluctuations follow technology fluctuations quite closely. This implies that any propagation mechanisms of the RBC model, besides the technology shock, is extremely weak.

20. Recall the Frisch-Slutsky paradigm.

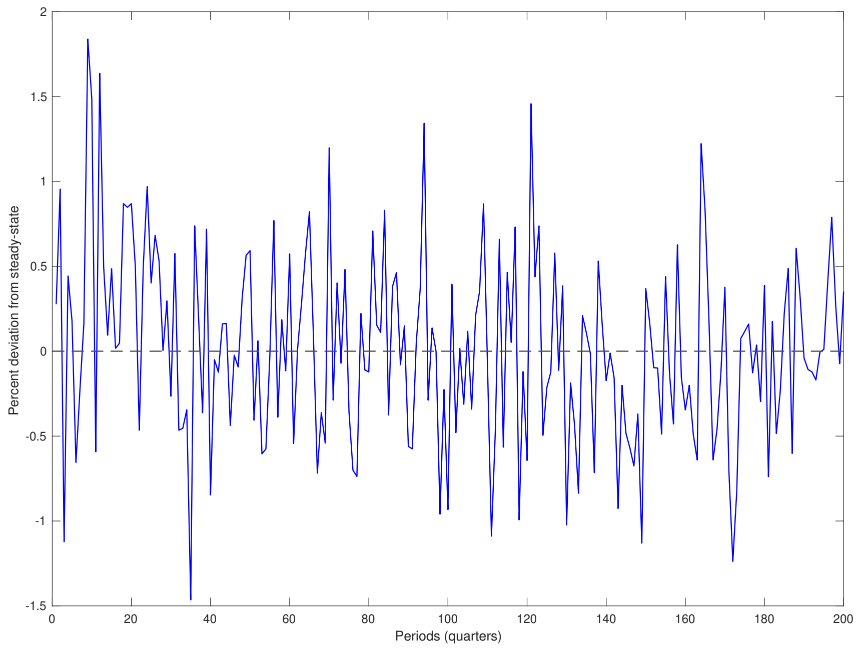
Figure 6.2: RBC Cycles Rely Heavily on Technology Fluctuations



Cogley and Nason (1995) noted another fact about business cycles that the RBC model does not match: output growth is positively autocorrelated (albeit not very autocorrelated – an autocorrelation coefficient of 0.34 – but still statistically significant). But RBC models do not generate this pattern (see Figure 6.3). They can only do so if one simulates a technology process that has a positively autocorrelated growth rate.

Cogley and Nason relate this back to the IRFs generated by RBC models. As you can see from Figure 6.4, the response of output to the technology shock pretty much matches the response of technology itself. Cogley and Nason argue that one needs to instead have "hump-shaped" responses to shocks – a growth rate increase needs to be followed by another growth rate increase – if a model is to match the facts about autocorrelated output growth. The responses to technology shocks do not deliver this. Also, while we don't have other shocks in the model (e.g. government spending shocks), Cogley and Nason show that RBC models don't generate hump-shaped responses for these either.

Figure 6.3: RBCs Do Not Generate Positively Autocorrelated Output Growth ($\rho = 0$)



The key takeaway from the Cogley and Nason critique is that RBC models follow the notion of "you are what you eat" or WYGIWYPI – what you get is what you put in. In other words, if you put into the neoclassical propagation mechanism a volatile and very persistent productivity shock then output fluctuations will also be very volatile and persistent. But if you put in random productivity shocks then the model will provide basically random output fluctuations. In other words, the neoclassical models reliance on capital accumulation as a propagation mechanism adds very little persistence. Is this a problem?

If we could be sure that productivity shocks really were very volatile and persistent, then the fact that the Kydland and Prescott or Hansen model does not not provide a propagation would not be a problem. The problem is there is very little evidence in favour of aggregate technology shocks: i) if business cycles are caused by productivity shocks how do we explain recessions – technical regressions? Is that plausible? (ii) different industries use different technologies. Why should all industries simultaneously experience a positive productivity shock? Are aggregate technology shocks realistic? (iii) are these shocks just oil price shifts? (iv) according to RBC theories the Solow residual should be exogenous. That is, it should not be influenced by any other variable such as monetary policy, government expenditure, and so on. For the US there is strong evidence that this is not the case, that the Solow residual is predictable and further that it is predictable by demand variables. If this is the case then it cannot be interpreted as a pure productivity shock.

What about other moments and correlations? Remember Table 1.2? Let's compare the US

Figure 6.4: IRFs to a Technology Shock

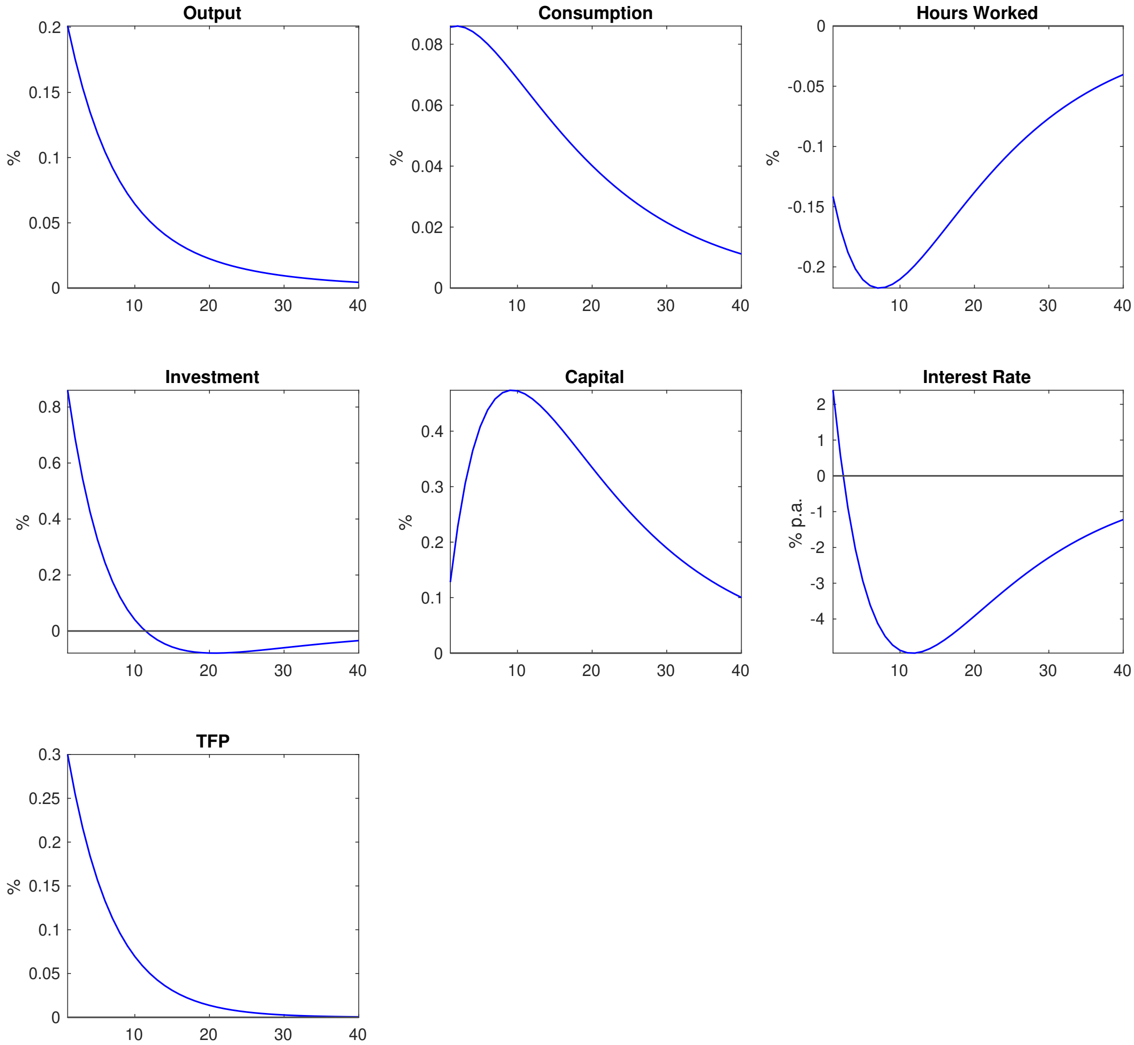


Table 6.1: Model and Data Moments

RBC Model Moments

Series	SD	Rel.SD	Corr w/ Y_t	Autocorr	Corr w/ Y_{t-4}	Corr w/ Y_{t+4}
Output	0.017	1.00	1.00	0.72	0.13	0.13
Consumption	0.006	0.35	0.95	0.78	0.34	-0.03
Investment	0.056	3.29	0.99	0.71	0.04	0.20
Hours	0.007	0.41	0.98	0.71	0.00	-0.23
Productivity	0.010	0.59	0.99	0.74	-0.50	0.06
Wage	0.010	0.59	0.99	0.74	0.74	0.06
Interest Rate	0.001	0.03	0.96	0.71	-0.05	0.26
TFP	0.012	0.69	0.99	0.72	0.11	0.15

US Business Cycle Moments

Series	SD	Rel.SD	Corr w/ Y_t	Autocorr	Corr w/ Y_{t-4}	Corr w/ Y_{t+4}
Output	0.017	1	1.00	0.85	0.07	0.11
Consumption	0.009	0.53	0.76	0.79	0.07	0.22
Investment	0.047	2.76	0.79	0.87	-0.10	0.26
Hours	0.019	1.12	0.88	0.90	0.29	-0.03
Productivity	0.011	0.65	0.42	0.72	-0.50	0.35
Wage	0.009	0.53	0.10	0.73	-0.10	0.10
Interest Rate	0.004	0.24	0.00	0.42	0.27	-0.25
Price Level	0.009	0.53	-0.13	0.91	0.09	-0.41
TFP	0.012	0.71	0.76	0.75	-0.34	0.34

Moments generated by model calibrated and simulated by Sims (2017). All series are HP filtered, with the data from 1948Q1 to 2010Q3.

business cycle moments with the moments generated by the RBC model. For consistency I will refer to the data and RBC model from Sims (2017).

As we saw from the simulation plots, the model does a good job at matching the volatilities of output, consumption, and investment, and we can see from the above table that it also does well with labour productivity and TFP volatility. The RBC model also does well with own autocorrelations – the series are all persistent with first order autocorrelation coefficients typically in the neighbourhood of 0.75. Lastly, the model captures the fact that most quantity series are quite procyclical, though these correlations are too high in the model relative to the data.

Where the model really struggles with are with factor prices. Look at wages and interest rates in the data – there is almost no correlation with output – yet, in the RBC model, wages and interest rates are highly correlated with output: Almost one-to-one. There is some evidence which suggests that aggregate wage data understates the procyclicality of wages due to a composition bias,[21] so this could be forgiven. But that doesn't explain the discrepancy for interest rates. The model also gets the relative standard deviation of interest rates to output standard deviation wrong too (the second columns): the model doesn't generate enough volatility in interest rates. Finally, the RBC

21. See "Measuring the Cyclicality of Real Wages: How Important is Composition Bias" by Solon, Barsky, and Parker (1994).

model does not generate enough volatility in work hours. In the data, hours are actually slightly more volatile than output, but in the RBC model hours are about half as volatile as output.

6.9 Comments and key readings

RBC analysis has been very controversial but also extremely influential. As is often the case with the neoclassical program, it is important to discriminate between methodological innovations and economic theories. The RBC program instigated by Prescott has been controversial for three reasons: (i) reliance on productivity shocks to explain the business cycle; (ii) use of competitive equilibrium models which satisfy the conditions of the Fundamental Welfare Theorems implying business cycles are optimal; and (iii) the eschewing of econometrics in favour of calibration. Another key feature is the use of computer simulations to assess theoretical models. It is now more than 30 years since the seminal RBC paper of Kydland and Prescott. This paper seems to have had three long-run impacts: i) a reassessment of the relative roles of supply and demand shocks in causing business cycles ii) widespread use of computer simulations to assess macroeconomic models iii) widespread use of non-econometric tools to assess the success of a theory. The RBC program is still a very active research area but current models are far more sophisticated in their market structure and while they still have an important role for productivity shocks, additional sources of uncertainty are allowed.

In addition to the Cogley and Nason critique, RBC models have also been criticised by Jordi Galí – and other Keynesian economists – for failing to explain the labour market response to technology shocks. Galí used VARs to show that hours worked tends to decline after a positive technology shock in strong contrast to the RBC model's predictions.

There are currently a number of branches of research aimed at fixing the deficiencies of the RBC approach. Some of them involve putting extra bells and whistles on the basic market-clearing RBC approach: Examples include variable capital utilisation, lags in investment projects, habit persistence in consumer utility, indivisible labour, and so on. Adding these elements tends to strengthen the propagation mechanism element of the model. But they ultimately fall short in reconciling the RBC's performance with the data.

The second approach is to depart more systematically from the basic RBC approach by adding rigidities and frictions into the model, such as sticky prices and wages. We will explore this in more depth when we move onto the New Keynesian DSGE model.

The literature is abundant with papers on RBC research. The key readings are listed below.

McCandless (2008) *ABCs of RBCs*: As the title of the book suggests, it's entirely dedicated to the RBC model, and also extends the basic model to improve its performance. McCandless even has a couple chapters introducing Keynesian-like assumptions, and a simple open-economy model.

Romer (2012) *Advanced Macroeconomics*: Chapter 5 gives a complete walkthrough of the baseline RBC model, and even provides an analytical solution by assuming that $\delta = 1$. Romer also provides some background on the RBC model, as well as explaining the merits and weaknesses of the RBC model.

Kydland and Prescott (1982) "Time to Build and Aggregate Fluctuations": The progenitor of the RBC revolution. The paper is extremely dated, and quite difficult to read. It presents the baseline RBC model with technology shocks. They found that simulated data from their model show the same patterns of volatility, persistence, and co-movement as are present in US data. The paper surprised macroeconomists as the paper presented a model with no money, nominal frictions, or a policy institution (no monetary or fiscal policy).

Hansen (1985) "Indivisible Labor and the Business Cycle": The standard model by Kydland and Prescott (1982) featured "divisible labour": Households voluntarily choose the amount of hours they work. Divisible labour households willingly substitute leisure time between periods in response to changes in factor prices. However, in the data, sufficient intertemporal substitution of leisure was not found (Ashenfelter, 1984; Hall, 1988). As such, the model cannot explain large fluctuations of hours worked, existence of unemployed workers, nor fluctuations in unemployment. Also, the model could not explain small fluctuations of productivity and wages relative to hours worked. Hansen introduced indivisible labor: In a certain period, indivisible labour households either work full time or not work at all – they are not able to work an intermediate amount of hours. Which households work full time or not is determined by a lottery. The results are displayed in Table 6.2. The Hansen model became the de-facto RBC model, but still featured a lot of the weaknesses we discussed.

Table 6.2: Indivisible Labour Improves RBC Model Performance

Standard deviations in percent (a) and correlations with output (b) for U.S. and artificial economies.

	Quarterly U.S. time series[a] (55,3–84,1)		Economy with divisible labor[b]		Economy with indivisible labor[b]	
Series	(a)	(b)	(a)	(b)	(a)	(b)
Output	1.76	1.00	1.35 (0.16)	1.00 (0.00)	1.76 (0.21)	1.00 (0.00)
Consumption	1.29	0.85	0.42 (0.06)	0.89 (0.03)	0.51 (0.08)	0.87 (0.04)
Investment	8.60	0.92	4.24 (0.51)	0.99 (0.00)	5.71 (0.70)	0.99 (0.00)
Capital stock	0.63	0.04	0.36 (0.07)	0.06 (0.07)	0.47 (0.10)	0.05 (0.07)
Hours	1.66	0.76	0.70 (0.08)	0.98 (0.01)	1.35 (0.16)	0.98 (0.01)
Productivity	1.18	0.42	0.68 (0.08)	0.98 (0.01)	0.50 (0.07)	0.87 (0.03)

Source: Hansen (1985).

Mehra and Prescott (1985) "The Equity Premium: A Puzzle": RBC models are successful at mimicking the cyclical behaviour of level quantities. But this paper shows that utility specifications in RBC models have counterfactual implications for asset prices. These utility specifications are not consistent with the difference between the average return to stocks and bonds. Mehra and Prescott follow this research up in their 2003 piece.

Greenwood, Hercowitz, and Huffman (1988) "Investment, Capacity Utilization, and the Real Business Cycle": Attempts to explain or correct the baseline RBC model. Exogenous technology shocks proposed by Kydland and Prescott (1982) are far too large in reality. In response, GHH propose that the Solow residual contains an endogenous component such as capacity utilisation. By doing so, a small technology shock which matches the data can still have a large impact on the macroeconomy.

King, Plosser, and Rebelo (1988) "Production, Growth and Business Cycles": Uses a simplified version of the Kydland and Prescott model and drops non-central ideas such as "time-to-build" in investment, non-separable utility in leisure, and technology shocks that include both a permanent and transitory component. The model reproduces the first-order features of US business cycles. Consumption, investment, and hours worked are all procyclical. Consumption is less volatile than output, investment is much more volatile than output, and hours worked are only slightly less volatile than output. Furthermore, recessions in the model last for about one year, just as in US data.

Benhabib, Rogerson, and Wright (1991) "Homework in Macroeconomics: Household Production and Aggregate Fluctuations": According to the baseline RBC model, the correlation between consumption and labour is high. Yet, in reality, this does not seem to be the case. Household labour is not included in the official statistics, such as housekeeping or child-rearing, but this may be important for the determination of macroeconomic variables. BRW's approach was to include home production in the RBC model, and make consumption an aggregate of both consumption of market goods and consumption of home production goods. The model with home production manages to break the strong positive correlation between consumption and labour, as well as labour and wages.

Cogley and Nason (1995) "Output Dynamics in Real-Business-Cycle Models": We've covered this paper when we assessed the RBC model. It's one of the more scathing criticisms of the RBC framework.

Merz (1995) "Search in the Labor Market and the Real Business Cycle": The simple RBC model cannot explain unemployment. In the real world, workers don't use their entire time endowment to work – they optimally choose between work and leisure. Merz used search and match models to explain unemployment. We will look at this further when we look at labour markets.

Galí (1999) "Technology, Employment, and the Business Cycle: Do Technology Shocks Explain Aggregate Fluctuations?": We've covered Galí's criticisms too. This was essentially the nail in the coffin for the RBC model, and fuelled the New Keynesian framework.

King and Rebelo (1999) "Resuscitating Real Business Cycles": The title is quite self explanatory. As taken from the abstract of the paper: This chapter exposits the basic RBC model and shows that it requires large technology shocks to produce realistic business cycles. While Solow residuals are sufficiently volatile, these imply frequent technological regress. Productivity studies permitting unobserved factor variation find much smaller technology shocks, suggesting the imminent demise of RBC models. However, they show that greater factor variation also dramatically amplifies shocks: a RBC model with varying capital utilisation yields realistic business cycles from small, nonnegative changes in technology.

Rebelo (2005) "Real Business Cycle Models: Past, Present and Future": This paper gives a comprehensive literature review of RBC research.

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7 Solving DSGE Models

7.1 Introduction

The solution of many discrete time DSGE macroeconomic models is a system of non-linear difference equations. One method for approximating the solution to these models is by log-linearising the system of equations about a point (typically the steady state), thereby translating the system of non-linear difference equations into a system of (approximately) linear difference equations. In this chapter, we describe how to arrive at the approximate policy functions/decisions rules once the system of equations has been transformed into a log-linearised system.

The method that we are primarily going to use for solving linear DSGE models is known as the method of undetermined coefficients. It was originally presented by McCallum (1983) and developed by Christiano (2002). What is probably the clearest exposition of the method is by Uhlig (1998), although he used a solution technique different from that of Christiano. Basically, a linear form for the solution is assumed and the method finds the coefficients for the solution of this form. The assumption of a linear form for the solution is not a very great jump, since linear models generally provide linear solutions.

The method of undetermined coefficients was not the first method in the literature for solving rational expectations models. The first method in economics for solving these linear rational expectations models comes from Blanchard and Kahn (1980), who used techniques that were in the engineering literature and applied them to macroeconomic models. Christiano uses solution techniques similar to those of Blanchard and Kahn for solving his undetermined coefficients problems. A good expanded explanation of these methods can be found in Blake and Fernandez-Corugedo (2010).

7.2 The general form and Blanchard-Kahn condition

Let $\boldsymbol{X}_t$ be an $(n+m)\times 1$ vector of variables expressed as expressed as percentage deviations from steady state. Let n be the number of "jump" or "forward-looking" variables, while m is the number of states or predetermined variables. In a deterministic growth model, for example, $n=1$ (consumption) and $m=1$ (the capital stock), while in the stochastic growth model $n=1$ (consumption) and $m=2$ (capital stock and TFP). We partition the vector of variables into two parts: $\boldsymbol{X}_{1,t}$ is an $n\times 1$ vector containing the jump variables, while $\boldsymbol{X}_{2,t}$ is an $m\times 1$ vector containing the state variables. The linearised solution takes the form:

$$\begin{aligned} \mathbb{E}_t \begin{bmatrix} \boldsymbol{X}_{1,t+1} \\ \boldsymbol{X}_{2,t+1} \end{bmatrix}_{(n+m)\times 1} &= \underset{(n+m)\times(n\times m)}{\boldsymbol{B}} \begin{bmatrix} \boldsymbol{X}_{1,t} \\ \boldsymbol{X}_{2,t} \end{bmatrix}_{(n\times m)\times 1} \\ \Leftrightarrow \mathbb{E}_t \boldsymbol{X}_{t+1} &= \boldsymbol{B}\boldsymbol{X}_t. \end{aligned} \tag{7.1}$$

We can typically derive a closed form expression for $\boldsymbol{B}$ in terms of the underlying parameters of the model once it has been log-linearised (which we have previously done). But this does not mean that we have the solution to the model. $\boldsymbol{B}$ tells us how the variables in the system will evolve given an initial starting point. But we only have the initial starting point for the state variables – we do not know where to "start" the jump variables. We have to work harder to figure that out, essentially imposing a terminal condition of non-explosion. The rest of what we do in this chapter is working out how to find that starting position for the non-predetermined variables.

Recall the definition of eigenvalues and eigenvectors. An eigenvalue is a scalar, λ, and an eigenvector is a vector, $\boldsymbol{e}$, which jointly satisfy:

$$\begin{aligned}\boldsymbol{B}\boldsymbol{e}_i &= \lambda_i \boldsymbol{e}_i, \\ (\boldsymbol{B} - \boldsymbol{\lambda I})\boldsymbol{e}_i &= \boldsymbol{0}, \quad i = 1, ..., n+m.\end{aligned}$$

Unless you've made a mistake, there will be the same number of distinct eigenvalues as there are rows/columns in the square matrix $\boldsymbol{B}$ (in this case there will be $n+m$ eigenvalues), some of which may be complex. There will also be the same number of distinct eigenvectors as there are rows/columns of $\boldsymbol{B}$. Index these eigenvalues/eigenvectors by $k = 1, ..., n+m$. The above definition will hold for each $k = 1, ..., n+m$. In other words:

$$\begin{aligned}\boldsymbol{B}\boldsymbol{e}_1 &= \lambda_1 \boldsymbol{e}_1, \\ \boldsymbol{B}\boldsymbol{e}_2 &= \lambda_2 \boldsymbol{e}_2, \\ &\vdots \\ \boldsymbol{B}\boldsymbol{e}_{n+m} &= \lambda_{n+m} \boldsymbol{e}_{n+m},\end{aligned}$$

which means that we can write these up as follows:

$$\boldsymbol{B}\begin{bmatrix} e_{1,1} & e_{2,1} & \cdots & e_{n+m,1} \\ e_{1,2} & e_{2,2} & \cdots & e_{n+m,2} \\ \vdots & \vdots & \ddots & \vdots \\ e_{1,n+m} & e_{2,n+m} & \cdots & e_{n+m,n+m} \end{bmatrix} = \begin{bmatrix} e_{1,1} & e_{2,1} & \cdots & e_{n+m,1} \\ e_{1,2} & e_{2,2} & \cdots & e_{n+m,2} \\ \vdots & \vdots & \ddots & \vdots \\ e_{1,n+m} & e_{2,n+m} & \cdots & e_{n+m,n+m} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_{n+m} \end{bmatrix}.$$

Use the following notation to clean things up a bit:

$$\boldsymbol{P} = \begin{bmatrix} e_{1,1} & e_{2,1} & \cdots & e_{n+m,1} \\ e_{1,2} & e_{2,2} & \cdots & e_{n+m,2} \\ \vdots & \vdots & \ddots & \vdots \\ e_{1,n+m} & e_{2,n+m} & \cdots & e_{n+m,n+m} \end{bmatrix},$$

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_{n+m} \end{bmatrix},$$

and using this notation, we have:

$$\begin{aligned} \boldsymbol{BP} &= \boldsymbol{P\Lambda}, \\ \boldsymbol{B} &= \boldsymbol{P\Lambda P}^{-1}. \end{aligned} \tag{7.2}$$

Note that we can arrange the eigenvalues and eigenvectors in whatever order you want, so long as the k-th column of $\boldsymbol{P}$ corresponds with the k-th eigenvalue which occupies the (k, k) position of $\boldsymbol{\Lambda}$. As such, it is helpful to "order" the eigenvalues from smallest to largest.[1] More generally, let

$$\boldsymbol{\Lambda} = \begin{bmatrix} \underset{S\times S}{\boldsymbol{\Lambda}_1} & \boldsymbol{O} \\ \boldsymbol{O} & \underset{U\times U}{\boldsymbol{\Lambda}_2} \end{bmatrix},$$

where $\boldsymbol{\Lambda}_1$ is a $S \times S$ diagonal matrix containing the S stable eigenvalues, while $\boldsymbol{\Lambda}_2$ is a $U \times U$ diagonal matrix containing the U unstable eigenvalues (obviously, $S + U = n + m$, but neither n nor m are necessarily guaranteed to equal S or U, respectively).

Using the eigenvalue/eigenvector decomposition of $\boldsymbol{B}$, we can rewrite the system as follows:

$$\mathbb{E}_t \boldsymbol{X}_{t+1} = \boldsymbol{P\Lambda P}^{-1} \boldsymbol{X}_t. \tag{7.3}$$

Premultiply each side by $\boldsymbol{P}^{-1}$ to get:

$$\mathbb{E}_t \boldsymbol{P}^{-1} \boldsymbol{X}_{t+1} = \boldsymbol{\Lambda P}^{-1} \boldsymbol{X}_t,$$

and then define the auxiliary vector $\boldsymbol{Z}_t$ as follows:

$$\boldsymbol{Z}_t = \boldsymbol{P}^{-1} \boldsymbol{X}_t,$$

and so we have

$$\begin{aligned} \mathbb{E}_t \boldsymbol{Z}_{t+1} &= \boldsymbol{\Lambda Z}_t \qquad (7.4) \\ \Leftrightarrow \mathbb{E}_t \begin{bmatrix} \boldsymbol{Z}_{1,t+1} \\ \boldsymbol{Z}_{2,t+1} \end{bmatrix} &= \begin{bmatrix} \boldsymbol{\Lambda}_1 & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{\Lambda}_2 \end{bmatrix} \begin{bmatrix} \boldsymbol{Z}_{1,t} \\ \boldsymbol{Z}_{2,t} \end{bmatrix}. \end{aligned}$$

1. In terms of absolute value, that is. If there are complex parts of the eigenvalues, order them by modulus, where the modulus is the square root of the sum of squared non-complex and complex components. e.g. If $y = x + zi$, then the modulus is $\sqrt{x^2 + z^2}$. If $z = 0$, then the modulus is just the absolute value.

We've partitioned $\boldsymbol{Z}_t$ into two parts: $\boldsymbol{Z}_{1,t}$ is partitioned as the first S variables in $\boldsymbol{Z}_t$, while $\boldsymbol{Z}_{2,t}$ are the second U elements of $\boldsymbol{Z}_t$. Because we've effectively rewritten this as a VAR(1) process with a diagonal coefficient matrix, $\boldsymbol{Z}_{1,t}$ and $\boldsymbol{Z}_{2,t}$ evolve independently of one another. We can write the expected values updating forward in time as:

$$\mathbb{E}_t \boldsymbol{Z}_{1,t+T} = \boldsymbol{\Lambda}_1^T \boldsymbol{Z}_{1,t}, \tag{7.5}$$

$$\mathbb{E}_t \boldsymbol{Z}_{2,t+T} = \boldsymbol{\Lambda}_2^T \boldsymbol{Z}_{2,t}. \tag{7.6}$$

Because the eigenvalues in $\boldsymbol{\Lambda}_1$ are all stable (absolute value less than 1), $\boldsymbol{\Lambda}_1^T \to 0$ as $T \to \infty$. The same does not hold true for the second expression, which contains the explosive eigenvalues. Because the eigenvalues in $\boldsymbol{\Lambda}_2$ are all unstable, $\mathbb{E}_t \boldsymbol{Z}_{2,t+T} \to \infty$ as T grows, unless $\boldsymbol{Z}_{2,t} = 0$. But we cannot let $\boldsymbol{Z}_{2,t+T} \to \infty$ while simultaneously being consistent with the transversality conditions and/or feasibility constraints.

For further clarity, let's write out what $\boldsymbol{P}^{-1}$ is:

$$\underset{(S+U)\times(n+m)}{\boldsymbol{P}^{-1}} = \begin{bmatrix} \underset{S\times n}{\boldsymbol{G}_{11}} & \underset{S\times m}{\boldsymbol{G}_{12}} \\ \underset{U\times n}{\boldsymbol{G}_{21}} & \underset{U\times m}{\boldsymbol{G}_{22}} \end{bmatrix}.$$

Since $S + U = n + m$, this is obviously still a square matrix, but the individual partitions need not necessarily be square matrices. Recall that there are S stable eigenvalues and U unstable ones, while there are n jump variables and m state variables. Let's write out in long hand what the $\boldsymbol{Z}$'s are:

$$\underset{S\times 1}{\boldsymbol{Z}_{1,t}} = \underset{S\times n}{\boldsymbol{G}_{11}} \underset{n\times 1}{\boldsymbol{X}_{1,t}} + \underset{S\times m}{\boldsymbol{G}_{12}} \underset{m\times 1}{\boldsymbol{X}_{2,t}},$$

$$\underset{U\times 1}{\boldsymbol{Z}_{2,t}} = \underset{U\times n}{\boldsymbol{G}_{21}} \underset{n\times 1}{\boldsymbol{X}_{1,t}} + \underset{U\times m}{\boldsymbol{G}_{22}} \underset{m\times 1}{\boldsymbol{X}_{2,t}}.$$

As noted above, the transversality/feasibility conditions require that $\boldsymbol{Z}_{2,t} = 0$. We can use this to then solve for the initial position of the jump variables $\boldsymbol{X}_{1,t}$ in terms of the given initial conditions of the states $\boldsymbol{X}_{2,t}$:

$$\boldsymbol{0} = \boldsymbol{G}_{21}\boldsymbol{X}_{1,t} + \boldsymbol{G}_{22}\boldsymbol{X}_{2,t},$$

and solving this yields

$$\boldsymbol{G}_{21}\boldsymbol{X}_{1,t} = -\boldsymbol{G}_{22}\boldsymbol{X}_{2,t}.$$

Provided that $\boldsymbol{G}_{21}$ is a square matrix, we can invert $\boldsymbol{G}_{21}$ and then we can solve this as:

$$\boldsymbol{X}_{1,t} = -\boldsymbol{G}_{21}^{-1}\boldsymbol{G}_{22}\boldsymbol{X}_{2,t}. \tag{7.7}$$

In other words, this is our linearised policy function. For a given state vector (i.e., given values of $\boldsymbol{X}_{2,t}$) this will tell us what the value of the jump variables need to be.

Now, what does it mean for $\boldsymbol{G}_{21}$ to be square/invertible? Recall that the dimension of $\boldsymbol{G}_{21}$ is $U \times n$, where U is the number of unstable eigenvalues and n is the number of jump variables. Put differently, we must have an equal number of unstable eigenvalues as we do jump variables – this is known as the Blanchard-Kahn condition, and it is required for saddle path stability. If we don't have enough unstable eigenvalues, there will be an infinite number of solutions. If we have too many unstable eigenvalues, there will be no solution.

7.2.1 Example: Deterministic growth model

Consider the Robinson Crusoe model (non-stochastic neoclassical growth model) with CRRA preferences, Cobb-Douglas production, and the level of technology normalised to unity.[2] It can be reduced to a system of non-linear difference equations:

$$
\begin{aligned}
C_t^{-\sigma} &= \beta C_{t+1}^{-\sigma}\Big(\underbrace{\alpha K_{t+1}^{\alpha-1} + 1 - \delta}_{R_{t+1}}\Big), \\
K_{t+1} &= K_t^{\alpha} - C_t + (1-\delta)K_t.
\end{aligned}
$$

Log-linearisation of these equations about the steady state yields:

$$
\begin{aligned}
-\sigma \ln C_t &= \ln\beta - \sigma \ln C_{t+1} + \ln R_{t+1} \\
-\frac{\sigma}{\bar{C}} dC_t &= -\frac{\sigma}{\bar{C}} dC_{t+1} + \frac{1}{\bar{R}} dR_{t+1} \\
-\sigma \frac{C_t - \bar{C}}{\bar{C}} &= -\sigma \frac{C_{t+1} - \bar{C}}{\bar{C}} + \frac{R_{t+1} - \bar{R}}{\bar{R}} \\
-\sigma \hat{C}_t &= -\sigma \hat{C}_{t+1} + \hat{R}_{t+1} \\
\Leftrightarrow \hat{C}_t &= \hat{C}_{t+1} - \frac{\beta(\alpha-1)\bar{r}^k}{\sigma} \hat{K}_{t+1},
\end{aligned} \tag{7.8}
$$

and[3]

$$
\begin{aligned}
\hat{K}_{t+1} &= \left(\alpha \bar{K}^{\alpha-1} + 1 - \delta\right) \hat{K}_t + (-1)\frac{\bar{C}}{\bar{K}} \hat{C}_t \\
&= \left[\bar{r}^k + 1 - \delta\right] \hat{K}_t - \frac{\bar{C}}{\bar{K}} \hat{C}_t \\
\hat{K}_{t+1} &= \frac{1}{\beta} \hat{K}_t - \frac{\bar{C}}{\bar{K}} \hat{C}_t,
\end{aligned} \tag{7.9}
$$

2. Unlike previous sections, I assume a start of period notation for capital. This makes the exposition slightly easier. Functionally, it's the same as using the end of period notation.

3. Recall the Taylor approximation method for the multivariate case:

$$
\hat{X}_t = f_X(\bar{X}, \bar{Y})\hat{X}_{t-1} + f_Y(\bar{X}, \bar{Y})\frac{\bar{Y}}{\bar{X}}\hat{Y}_t.
$$

where $\bar{r}^k = \alpha \bar{K}^{\alpha-1}$ (the steady state marginal product of capital), $\bar{R} = \beta^{-1}$, and $\frac{\bar{C}}{\bar{K}}$ is the steady state ratio of consumption to capital, both of which are functions of underlying parameters of the model. This can be re-arranged into the VAR(1) form, $\mathbb{E}_t \boldsymbol{X}_{t+1} = \boldsymbol{B}\boldsymbol{X}_t$, as:

$$\mathbb{E}_t \begin{bmatrix} \hat{C}_{t+1} \\ \hat{K}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 - \frac{\bar{C}}{\bar{K}} \frac{\beta(\alpha-1)\bar{r}^k}{\sigma} & \frac{(\alpha-1)\bar{R}}{\sigma} \\ -\frac{\bar{C}}{\bar{K}} & \frac{1}{\beta} \end{bmatrix} \begin{bmatrix} \hat{C}_t \\ \hat{K}_t \end{bmatrix}.$$

We assume the following parameterisation: $\sigma = 1$, $\beta = 0.95$, $\delta = 0.1$, and $\alpha = 0.33$. These values then imply that $\bar{K} = 3.16$ and $\bar{C} = 1.146$. The numerical values of this matrix are easily seen to be:

$$\boldsymbol{B} = \begin{bmatrix} 1.0352 & -0.1023 \\ -0.3625 & 1.0526 \end{bmatrix}.$$

The MATLAB function "`[lam,V,j]=eig_order(M);`" will produce a diagonal matrix of eigenvalues ordered from smallest to largest (this is the output matrix "`lam`") and the matrix of eigenvectors corresponding with these eigenvalues (the output matrix "`V`" will be the matrix of eigenvectors). The output "`j`" is the index of the first unstable eigenvalue. Remember that we can write $\boldsymbol{B}$ as

$$\boldsymbol{B} = \boldsymbol{P}\boldsymbol{\Lambda}\boldsymbol{P}^{-1},$$

and using MATLAB, the eigenvalues of $\boldsymbol{B}$ come out to be 0.85 and 1.24, so the Blanchard-Kahn condition for saddle path stability are satisfied (i.e., one explosive root, one stable root). We find that $\boldsymbol{P}^{-1}$ is:

$$\boldsymbol{P}^{-1} = \begin{bmatrix} -1.0759 & -0.5462 \\ 1.0547 & -0.5861 \end{bmatrix},$$

and our matrix $\boldsymbol{Z}_t = \boldsymbol{P}^{-1}\boldsymbol{X}_t$ takes the following form:

$$\begin{aligned} Z_{1,t} &= -1.0759\hat{C}_t - 0.5462\hat{K}_t \\ Z_{2,t} &= 1.0547\hat{C}_t - 0.5861\hat{K}_t. \end{aligned}$$

Using eigenvalue decomposition, we know that (with only two variables the diagonal matrices of eigenvalues are just scalars):

$$\begin{aligned} Z_{1,t+T} &= \lambda_1^T Z_{1,t}, \\ Z_{2,t+T} &= \lambda_2^T Z_{2,t}. \end{aligned}$$

Satisfaction of the transversality and feasibility conditions requires that $Z_{2,t} = 0$. This means our linearised policy function is:

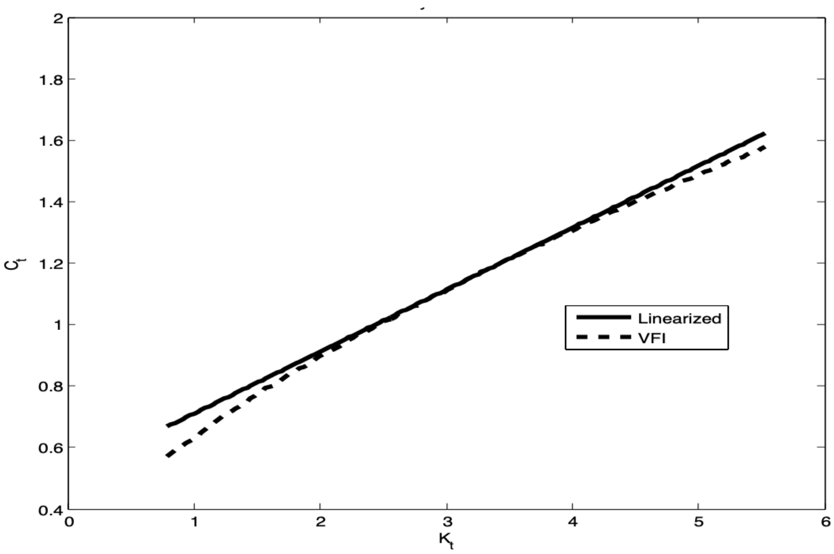
$$\hat{C}_t = \frac{0.5861}{1.0547}\hat{K}_t = 0.5557\hat{K}_t$$

As the above expression is log-linear, we need to adjust to get the policy function in terms of levels:

$$\begin{aligned}\frac{C_t - \bar{C}}{\bar{C}} &= 0.5557\frac{K_t - \bar{K}}{\bar{K}} \\ C_t - \bar{C} &= 0.5557\frac{\bar{C}}{\bar{K}}K_t - 0.5557\bar{C} \\ C_t &= 0.4443\bar{C} + 0.5557\frac{\bar{C}}{\bar{K}}K_t.\end{aligned}$$

Below is a plot of this linearised policy function and the policy function retrieved for the same parameterisation of the model using value function iteration. As you can see, the linearised policy function performs pretty well, especially near the steady state. The linear approximation grows worse as σ increases (the policy function becomes more concave).

Figure 7.1: Policy Functions



Source: Sims (2017)

7.2.2 Example: Stochastic growth model

Now consider the model with stochastic TFP shocks. The non-linear system of difference equations can be written as:

$$\begin{aligned}C_t^{-\sigma} &= \beta\mathbb{E}_t C_{t+1}^{-\sigma}\left(\alpha A_{t+1}K_{t+1}^{\alpha-1} + 1 - \delta\right), \\ K_{t+1} &= A_t K_t^{\alpha} - C_t + (1-\delta)K_t, \\ \ln A_{t+1} &= \rho\ln A_t + \varepsilon_t,\end{aligned}$$

where ε_t is a white noise process, and we assume that $\bar{A} = 1$, which means that the mean of the log of technology is zero. One can show that the log-linearised equations are:

$$\begin{aligned}
\hat{C}_t &= \hat{C}_{t+1} - \frac{\beta \bar{r}^k}{\sigma} \left[(\alpha - 1)\hat{K}_{t+1} + \hat{A}_{t+1} \right], \\
\hat{K}_{t+1} &= \bar{K}^{\alpha-1}\hat{A}_{t+1} - \frac{\bar{C}}{\bar{K}}\hat{C}_t + \frac{1}{\beta}\hat{K}_t, \\
\hat{A}_{t+1} &= \rho \hat{A}_t + \varepsilon_t.
\end{aligned}$$

Isolating the $t+1$ variables, to get the VAR(1) form, we get:

$$\mathbb{E}_t \begin{bmatrix} \hat{C}_{t+1} \\ \hat{K}_{t+1} \\ \hat{A}_{t+1} \end{bmatrix} = \begin{bmatrix} 1 - \frac{\beta(\alpha-1)\bar{r}^k}{\sigma}\frac{\bar{C}}{\bar{K}} & \frac{(\alpha-1)\bar{r}^k}{\sigma} & \frac{\beta \bar{r}^k(\rho + (\alpha-1)\bar{K}^{\alpha-1})}{\sigma} \\ -\frac{\bar{C}}{\bar{K}} & \frac{1}{\beta} & \bar{K}^{\alpha-1} \\ 0 & 0 & \rho \end{bmatrix} \begin{bmatrix} \hat{C}_t \\ \hat{K}_t \\ \hat{A}_t \end{bmatrix}.$$

Using the same parameterisation as in the deterministic example with $\rho = 0.95$, we find that the eigenvalues of this matrix are 0.8512, 0.95, and 1.2367, so the Blanchard-Kahn conditions are satisfied. The inverse of the matrix of eigenvectors are seen to be:

$$\boldsymbol{P}^{-1} = \begin{bmatrix} -1.0759 & -0.5462 & 3.5671 \\ 0 & 0 & 3.4172 \\ 1.0547 & -0.5861 & -0.6041 \end{bmatrix},$$

and the components of the $\boldsymbol{Z}$ matrix are then:

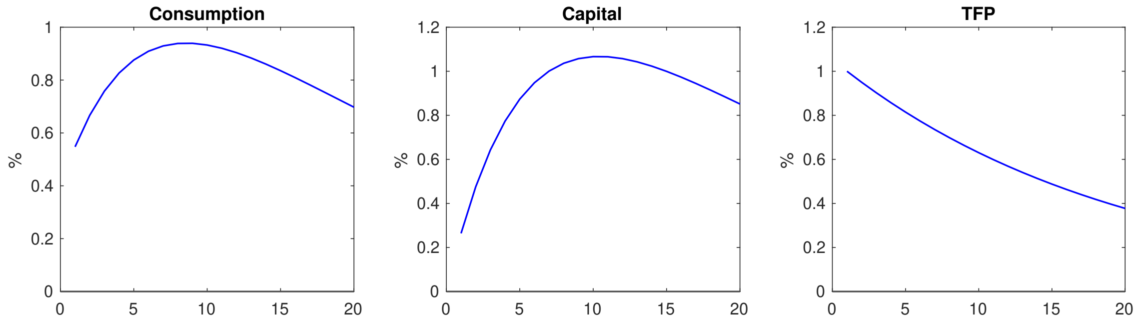
$$\begin{aligned}
\underset{2\times 1}{\boldsymbol{Z}_{1,t}} &= \begin{bmatrix} -1.0759 \\ 0 \end{bmatrix} \hat{C}_t + \begin{bmatrix} -0.5462 & 3.5671 \\ 0 & 3.4172 \end{bmatrix} \begin{bmatrix} \hat{K}_t \\ \hat{A}_t \end{bmatrix}, \\
\underset{1\times 1}{Z_{2,t}} &= 1.0547\hat{C}_t + \begin{bmatrix} -0.5861 & -0.6041 \end{bmatrix} \begin{bmatrix} \hat{K}_t \\ \hat{A}_t \end{bmatrix}.
\end{aligned}$$

Stability requires that $Z_{2,t} = 0$ since that is associated with the explosive eigenvalue. We can then solve for the policy function as:

$$\begin{aligned}
\hat{C}_t &= -\frac{1}{1.0547} \begin{bmatrix} -0.5861 & -0.6041 \end{bmatrix} \begin{bmatrix} \hat{K}_t \\ \hat{A}_t \end{bmatrix}, \\
&= 0.5557\hat{K}_t + 0.5728\hat{A}_t.
\end{aligned}$$

Given this policy function and an initial condition for $\hat{K}_t$, we can shock $\hat{A}_t$ and then let the system play out. Below (Figure are the impulse responses to a one percent shock to technology.

Figure 7.2: IRFs to a Technology Shock



7.2.3 Dealing with static variables

Static variables are variables that only appear in the model at t and are denoted by subscript t. They are not explicitly forward-looking (jump variables) or explicitly backward-looking (state variables), though these variables are often implicitly forward-looking through their dependence on jump variables (like consumption). Normally, before we computationally solve the model, we eliminate static variables by writing them in terms of the jump and state variables. Of course, some static variables are easier to eliminate than others. The easy ones are variables which are essentially just log-linear combinations of the jump and state variables.

This is easier to understand through a demonstration. Consider output and the production function,

$$\begin{aligned} Y_t &= A_t K_t^{\alpha} \\ \implies \hat{Y}_t &= \hat{A}_t + \alpha \hat{K}_t, \end{aligned}$$

which of course nothing but a combination of state variables.

Another easy one is investment. From the log-linearised aggregate resource constraint, we know that:

$$\hat{Y}_t = \frac{\bar{C}}{\bar{Y}} \hat{C}_t + \frac{\bar{I}}{\bar{Y}} \hat{I}_t,$$

where $\bar{I} = \delta \bar{K}$. Do a bit of rearranging to just get $\hat{I}_t$:

$$\hat{I}_t = \frac{\bar{Y}}{\bar{I}} \hat{Y}_t - \frac{\bar{C}}{\bar{I}} \hat{C}_t.$$

In other words, once we know $\hat{C}_t$ and $\hat{I}_t$, we have $\hat{Y}_t$. It is this reason that sometimes these variables are called "redundant" variables, because they are simply linear combinations of jump variables and state variables.

Some static variables are harder to deal with. For example, suppose that labour N_t is variable, and suppose that the period utility function – or felicity function – takes the form:

$$U(C_t, N_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \psi \frac{N_t^{1+\theta}}{1+\theta},$$

and the production function for this economy takes the form:

$$Y_t = A_t K_t^{\alpha} N_t^{1-\alpha}.$$

One can show that the consumption Euler equation and capital accumulation equation takes the following form:

$$\begin{aligned} C_t^{-\sigma} &= \beta \mathbb{E}_t C_{t+1}^{-\sigma} \left(\alpha A_{t+1} K_{t+1}^{\alpha-1} N_{t+1}^{1-\alpha} + 1 - \delta \right), \\ K_{t+1} &= A_t K_t^{\alpha} N_t^{1-\alpha} - C_t + (1-\delta) K_t. \end{aligned}$$

The question is: how do we deal with the N_t? The answer is that we have to use the first order condition for optimal labour supply. We can show that it takes the following form:

$$\psi N_t^{\theta} = C_t^{-\sigma} (1-\alpha) A_t K_t^{\alpha} N_t^{-\alpha}.$$

In words, this says that the marginal disutility from work equals the marginal utility of consumption times the real wage (the marginal product of labour). What you can do is to then solve for N_t in terms of the jump and state variables:

$$\begin{aligned} N_t^{\theta+\alpha} &= \frac{1}{\psi} C_t^{-\sigma} (1-\alpha) A_t K_t^{\alpha} \\ N_t &= \left(\frac{1}{\psi} C_t^{-\sigma} (1-\alpha) A_t K_t^{\alpha} \right)^{\frac{1}{\theta+\alpha}}. \end{aligned}$$

Given this, we can substitute this whenever N_t shows up in the first order conditions and you're back to the kind of system we previously had, though it is more complicated. In practice, the easier thing to do is often to log-linearise all the equations first, and then eliminate the log-linearised $\hat{N}_t$.

7.2.4 Getting the dynamics right

Suppose we want to construct impulse responses or simulate data from the linearised model. As an example, suppose that we take the deterministic growth model and want to compute what happens in expectation if the capital stock starts out below the steady state. The simple thing to do would be to start at some $\hat{K}_0$, set $\hat{C}_0 = -\boldsymbol{G}_{21}^{-1} \boldsymbol{G}_{22} \hat{K}_0$, and then trace out expected future dynamics as:

$$\begin{bmatrix} \hat{C}_t \\ \hat{K}_t \end{bmatrix} = \boldsymbol{B}^t \begin{bmatrix} \hat{C}_0 \\ \hat{K}_0 \end{bmatrix} = \boldsymbol{B}^t \begin{bmatrix} -\boldsymbol{G}_{21}^{-1} \boldsymbol{G}_{22} \hat{K}_0 \\ \hat{K}_0 \end{bmatrix}.$$

This is analytically correct, but is prone to numerical problems. Why? Recall the whole idea of saddle path stability. If you even slightly deviate from the policy function, the system eventually explodes (due to the presence of unstable eigenvalues/roots in $\boldsymbol{B}$). In practice, there will be small numerical errors in the policy function $-\boldsymbol{G}_{21}^{-1}\boldsymbol{G}_{22}$. Like, numerical errors to several decimal places – but the system still can't tolerate these – particularly at longer horizons. If we do the exercise, everything will look great for about 100 periods, but out at longer horizons the system starts to explode.

There is a straightforward way of dealing with this and avoiding the potential for explosion that results from small numerical errors. Consider the general case. Decompose $\boldsymbol{B}$ into blocks:

$$\boldsymbol{B} = \begin{bmatrix} \underset{n\times n}{\boldsymbol{B}_{11}} & \underset{n\times m}{\boldsymbol{B}_{12}} \\ \underset{m\times n}{\boldsymbol{B}_{21}} & \underset{m\times m}{\boldsymbol{B}_{22}} \end{bmatrix},$$

write out the original system out in long hand using this notation:

$$\begin{aligned} \mathbb{E}_t \boldsymbol{X}_{1,t+1} &= \boldsymbol{B}_{11}\boldsymbol{X}_{1,t} + \boldsymbol{B}_{12}\boldsymbol{X}_{2,t}, \\ \mathbb{E}_t \boldsymbol{X}_{2,t+1} &= \boldsymbol{B}_{21}\boldsymbol{X}_{1,t} + \boldsymbol{B}_{22}\boldsymbol{X}_{2,t}. \end{aligned}$$

Now, plug in the policy function to eliminate $\boldsymbol{X}_{1,t}$ in both expressions:

$$\begin{aligned} \mathbb{E}_t \boldsymbol{X}_{1,t+1} &= (-\boldsymbol{B}_{11}\boldsymbol{G}_{21}^{-1}\boldsymbol{G}_{22} + \boldsymbol{B}_{12})\boldsymbol{X}_{2,t}, \\ \mathbb{E}_t \boldsymbol{X}_{2,t+1} &= (-\boldsymbol{B}_{21}\boldsymbol{G}_{21}^{-1}\boldsymbol{G}_{22} + \boldsymbol{B}_{22})\boldsymbol{X}_{2,t}. \end{aligned}$$

Define a new matrix, $\boldsymbol{A}$, as follows:

$$\boldsymbol{A} = \begin{bmatrix} \underset{n\times n}{\boldsymbol{O}} & -\boldsymbol{B}_{11}\boldsymbol{G}_{21}^{-1}\boldsymbol{G}_{22} + \boldsymbol{B}_{12} \\ \underset{m\times n}{\boldsymbol{O}} & -\boldsymbol{B}_{21}\boldsymbol{G}_{21}^{-1}\boldsymbol{G}_{22} + \boldsymbol{B}_{22} \end{bmatrix}.$$

Then, write the system as:

$$\mathbb{E}_t \begin{bmatrix} \boldsymbol{X}_{1,t+1} \\ \boldsymbol{X}_{2,t+1} \end{bmatrix} = \boldsymbol{A} \begin{bmatrix} \boldsymbol{X}_{1,t} \\ \boldsymbol{X}_{2,t} \end{bmatrix}.$$

Effectively what this does is imposes the policy function so that you can write the AR coefficient matrix with only coefficients on $\boldsymbol{X}_{2,t}$, the vector of states. This turns out to eliminate the problem. You can then proceed as follows – you can start the system as some arbitrary value of the state, start the controls at the appropriate place given the policy function, and then iterate forward using $\boldsymbol{A}$ instead of $\boldsymbol{B}$.

7.3 Solving the RBC model

We just discussed how static and redundant variables need to be "eliminated" to solve for the linearised policy functions. This is correct and can be done by hand, but it is algebraically intense and annoying. Below we discuss a way in which to do this that just involves manipulation of a few matrices. Recall that our RBC model is comprised of the following log-linearised set of equations:[4]

$$
\begin{aligned}
\hat{C}_t &= \mathbb{E}_t \hat{C}_{t+1} - \frac{1}{\sigma} \mathbb{E}_t \hat{R}_{t+1}, \\
\hat{K}_{t+1} &= \left[\frac{\alpha\delta}{\beta^{-1} + \delta - 1} \right] \hat{I}_t + (1-\delta)\hat{K}_t, \\
\hat{A}_t &= \rho \hat{A}_{t-1} + \varepsilon_t, \\
\hat{N}_t &= \hat{Y}_t - \sigma \hat{C}_t, \\
\hat{Y}_t &= \hat{A}_t + \alpha \hat{K}_t + (1-\alpha)\hat{N}_t, \\
\hat{Y}_t &= \left[1 - \frac{\alpha\delta}{\beta^{-1} + \delta - 1} \right] \hat{C}_t + \left[\frac{\alpha\delta}{\beta^{-1} + \delta - 1} \right] \hat{I}_t, \\
\hat{w}_t &= \hat{A}_t + \alpha \hat{K}_t - \alpha \hat{N}_t, \\
\hat{R}_t &= [1 - \beta(1-\delta)] \left[\hat{Y}_t - \hat{K}_t \right].
\end{aligned}
$$

Let's stack all of these up into vectors. Let:

$$
\boldsymbol{X}_t = \begin{bmatrix} \hat{C}_t \\ \hat{K}_t \\ \hat{A}_t \\ \hat{N}_t \\ \hat{Y}_t \\ \hat{I}_t \\ \hat{w}_t \\ \hat{R}_t \end{bmatrix},
$$

and note that we haven't ordered these randomly. We've started with the forward-looking jump variable, then the two state variables, and then the redundant/static variables. We can write out

4. I've added an equation for wages. It's not necessary, but we may as well include both factor prices. Also, I've adjusted the timing of capital here.

the log-linearised conditions in matrix form as:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sigma} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \mathbb{E}_t \begin{bmatrix} \hat{C}_{t+1} \\ \hat{K}_{t+1} \\ \hat{A}_{t+1} \\ \hat{N}_{t+1} \\ \hat{Y}_{t+1} \\ \hat{I}_{t+1} \\ \hat{w}_{t+1} \\ \hat{R}_{t+1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-\delta & 0 & 0 & 0 & \frac{\alpha\delta}{\beta^{-1}+\delta-1} & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 & 0 & 0 & 0 \\ -\sigma & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & \alpha & 1 & (1-\alpha) & -1 & 0 & 0 & 0 \\ \left[1 - \frac{\alpha\delta}{\beta^{-1}+\delta-1}\right] & 0 & 0 & 0 & -1 & \left[\frac{\alpha\delta}{\beta^{-1}+\delta-1}\right] & 0 & 0 \\ 0 & \alpha & 1 & -\alpha & 0 & 0 & -1 & 0 \\ 0 & -\left[1-\beta(1-\delta)\right] & 0 & 0 & \left[1-\beta(1-\delta)\right] & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \hat{C}_t \\ \hat{K}_t \\ \hat{A}_t \\ \hat{N}_t \\ \hat{Y}_t \\ \hat{I}_t \\ \hat{w}_t \\ \hat{R}_t \end{bmatrix},$$

or, more compactly as:

$$\boldsymbol{A}_0 \mathbb{E}_t \boldsymbol{X}_{t+1} = \boldsymbol{D}_0 \boldsymbol{X}_t.$$

There are a lot of rows of zeros in the $\boldsymbol{A}_0$ coefficient matrix – these rows correspond to the redundant variables. Note, however, that we can decompose these matrices as follows: Let n be the number of jump variables (here it's one, $\hat{C}_t$), m be the number of states (here it's two, $\hat{K}_t$ and $\hat{A}_t$), and q be the number of redundant/static variables (here five). We can then write:

$$\boldsymbol{A}_0 = \begin{bmatrix} \underset{(n+m)\times(n+m)}{\boldsymbol{A}_{11}} & \underset{(n+m)\times q}{\boldsymbol{A}_{12}} \\ \underset{q\times(n+m)}{\boldsymbol{O}} & \underset{q\times q}{\boldsymbol{O}} \end{bmatrix},$$

$$\boldsymbol{D}_0 = \begin{bmatrix} \underset{(n+m)\times(n+m)}{\boldsymbol{D}_{11}} & \underset{(n+m)\times q}{\boldsymbol{D}_{12}} \\ \underset{q\times(n+m)}{\boldsymbol{D}_{21}} & \underset{q\times q}{\boldsymbol{D}_{22}} \end{bmatrix}.$$

Now, let $\boldsymbol{Y}_t$ be the $(n+m)\times 1$ vector of jump and state variables, and $\boldsymbol{X}_t$ be the $q \times 1$ vector of redundant variables. We can write this out as:

$$\begin{bmatrix} \underset{(n+m)\times(n+m)}{\boldsymbol{A}_{11}} & \underset{(n+m)\times q}{\boldsymbol{A}_{12}} \\ \underset{q\times(n+m)}{\boldsymbol{O}} & \underset{q\times q}{\boldsymbol{O}} \end{bmatrix} \mathbb{E}_t \begin{bmatrix} \boldsymbol{Y}_{t+1} \\ \boldsymbol{X}_{t+1} \end{bmatrix} = \begin{bmatrix} \underset{(n+m)\times(n+m)}{\boldsymbol{D}_{11}} & \underset{(n+m)\times q}{\boldsymbol{D}_{12}} \\ \underset{q\times(n+m)}{\boldsymbol{D}_{21}} & \underset{q\times q}{\boldsymbol{D}_{22}} \end{bmatrix} \begin{bmatrix} \boldsymbol{Y}_t \\ \boldsymbol{X}_t \end{bmatrix}.$$

From this, we can see that:

$$\boldsymbol{O} = \boldsymbol{D}_{21}\boldsymbol{Y}_t + \boldsymbol{D}_{22}\boldsymbol{X}_t,$$

and since $\boldsymbol{D}_{22}$ is a square matrix, we can (in principle) invert it, so we have:

$$\boldsymbol{X}_t = -\boldsymbol{D}_{22}^{-1}\boldsymbol{D}_{21}\boldsymbol{Y}_t.$$

In other words, we can write the vector redundant variables as a linear combination of the jump and state variables! Note that the dimension of $-\boldsymbol{D}_{22}^{-1}\boldsymbol{D}_{21}\boldsymbol{Y}_t$ is $q \times 1$. Hence, we can write:

$$\begin{bmatrix} \underset{(n+m)\times(n+m)}{\boldsymbol{A}_{11}} & \underset{(n+m)\times q}{\boldsymbol{A}_{12}} \\ \underset{q\times(n+m)}{\boldsymbol{O}} & \underset{q\times q}{\boldsymbol{O}} \end{bmatrix} \mathbb{E}_t \begin{bmatrix} \boldsymbol{Y}_{t+1} \\ -\boldsymbol{D}_{22}^{-1}\boldsymbol{D}_{21}\boldsymbol{Y}_{t+1} \end{bmatrix} = \begin{bmatrix} \underset{(n+m)\times(n+m)}{\boldsymbol{D}_{11}} & \underset{(n+m)\times q}{\boldsymbol{D}_{12}} \\ \underset{q\times(n+m)}{\boldsymbol{D}_{21}} & \underset{q\times q}{\boldsymbol{D}_{22}} \end{bmatrix} \begin{bmatrix} \boldsymbol{Y}_t \\ -\boldsymbol{D}_{22}^{-1}\boldsymbol{D}_{21}\boldsymbol{Y}_t \end{bmatrix},$$

or:

$$(\boldsymbol{A}_{11} - \boldsymbol{A}_{12}\boldsymbol{D}_{22}^{-1}\boldsymbol{D}_{21})\mathbb{E}_t\boldsymbol{Y}_{t+1} = (\boldsymbol{D}_{11} - \boldsymbol{D}_{12}\boldsymbol{D}_{22}^{-1}\boldsymbol{D}_{21})\boldsymbol{Y}_t.$$

The dimensions work out:

- $\boldsymbol{A}_{11}$ is $(n+m)\times(n+m)$;
- $\boldsymbol{A}_{12}$ is $(n+m)\times q$;
- $\boldsymbol{D}_{22}^{-1}\boldsymbol{D}_{21}$ is $q \times (n+m)$;
- Hence, $\boldsymbol{A}_{11} - \boldsymbol{A}_{12}\boldsymbol{D}_{22}^{-1}\boldsymbol{D}_{21}$ is $(n+m)\times(n+m)$;
- $\boldsymbol{D}_{11}$ is $(n+m)\times(n+m)$;
- $\boldsymbol{D}_{12}$ is $(n+m)\times q$;
- $\boldsymbol{D}_{22}^{-1}\boldsymbol{D}_{21}$ is $q \times (n+m)$;
- Hence, $\boldsymbol{D}_{11} - \boldsymbol{D}_{12}\boldsymbol{D}_{22}^{-1}\boldsymbol{D}_{21}$ is $(n+m)\times(n+m)$.

Since these are both square, we can invert to form:

$$\mathbb{E}_t\boldsymbol{Y}_{t+1} = \boldsymbol{B}\boldsymbol{Y}_t,$$

where $\boldsymbol{B} = (\boldsymbol{A}_{11} - \boldsymbol{A}_{12}\boldsymbol{D}_{22}^{-1}\boldsymbol{D}_{21})^{-1}(\boldsymbol{D}_{11} - \boldsymbol{D}_{12}\boldsymbol{D}_{22}^{-1}\boldsymbol{D}_{21})$. In other words, what we've done here is system reduction – we've reduced the system back to the VAR(1) in only the jumps and states, and given $\boldsymbol{B}$ can solve for the policy function mapping the states into the jump variables exactly as before. Given this new matrix $\boldsymbol{B}$ only in the system of jump and state variables, we can find the policy function just as before.

Recall, that in state space form in terms of jump and state variables, we can write the state as:

$$\mathbb{E}_t \boldsymbol{X}_{2,t+1} = \boldsymbol{B}_{21}\boldsymbol{X}_{1,t} + \boldsymbol{B}_{22}\boldsymbol{X}_{2,t},$$

where $\boldsymbol{X}_{1,t}$ was the $n \times 1$ vector of jump variables and $\boldsymbol{X}_{2,t}$ were the $m \times 1$ vector of states. Using the policy function mapping the states into the jumps, we can write this as:

$$\mathbb{E}_t \boldsymbol{X}_{2,t+1} = (\boldsymbol{B}_{21}\boldsymbol{\Phi} + \boldsymbol{B}_{22})\boldsymbol{X}_{2,t},$$

where $\boldsymbol{\Phi} = -\boldsymbol{G}_{21}^{-1}\boldsymbol{G}_{22}$, $\boldsymbol{X}_{1,t} = \boldsymbol{\Phi}\boldsymbol{X}_{2,t}$, and the $\boldsymbol{G}$ matrices correspond to different blocks of the inverse matrix of eigenvectors of $\boldsymbol{B}$ appropriately sorted. Now, we we may want to write this expression without expectation operators and instead with shocks. We know that:

$$\boldsymbol{X}_{2,t} = (\boldsymbol{B}_{21}\boldsymbol{\Phi} + \boldsymbol{B}_{22})\boldsymbol{X}_{2,t-1} + \boldsymbol{H}_0\boldsymbol{\varepsilon}_t,$$

where $\boldsymbol{\varepsilon}_t$ is a $k \times 1$ vector shocks (in the baseline RBC model $k = 1$, the TFP shock), and $\boldsymbol{H}_0$ is $m \times k$. In the baseline RBC model if the elements of the states are capital and productivity, we know that $\boldsymbol{H}_0 = \begin{bmatrix} 0 & 1 \end{bmatrix}$. Since we know that $\boldsymbol{X}_{1,t} = \boldsymbol{\Phi}\boldsymbol{X}_{2,t}$, we can write:

$$\boldsymbol{X}_{1,t} = \boldsymbol{\Phi}(\boldsymbol{B}_{21}\boldsymbol{\Phi} + \boldsymbol{B}_{22})\boldsymbol{X}_{2,t-1} + \boldsymbol{\Phi}\boldsymbol{H}_0\boldsymbol{\varepsilon}_t,$$

and we can stack to write:

$$\begin{bmatrix} \boldsymbol{X}_{1,t} \\ \boldsymbol{X}_{2,t} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Phi}(\boldsymbol{B}_{21}\boldsymbol{\Phi} + \boldsymbol{B}_{22}) \\ \boldsymbol{B}_{21}\boldsymbol{\Phi} + \boldsymbol{B}_{22} \end{bmatrix} \boldsymbol{X}_{2,t-1} + \begin{bmatrix} \boldsymbol{\Phi}\boldsymbol{H}_0 \\ \boldsymbol{H}_0 \end{bmatrix} \boldsymbol{\varepsilon}_t.$$

Now, we need to get the redundant/static variables back in. Recall that we can write:

$$\boldsymbol{X}_t = -\boldsymbol{D}_{22}^{-1}\boldsymbol{D}_{21}\boldsymbol{Y}_t,$$

and let's define $\boldsymbol{\Psi} = -\boldsymbol{D}_{22}^{-1}\boldsymbol{D}_{21}$. This matrix is $q \times (n + m)$. Let's decompose it as follows:

$$\boldsymbol{\Psi} = \begin{bmatrix} \underset{q \times n}{\boldsymbol{\Psi}_{11}} & \underset{q \times m}{\boldsymbol{\Psi}_{12}} \end{bmatrix}.$$

In other words, we can write the redundant/state variables as:

$$\boldsymbol{X}_t = \boldsymbol{\Psi}_{11}\boldsymbol{X}_{1,t} + \boldsymbol{\Psi}_{12}\boldsymbol{X}_{2,t},$$

But using the policy function, we have:

$$\boldsymbol{X}_t = (\boldsymbol{\Psi}_{11}\boldsymbol{\Phi} + \boldsymbol{\Psi}_{12})\boldsymbol{X}_{2,t},$$

and lagging $\boldsymbol{X}_{2,t}$ we have:

$$\boldsymbol{X}_t = (\boldsymbol{\Psi}_{11}\boldsymbol{\Phi} + \boldsymbol{\Psi}_{12})(\boldsymbol{B}_{21}\boldsymbol{\Phi} + \boldsymbol{B}_{22})\boldsymbol{X}_{2,t-1} + (\boldsymbol{\Psi}_{11}\boldsymbol{\Phi} + \boldsymbol{\Psi}_{12})\boldsymbol{H}_0\boldsymbol{\varepsilon}_t.$$

Hence, we can characterise the solution as:

$$\boldsymbol{W}_t = \boldsymbol{F}\boldsymbol{W}_{t-1} + \boldsymbol{J}\boldsymbol{\varepsilon}_t, \tag{7.10}$$

where

$$\boldsymbol{W}_t = \begin{bmatrix} \boldsymbol{X}_{1,t} \\ \boldsymbol{X}_{2,t} \\ \boldsymbol{X}_t \end{bmatrix}, \; \boldsymbol{F} = \begin{bmatrix} \boldsymbol{\Phi}(\boldsymbol{B}_{21}\boldsymbol{\Phi} + \boldsymbol{B}_{22}) \\ \boldsymbol{B}_{21}\boldsymbol{\Phi} + \boldsymbol{B}_{22} \\ (\boldsymbol{\Psi}_{11}\boldsymbol{\Phi} + \boldsymbol{\Psi}_{12})(\boldsymbol{B}_{21}\boldsymbol{\Phi} + \boldsymbol{B}_{22}) \end{bmatrix}, \; \boldsymbol{J} = \begin{bmatrix} \boldsymbol{\Phi}\boldsymbol{H}_0 \\ \boldsymbol{H}_0 \\ (\boldsymbol{\Psi}_{11}\boldsymbol{\Phi} + \boldsymbol{\Psi}_{12})\boldsymbol{H}_0 \end{bmatrix}.$$

We can then use this formulation to produce impulse responses and model simulations.

7.4 Comments and key readings

As observed, solving DSGE models can be algebraically intense, and keeping track of all the matrix manipulations can be tricky. In reality, we will rarely do this by hand – software is getting increasingly better at solving models using the methods we just discussed. Something like Dynare will solve your model, simulate shocks, plot impulse responses, and so on very easily.

Key readings for this chapter are, once again, far and wide. But a few to focus on would be McCandless (2008), Uhlig (1998), and the fantastic set of notes by Eric Sims (2017). Just be aware that everyone uses different notation for their matrices and decompositions, and there are many different approaches to getting the same result.

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8 Modelling the Labour Market

8.1 Introduction

In the previous sections we saw that while the baseline RBC model did a decent job of explaining some business cycle facts, its main failures were due to the labour market. In particular, the model fails to convincingly account for the large fluctuations in employment over the business cycle in the absence of any movements in wages. Exactly where to progress from this point is not clear and a plethora of different approaches exist. In this section we introduce three extensions to the baseline RBC model, and they primarily focus on labour market dynamics and introducing unemployment. We choose these approaches for the following reasons: (i) they are all choice theoretic, (ii) they all assume Rational Expectations, and (iii) they each differ in their welfare implications. We focus on three models: Hansen's indivisible labour with lotteries RBC model; the Shapiro and Stiglitz (1984) efficiency wages model; and, the Diamond-Mortensen-Pissarides search and match model.

8.2 Hansen's RBC model with indivisible labour

8.2.1 Motivation

Quick recap: the standard RBC models proposed by Kydland and Prescott, and Lucas and Prescott motivated that under the assumption of competitive markets, shocks to productivity lead to changes in economic growth. Standard RBC models feature "divisible labour" – households voluntarily choose the amount of hours they work. This has long been a point of criticism of these models (which we covered in the previous chapters).

Divisible labour households willingly substitute leisure time between periods in response to changes in factor prices (wages and interest rates). However, sufficient intertemporal substitution of leisure was not found to support this claim (Ashenfelter, 1984; Hall, 1988). Thus, the standard model cannot explain large fluctuations of hours worked, existence of unemployed workers, or fluctuations in unemployment. The model also cannot explain small fluctuations in productivity and wages relative to hours worked. Kydland and Prescott attempted to explain this puzzle, but were unable to account for these observations.

Hansen (1985) introduced "indivisible labour" into the standard RBC model, and the model quickly become a standard model used by RBC researchers. The idea indivisible labour was that in a certain period, households either work full time or they do not work at all – they are unable to work an intermediate amount of hours. Fluctuations in aggregate labour hours arise from households entering and leaving unemployment, which was a consistent feature of US post-war labour market data. The model could account for large aggregate fluctuations in hours worked, relative to productivity, while also having a small intertemporal elasticity of substitution for leisure on the part of individuals. This follows because the utility function of the representative agent implies an elasticity of substitution between leisure in different periods that is infinite.[1]

1. Note that in the this model utility functions of the representative agent and the utility function of the individual

To rephrase, standard RBC models focused on the "intensive margin" – households that adjust their labour supply with respect to factor prices – however not much focus was on the "extensive margin" – households entering in or out of employment. This was a big oversight (e.g., female labour supply). Consider the following decomposition from Hansen's paper:

$$\text{var}(\ln H_t) = \underbrace{\text{var}(\ln h_t)}_{\approx 20\%} + \underbrace{\text{var}(\ln N_t)}_{\approx 55\%} + \underbrace{2\,\text{cov}(\ln h_t, \ln N_t)}_{\approx 25\%},$$

where H_t is total hours worked, h_t is average hours worked, and N_t is number of individuals at work. Most households either work full time or not at all. This may be due to the presence of non-convexities either in individual preferences for leisure or in the production technology. For example, marginal productivity of labour could be high early in the week and it could be low late in the week, and this would imply a convex production function at first and then concave after. Hansen's model assumes non-convexity based on the property of preferences, and that individual households have preferences defined at two levels: full time work or no work at all. Thus, individuals can only adjust labour supply along the extensive margin.

8.2.2 The baseline model

Hansen's paper follows an inductive reasoning argument: the real economy has fluctuations along both the extensive and intensive margins, so compare two models – one with an intensive margin and one with an extensive margin. Thus, we can determine the importance of non-convexities for explaining labour variance. If both economies exhibit similar cyclical behaviour, then a model (or the real world) that incorporates both margins would also exhibit similar behaviour.

The following equations characterise the two economies:

$$f(\lambda_t, k_t, h_t) = \lambda_t k_t^{\theta} h_t^{1-\theta}, \tag{8.1}$$

$$c_t + i_t \leq f(\lambda_t, k_t, h_t), \tag{8.2}$$

$$k_{t+1} = (1-\delta)k_t + i_t, \;\; \delta \in (0,1), \tag{8.3}$$

$$\lambda_{t+1} = \gamma\lambda_t + \varepsilon_{t+1}, \tag{8.4}$$

and where for simplicity, a single firm is assumed to exist. The technology shock follows a first-order Markov process, where the ε_t's are IID with distribution F. F has mean $1-\gamma$, and the unconditional mean of λ_t is equal to 1.

The base (divisible labour) model has a continuum of infinitely lived household along the closed set $[0,1]$ which populate the economy, where they maximise the following:

$$\mathbb{E}_t \sum_{t=0}^{\infty} \beta^t u(c_t, l_t),$$

household are different.

where $\beta \in (0, 1)$ is the discount factor and $l_t = 1 - h_t$. Utility in period t is given by:

$$u(c_t, l_t) = \log c_t + A \log l_t, \;\; A > 0, \tag{8.5}$$

and is subject to the following budget constraint:

$$c_t + i_t \leq w_t h_t + r_t k_t \tag{8.6}$$

and the law of motion of capital (8.3). The FOCs for the firm's maximisation problem imply that the wage and interest rate each period are equal to the marginal productivity of labour and capital, respectively. No externalities or distortions characterise the economy, so the competitive equilibrium can be considered as a Pareto optimum. Pareto optimum is the solution of maximising expected welfare of the representative agent subject to technology constraints. This completes the specification of the base model.

8.2.3 Economy with indivisible labour

For the indivisible labour model we apply a restriction that individuals can work full time, h_0, or not at all. There are some slight mathematical challenges: feasible equilibrium requires consumption possibilities set to be convex; and, trading work hours implies consumption possibilities set to be non-convex. Hansen resolves this by introducing a lottery to convexify the set – households choose lotteries to work or not rather than hours. Thus, households choose a probability of working, α_t.

The introduction of the lottery also implies that firms offer complete income insurance to households. Firms and households make and trade a contract that the households works h_0 with probably α_t, and that the household gets paid whether it works or not. All households are ex-ante identical (face the same α_t) but differ ex-post depending on outcome of lottery.

The utility function is now:

$$u(c_t, a_t) = \log c_t + A\alpha_t \log(1 - h_0), \tag{8.7}$$

and per capita hours worked in period t is given as:

$$h_t = \alpha h_0. \tag{8.8}$$

Other features of this economy are identical to the base model, described by (8.1)-(8.4).

Firms employ labour until:

$$f_h(\lambda_t, k_t, h_t) = w_t.$$

Due to the lottery system, the household budget constrain is slightly different compared to the baseline model:

$$c_t + i_t \leq w_t \alpha_t h_0 + r k_t. \tag{8.9}$$

Thus welfare is maximised by the following:

$$\mathbb{E}_t \sum_{t=0}^{\infty} \beta^t u(c_t, \alpha_t),$$

subject to (8.3) and (8.9).

8.2.4 Solving the model

As discussed earlier, a key property of this model is that the elasticity of substitution between leisure in different periods for the representative agent is infinite. To derive, begin by substituting $h_t = 1 - l_t$ into (8.8) to get:

$$\alpha_t = \frac{1 - l_t}{h_0},$$

and substituting α_t into (8.7) gets:

$$\begin{aligned} u(c_t, a_t) &= \ln c_t + A\frac{(1 - l_t)}{h_0} \ln(1 - h_0) \\ &= \ln c_t + \underbrace{\frac{A}{h_0} \ln(1 - h_0)}_{=\text{constant}} - \frac{Al_t}{h_0} \ln(1 - h_0) \\ &= \ln c_t + Bl_t, \end{aligned}$$

where we ignore the constant term, and $B = -\frac{A}{h_0} \log(1-h_0)$. This utility function is linear in leisure which implies an infinite elasticity of substitution between leisure in different periods. This follows no matter how small this elasticity is for the individuals populating the economy. Therefore, the elasticity of substitution between leisure in different periods for the aggregate economy is infinite and independent of the willingness of individuals to substitute leisure across time.[2]

Hansen then goes onto solve the model by using the method of linear quadratic dynamic programming following Kydland and Prescott's procedure.[3] Key calibration parameters are $\delta = 0.025$, $\beta = 0.99$, and $A = 2$, which implies hours worked in the steady state (divisible labour case) is close to 1/3. h_0 was found to be 0.53, by equating hours of work in the steady state for both models equal to one another.

8.2.5 Key results

The table below presents the key findings from the Hansen paper. We've seen it before in Figure 6.2, but it's worth repeating the key points point here.

2. This was originally shown by Rogerson (1988). This result depends on the utility function being additively separable across time.

3. Linear quadratic dynamic programming is beyond the scope of this course. It's an alternative to the method of Blanchard and Kahn and the method of undetermined coefficients. McCandless (2008) (chapter 7) provides a good treatment.

Table 8.1: COMPARISON OF BASELINE RBC MODEL AND INDIVISIBLE LABOUR MODEL

Standard deviations in percent (a) and correlations with output (b) for U.S. and artificial economies.

	Quarterly U.S. time series[a] (55,3–84,1)		Economy with divisible labor[b]		Economy with indivisible labor[b]	
Series	(a)	(b)	(a)	(b)	(a)	(b)
Output	1.76	1.00	1.35 (0.16)	1.00 (0.00)	1.76 (0.21)	1.00 (0.00)
Consumption	1.29	0.85	0.42 (0.06)	0.89 (0.03)	0.51 (0.08)	0.87 (0.04)
Investment	8.60	0.92	4.24 (0.51)	0.99 (0.00)	5.71 (0.70)	0.99 (0.00)
Capital stock	0.63	0.04	0.36 (0.07)	0.06 (0.07)	0.47 (0.10)	0.05 (0.07)
Hours	1.66	0.76	0.70 (0.08)	0.98 (0.01)	1.35 (0.16)	0.98 (0.01)
Productivity	1.18	0.42	0.68 (0.08)	0.98 (0.01)	0.50 (0.07)	0.87 (0.03)

Source: Hansen (1985).

The indivisible labour economy displays much more fluctuations than the baseline RBC model. The standard RBC model undershoots fluctuations and volatility drastically, while the indivisible labour model overshoots slightly – so things look promising. Observe the ratio of the standard deviation in hours worked to the standard deviation of productivity: the standard RBC model has a ratio of 1, while the indivisible model has a ratio of 2.7. For the record, the US economy has a ratio of 1.4. Hansen's paper seemingly addressed a lot of the weaknesses in the original Kydland and Prescott (1982) model, who despite easing intertemporal substitution of leisure, struggled to get the baseline RBC model's ratio above 1.17. Conversely, the indivisible labour model delivers – as expected considering the assumptions – a high ratio.

However, the Hansen model still suffers from a lot of weaknesses that have plagued RBC models (see the chapter on the RBC model for an overview). Primarily, the propagation mechanism is still weak – productivity shocks are still too highly correlated with output fluctuations, and there is still too high a correlation between hours worked and wages.

Nevertheless, the Hansen model was a step in the right direction, and the model became a mainstream model in classrooms since its introduction.

8.3 Efficiency wages and the Shapiro-Stiglitz model

8.3.1 Motivation

Now we delve a bit deeper into mechanisms that can explain unemployment. If there is unemployment in a Walrasian labour market, unemployed workers immediately bid the wage down until supply and demand are in balance. Theories of unemployment can therefore be classified according to their view of why this mechanism fails to operate. Concretely, consider an unemployed worker who offers to work for a firm for slightly less (or more) than the firm is currently paying, and who is otherwise identical to the firm's current workers. The firm may not want to offer a different wage, however, due to there being costs and benefits to paying lower (or higher) wages. Theories in which there is a cost as well as a benefit to the firm of paying different wages are known as efficiency wage theories.

First, and most simply, a higher wage, for example, can increase workers' food consumption, and thereby cause them to better nourished and more productive. Obviously this possibility is not important in developed economies. Nonetheless, it provides a concrete example of an advantage of paying a higher wage. For that reason, it is often a useful reference point.

Second, a higher wage can increase workers' effort in situations where the firm cannot monitor them perfectly. In a Walrasian labour market, workers are indifferent about losing their jobs, since identical jobs are immediately available. thus, if the only way that firms can punish workers who exert low effort is bu firing them, workers in such a labour market have no incentive to exert effort. But if a firm pays more than the market-clearing wage, its jobs are valuable. Thus, its workers may choose to exert effort instead of shirking.

Third, paying a higher wage can improve workers' ability along dimensions the firm cannot observe. Specifically, if higher ability workers have higher reservation wages, offering a higher wage raises the average quality of the applicant pool, and thus raises the average ability of the workers the firm hires (Weiss, 1980).

Finally, a high wage can build loyalty among workers and hence induce higher effort; conversely, a low wage can cause anger and desire for revenge, and thereby lead to shirking or sabotage. Akerlof and Yellen (1990) present extensive evidence that workers' effort is affected by such forces as anger, jealousy, and gratitude. For example, they describe studies showing that workers who believe they are underpaid sometimes perform their work in ways that are harder for them in order to reduce their employers' profits.

8.3.2 A simple efficiency wage model

We now turn to a model of efficiency wages. There is a large number, N, of identical competitive firms. The representative firm seeks to maximise its profits, which are given by:

$$\pi = Y - wL, \tag{8.10}$$

where Y is the firm's output, w is the wage that it pays, and L is the amount of labour it hires. A firm's output depends on the number of workers it employs and on their effort. For simplicity, we assume the firm's production technology is:

$$Y = F(eL),\ F'(\cdot) > 0,\ F''(\cdot) < 0,$$

where e denotes workers' effort. The crucial assumption of efficiency wage models is that effort depends positively on the wage the firm pays:

$$e = e(w),\ e'(\cdot) > 0.$$

We also assume that there are $\bar{L}$ identical workers, each of whom supplies 1 unit of labour inelastically.

The problem facing the representative firm is:

$$\underset{L,w}{\operatorname{argmax}} \ \{F(e(w)L) - wL\}.$$

If there are unemployed workers, the firm can choose the wage freely. If unemployment is zero, on the other hand, the firm must pay at least the wage paid by other firms. When the firm is unconstrained, the FOCs for L and w are:

$$F'(e(w)L)e(w) - w = 0, \tag{8.11}$$

$$F'(e(w)L)e'(w)L - L = 0. \tag{8.12}$$

We can rewrite the first FOC (8.11) as:

$$F'(e(w)L) = \frac{w}{e(w)},$$

and by substituting this into (8.12) we get:

$$\frac{w}{e(w)}e'(w)L = L,$$

and then divide by L to get:

$$\frac{w}{e(w)}e'(w) = 1. \tag{8.13}$$

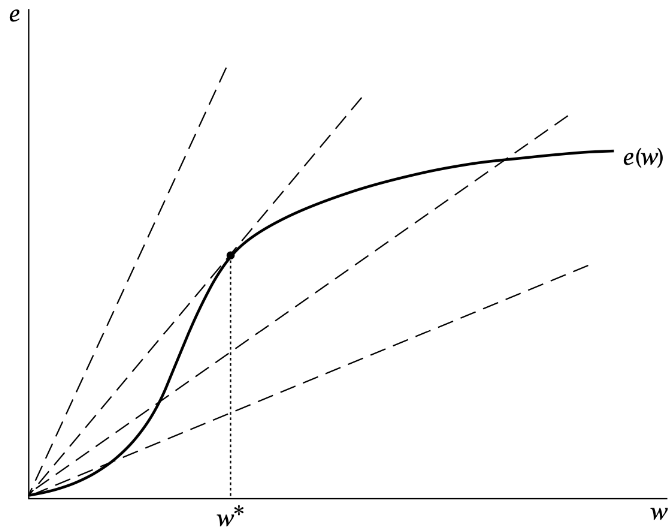
What this equation states is that at the optimum, the elasticity of effort with respect to the wage is 1. To understand this condition, note that output is a function of the quantity of effective labour, eL. The firm therefore wants to hire effective labour as cheaply as possible. When the firm hires a worker, it obtains $e(w)$ units of effective labour at a cost of w. Thus, the cost per unit of effect labour is $w/e(w)$. When the elasticity of $e$with respect to w is 1, a marginal change in w has no effect on this ratio; thus this is the FOC of the problem of choosing w to minimise the cost of effective labour. The wage satisfying (8.13) is known as the efficiency wage.

Figure 8.1 shows the choice of w graphically in (w, e) space. The rays coming out of the origin are lines where the ratio of e to w is constant; the ratio is larger on the higher rays. Thus, the firm wants to choose w to attain a high a ray as possible. This occurs where the $e(w)$ function is just tangent to one of the rays – that is, where the elasticity of e with respect to w is 1.

Note also that the FOC (8.11) states that the firm hires workers until the marginal product of effective labour equals its cost. This is analogous to the condition in a standard labour demand problem that the firm hires labour up to the point where the marginal product equals the wage.

Equations (8.11) and (8.13) describe the behaviour of a single firm. Scaling up to the economy wide equilibrium is straightforward. Let w^* and L^* denote the values of w and L that satisfy (8.11) and (8.13). Since firms are identical, each firm chooses these same values of w and L. Total labour demand is therefore NL^*. If labour supply, $\bar{L}$, exceeds this amount, firms are unconstrained in

Figure 8.1: Determination of the Efficiency Wage



Source: Romer (2012).

their choice of w. In this case, the wage is w^*, employment is NL^*, and there is unemployment of amount $\bar{L} - NL^*$. If NL^* exceeds $\bar{L}$, firms are constrained, and the wage is bid up to the point where demand and supply are in balance, and there is no unemployment.

This simple model shows how efficiency wages can give rise to unemployment. In addition, the model implies that the real wage is unresponsive to demand shifts. Suppose the demand for labour increases. Since the efficiency wage, w^*, is determined entirely by the properties of the effort function, $e(\cdot)$, there is no reason for firms to adjust their wages. Thus the model provides a candidate explanation of why shifts in labour demand lead to large movements in employment and small changes in the real wage. In addition, the fact that the real wage and effort do not change implies that the cost of a unit of effective labour does not change. As a result, in a model with price setting firms, the incentive to adjust prices is small.

Unfortunately, these results are less promising than they appear. The key difficulty is that they apply not just to the short-run but to the long-run too: the model implies that as economic growth shifts the demand for labour outward, the real wage remains unchanged and unemployment trends downward. Eventually, unemployment reaches zero, at which further increases in demand lead to increases in the real wage. In practice, however, we observe no clear trend in unemployment over extended periods. In other words, shifts in labour demand in the short-run only affect employment and not the real wage; and in the long-run it only seems to affect only real wages. This simple model does not explain this pattern.

8.3.3 Assumptions of the Shapiro-Stiglitz model

We now look at the Shapiro-Stiglitz model, which focuses on firms' monitoring ability (or lack thereof). Presenting a formal model of imperfect monitoring serves three purposes. First, it allows us to investigate whether this idea holds up under scrutiny. Second, it permits us to analyse additional questions. For example, only with a formal model can we ask whether government policies can improve welfare. Third, the mathematical tools the model employs are useful in other settings.

We first assume that the economy consists of a large number of workers, $\bar{L}$, and a large number of firms, N. Workers maximise their expected discounted utilities, and firms maximise their expected discounted profits. The model is set in continuous time. For simplicity, this analysis focuses on steady states.

Consider workers first. The representative worker's lifetime utility is:

$$U = \int_{t=0}^{\infty} \exp(-\rho t) u(t) dt, \;\; \rho > 0,$$

where $u(t)$ is instantaneous utility at time t, and ρ is the discount rate. Instantaneous utility is:

$$u(t) = \begin{cases} w(t) - e(t) & \text{if employed,} \\ 0 & \text{if unemployed,} \end{cases}$$

where w is the wage and e is the worker's effort. There are only two possible effort levels, $e = 0$ and $e = \bar{e}$. Thus, at any moment a worker must be in one of three states: employed and exerting effort (denoted E), employed and shirking (denoted S), or unemployed (denoted U).

A key ingredient of the model is its assumptions concerning workers' transitions between the three states. First, there is an exogenous rate at which jobs end. Specifically, if a worker begins working a job at some time, t_0 (and if the worker exerts effort), the probability that the worker is still employed in the job at some time later, t, is:

$$P(t) = \exp(-b(t - t_0)), \;\; b > 0. \tag{8.14}$$

This equation implies that $P(t + \tau)/P(t)$ equals $\exp(-b\tau)$, and thus that it is independent of t: if a worker is employed at some time, the probability that she is still employed at time τ later is $\exp(-b\tau)$ regardless of how long the worker has already been employed. This assumption that job breakups follow a Poisson process simplifies the analysis greatly, because it implies that there is no need to keep track of how long workers have been in their jobs.

An equivalent way to describe the process of job breakup is to say that it occurs with probability b per unit time, or to say that the hazard rate for job breakup is b. That is, the probability that an employed worker's job ends in the next dt units of time approaches $b \cdot dt$ as $dt \to 0$. To see that our assumptions imply this, note that (8.14) implies $P'(t) = -bP(t)$.

The second assumption concerning workers' transitions between states is that firms' detection of workers who are shirking is also a Poisson process. Specifically, detection occurs with probability q per unit time. q is exogenous, and detection is independent of job breakups. workers who are caught shirking are fired. Thus, if a worker is employed but shirking, the probability that she is still employed time τ later is $\exp(-q\tau)\exp(-b\tau)$, the probability that the worker has not been caught and fired times the probability that the job has not ended exogenously.

Third, unemployed workers find employment at rate a per unit time. Each worker takes a as given. In the economy as a whole, however, a is determined endogenously. When firms want to hire workers, they choose workers at random out of the pool of unemployed workers. Thus a is determined by the rate at which firms are hiring (which is determined by the number of employed workers and the rate at which jobs end) and the number of unemployed workers. Because workers are identical, the probability of finding a job does not depend on how workers become unemployed or on how long they are unemployed.

Firms' behaviour is simple. A firm's profits at time t are:

$$\pi(t) = F(\bar{e}L(t)) - w(t)\left[L(t) + S(t)\right],\ F'(\cdot) > 0,\ F''(\cdot) < 0, \tag{8.15}$$

where L is the number of employees who are exerting effort and S is the number who are shirking. The problem facing the firm is to set w sufficiently high that its workers do not shirk, and to choose L. Because the firm's decisions at any date affect profits only at that date, there is no need to analyse the present value of profits: the firm chooses w and L at each moment to maximise the instantaneous flow of profits.

The final assumption of the model is:

$$\begin{aligned} \bar{e}F'\left(\frac{\bar{e}\bar{L}}{N}\right) &> \bar{e}, \\ \implies F'\left(\frac{\bar{e}\bar{L}}{N}\right) &> 1. \end{aligned}$$

This condition states that if each firm hires $1/N$ of the labour force, the marginal product of labour exceeds the cost of exerting effort. Thus, in the absence of imperfect monitoring, there is full employment.

8.3.4 The values of states

Let V_i denote the value of being in state i (for $i = (E, S, U)$). That is, V_i is the expected value of discounted lifetime utility from the present moment forward of a worker who is in state i. Because transitions among states follow Poisson processes, the V_i's do not depend on how long the worker has been in the current state or on the worker's prior history. And because we are focusing on steady states, the V_i's are constant over time.

To find V_E, V_S, and V_U, it is not necessary to analyse the various paths the worker may follow over the infinite future. Instead we can use dynamic programming. The central idea of dynamic programming is to look at only a brief interval of time and use the V_i's themselves to summarise what occurs after the end of the interval.[4] Consider first a worker that is employed and exerting effort at time 0. Suppose temporarily that time is divided into intervals of length Δt, and that a worker who loses her job during one interval cannot begin to look for a new job until the beginning of the next interval. Let $V_E(\Delta t)$ and $V_U(\Delta t)$ denote the values of employment and unemployment as of the beginning of an interval under this assumption. In a moment we will let $\Delta t \to 0$. When we do this, the constraint that a worker who loses her job during an interval cannot find a new job during the remainder of that interval becomes irrelevant. Thus $V_E(\Delta t) \to V_E$,

If a worker is employed in a job paying a wage of w, $V_E(\Delta t)$ is given by:

$$\begin{aligned} V_E(\Delta t) = &\int_{t=0}^{\Delta t} \exp(-bt)\exp(-\rho t)(w - \bar{e})dt \\ &+ \exp(-\rho \Delta t)\left[\exp(-b\Delta t)V_E(\Delta t) + (1 - \exp(-b\Delta t))V_U(\Delta t)\right]. \end{aligned} \tag{8.16}$$

The first term of this equation reflects utility during the interval $(0, \Delta t)$. The probability that the worker is still employed at time t is $\exp(-bt)$. if the worker is employed, flow utility is $w - \bar{e}$. Discounting this back to time 0 yields an expected contribution to lifetime utility of $\exp(-(\rho+b)t)(w-\bar{e})$. The second term reflects utility after Δt. At time Δt, the worker is employed with probability $\exp(-b\Delta t)$ and unemployed with probability $1 - \exp(-b\Delta t)$. Combining these probabilities with the V's and discounting yields the second term.

If we compute the integral in (8.16), we can rewrite the equation as:

$$\begin{aligned} V_E(\Delta t) = &\frac{1}{\rho + b}\left(1 - \exp(-(\rho + b)\Delta t)\right)(w - \bar{e}) \\ &+ \exp(-\rho \Delta t)\left[\exp(-b\Delta t)V_E(\Delta t) + (1 - \exp(-b\Delta t))V_U(\Delta t)\right], \end{aligned}$$

and solving this expression for $V_E(\Delta t)$ gives:

$$V_E(\Delta t) = \frac{1}{\rho + b}(w - \bar{e}) + \frac{1}{1 - \exp(-(\rho + b)\Delta t)}\exp(-\rho\Delta t)(1 - \exp(-b\Delta t))V_U(\Delta t). \tag{8.17}$$

As described above, V_E equals the limit of $V_E(\Delta t)$ as $\Delta t \to 0$. Similarly, V_U equals the limit of $V_U(\Delta t)$ as $t \to 0$. To find this limit, apply L'Hopital's rule to (8.17). This yields:

$$V_E = \frac{1}{\rho + b}\left[(w - \bar{e}) + bV_U\right]. \tag{8.18}$$

Intuitively: Think of an asset that pays dividends at rate $w - \bar{e}$ per unit time when the worker is employed and no dividends when the worker is unemployed. In addition, assume that the asset is

4. Here we look dynamic programming in a continuous context. We previously looked at the discrete time case, where we only looked at one period ahead. See Ljungqvist and Sargent (2018) for a proper treatment of dynamic programming.

being priced by risk-neutral investors with required rate of return ρ. Since the expected present value of lifetime dividends of this asset is the same as the worker's expected present value of lifetime utility, the asset's price must be V_E when the worker is employed and V_U when the worker is unemployed. For the asset to be held, it must provide an expected rate of return of ρ. That is, its dividends per unit time, plus any expected capital gains or losses per unit time, must equal ρV_E. When the worker is employed, dividends per unit time are $w - \bar{e}$, and there is a probability b per unit time of a capital loss of $V_E - V_U$. Thus,

$$\rho V_E = (w - \bar{e}) - b(V_E - V_U), \tag{8.19}$$

and rearranging this expression yields (8.18).

If the worker is shirking, the "dividend" is w per unit time, and the expected capital loss is $(b+q)(V_S - V_U)$ per unit time. Thus, reasoning parallel to that used to derive (8.19) implies:

$$\rho V_S = w - (b+q)(V_S - V_U). \tag{8.20}$$

Finally, if the worker is unemployed, the dividend is zero and the expected capital gain (assuming that firms pay sufficiently high wages that employed workers exert effort) is $a(V_E - V_U)$ per unit time. Thus:

$$\rho V_U = a(V_E - V_U). \tag{8.21}$$

8.3.5 The no-shirking condition

The firm must pay enough that $V_E \geq V_S$; otherwise, its workers exert no effort and produce nothing. At the same time, since effort cannot exceed $\bar{e}$, there is no need to pay any excess over the minimum needed to induce effort. Thus the firm chooses w so that V_E equals V_S:

$$V_E = V_S.$$

Substitute in our expressions for V_E and V_S from (8.19) and (8.20) to yield:

$$\begin{aligned} (w - \bar{e}) - b(V_E - V_U) &= w - (b+q)(V_S - V_U) \\ \Leftrightarrow V_E - V_U &= \frac{\bar{e}}{q}. \end{aligned} \tag{8.22}$$

This equation implies that firms set wages high enough that workers strictly prefer employment to unemployment. Thus workers obtain rents. The size of the premium is increasing in the cost of exerting effort, $\bar{e}$, and decreasing in firms' efficacy in detecting shirkers, q.

The next step is to find what the wage must be for the rent to employment to equal $\bar{e}/q$. Equations (8.19) and (8.21) imply:

$$\rho(V_E - V_U) = (w - \bar{e}) - (a+b)(V_E - V_U). \tag{8.23}$$

It follows that for $V_E - V_U$ to equal $\bar{e}/q$, the wage must satisfy

$$w = \bar{e} + (a + b + \rho)\frac{\bar{e}}{q}. \tag{8.24}$$

Thus, the wage needed to induce effort is increasing in the cost of effort $\bar{e}$, the ease of finding jobs a, the rate of job breakup b, and the discount rate ρ, and decreasing in the probability that shirkers are detected q.

Next, write the rate at which the unemployed find jobs a in terms of employment per firm L. Use the fact that, since the economy is in steady state, movements along the extensive margin balance. The number of workers becoming unemployed per unit time is N (the number of firms) times L (the number of workers per firm) times b (the rate of job breakup). The number of unemployed workers finding jobs is $\bar{L} - NL$ times a. Equating these two quantities yields:

$$a = \frac{NLb}{\bar{L} - NL}, \tag{8.25}$$

which implies:

$$a + b = \frac{\bar{L}}{\bar{L} - NL}b,$$

and substituting this into (8.24) yields:

$$w = \bar{e} + \left(\rho + \frac{\bar{L}}{\bar{L} - NL}b\right)\frac{\bar{e}}{q}, \tag{8.26}$$

also known as the no-shirking condition (NSC). It shows, as a function of the levels of employment, the wage that firms must pay to induce workers to exert effort. When more workers are employed, there are fewer unemployed workers and more workers leaving their jobs; thus it is easier for unemployed workers to find employment. The wage needed to deter shirking is therefore an increasing function of employment. At full employment, unemployed workers find work instantly, and so there is no cost to being fired and no wage that can deter shirking. The set of points in (NL, w) space satisfying the NSC are shown in Figure 8.2.

8.3.6 Closing the model

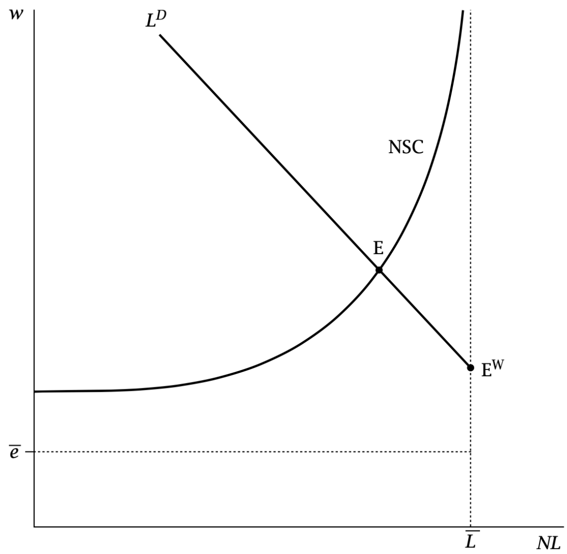
Firms hire workers up to the point where the marginal product of labour equals the wage. Equation (8.15) implies that when its workers are exerting effort, a firm's flow profits are:

$$\pi(t) = F(\bar{e}L) - wL.$$

Thus, the condition for the marginal product of labour equaling the wage is:

$$\frac{\partial \pi(t)}{\partial L} = F'(\bar{e}L)\bar{e} - w = 0$$

Figure 8.2: The Shapiro-Stiglitz Model



Source: Romer (2012).

$$\implies F'(\bar{e}L)\bar{e} = w. \tag{8.27}$$

The set of points satisfying (8.27) (which is a simple labour demand curve) is also shown in Figure 8.2.

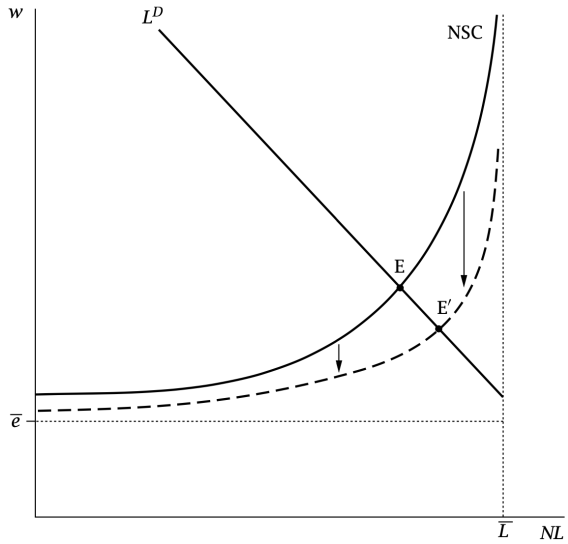
Labour supply is horizontal at $\bar{e}$ up to the number of workers $\bar{L}$, and then vertical. In the absence of imperfect monitoring, equilibrium occurs at the intersection of labour demand and supply. Our assumption that the marginal product of labour at full employment exceeds the disutility of effort ($F'(\bar{e}\bar{L}/N) > 1$) implies that this intersection occurs in the vertical part of the labour supply curve. The Walrasian equilibrium is shown as point E^W in the diagram.

With imperfect monitoring, equilibrium occurs at the intersection of the labour demand demand curve (Equation (8.27)) and the NSC (Equation (8.26)). This is shown as Point E in the diagram. At the equilibrium, there is unemployment. Unemployed workers strictly prefer to be employed at the prevailing wage and exert effort than to remain unemployed. Nonetheless, they cannot bid the wage down: firms know that if they hire additional workers at slightly less than the prevailing wage, the workers will prefer shirking to exerting effort. Thus the wage does not fall, and the unemployment remains.

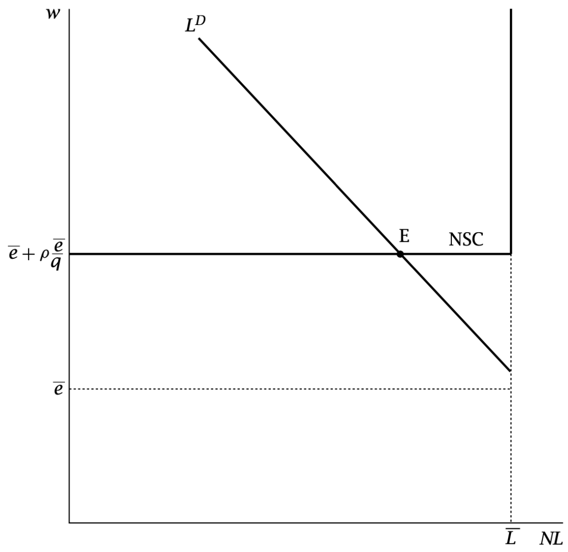
Two examples may help to clarify the workings of the model. First, a rise in q – an increase in the probability per unit time that shirker is detected – shifts the no-shirking locus down and does not affect the labour demand curve. This is shown in Figure 8.3a. The real wage falls and employment rises. As q approaches infinity, the probability that a shirker is detected in any finite length of time approaches 1. As a result, the no-shirking wage approaches $\bar{e}$ for any level of employment less than full employment. Thus, the economy approaches the Walrasian equilibrium.

Second, if there is no turnover ($b = 0$), unemployed workers are never hired. As a result, the no-shirking wage in this case is $\bar{e} + \rho\bar{e}/q$. Intuitively, the gain from shirking relative to exerting effort is $\bar{e}$ per unit time. The cost is that there is probability q per unit time of becoming permanently unemployed and thereby losing the discounted surplus from the job, which is $(w - \bar{e})/\rho$. Equating the cost and benefit gives $w = \bar{e} + \rho\bar{e}/q$. This is shown in Figure 8.3b.

Figure 8.3: Changes in the Shapiro-Stiglitz Model



(a) The Effects of $q \uparrow$

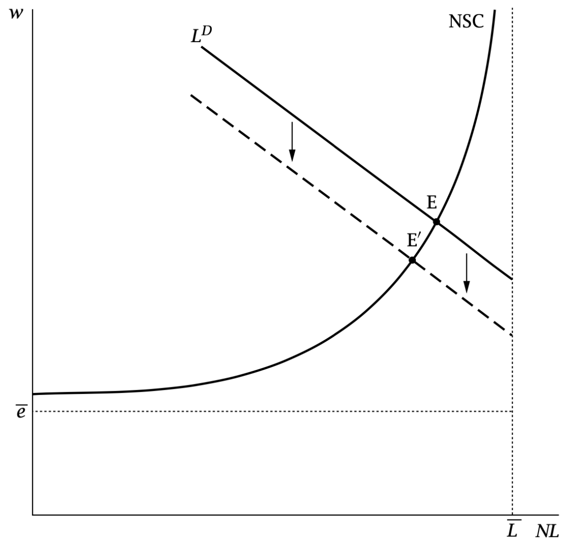


(b) No Turnover ($b = 0$)

Source: Romer (2012).

8.3.7 Implications of the Shapiro-Stiglitz model

Figure 8.4: Fall in Labour Demand



Source: Romer (2012).

The model implies that there is equilibrium unemployment and suggests various factors that are likely to influence it. Thus the model has some promise as a candidate explanation of unemployment. Unfortunately, the model is so stylised that it is difficult to determine what level of unemployment it predicts.

With regard to short-run fluctuations, consider the impact of a fall in labour demand, shown in Figure 8.4. w and L move down along the NSC locus. Since labour supply is perfectly inelastic, employment necessarily responds more than it would without imperfect monitoring. Thus, the model suggests one possible reason that wages may respond less to demand driven output fluctuations than they would if workers were always on their labour supply curves.

Unfortunately, however, this effect appears to be quantitatively small. When unemployment is lower, a worker who is fired can find a new job more easily, and so the wage needed to prevent shirking is higher; this is the reason the NSC locus slopes up. Attempts to calibrate the model suggest that the locus is quite steep at the levels of unemployment we observe. That is, the model implies that the impact of a shift in labour demand falls mainly on wages and relatively little on employment.

Finally, the model implies that the decentralised equilibrium is inefficient. To see this, note that the marginal product of labour at full employment, $F'(\bar{e}\bar{L}/N)\bar{e}$, exceeds the cost to workers of supply effort, $\bar{e}$. Thus the first-best allocation is for everyone to be employed and exert effort. Of course, the government cannot bring this about by simply dictating that firms move down the labour demand curve until full employment is reached: this policy causes workers to shirk, and thus results in zero output. But Shapiro and Stiglitz note that wage subsidies financed by lump-sum taxes or profits taxes improve welfare. This policy shifts the labour demand curve up, and thus increases the wage and employment along the NSC. Since the value of the additional output exceeds the opportunity cost of producing it, overall welfare rises. How the gain is divided between workers and firms depends on how the wage subsidies are financed.

8.4 Models of search and match

8.4.1 Motivation

The final departure of the labour market from Walrasian assumptions that we consider is the simple fact that workers and jobs are heterogeneous. In a frictionless labour market, firms are indifferent about losing their workers, since identical workers are costlessly available at the same wage; likewise, workers are indifferent about losing their jobs. These implications are obviously not accurate descriptions of actual labour markets.

When workers and jobs are highly heterogeneous, the labour market has little resemblance to a Walrasian market. Rather than meeting in centralised markets where employment and wages are determined by the intersections of supply and demand curves, workers and firms meet in a decentralised, one-on-one fashion, and engage in a costly process of trying to match up idiosyncratic preferences, skills, and needs. Since this process is not instantaneous, it results in some unemployment. In addition, it may have implications for how wages and employment respond to shocks.

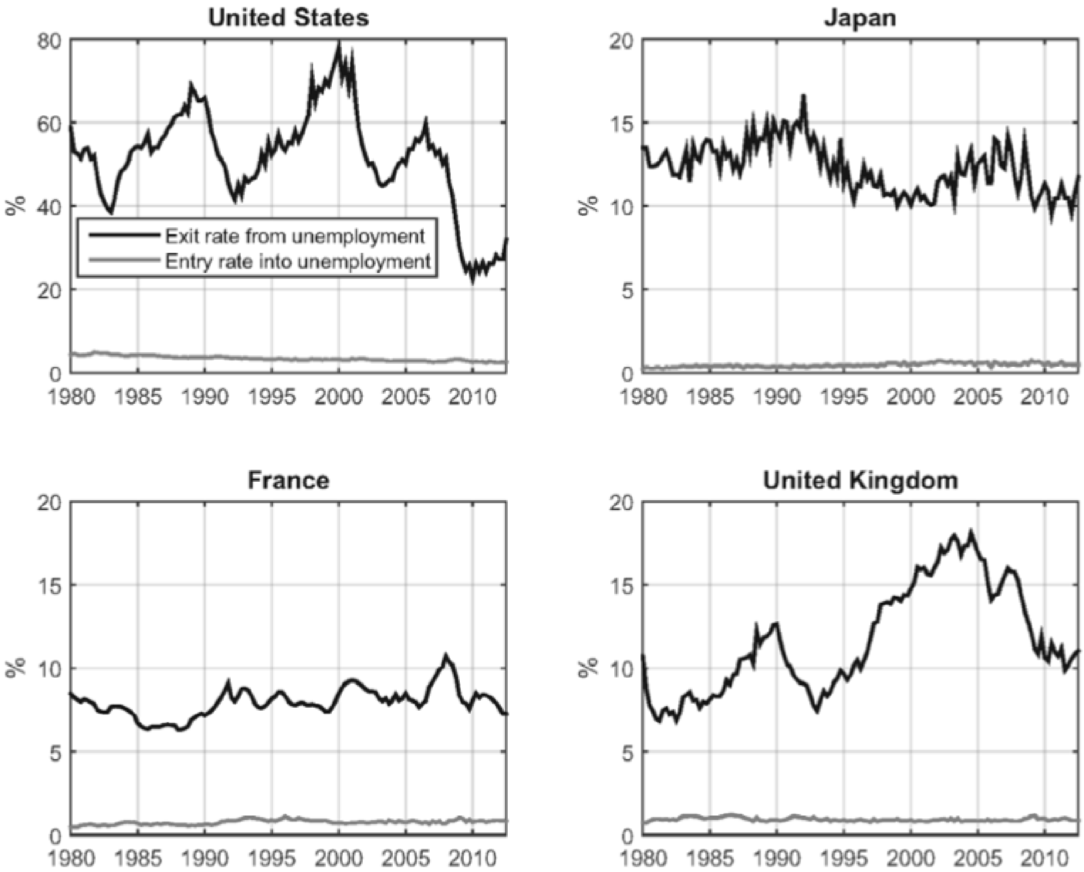
In this section, we present a model of firm and worker heterogeneity and the matching process. Because modelling heterogeneity requires abandoning many of our usual tools, even a basic model is relatively complicated. As a result, the model here only introduces some of the issues involved. This class of models is known collectively as the Diamond-Mortensen-Pissarides search and match (SAM) model.

SAM models have become common as a means of understanding the macroeconomics of the labour market. One reason for this is a number of empirical studies which have examined the behaviour of labour markets over the business cycle. These studies reveal a number of different phenomena such as:

- Even in recessions, large numbers of firms have unfilled vacancies and in booms some firms are laying off workers.
- In every period there are large gross labour market flows; movements in job creation and job destruction. The change in unemployment reflects net flows only (i.e., job destruction less job creation) and so is only a part of the overall labour market story.
- Job creation is slightly procyclical but job destruction is strongly counter-cyclical with big spikes in recessions. In other words, big increases in unemployment are caused by occasional large periods of job destruction. The fact that job creation and job destruction have different

cyclical properties suggests that labour market allocations are not well coordinated – it takes several periods for the unemployed to find vacancies.

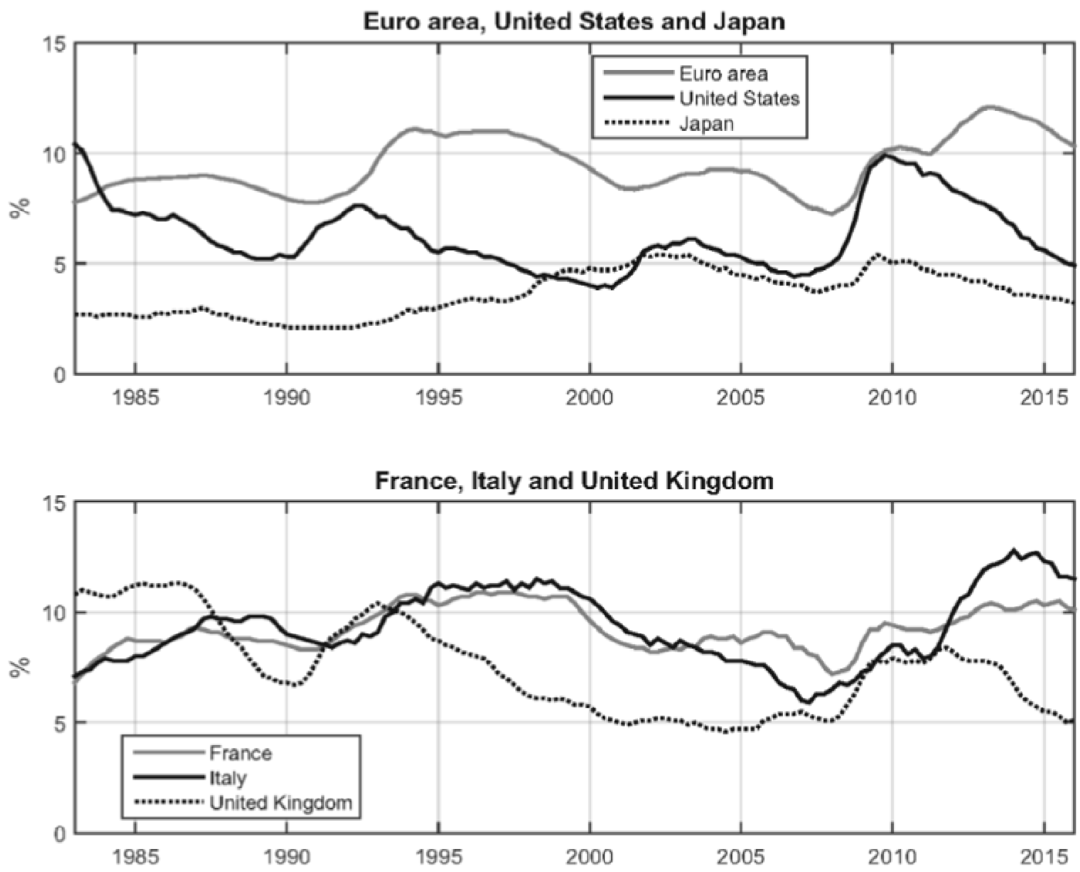
Figure 8.5: Exit and Entry Rates to/from Unemployment



These facts imply that it is probably flows in and out of the labour market that are particularly important for understanding the cyclical behaviour of unemployment. Notice from Figure 8.5 that the exit rate from unemployment (we will define this shortly) is more cyclical than the entry rate into unemployment, and that to a first order approximation, the entry rate seems to be constant across time.

Finally, one other motivation for studying SAM models is that it allows us to consider externalities that may arise in the labour market. SAM models inherently present frictions into the labour market, which may produce some inefficiencies that we see in the real world. For example, the wage bargaining process itself may be inefficient: the process may not internalise the effect of wages on the unemployed because they are not represented in the bargained contract between a hired worker and a firm (commonly known as the insider-outsider dilemma). A typical Walrasian

Figure 8.6: Historical Unemployment Rates



labour market does not take into account these real frictions, and is unable to give much guidance in terms of policy recommendations to alleviate unemployment. By introducing labour market frictions, our model is able to capture one aspect of real life phenomena, and we can begin to consider optimal policy.

8.4.2 Definitions

Recall the definition of an unemployed individual: an individual who is without work, currently able to work, and seeking work. Figure 8.6 plots unemployment rates for a selection of developed economies. As previously stated, to understand the changes in the unemployment rate, we need to consider worker flows.

Let's define the entry rate into unemployment as δ_t, where

$$\delta_t = \frac{\text{flow of employed workers becoming unemployed during period } t}{\text{employed at the beginning of period } t}.$$

In other words, δ_t is the average probability of a worker becoming unemployed in period t. Next, let's define the exit rate from unemployment, p_t, as:

$$p_t = \frac{\text{flow of unemployed workers becoming employed during period } t}{\text{unemployed at the beginning of period } t},$$

and so p_t is the average probability of an unemployed person becoming employed in period t.

Linking the rate of unemployment with worker flows we get:

$$\begin{aligned} u_t - u_{t-1} &= \delta_t(1 - u_{t-1}) - p_t u_{t-1} \\ \Leftrightarrow \Delta u_t &= \text{gross entry flow} - \text{gross exit flow}, \end{aligned} \tag{8.28}$$

where u_t is the unemployment rate. Using $\bar{\delta}$ and $\bar{p}$ to denote the mean transition rates to/out of unemployment, from (8.28), the mean unemployment rate is:

$$\begin{aligned} 0 &= \bar{\delta} - \bar{\delta}\bar{u} - \bar{p}\bar{u} \\ \bar{u} &= \frac{\bar{\delta}}{\bar{\delta} + \bar{p}}. \end{aligned} \tag{8.29}$$

Figure 8.7: Average Entry and Exit Rates of Unemployment

So, we can conclude that entry and exit rates are quite informative on the conditions of the labour market. As an example, consider Figure 8.7 which shows the different average entry and exit rates of unemployment for a selection of countries. We can infer that the US tends to have a very flexible labour market, where it's quite easy to find and lose a job. Conversely, an economy like Italy seems to have a rigid labour market, where it's difficult to find jobs, but job duration seems to be quite long.

8.4.3 The matching function

Let the number of matches in each period be given by the matching function:

$$m(v_t, u_t) = v_t^{1-\xi} u_t^{\xi}, \quad 0 < \xi < 1, \tag{8.30}$$

where v_t is the number of job vacancies, and n_t and u_t is the number of employed and unemployed workers, respectively:

$$u_t = 1 - (1 - \delta) n_{t-1}. \tag{8.31}$$

The properties of the matching function are crucial to the model. In principle, it need not have constant returns to scale. When it exhibits increasing returns, there are thick-market effects: increases in the resources devoted to search make the matching process operate more effectively, in the sense that it yields more output (matches) per unit of input (unemployment and vacancies). When the matching function has decreasing returns, there are crowding effects.

The prevailing view, however, is that in practice constant returns is a reasonable approximation. For a large economy, over a relevant range, the thick-market and crowding effects may be relatively unimportant or may roughly balance. Empirical efforts to estimate the matching function have found no strong evidence of departures from constant returns.[5]

The assumption of constant returns implies that a single number, the ratio of vacancies to unemployment, summarises the tightness of the labour market. Define $\theta_t = v_t/u_t$ and note that constant returns imply:

$$\frac{m(v_t, u_t)}{v_t} = m\left(1, \frac{1}{v_t/u_t}\right) = q\left(\frac{v_t}{u_t}\right) = q(\theta_t), \tag{8.32}$$

which is the matching rate for vacancies (probability of filling a vacancy), and:

$$\frac{m(v_t, u_t)}{u_t} = m\left(\frac{v_t}{u_t}, 1\right) = p\left(\frac{v_t}{u_t}\right) = p(\theta_t), \tag{8.33}$$

which is the matching rate for the unemployed (probability of finding a job).

Our assumption that $m(v_t, u_t)$ exhibits constant returns and that it is increasing both arguments imply that $m(\theta_t)$ is increasing in θ_t, but that the increase is less than proportional. Thus, when the labour market is tighter (when θ_t is greater), the job-finding rate is higher and the vacancy filling rate is lower.

When macroeconomists want to assume a functional form for the matching function, they almost universally assume that it is Cobb-Douglas. We will take that approach here too.

5. See for example “The Beveridge Curve” by Blanchard et al. (1989).

8.4.4 Employment accumulation

Each period a fraction, δ, of workers lose (exogenously) their job, and each period a fraction, $q(\theta_t)$, of vacancies are filled. Let us denote the law of motion of employment as:

$$n_t = (1-\delta)n_{t-1} + q(\theta_t)v_t, \tag{8.34}$$

where the first term on the RHS – jobs that survive separation – is a stock, and the second term – new jobs created – is a flow object. Note that we use $q(\theta_t)$ and not $p(\theta_t)$ for the law of motion here.

8.4.5 Unemployment dynamics

Let's turn to look at what determines the unemployment rate. In this model, members of the households are either employed or unemployed, and so we have:

$$\begin{aligned} \text{Labour Force} &= \text{Employed} + \text{Unemployed} \\ \Leftrightarrow \bar{L} &= n_t + u_t, \end{aligned}$$

where we assume that the labour force is assumed to be constant, and no households are outside the labour force. Thus, in the model, the rates of employment and unemployment are related as:

$$1 = n_t^r + u_t^r,$$

and the unemployment rate is defined as:

$$u_t^r = 1 - n_t^r. \tag{8.35}$$

What about unemployment in the long-run? Begin by considering the stock of employed in the model given by the law of motion of employment (8.34):

$$n_t = (1-\delta)n_{t-1} + q(\theta_t)v_t,$$

and note that job creation can alternatively be expressed as:

$$q(\theta_t)v_t = p(\theta_t)u_t.$$

In words, this just says that the entry matching rate for vacancies multiplied by the number of vacancies is equal to the matching rate of employment multiplied by the number of unemployed. This is intuitive. So, we can rewrite the law of motion for employment as:

$$n_t = (1-\delta)n_{t-1} + p(\theta_t)u_t. \tag{8.36}$$

Using (8.36), and normalising for the labour force, and evaluating at the steady state we have:

$$\delta n^r = p(\theta)u^r, \tag{8.37}$$

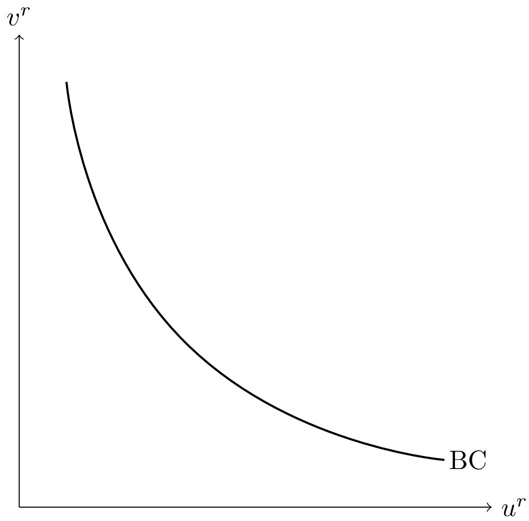
and since $n^r = 1 - u^r$ (from (8.35)), Equation (8.37) yields:

$$u^r = \frac{\delta}{\delta + p(\theta)}, \tag{8.38}$$

which is the long-run rate of unemployment, and it is determined by the job separation rate, δ, and the job creation rate, $p(\theta)$. Note that fiscal policy and labour market institutions are critical as they both affect $p(\theta)$ in this model. If the model had endogenous job separation, then δ would also be affected by policy and institutions.

Plotting (8.38) in (u^r, v^r) space gives us what is known as the Beveridge curve[6]:

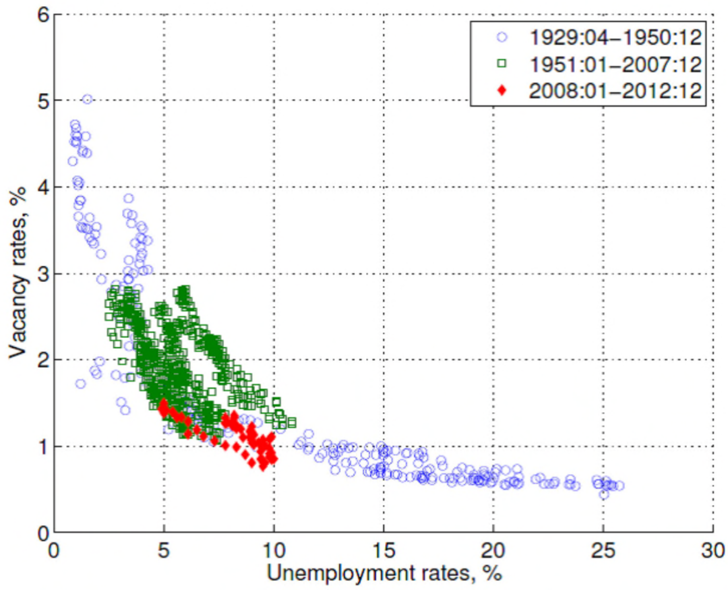
Figure 8.8: The Beveridge Curve



The Beveridge curve is fairly supported by the data as shown in Figure 8.9. In recent years, there has been some discussion about the accuracy of the Beveridge curve (about whether or not it's a "curve"). Such discussion is beyond the scope of this course, however.

6. Fun fact for the Oxonians: Baron William Henry Beveridge was Master of University College.

Figure 8.9: Empirical Evidence for the Beveridge Curve (US data)



8.4.6 Households and firms

Our assumptions here are fairly standard. Households provide labour to the firm, and maximise their utility subject to a budget constraint. Firms maximise profits, and using labour inputs n_t to produce goods with a simple production technology:

$$y_t = a_t n_t^{\alpha}, \quad 0 < \alpha < 1. \tag{8.39}$$

Firms recruit workers by posting vacancies v_t at a flat cost κ, and face matching frictions $m(v_t, u_t)$.

Because a firm and a worker that meet are collectively better off if the firm hires the worker, they would be forgoing a mutually advantageous trade if the firm did not hire the worker. Thus the assumption that all meetings lead to hires is reasonable. But this does not uniquely determine the wage. The wage must be high enough such that the worker wants to work for the job, and low enough such that the firm wants to hire the worker. Because there is strictly positive surplus from the match, there is a range of wages that satisfy these requirements. Workers and firms bargain and negotiate wages in each period according to Nash Bargaining.

To explain Nash Bargaining, let $\mathcal{S}_t$ denote the sum of firm's surplus, $\mathcal{J}_t$, and the worker's surplus, $\mathcal{W}_t$:

$$\mathcal{S}_t = \mathcal{J}_t + \mathcal{W}_t.$$

The worker takes a fraction η of the total surplus:

$$\mathcal{W}_t = \eta \mathcal{S}_t,$$

and so by defining surpluses $\mathcal{J}_t$ and $\mathcal{W}_t$, we can derive the established wage, w_t (we will show this soon).

8.4.7 The social planner's problem

There are two ways to go about solving for equilibrium – a centralised solution via the Ramsey social planner, and through a competitive, decentralised process. Let's first solve the Ramsey planner's problem, and then compare it to the decentralised equilibrium. The results are quite illuminating.

The Ramsey planner chooses $\{c_t, n_t, v_t\}_{t=0}^{\infty}$ to maximise utility:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \ln c_{t+s},$$

subject to the economy wide resource constraint:

$$y_t = c_t + \kappa v_t,$$

and the law of motion of employment (8.34):

$$n_t = (1-\delta)n_{t-1} + q(\theta_t)v_t.$$

The Ramsey planner knows:

- the production technology (8.39): $y_t = a_t n_t^{\alpha}$;
- the matching function (8.30): $m(v_t, u_t) = v_t^{1-\xi} u_t^{\xi}$;
- the transition probabilities (8.32)) and (8.33: $q(\theta_t) = \frac{m(v_t,u_t)}{v_t} = \theta_t^{-\xi}$ and $p(\theta_t) = \frac{m(v_t u_t)}{u_t} = \theta_t^{1-\xi}$, where $\theta_t = v_t/u_t$; and
- the amount of unemployed before hiring (8.31): $u_t = 1-(1-\delta)n_{t-1}$,

So, we can rewrite the constraints for the Ramsey planner as:

$$a_t n_t^{\alpha} = c_t + \kappa v_t, \tag{8.40}$$

and

$$\begin{aligned} n_t &= (1-\delta)n_{t-1} + \theta_t^{-\xi} v_t \\ &= (1-\delta)n_{t-1} + \left(\frac{v_t}{u_t}\right)^{-\xi} v_t \\ &= (1-\delta)n_{t-1} + \left(\frac{v_t}{1-(1-\delta)n_{t-1}}\right)^{-\xi} v_t. \end{aligned} \tag{8.41}$$

Note that the transition probabilities $q(\theta_t)$ and $p(\theta_t)$ are endogenous to the problem of the social planner. In other words, the Ramsey planner internalises/considers the effect of changes in vacancies and unemployment on the labour market transition probabilities and their effect on the equilibrium of the economy – the Ramsey planner internalises the effect of any potential search externalities on the equilibrium.

The Lagrangian for the Ramsey planner is:

$$\mathcal{L} = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left\{ \ln c_{t+s} + \lambda_{1,t+s} \left(a_{t+s} n_{t+s}^{\alpha} - c_{t+s} - \kappa v_{t+s} \right) + \lambda_{2,t+s} \begin{bmatrix} (1-\delta) n_{t-1+s} \\ + \left(\frac{v_{t+s}}{1-(1-\delta)n_{t-1+s}} \right)^{-\xi} v_{t+s} \\ -n_{t+s} \end{bmatrix} \right\},$$

where $\lambda_{1,t}$ and $\lambda_{2,t}$ are the Lagrangian multipliers, and the social planner chooses $\{c_t, n_t, v_t\}_{t=0}^{\infty}$. We know how to solve this problem – rewrite it as:

$$\begin{aligned} \mathcal{L} = \ln c_t + \lambda_{1,t} \left(a_t n_t^{\alpha} - c_t - \kappa v_t \right) + \lambda_{2,t} \left[(1-\delta) n_{t-1} + \left(\frac{v_t}{1-(1-\delta)n_{t-1}} \right)^{-\xi} v_t - n_t \right] \\ + \beta \mathbb{E}_t \left[\lambda_{2,t+1} \left[(1-\delta) n_t + \left(\frac{v_{t+1}}{1-(1-\delta)n_t} \right)^{-\xi} v_{t+1} - n_{t+1} \right] \right], \end{aligned} \tag{8.42}$$

and our FOCs are:

$$\frac{\partial \mathcal{L}}{\partial c_t} = \frac{1}{c_t} - \lambda_{1,t} = 0, \tag{8.43}$$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial n_t} &= \alpha \lambda_{1,t} a_t n_t^{\alpha-1} - \lambda_{2,t} \\ &+ \beta \mathbb{E}_t \lambda_{2,t+1} \left[(1-\delta) - \xi \left(\frac{v_{t+1}}{1-(1-\delta)n_t} \right)^{-\xi-1} v_{t+1} \left(\frac{(1-\delta) v_{t+1}}{(1-(1-\delta)n_t)^2} \right) \right] = 0, \end{aligned} \tag{8.44}$$

$$\frac{\partial \mathcal{L}}{\partial v_t} = -\kappa \lambda_{1,t} + \lambda_{2,t} \left[\left(\frac{v_t}{1-(1-\delta)n_{t-1}} \right)^{-\xi} - \xi \left(\frac{v_t}{1-(1-\delta)n_{t-1}} \right)^{-\xi-1} \frac{v_t}{1-(1-\delta)n_{t-1}} \right] = 0. \tag{8.45}$$

We can do some re-arranging and use our definitions of θ_t, $p(\theta_t)$, and $q(\theta_t)$ to get:

$$\lambda_{1,t} = \frac{1}{c_t}, \tag{8.46}$$

$$\frac{\lambda_{2,t}}{\lambda_{1,t}} = \alpha \frac{y_t}{n_t} + (1-\delta)\beta \mathbb{E}_t \frac{\lambda_{2,t+1}}{\lambda_{1,t}} \left[1 - \xi p(\theta_{t+1}) \right], \tag{8.47}$$

$$\frac{\kappa}{q(\theta_t)} = \frac{\lambda_{2,t}}{\lambda_{1,t}} (1-\xi), \tag{8.48}$$

which describe the marginal utility of consumption, the marginal benefit of an additional worker, and the marginal benefit of posting a vacancy.

Combining our FOCs, we can derive an equilibrium condition. Begin by multiplying equation (8.47) with $\lambda_{1,t}/\lambda_{2,t}$ to obtain:

$$1 = \frac{\lambda_{1,t}}{\lambda_{2,t}} \alpha \frac{y_t}{n_t} + (1-\delta)\beta \mathbb{E}_t \frac{\lambda_{2,t+1}}{\lambda_{2,t}} \left[1 - \xi p(\theta_{t+1})\right], \tag{8.49}$$

and note that from (8.48) we have:

$$\frac{\lambda_{1,t}}{\lambda_{2,t}} = \frac{1-\xi}{\kappa/q(\theta_t)}, \tag{8.50}$$

and so we combine equations (8.49) and (8.50) to get:

$$\frac{\kappa}{q(\theta_t)} = (1-\xi)\alpha \frac{y_t}{n_t} + (1-\delta)\beta \frac{\kappa}{q(\theta_t)} \mathbb{E}_t \frac{\lambda_{2,t+1}}{\lambda_{2,t}} \left[1 - \xi p(\theta_{t+1})\right]. \tag{8.51}$$

Note that from (8.50) we also have:

$$\lambda_{2,t} = \frac{[\kappa/q(\theta_t)]}{1-\xi} \lambda_{1,t},$$

and so we can write (8.51) as:

$$\begin{aligned} \frac{\kappa}{q(\theta_t)} &= (1-\xi)\alpha \frac{y_t}{n_t} + (1-\delta)\beta \frac{\kappa}{q(\theta_t)} \mathbb{E}_t \frac{\lambda_{1,t+1} \left[\kappa/q(\theta_{t+1})\right]}{\lambda_{1,t} \left[\kappa/q(\theta_t)\right]} \left[1 - \xi p(\theta_{t+1})\right] \\ \Leftrightarrow \frac{\kappa}{q(\theta_t)} &= (1-\xi)\alpha \frac{y_t}{n_t} + (1-\delta)\beta \mathbb{E}_t \frac{\lambda_{1,t+1}}{\lambda_{1,t}} \left[1 - \xi p(\theta_{t+1})\right] \frac{\kappa}{q(\theta_{t+1})}. \end{aligned} \tag{8.52}$$

This equation is known as the job creation (JC) condition. There are a few things to note with the JC condition that we just derived. First, note that $\beta \mathbb{E}_t \frac{\lambda_{1,t+1}}{\lambda_{1,t}}$ is the stochastic discount factor or pricing kernel, $M_{t,t+1}$, that we used when we solved the firm problem in the decentralised RBC model – remember that $\lambda_{1,t}$ is nothing but the marginal utility from consumption. Secondly, the term $\left[1 - \xi p(\theta_{t+1})\right] \kappa/q(\theta_{t+1})$ represents the net future benefit of a match and vacancy as the term $\xi p(\theta_{t+1})/q(\theta_{t+1}) = \xi \theta_{t+1}$ corrects for foregone search costs.

Now, the question we wish to ask is: Under what condition does a decentralised market economy replicate the Ramsey social planner allocation?

8.4.8 The decentralised equilibrium

Recall that the decentralised economy is such that each agent maximises its own objective function subject to its own constraint. We anticipate that externalities may arise since each agent fails to internalise the effect of its own choice on the economy.

Let's begin with the firms' problem: Each firm is small and takes transition probabilities as given, so the representative firm's problem is:

$$\underset{\{n_t, v_t\}_{t=0}^{\infty}}{\text{argmax}}\ \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \lambda_{1,t+s} \left(y_{t+s} - w_{t+s} n_{t+s} - \kappa v_{t+s} \right), \tag{8.53}$$

subject to:

$$n_t = (1-\delta) n_{t-1} + q(\theta_t) v_t, \tag{8.54}$$

$$y_t = a_t n_t^{\alpha}. \tag{8.55}$$

So, the firm wants to maximise profits subject to the law of motion of employment and its production technology.

Now, use the law of motion of employment (8.54) to solve for v_t and then substitute it and (8.55) into the firm's objective function (8.53) to get:

$$\underset{\{n_t\}_{t=0}^{\infty}}{\text{argmax}}\ \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \lambda_{1,t+s} \left(a_{t+s} n_{t+s}^{\alpha} - w_{t+s} n_{t+s} - \frac{\kappa}{q(\theta_{t+s})} \left[n_{t+s} - (1-\delta) n_{t-1+s} \right] \right),$$

which gives us the following FOC:

$$\frac{\kappa}{q(\theta_t)} = \alpha \frac{y_t}{n_t} - w_t + (1-\delta)\beta \frac{\lambda_{1,t+1}}{\lambda_{1,t}} \frac{\kappa}{q(\theta_{t+1})}. \tag{8.56}$$

This equation is the JC condition for the decentralised economy. In words, it states that the expected cost of posting a vacancy (the LHS) is equal to the expected benefits that the additional vacancy takes into production (RHS). Note that no where in the solution has the firm considered the effects of changes in vacancies and unemployment on the labour market transition probabilities – the transition probabilities are entirely exogenous to the solution. Furthermore, note that the real wage w_t enters the JC condition for the firm (the Ramsey planner isn't affected by prices), and so the amount of vacancies that are posted depends on how the wage splits the surplus between workers and firms when they match.

As previously mentioned, workers and firms negotiate wages in each period according to Nash Bargaining. Recall that $\mathcal{J}_t$ is the firm's surplus and $\mathcal{W}_t$ is the worker's surplus. According to Nash bargaining, the workers take a fraction η of the total surplus:

$$\begin{aligned} \mathcal{W}_t &= \eta(\mathcal{J}_t + \mathcal{W}_t) = \eta \mathcal{S}_t \\ \implies (1-\eta)\mathcal{W}_t &= \eta \mathcal{J}_t, \end{aligned} \tag{8.57}$$

where (8.57) is called the Nash Bargaining sharing rule. This rule can be formally derived from:

$$w_t = \arg\max_{w_t} \mathcal{W}_t(w_t)^{\eta} \mathcal{J}_t(w_t)^{1-\eta},$$

with $\partial \mathcal{W}_t(w_t)/\partial w_t > 0$ and $\partial \mathcal{J}_t/\partial w_t < 0$. The solution is given by:

$$\begin{aligned} 0 &= \frac{\partial}{\partial w_t} \mathcal{W}_t(w_t)^\eta \mathcal{J}_t(w_t)^{1-\eta} \\ 0 &= \eta \mathcal{W}_t^{\eta-1} \mathcal{J}_t^{1-\eta} - (1-\eta)\mathcal{W}_t^\eta \mathcal{J}_t^{-\eta} \\ \implies (1-\eta)\mathcal{W}_t &= \eta \mathcal{J}_t. \end{aligned}$$

Define the worker's surplus from working as:

$$\mathcal{W}_t = w_t - b + (1-\delta)\beta \frac{\lambda_{1,t+1}}{\lambda_{1,t}} \left[1 - p(\theta_{t+1})\right] \mathcal{W}_{t+1}, \tag{8.58}$$

where b can be thought of as some unemployment benefit – or any form of value when the household is unemployed. In words this equation is a law of motion of sorts for worker surplus. It states that the surplus in period t is comprised of wages, unemployment benefits, and a discounted value of the surplus in $t+1$ discounted by the relevant probabilities (and the stochastic discount factor, of course).

The firm's surplus from a match is simply what we got from solving its problem (8.56):

$$\mathcal{J}_t = \frac{\kappa}{q(\theta_t)} = \alpha \frac{y_t}{n_t} - w_t + (1-\delta)\beta \frac{\lambda_{1,t+1}}{\lambda_{1,t}} \underbrace{\frac{\kappa}{q(\theta_{t+1})}}_{\mathcal{J}_{t+1}},$$

which we can also think of as a sort of arbitrage condition.

To derive the wage from Nash bargaining, recall that we had:

$$(1-\eta)\mathcal{W}_t = \eta \mathcal{J}_t,$$

which we also assumes holds for period $t+1$ too. Now, let's substitute in the expressions for $\mathcal{W}_t$ and $\mathcal{J}_t$ to get:

$$(1-\eta)\left[w_t - b + (1-\delta)\beta \frac{\lambda_{1,t+1}}{\lambda_{1,t}} \left[1 - p(\theta_{t+1})\right] \mathcal{W}_{t+1}\right] = \eta \left[\alpha \frac{y_t}{n_t} - w_t + (1-\delta)\beta \frac{\lambda_{1,t+1}}{\lambda_{1,t}} \mathcal{J}_{t+1}\right], \tag{8.59}$$

and recall that:

$$\mathcal{J}_t = \frac{\kappa}{q(\theta_t)}, \tag{8.60}$$

and that the Nash Bargaining condition implies:

$$\begin{aligned} \mathcal{W}_t &= \frac{\eta}{1-\eta} \mathcal{J}_t \\ \implies \mathcal{W}_t &= \frac{\eta}{1-\eta} \frac{\kappa}{q(\theta_t)}. \end{aligned} \tag{8.61}$$

Substitute in (8.61) into (8.59) to get:

$$(1-\eta)\left[w_t - b + (1-\delta)M_{t,t+1}\left[1-p(\theta_{t+1})\right]\frac{\eta}{1-\eta}\frac{\kappa}{q(\theta_{t+1})}\right] = \eta\left[\alpha\frac{y_t}{n_t} - w_t + (1-\delta)M_{t,t+1}\frac{\kappa}{q(\theta_{t+1})}\right],$$

and rearrange:

$$(1-\eta)w_t - (1-\eta)b + (1-\delta)M_{t,t+1}\left[1-p(\theta_{t+1})\right]\eta\frac{\kappa}{q(\theta_{t+1})} = \eta\alpha\frac{y_t}{n_t} - \eta w_t + \eta(1-\delta)M_{t,t+1}\frac{\kappa}{q(\theta_{t+1})}$$

$$\begin{aligned} w_t &= (1-\eta)b + \eta\left[\alpha\frac{y_t}{n_t} + (1-\delta)M_{t,t+1}\frac{\kappa}{q(\theta_{t+1})} - (1-\delta)M_{t,t+1}\left[1-p(\theta_{t+1})\right]\frac{\kappa}{q(\theta_{t+1})}\right] \\ w_t &= (1-\eta)\underbrace{b}_{\text{Worker reservation wage}} + \eta\left[\underbrace{\alpha\frac{y_t}{n_t} + (1-\delta)M_{t,t+1}\frac{\kappa}{q(\theta_{t+1})}p(\theta_{t+1})}_{\text{Firm reservation wage}}\right], \end{aligned} \tag{8.62}$$

where $M_{t,t+1} = \beta\frac{\mathbb{E}_t\lambda_{1,t+1}}{\lambda_{1,t}}$ is the stochastic discount factor/pricing kernel. As we can see, the wage splits the surplus according to the bargaining power of the worker, η. If the firm has all the bargaining power ($\eta \to 0$), the real wage is set equal to the minimum the worker would accept to work (b). Conversely, if the worker has all the bargaining power ($\eta \to 1$), the real wage is set equal to the outside option for the firm.

Note also that (8.62) contains the term $p(\theta_{t+1})/q(\theta_{t+1}) = \theta_{t+1}$ which is nothing but labour market tightness. This implies that the tighter the labour market, the higher the wage. Why? A higher θ means that there are several vacancies for a given number of unemployed workers. The labour market is tight, and a firm has to "compete" with other firms to attract a worker. But since recruiting costs are higher, this means that firms are prepared to pay a higher wage in order to find a match.

Also note that the bargained wage (8.62) depends on parameters which are influenced by fiscal and labour market policy: the job separation rate δ, the job creation rate $p(\theta_{t+1})$, and – perhaps most critically – the outside option of working b. The higher b is, the higher the wage.

Finally, to solve for the decentralised equilibrium, use the bargained wage (8.62) in the JC condition (8.56):

$$\begin{aligned} \frac{\kappa}{q(\theta_t)} &= \alpha\frac{y_t}{n_t} - \left[(1-\eta)b + \eta\left(\alpha\frac{y_t}{n_t} + (1-\delta)M_{t,t+1}\kappa\frac{p(\theta_{t+1})}{q(\theta_{t+1})}\right)\right] + (1-\delta)M_{t,t+1}\frac{\kappa}{q(\theta_{t+1})} \\ \implies \frac{\kappa}{q(\theta_t)} &= (1-\eta)\left[\alpha\frac{y_t}{n_t} - b\right] - (1-\delta)M_{t,t+1}\left[1-\eta p(\theta_{t+1})\right]\frac{\kappa}{q(\theta_{t+1})}. \end{aligned} \tag{8.63}$$

8.4.9 The Ramsay planner equilibrium vs the market equilibrium

Are the allocations in the decentralised economy efficient? Compare the JC conditions in equilibrium for the Ramsey planner and the decentralised economy. The JC condition for the Ramsey planner (8.52) is

$$\frac{\kappa}{q(\theta_t)} = (1-\xi)\alpha\frac{y_t}{n_t} + (1-\delta)M_{t,t+1}\left[1-\xi p(\theta_{t+1})\right]\frac{\kappa}{q(\theta_{t+1})},$$

and for the market economy (assuming $b=0$) (8.63),

$$\frac{\kappa}{q(\theta_t)} = (1-\eta)\alpha\frac{y_t}{n_t} - (1-\delta)M_{t,t+1}\left[1-\eta p(\theta_{t+1})\right]\frac{\kappa}{q(\theta_{t+1})}.$$

Comparing the two JC conditions, the decentralised economy equilibrium is Pareto efficient if and only if

$$\eta = \xi.$$

In other words, when the worker's bargaining power is equal to the elasticity of the matching function with respect to unemployment (recall that $m(v_t, u_t) = v_t^{1-\xi}u_t^{\xi}$). This is known as the Hosios Condition.

8.4.10 The Hosios Condition

Is the Hosios Condition feasible? Yes, in principle, since $0 \leq \{\eta, \xi\} \leq 1$. However, it might not be satisfied in practice, as the bargaining power η might not be the same as ξ. These parameters are both very different objects. The parameter η is determined without reference to the matching function, while ξ can be interpreted as the relative weight of unemployment to matches (or, $1-\xi$ could be the relative contribution of vacancies to matches).

If ξ is high, then unemployment in the economy contributes more to generating matches than vacancies do. This implies that in equilibrium, high unemployment improves labour market efficiency, which in turn implies that the wage bargaining parameter η would be high (since high wages lead to high unemployment). This argument identifies a positive relation between ξ and η rather than absolute equality.

Recall that we talked about the insider-outsider dilemma. The wage bargaining process has two types of agents: i) Insiders: hired workers and firms that fill a vacancy, and ii) Outsiders: unemployed workers and firms with unfilled vacancies. The wage bargaining process reflects the interests of the insiders, and not the outsiders. For firms, their externality is that if hiring is too hire it makes the outside firms worse-off (tighter labour market, low vacancy filling probability). For workers, their externality is that if the labour market is loose, it makes unemployed workers outside the bargaining contract worse-off (slack labour market, low job finding probability).

In other words, the insiders ignore the interests of the outsiders, yet the wage decisions affect the allocations of the outsiders. If unemployed workers and firms without matches were given the chance to participate in the bargaining process, they would choose a wage rule that delivers the allocations chosen by the Ramsey planner.

We can use the JC condition and the Beveridge curve to visualise the efficient level of unemployment in (u, v) space. Recall that the decentralised JC condition (8.63):

$$\frac{\kappa}{q(\theta_t)} = (1-\eta)\alpha\frac{y_t}{n_t} - (1-\delta)M_{t,t+1}\left[1-\eta p(\theta_{t+1})\right]\frac{\kappa}{q(\theta_{t+1})},$$

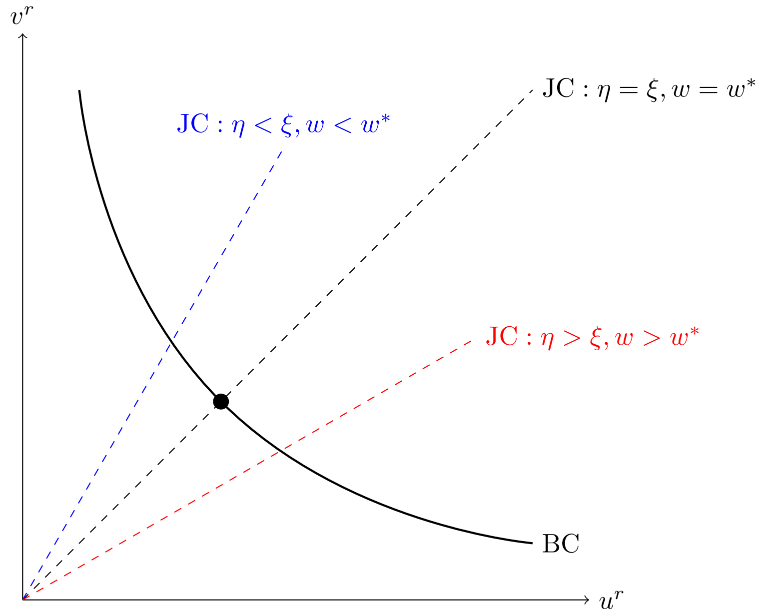
and at the steady state it becomes:

$$\frac{\kappa}{q(\theta)} = (1-\eta)\alpha\frac{y}{n} + \beta(1-\delta)\left[1-\eta p(\theta)\right]\frac{\kappa}{q(\theta)}.$$

Plotting the JC condition and Beveridge curve we get Figure 8.10. How do the plots of the Ramsey planner equilibrium and market equilibrium differ?

- If $\eta = \xi$: the market wage is equal to the Pareto optimum wage, and the amount of vacancies and unemployment are efficient;
- If $\eta < \xi$: the market wage is lower than the Pareto optimum wage, and therefore vacancy posting is suboptimally high and unemployment is suboptimally low; and
- If $\eta > \xi$: the market wage is suboptimally higher than the Pareto optimum wage, and therefore vacancy posting is suboptimally low and unemployment is suboptimally high.

Figure 8.10: The Beveridge Curve and Job Creation Condition



8.4.11 Labour market policies

Fiscal policy and labour market policies can offset the inefficient distortions of wage bargaining.

For example, consider the duration of filling a vacancy:

$$\frac{1}{q(\theta_t)} = \theta_t^{\xi}.$$

If the duration of filling a vacancy is high then firms are causing more congestion to other firms posting vacancies. In other words, the vacancy posting rate is too high. In this case, it may be efficient to "tax" the firm with a higher wage, implemented by increasing η. Suppose the government levies a wage tax, τ, on firms so that the effective wage bill becomes $(1+\tau)w_t$. The JC condition for the market economy (8.56) becomes:

$$\frac{\kappa}{q(\theta_t)} = \alpha\frac{y_t}{n_t} - (1+\tau)w_t + (1-\delta)\beta\frac{\lambda_{1,t+1}}{\lambda_{1,t}}\frac{\kappa}{q(\theta_{t+1})}, \tag{8.64}$$

and the wage equation (8.62) becomes:

$$(1-\eta)\left\{w_t - b + (1-\delta)M_{t,t+1}\left[1-p(\theta_{t+1})\right]\frac{\eta}{1-\eta}\frac{\kappa}{q(\theta_{t+1})}\right\} = \eta\left\{\begin{array}{c}\alpha\frac{y_t}{n_t} - (1+\tau)w_t \\ +(1-\delta)M_{t,t+1}\frac{\kappa}{q(\theta_{t+1})}\end{array}\right\}$$

$$(1-\eta)\left\{(1-\delta)M_{t,t+1}\left[1-p(\theta_{t+1})\right]\frac{\eta}{1-\eta}\frac{\kappa}{q(\theta_{t+1})} - b\right\} + w_t\left(1+\tau\eta\right) = \eta\left\{\begin{array}{c}\alpha\frac{y_t}{n_t} \\ +(1-\delta)M_{t,t+1}\frac{\kappa}{q(\theta_{t+1})}\end{array}\right\},$$

and with a bit cleaning up:

$$w_t = \frac{1-\eta}{1+\tau\eta}b + \frac{\eta}{1+\tau\eta}\left[\alpha\frac{y_t}{n_t} + (1-\delta)M_{t,t+1}\kappa\frac{p(\theta_{t+1})}{q(\theta_{t+1})}\right]. \tag{8.65}$$

Using (8.65) (and assuming that $b=0$), and substituting it into the JC condition (8.64) yields:

$$\frac{\kappa}{q(\theta_t)} = \left(1 - \frac{1+\tau}{1+\eta\tau}\eta\right)\alpha\frac{y_t}{n_t} + (1-\delta)M_{t,t+1}\left[1 - \frac{1+\tau}{1+\tau\eta}\eta p(\theta_{t+1})\right]\frac{\kappa}{q(\theta_{t+1})}. \tag{8.66}$$

Recall that the Ramsey planner equilibrium (8.52) was:

$$\frac{\kappa}{q(\theta_t)} = (1-\xi)\alpha\frac{y_t}{n_t} + (1-\delta)\beta\mathbb{E}_t\frac{\lambda_{1,t+1}}{\lambda_{1,t}}\left[1-\xi p(\theta_{t+1})\right]\frac{\kappa}{q(\theta_{t+1})}.$$

Comparing the market equilibrium with taxes to the Ramsey planner equilibrium reveals that the efficiency condition is:

$$\xi = \frac{1+\tau}{1+\tau\eta}\eta,$$

which is associated with the tax equal to:

$$\tau = \frac{\xi-\eta}{\eta(1-\xi)}. \tag{8.67}$$

Equation (8.67) shows that if $\eta = \xi \implies \tau = 0$. In other words, the market equilibrium is efficient and there is no need to tax labour. But if $\eta < \xi$, then wages are too low which then implies a positive τ is optimal (a tax on labour) – there are too many job postings, so taxes should be put in place to disincentivise vacancy postings and increase unemployment. If $\eta > \xi$, then wages are set too high which implies $\tau < 0$ be optimal (subsidy on labour). A wage subsidy stimulates vacancy postings and decreases unemployment, achieving efficiency.

8.4.12 Endogenous job separation

Up until now our SAM model assumed the job separation rate δ to be exogenous. But we know that this parameter is in fact endogenous – it is determined by firms and it changes over the business cycle. In this section we allow firms to optimally choose the job separation rate to retain profitability. This endogenous job separation will be important to explain shifts in the Beveridge curve.

To illustrate the model with endogenous separation, we make the following assumptions: $\alpha = 1$, so that there is constant returns to scale for production in labour; there is no bargaining so $w_t = b$ (i.e., $\eta = 0$ in Nash Bargaining); and, there is no exogenous job separation $\delta = 0$.

Assume that the productivity of a firm i is comprised of aggregate productivity a and job specific productivity $\sigma\varepsilon_i$:

$$y_i = a_t n_{i,t} \sigma_t \varepsilon_{i,t}, \tag{8.68}$$

where $\sigma > 0$ and $\varepsilon_i \sim F(\cdot)$ which is common for all firms and has support $\varepsilon_i \in [\underline{\varepsilon}, \overline{\varepsilon}]$. In each period, there is a new draw of ε from $F(\cdot)$. Assume the first draw $\varepsilon = \overline{\varepsilon}$ to ensure that a new job is never destroyed, but it will not stay there.

The JC condition of the economy is similar to before:

$$\mathcal{J}(\overline{\varepsilon}) = \frac{\kappa}{q(\theta)}, \tag{8.69}$$

where $\mathcal{J}(\overline{\varepsilon})$ for the assumption $\varepsilon = \overline{\varepsilon}$. To characterise the JC condition in presence of endogenous separation, we need to further define $\mathcal{J}(\overline{\varepsilon})$. Our assumptions deliver the steady-state firm's surplus for an active job with productivity ε equal to:

$$\mathcal{J}(\varepsilon) = a\sigma\varepsilon - b + \beta \int_{\underline{\varepsilon}}^{\overline{\varepsilon}} \mathcal{J}(x) dF(x), \tag{8.70}$$

where x is the draw of the new job-specific productivity shock from $F(\varepsilon)$. Note that:

$$\mathbb{E}\left[\mathcal{J}(x)\right] = \int_{\underline{\varepsilon}}^{\overline{\varepsilon}} \mathcal{J}(x) dF(x) > 0, \tag{8.71}$$

which is the expected value of a job after a draw x from the distribution $F(\varepsilon)$. $\mathbb{E}[\mathcal{J}(x)]$ is positive since if negative the firm destroys the job and posts a vacancy (i.e., ex-ante $\mathbb{E}[\mathcal{J}(x)]$ is never negative). To obtain $\mathcal{J}(\overline{\varepsilon})$ in the JC condition (8.69) we need to determine $\mathbb{E}[\mathcal{J}(x)]$ and evaluate

$\mathcal{J}(\varepsilon)$ at $\bar{\varepsilon}$.

$\mathcal{J}(\varepsilon)$ is monotonic in ε, i.e.,

$$\frac{d\mathcal{J}(\varepsilon)}{d\varepsilon} = \frac{a\sigma}{1-\beta} > 0.$$

Job destruction is characterised by a cutoff rule:

$$\exists \varepsilon_d : \varepsilon < \varepsilon_d \rightarrow \text{Separation}.$$

The cutoff threshold ε_d is determined by the condition:

$$\mathcal{J}(\varepsilon_d) = 0, \tag{8.72}$$

and we can use this condition to determine our object of interest, $\mathbb{E}[\mathcal{J}(x)]$.

The value of the marginal job is:

$$\mathcal{J}(\varepsilon_d) = a\sigma\varepsilon_d - b + \beta\mathbb{E}[\mathcal{J}(x)].$$

From the job destruction condition 8.72 we have:

$$0 = a\sigma\varepsilon_d - b + \beta\mathbb{E}[\mathcal{J}(x)],$$

and thus we derive:

$$\mathbb{E}[\mathcal{J}(x)] = \frac{b - a\sigma\varepsilon_d}{\beta} > 0, \tag{8.73}$$

since ex-ante this is always positive. Therefore, the flow of profit of the marginal job, $a\sigma\varepsilon_d - b$, must be negative for the job to be destroyed.

What determines $\mathcal{J}(\bar{\varepsilon})$? Recall the steady state firm surplus with productivity ε (8.70):

$$\mathcal{J}(\varepsilon) = a\sigma\varepsilon - b + \beta\mathbb{E}[\mathcal{J}(x)],$$

and we just found $\mathbb{E}[\mathcal{J}(x)]$, so substitute that into (8.70) to get:

$$\mathcal{J}(\varepsilon) = a\sigma(\varepsilon - \varepsilon_d),$$

and if $\varepsilon = \bar{\varepsilon}$, $\mathcal{J}(\bar{\varepsilon})$ becomes:

$$\mathcal{J}(\bar{\varepsilon}) = a\sigma(\bar{\varepsilon} - \varepsilon_d),$$

and substitute this into the job creation condition (8.69) to get:

$$\begin{aligned} a\sigma(\bar{\varepsilon} - \varepsilon_d) &= \frac{\kappa}{q(\theta)} \\ \Leftrightarrow q(\theta) &= \frac{\kappa}{a\sigma(\bar{\varepsilon} - \varepsilon_d)}. \end{aligned} \tag{8.74}$$

So job separation matters for job creation and is important for job turnover. If the job separation increases ($\varepsilon_d \uparrow$), the probability of filling a vacancy increases (since there are unemployed workers) and therefore there is a larger market turnover (as we saw in the data).

Now, we turn to the Beveridge curve under endogenous job separation. The law of motion of employment becomes:

$$n_t = (1 - F(\varepsilon_d))n_{t-1} + q(\theta_t)v_t, \tag{8.75}$$

where $F(\varepsilon_d)$ is the endogenous separation rate. At the steady state (recall $n = 1 - u$ and $q(\theta)v = p(\theta)u$) we have:

$$\begin{aligned} n &= (1 - F(\varepsilon_d))n + q(\theta)v \\ \implies 1 - u &= (1 - F(\varepsilon_d))(1 - u) + p(\theta)u, \end{aligned} \tag{8.76}$$

which implies the steady state level of unemployment:

$$u = \frac{F(\varepsilon_d)}{F(\varepsilon_d) + p(\theta)}. \tag{8.77}$$

This equation is the Beveridge curve with endogenous job separation. An important implication of this Beveridge curve is that changes in the threshold of job-specific productivity shift it. For example, an increase in a decreases ε_d and therefore $F(\varepsilon_d)$ falls. The Beveridge curve shifts inward, and unemployment is lower for any given level of job creation. Recent research shows that endogenous job separation generates significant non-linearities in the fluctuations of labour market variables over the business cycle.[7]

How does endogenous job separation affect efficiency? The Hosios condition continues to hold[8], although the details are beyond the scope of this course. The key intuition is that job separation is a structural feature of the economy, equally internalised by the market economy and the Ramsey planner. When the Hosios condition holds, the market economy sets the same job-specific productivity threshold and bargaining is efficient as in an economy with the Ramsey social planner.

8.5 Comments and key readings

There is still no consensus about how to model the labour market in a neoclassical framework. Increasing attention is being placed on SAM models, but this research strategy is still too recent to assess its performance. A number of studies which compare labour market models suggest that the key to understanding the cyclical behaviour of labour markets is to try and understand which margins a firm can adjust freely and which the firm has to treat as fixed. In the search and match models, the firm can choose vacancies freely but inherits an employment stock. Understanding these rigidities and their causes and the optimal responses of firms and workers to these rigidities is clearly a crucial issue.

7. See Pizzinelli, Theodoridis, and Zanetti (2020).
8. See Mortensen and Pissarides (1999)

The other important thing to notice about most of the models in this lecture is that they seek to improve the performance of the RBC model by introducing additional shocks: the most promising SAM models have aggregate and allocative shocks. This is increasingly how RBC models are being developed, with widespread opinion being that productivity shocks alone cannot explain business cycle fluctuations. Essentially, these models are trying to explain the zero correlation between wages and employment over the business cycle by letting both the labour supply and demand curves shift.

Key readings were mentioned throughout this chapter. For indivisible labour see Hansen's 1985 *JME* paper, Rogerson (1988) "Indivisible Labor, Lotteries and Equilibrium" is also good, and the textbook *ABCs of RBCs* by McCandless (2008) gives a very thorough treatment. McCandless also goes through linear quadratic dynamic programming which was used by both Hansen (1985) and Kydland and Prescott (1982).

Advanced Macroeconomics by Romer (2012) gives a good rundown of efficiency wages, including the Shapiro-Stiglitz model. These notes were based primarily on Romer's material.

The literature on SAM models has increased at a rapid pace since Diamond, Mortensen, and Pissarides were awarded the Nobel prize in 2010. Specific papers referred to in these notes were "The Beveridge Curve" Blanchard et al. (1989), "Aggregate Demand Management in Search Equilibrium" Diamond (1982a), "Wage Determination and Efficiency in Search Equilibrium" Diamond (1982b), "Job Creation and Job Destruction in the Theory of Unemployment" Mortensen and Pissarides (1994), and *Equilibrium Unemployment Theory* by Pissarides (2000). Additionally, I would highly recommend Gertler and Trigari (2009) and Gertler, Sala, and Trigari (2008) for those interested in a model of SAM embedded in a New Keynesian framework. Finally, since initially writing these notes, the macro-labour literature has flourished – notable authors that have take a very quantitative and structural approach to modelling labour markets are Adrien Bilal, Simon Mongey, Gianluca Violante, and many more that I don't have time to list here.

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9 Real Dynamics in the RBC Model

9.1 Introduction

This is the final chapter in which we look at ways of amending and extending the baseline RBC model in order to improve its performance. We've looked at alternative ways of modelling the labour market (indivisible labour, efficiency wages, and search and match), and we saw how they significantly improve the performance of the RBC model. However, on their own, they are still not sufficient, so RBC models (and other current DSGE models) usually incorporate additional sources of dynamics to help them come close to the data. There is considerable controversy about this approach because each extra source of dynamics introduces additional free parameters into the model which can be criticised as being ad hoc. Lucas warned to "beware economists bearing free parameters," and there is some discomfort amongst macroeconomists that these additional sources of dynamics are not immune to the Lucas critique. In other words, there is inherent danger in adding mechanisms to match the data because these mechanisms may change once policy changes. With that said, in this section we will be looking at:

1. Habits: The assumption in these models is that consumers gain more utility from consuming in the current period if current consumption is close to that in the previous period. This means consumers tend to smooth consumption even more than before, which adds inertia and makes consumption smoothing an even stronger propagation mechanism.

2. Adjustment costs: Data from engineering studies suggests that it is not possible to adjust capital instantaneously – Rome wasn't build in a day. This is usually modelled as a cost of changing the capital stock or a cost of changing the level of investment. These two assumptions have different implications for the response of the economy to technology shocks.

3. Investment specific technological change: Recent work argues that aggregate technology shocks are not the prime impulse for business cycle dynamics. Instead, the key impulse is a shock to the rate at which output goods are converted into productive capital. It is argued that these shocks induce dynamics that are more consistent with observed data.

Some of the topics and concepts introduced in this chapter will be revisited when we look at macro-finance – the modifications we make to the RBC model here have important implications to explaining the equity premium puzzle. But without further ado, let's begin our last effort of patching up the RBC model.

9.2 Habits

We will first look at a general model of habits inspired by Abel (1990) and Galí (1994). It is assumed that the preferences of the representative agent are defined over consumption c_t relative to a preference parameter v_t:

$$\frac{1}{1-\sigma}\mathbb{E}_t\sum_{s=0}^{\infty}\beta^s\left(\frac{c_{t+s}}{v_{t+s}}\right)^{1-\sigma},$$

where the preference parameter v_t represents the individual's habit. The most common habit model defines v_t by:

$$v_t = \left(c_{t-1}^D C_{t-1}^{1-D}\right)^{\gamma},$$

with $\gamma > 0$ and $0 < D < 1$. c_{t-1} is individual consumption in the previous period and C_{t-1} is aggregate consumption in the previous period. The idea is that preferences are a function of a weighted average of what happened in the previous period.[1] If $D = 1$ then utility is defined as consumption relative to what the individual did in the previous period, which is arguably the most natural interpretation of what it means for a consumer to have a habit. This is known as the internal habit and means that the individual prefers their consumption in the current period to be as close as possible relative to their consumption in the previous period. In contrast, when $D = 0$, utility is defined as consumption relative to what happened on aggregate in the previous period. This is an external habit and implies that an individual prefers their consumption to be close to average consumption in the previous period – a sort of "catching up with the Joneses" effect. Whether habits are internal or external has important implications for the consumption Euler equation and for consumption and aggregate dynamics. If habits are internal then the consumer internalises the effect of current period consumption decisions on their habit next period (i.e., the consumer knows that its current period decision affects habits in future periods); if habits are external this effect is absent.

As an alternative, it is possible to define the habit relative to individual or aggregate consumption in the current period:

$$v_t = \left(c_t^D C_t^{1-D}\right)^{\gamma}.$$

These models are known as "keeping up with the Joneses" as habits and decisions are formed contemporaneously. The remainder of these notes will focus on "catching up" habits.

The marginal utility of consumption in the general habit model can be calculated by substituting the expression for v_t into the utility maximisation problem, and then differentiating with respect to c_t:

$$\begin{aligned}\frac{\partial U_t}{\partial c_t} &= \frac{\partial}{\partial c_t}\left\{\frac{1}{1-\sigma}\left[\frac{c_t}{\left(c_{t-1}^D C_{t-1}^{1-D}\right)^{\gamma}}\right]^{1-\sigma} + \frac{\beta}{1-\sigma}\mathbb{E}_t\left[\frac{c_{t+1}}{\left(c_t^D C_t^{1-D}\right)^{\gamma}}\right]^{1-\sigma}\right\}\\ &= \left(\frac{c_t}{\left(c_{t-1}^D C_{t-1}^{1-D}\right)^{\gamma}}\right)^{-\sigma}\frac{1}{\left(c_{t-1}^D C_{t-1}^{1-D}\right)^{\gamma}} + \beta\mathbb{E}_t\left[\frac{c_{t+1}}{\left(c_t^D C_t^{1-D}\right)^{\gamma}}\right]^{-\sigma}\left(-D\gamma\frac{c_{t+1}}{c_t^{1+D\gamma}C_t^{(1-D)\gamma}}\right)\end{aligned}$$

1. We could write utility as:

$$U_t = \frac{c_t^{1-\sigma}C_t^{\gamma}C_{t-1}^{\lambda}}{1-\sigma},$$

which would show effects of both "catching up and keeping up with the Joneses".

$$
\begin{aligned}
&= \left(\frac{c_t}{v_t}\right)^{-\sigma}\frac{1}{v_t} - \beta D\gamma\mathbb{E}_t\left[\left(\frac{c_{t+1}}{v_{t+1}}\right)^{-\sigma}\frac{c_{t+1}}{v_{t+1}}\frac{1}{c_t}\right]\\
\Leftrightarrow \lambda_t &= \underbrace{\left(\frac{c_t}{v_t}\right)^{1-\sigma}\frac{1}{c_t}}_{U_{c,t}} - \beta D\gamma\mathbb{E}_t\left[\underbrace{\left(\frac{c_{t+1}}{v_{t+1}}\right)^{1-\sigma}\frac{1}{v_{t+1}}}_{U_{v,t+1}}\frac{v_{t+1}}{c_t}\right],
\end{aligned} \tag{9.1}
$$

where to get from the third to the fourth line, I multiply the first term on the RHS by c_t/c_t and the second term by v_{t+1}/v_{t+1}. Note that (9.1) is nothing but the marginal utility of individual consumption. The marginal utility of consumption appears in the now-familiar intertemporal Euler equation:

$$
\mathbb{E}_t\left[\beta R_{t+1}\frac{\lambda_{t+1}}{\lambda_t}\right] = 1,
$$

which when log-linearised becomes:

$$
\hat{\lambda}_t = \mathbb{E}_t\hat{\lambda}_{t+1} + \mathbb{E}\hat{R}_{t+1}. \tag{9.2}
$$

In the time separable case $\gamma = 0$, $v_t = 1$, so $\lambda_t = c_t^{-\sigma}$ and $\hat{\lambda}_t = -\sigma\hat{c}_t$. As usual, the consumption Euler equation is:

$$
\hat{c}_t = \mathbb{E}_t\hat{c}_{t+1} - \frac{1}{\sigma}\mathbb{E}_t\hat{R}_{t+1}.
$$

When habits are external we have $\gamma > 0$, $D = 0$, $v_t = C_{t-1}^{\gamma}$, and so $\lambda_t = c_t^{-\sigma}C_{t-1}^{\gamma(\sigma-1)}$. This is the assumption in Smets and Wouters (2007). In equilibrium, $c_t = C_t$ so the marginal utility of consumption is

$$
\lambda_t = c_t^{-\sigma}c_{t-1}^{\gamma(\sigma-1)},
$$

and

$$
\hat{\lambda}_t = -\sigma\hat{c}_t + \gamma(\sigma-1)\hat{c}_{t-1}.
$$

The consumption Euler equation is then:

$$
\hat{c}_t = \frac{\gamma(\sigma-1)}{\gamma(\sigma-1)+\sigma}\hat{c}_{t-1} + \frac{\sigma}{\gamma(\sigma-1)+\sigma}\mathbb{E}_t\hat{c}_{t+1} - \frac{1}{\gamma(\sigma-1)+\sigma}\mathbb{E}_t\hat{R}_{t+1}. \tag{9.3}
$$

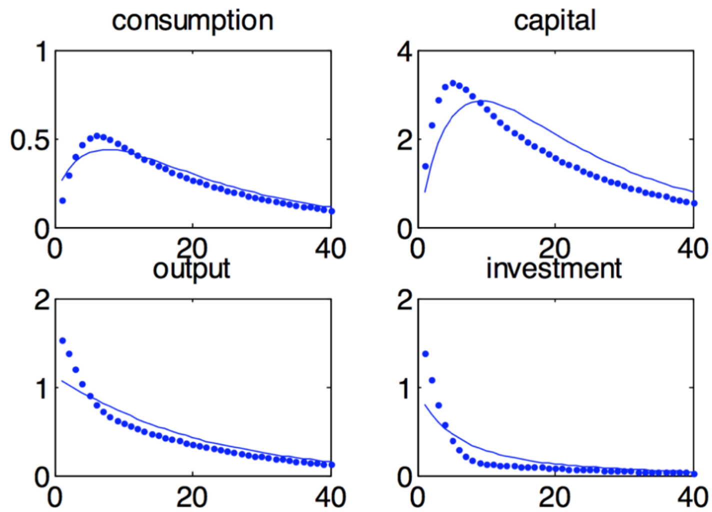
This is the consumption Euler equation under external habit formation. We have terms in $\hat{c}_{t-1}$ and $\mathbb{E}_t\hat{c}_{t+1}$ on the RHS so consumption decisions are both backward and forward looking. The backward-looking component is driven by the external habit, with the coefficients on the backward and forward looking terms summing to unity. Current consumption also reacts less to expectations of the real interest rate $\mathbb{E}_t\hat{R}_{t+1}$ under external habits – as the consumer prefers to keep consumption this period close to consumption in the previous period (the reference point for habits), and so the change in consumption for a given expected real interest rate will be less. Close examination of (9.3) shows that the habit terms drop out if preferences are logarithmic. This is a feature of our preferences specification, because with logarithmic preferences and $D = 0$, the per-period utility function becomes:

$$\ln\left(\frac{c_t}{v_t}\right) = \ln c_t - \gamma \ln C_{t-1},$$

and individual and aggregate (lagged) consumption are additively separable.

Adding external habits to an RBC model potentially changes its dynamics significantly. There are externalities in this model (running from individual to aggregate consumption) so we need to solve for the decentralised equilibrium rather than the Ramsey planner's problem. Figure 9.1 shows IRFs for a technology shock in models with (the dotted line) and without (the solid line) habits. The most obvious difference in the responses is that consumption does not increase as quickly on impact in the model with habits. Intuitively, the consumer is initially reluctant to increase consumption from its steady state value because that would involve moving away from their habits. Once they do start to increase consumption, though, there is an additional incentive to keep consumption high (the consumer has developed new habits) so consumption rises to a higher level under habits than without.

Figure 9.1: Response to Technology Shock in Model with (dotted line) and without (solid line) Habits



When habits are internal we have $\gamma > 0$, $D = 1$, $v_t = c_{t-1}^{\gamma}$. This is the assumption in Christiano, Eichenbaum, and Evans (2005). We can use (9.1) to write:

$$\lambda_t = c_t^{-\sigma} c_{t-1}^{\gamma(\sigma-1)} - \beta\gamma \mathbb{E}_t \left[c_{t+1}^{1-\sigma} c_t^{\gamma(\sigma-1)-1} \right],$$

which when log-linearised implies:

$$\hat{\lambda}_t = \frac{\gamma(\sigma-1)}{1-\beta\gamma}\hat{c}_{t-1} - \frac{\sigma + \beta\gamma(\gamma(\sigma-1)-1)}{1-\beta\gamma}\hat{c}_t + \frac{\beta\gamma(\sigma-1)}{1-\beta\gamma}\hat{c}_{t+1}.$$

Substituting into the log-linearised intertemporal Euler equation (9.2) gives the Euler equation under internal habits:

$$\hat{c}_t = \frac{\gamma(\sigma-1)}{\sigma+\beta\gamma(\gamma(\sigma-1)-1)+\gamma(\sigma-1)}\hat{c}_{t-1} + \frac{\beta\gamma(\sigma-1)+\sigma+\beta\gamma(\gamma(\sigma-1)-1)}{\sigma+\beta\gamma(\gamma(\sigma-1)-1)+\gamma(\sigma-1)}\mathbb{E}_t\hat{c}_{t+1}$$
$$- \frac{\beta\gamma(\sigma-1)}{\sigma+\beta\gamma(\gamma(\sigma-1)-1)+\gamma(\sigma-1)}\mathbb{E}_t\hat{c}_{t+2} - \frac{1-\beta\gamma}{\sigma+\beta\gamma(\gamma(\sigma-1)-1)+\gamma(\sigma-1)}\mathbb{E}_t\hat{R}_{t+1}.$$

The dynamics under internal habits are richer than those under external habits. we now have terms in $\hat{c}_{t-1}$, $\mathbb{E}_t\hat{c}_{t+1}$, and $\mathbb{E}_t\hat{c}_{t+2}$ as the consumer recognises that i) this period's consumption choice needs to be close to last period's consumption choice because of the habit, and ii) next period's consumption choice will become the habit in the following period. Once again, the sum of the coefficients on the three terms sum to unity.

In all these models of habits, we have worked with a representative agent and a single consumption good. This is not entirely satisfactory as one might imagine habits to work best at the individual product level. For example, it is easier to imagine the consumer becoming addicted to cigarettes or fried green tomatoes than become addicted to consumption per se. The paper by Ravn, Schmitt-Grohé, and Uribe (2006) introduces "deep habits" that are defined at product rather than aggregate level. This is an obvious step forward, though for analytical tractability they are forced to rely on external habits at the product level. In other words, a consumers becomes addicted to cigarettes because everyone else consumes them. This is not ideal but at present there are no tractable models of internal habits at the product level.

9.3 Adjustment costs

The view that adjustment costs are important in explaining economic fluctuations goes back to Tobin's historical q paper (1977), with its focus on the relationship between the market value of installed capital and the replacement cost of capital. Indeed, in the simple RBC model, with output costless to transform between consumption goods and productive capital, there is no reason for Tobin's q to deviate from unity. Models with adjustment costs break this feature by assuming that it takes resources to transform output goods into productive capital. Exactly what form these costs take is essentially an engineering question and depends on the mechanics of production and management processes.

We begin by looking at a model where the firm finds it costly to change the level of investment from one period to another. This investment adjustment cost model is very popular and is used in Christiano, Eichenbaum, and Evans (2005) and Smets and Wouters (2007). One possible motivation is that the firm has an investment department which has a certain capacity to plan and implement investments. If actual investments deviate from this norm, then costs are incurred. To see this in a simple model, define $i_t = A_t k_t^\alpha - c_t$ as the quantity of output made available for transformation into productive capital. In a model without adjustment costs the increase in productive capital would simply be i_t. But in the investment adjustment cost model, it is assumed that only a fraction, $1 - s(i_t/i_{t-1})$, of the output made available actually gets transformed into productive capital.

The function $s(i_t/i_{t-1})$ is assumed to satisfy $s(1) = s'(1) = 0$ and $s''(1) = \kappa$. Under these restrictions we have that the investment adjustment cost is an increasing convex function of how much this period's investment deviates from last period's investment. In the steady state $i_t = i_{t-1}$ so there are no steady state adjustment costs. There are no externalities in the model with adjustment costs so equilibrium can be solved as a Ramsey planner's problem:

$$\max \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \frac{c_{t+s}^{1-\sigma}}{1-\sigma},$$

subject to

$$\begin{aligned} k_{t+1} &= (1-\delta)k_t + \left[1 - s\left(\frac{i_t}{i_{t-1}}\right)\right] i_t, \\ i_t &= A_t k_t^{\alpha} - c_t. \end{aligned}$$

We solve this problem the standard way. Use a Lagrangian with λ_t as the multiplier on the resource constraint and μ_t as the multiplier on the investment equation. The FOC with respect to i_t is:

$$\lambda_t \left[1 - s\left(\frac{i_t}{i_{t-1}}\right)\right] - \lambda_t s'\left(\frac{i_t}{i_{t-1}}\right)\frac{i_t}{i_{t-1}} + \beta \mathbb{E}_t \lambda_{t+1} s'\left(\frac{i_{t+1}}{i_t}\right)\left(\frac{i_{t+1}}{i_t}\right)^2 = \mu_t, \tag{9.4}$$

which is a non-linear equation in λ_t, λ_{t+1}, μ_t, i_{t-1}, i_t, and i_{t+1}. To take a log-linear approximation, it is useful to write:

$$\frac{\mu_t}{\lambda_t} = 1 - s\left(\frac{i_t}{i_{t-1}}\right) - s'\left(\frac{i_t}{i_{t-1}}\right)\frac{i_t}{i_{t-1}} + \beta \mathbb{E}_t \frac{\lambda_{t+1}}{\lambda_t} s'\left(\frac{i_{t+1}}{i_t}\right)\left(\frac{i_{t+1}}{i_t}\right)^2,$$

and use the properties of $s(i_t/i_{t-1})$ to give the log-linearised FOC for investment:

$$\hat{i}_t = \frac{1}{1+\beta}\hat{i}_{t-1} + \frac{\beta}{1+\beta}\mathbb{E}_t \hat{i}_{t+1} + \frac{1}{\kappa(1+\beta)}\hat{q}_t, \tag{9.5}$$

where q_t is Tobin's q and is the shadow price of installed capital, $\hat{q} = \hat{\lambda}_t - \hat{\mu}_t$. The presence of investment adjustment costs introduces inertia in investment, as reflected by the lagged investment term. The investment decision also becomes forward looking, as it is costly to change the level of investment. The larger the parameter κ, the less sensitive current investment is to the shadow value of installed capital.

An alternative specification is the capital adjustment costs model that assumes the fraction of output available for investment that is transformed into productive capital is $1 - s(i_t/k_t)$. The function $s(\cdot)$ is assumed to satisfy $s(\omega) = s'(\omega) = 0$ and $s''(\omega) = \varepsilon$, where ω is the steady state investment to capital ratio. This capital adjustment cost model can be motivated by thinking that it is increasingly difficult for a firm to make large than small investments. The cost therefore depends on the quantity of investment each period rather than the change in investment. Dividing through by k_t provides a useful scaling of the function and ensures neat and intuitive FOCs. In steady state $\omega = i_t/k_t$ so there are no steady state adjustment costs. The Ramsey planner solves:

$$\max \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \frac{c_{t+s}^{1-\sigma}}{1-\sigma},$$

subject to:

$$\begin{aligned} k_{t+1} &= (1-\delta)k_t + \left[1 - s\left(\frac{i_t}{k_t}\right)\right] i_t, \\ i_t &= A_t k_t^{\alpha} - c_t. \end{aligned}$$

Setting up the Lagrangian, the FOC with respect to i is:

$$\lambda_t \left[1 - s\left(\frac{i_t}{k_t}\right)\right] - \lambda_t s'\left(\frac{i_t}{k_t}\right)\frac{i_t}{k_t} = \mu_t. \tag{9.6}$$

Tobin's q is again the shadow price of installed capital, $q_t = \lambda_t/\mu_t$, and so:

$$q_t \left[1 - s\left(\frac{i_t}{k_t}\right) - s'\left(\frac{i_t}{k_t}\right)\frac{i_t}{k_t}\right] = 1,$$

which when log-linearised becomes:

$$\hat{i}_t = \hat{k}_t + \frac{1}{\varepsilon\omega^2}\hat{q}_t. \tag{9.7}$$

In contrast to investment adjustment costs, investment now responds immediately to movements in the current shadow price of capital, $\hat{q}_t$. Hence, capital adjustment costs are not in themselves able to generate the sort of inertia in investments observed in the data.

9.4 Investment specific productivity shocks

The final mechanism we examine to change the real dynamics of the model is investment specific productivity shocks. The productivity shocks in all the models we have examined so far have been neutral in the sense that they increase the productivity of resources when making either consumption or productive capital. This improvement in productivity has been behind the IRFs which show consumption, capital, labour, output, and investment all rising after a technology shock. However, there is evidence that productivity shocks are not as neutral as this. In particular, there may be shocks which affect the ability of firms to change output goods into productive capital. These investment specific technology shocks impact on the relative efficiency with which firms can transform output into consumption goods and productive capital. After a positive shock to investment specific technology we would expect to see the economy shift from production of consumption goods to production of productive capital. Investment specific technology shocks do not introduce externalities, so we can illustrate their effects by solving the Ramsey planner's problem:

$$\max \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s U(c_{t+s}, l_{t+s}),$$

subject to:

$$\begin{aligned} c_t + i_t &= A_t k_t^{\alpha} l_t^{1-\alpha}, \\ k_{t+1} &= (1-\delta)k_t + v_t i_t, \\ \log A_{t+1} &= \rho \log A_t + \varepsilon_{A,t}, \\ \log v_{t+1} &= \tau \log v_t + \varepsilon_{v,t}, \end{aligned}$$

where A_t is the neutral productivity shock and v_t is the investment specific productivity shock. The IRFs of this economy are shown below:

Figure 9.2: Response to Neutral Technology Shock ($\varepsilon_{A,t}$)

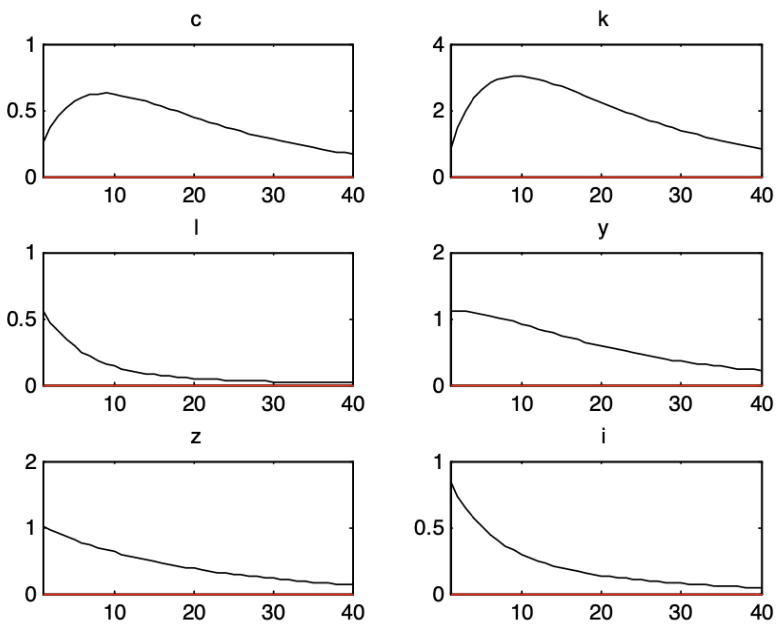
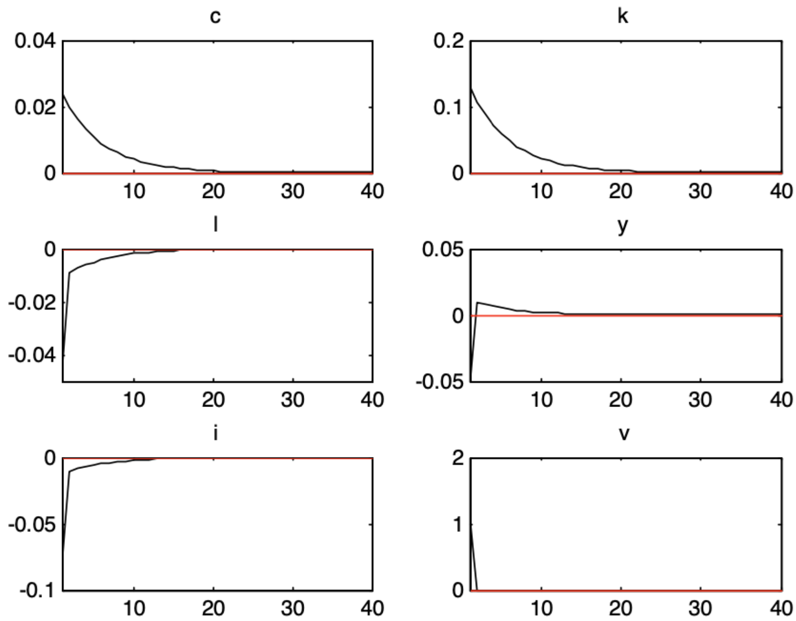


Figure 9.3: RESPONSE TO INVESTMENT-SPECIFIC SHOCK ($\varepsilon_{v,t}$)



The response to a neutral technology shock is as in the baseline RBC model, with labour, output, investment, capital, and consumption all rising after a positive shock.

The response of the model to an investment-specific technology shock is more nuanced: $v_t i_t$ rises but investment i_t falls as it becomes temporarily more efficient to turn output goods into productive capital than consumption goods. That consumption and investment move in opposite directions has the potential to better fit observed data. Indeed, the paper by Fisher (2006) shows that a combination of neutral and investment specific technology shocks can explain 73% of the variation in hours and 44% of the variation in output before in output afterwards. These numbers are considerably higher than in corresponding models with only neutral technology shocks, and have reignited the debate on the relative role of supply and demand shocks in driving the business cycle. The paper by Justiniano and Primiceri (2008) further contributes to this discussion by estimating a DSGE model with stochastic volatility and finds that reduced investment specific technology shocks play a very important role in the Great Moderation of US business cycles seen from 1984 to 2007.

9.5 Comments and key readings

This concludes our focus on improving the performance of the baseline RBC model. The subsections in this chapter are quite self-contained, and have the relevant references within. However, for convenience, the main references are: "Asset Prices under Habit Formation and Catching up with the Joneses" by Abel (1990), "Keeping Up with the Joneses: Consumption Externalities, Portfolio Choice, and Asset Prices" by Galí (1994), and "Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy" by Christiano, Eichenbaum, and Evans (2005). *Advanced Macroeconomics* by Romer (2012) also offers a good treatment of the dynamics presented in this chapter.

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Part II

Monetary Economics

10 Building a Monetary DSGE Model

10.1 Introduction

During the years following the papers of Kydland and Prescott (1982), Hansen (1985), Prescott (1986), and Lucas (1987), RBC theory provided the main reference framework for the analysis of economic fluctuations. It ushered in a revolution in macroeconomics, which was both methodological and conceptual.

As we've seen, RBC models introduced macroeconomists to DSGE models – giving us techniques and toolkits from other much more quantitatively demanding disciplines such as engineering and computer science. This was made possible following the Rational Expectations revolution ushered in by macroeconomists such as Muth, Lucas, Prescott, Sargent, and Wallace. DSGE models featured optimally acting agents, whose behaviour was able to be aggregated in order to construct a system of equilibrium equations. These models allowed experiments to be conducted, essentially allowing macroeconomists to undertake counterfactual analysis of the economy subject to a variety of shocks. As Galí (2015) points out, the most striking dimension of the RBC revolution, however, was conceptual. It rested on three basic claims:

- The efficiency of business cycles: The bulk of economic fluctuations observed in industrialised countries could be interpreted as an equilibrium outcome resulting from the economy's response to exogenous variations in real forces (most importantly, technology), in an environment characterised by perfect competition and frictionless markets. According to that view, cyclical fluctuations did not necessarily signal an inefficient allocation of resources. In fact, the fluctuations generated by the standard RBC model were fully optimal. That view had an important corollary: Stabilisation policies may not be necessary or desirable, and they could even be counterproductive! This was in stark contrast with the conventional interpretation, tracing back to Keynes (1936), of recessions as periods with an inefficiently low utilisation of resources that could be brought to an end by means of economic policies aimed at expanding aggregate demand.

- The importance of technology shocks as a source of economic fluctuations: This claim was derived from the ability of the baseline RBC model to generate "realistic" looking fluctuations in output and other macroeconomic variables, even when variations in total factor productivity – calibrated to match properties of the Solow residual – are assumed to be the only exogenous driving force of the model. Such an interpretation of economic fluctuations was in stark contrast with the traditional view of technological change as a source of long term growth, unrelated to business cycles.

- The limited role of monetary factors: Most importantly, RBC theory sought to explain economic fluctuations with no reference to monetary factors, even abstracting from the existence of money in the models.

We have examined each of these claims in the previous sections, and called into question their validity. While the RBC model provided us with a good training ground to familiarise ourselves with DSGE models, the RBC model itself had many shortcomings. Most notably, and as stated by the Galí quote above, the RBC model had little relevance for the analysis of macroeconomic policy. We could add monetary policy – as was done by papers such as Cooley and Hansen (1989) – but, in the absence of any price or wage frictions, it would have no real effects. Monetary policy could not change the real interest rate or influence real output. This is concerning for us because if you recall our Kaldor stylised facts, and the characteristics of business cycles, some variables appear to be more "sticky" than others, and the RBC model failed to explain these key empirical findings. So, the first order of business is to incorporate frictions or rigidities, so that monetary policy has a role to play, and that by doing so, perhaps we can build a model which can better explain the business cycle and other empirical findings.

In other words, to allow for a realistic model of business cycles and monetary policy, we need a framework in which prices don't simply follow the money supply, and nominal interest rates and inflation don't just move together one-for-one. In this kind of "Keynesian" model, prices are sticky, so real interest rates can be influenced by the central bank. Real interest rates can affect the performance of the economy, which in turn influences inflation via a Phillips Curve relationship.

With that rather lengthy recap of the RBC model, it's now time to depart the RBC theory framework, and embark on a journey which will eventually lead us to the New Keynesian framework. Before formally developing a New Keynesian model, however, we will go over the Lucas Critique, build a couple classical models in which we incorporate money and inflation, and examine the empirical facts that our New Keynesian model has to match. Much like when we first started with the RBC model, we need to observe the data to give us some direction and motivation for what kind of model we want to build. Let's get started.

10.2 The Lucas Critique

The Lucas Critique (Lucas, 1976) is an important philosophical point that forms the basis of much of modern macroeconomics. From Keynes until the mid-1970s, macroeconomics looked very different to what it does now. On the theoretical side, people used variants of a textbook IS-LM (investment-saving liquidity-money) model. That model did not take agent optimisation, dynamics, or expectations formation very seriously. On the empirical side, people used "large scale" macroeconometric models. These were essentially systems of simultaneous equations featuring aggregate variables – many of the larger models would feature hundreds of variables. The design of these macroeconometric models was based on fit and forecasting – essentially regressions – with little attention paid to any underlying theory or actual economics. There was no microfoundation, and agents' behaviour was postulated to be based on adaptive expectations, which were essentially ad-hoc.

The essential gist of Lucas' Critique[1] is that it is fraught with hazard to try and predict the effects of a policy change based on correlations (regression coefficients) based on historical data. We say that a parameter is "structural" if it is invariant to the rest of the economic environment, and in particular the policy environment. A parameter is "reduced form" if it is not invariant to the environment, or more generally if that parameter cannot be mapped back into some economic primitive. We'll consider two examples to make this point.

10.2.1 Example: Simple consumption saving model

Consider a very simple two period consumption saving model with a fixed real interest rate and no uncertainty. The household takes income flows to be exogenous. It solves the following problem:

$$\max_{C_t, C_{t+1}} \left\{ \frac{C_t^{1-\sigma} - 1}{1-\sigma} + \beta \frac{C_{t+1}^{1-\sigma} - 1}{1-\sigma} \right\},$$

subject to:

$$C_t + \frac{C_{t+1}}{1+r} = Y_t + \frac{Y_{t+1}}{1+r}.$$

The FOC, or Euler equation, is of course:

$$C_t^{-\sigma} = \beta(1+r)C_{t+1}^{-\sigma}.$$

There are two structural parameters here: β and σ, which govern how heavily you discount future utility flows and how much curvature there is in the utility function, respectively. Let's assume that $\sigma = 1$ (which means the utility function collapses to logarithmic utility via L'Hopital's Rule). We can then derive a consumption function that looks like:

$$C_t = \frac{1}{1+\beta}\left(Y_t + \frac{Y_{t+1}}{1+r}\right).$$

Here the marginal propensity to consume (MPC) is the partial derivative of C_t with respect to Y_t, which is $\frac{1}{1+\beta}$. This just a transformation of a structural parameter, and so we could consider the MPC itself to actually be structural.

Now, suppose an econometrician estimates a regression of consumption on income:

$$C_t = \alpha + \gamma Y_t + u_t.$$

This regression is misspecified in the sense that it omits Y_{t+1} – this is an error term in the regression. If current income is uncorrelated with future income, Y_t would be uncorrelated with the error

1. The Lucas Critique also has significant implications and ramifications for disciplines outside of economics. But as far as I know, other social sciences have yet to have their Lucas Critique moment of revelation.

term, and we could get $\gamma = \frac{1}{1+\beta}$ (at least in a large enough sample). But what if current income is correlated with future income (i.e. income is persistent)? Then there is an omitted variable; Y_t will be positively correlated with the error term, which will mean that you will get an upward biased estimate of γ.

Suppose that in the past changes in come have been very persistent – meaning that when Y_t changes, Y_{t+1} changes by almost the same amount. The consumption function derived from the theory would suggest that consumption would then react roughly one-for-one with changes in income. Suppose an econometrician goes and estimates this equation and comes back with a large estimate of γ (close to 1, say). He then goes to a policy adviser and says reports that the MPC is close to 1. This implies that giving households more income (say, through a tax cut), will cause households to spend most of their additional income. Suppose the policy maker did give households an extra dollar of income through a tax cut. The economic theory tells us that raising household income will cause them to increase their consumption by only $\frac{1}{1+\beta}$. If $\beta = 0.99$, say, then this means that the additional consumption will only be around $\frac{1}{2}$. This is smaller than the results estimated from the regression, which suggests that the MPC is close to 1. In this example, using the correlation between income and consumption estimated from past data (when income changes were very persistent) is not informative about what will happen if you consider a temporary change in income.

10.2.2 Example: The Phillips Curve

Consider another example, which was really the thing that Lucas was criticising. As we will see later in the course, it is possible to derive a "Phillips Curve" which shows some relationship between economic activity, inflation, and expected inflation:

$$\pi_t = \theta(u_t - u^N) + \beta\mathbb{E}_t\pi_{t+1}, \tag{10.1}$$

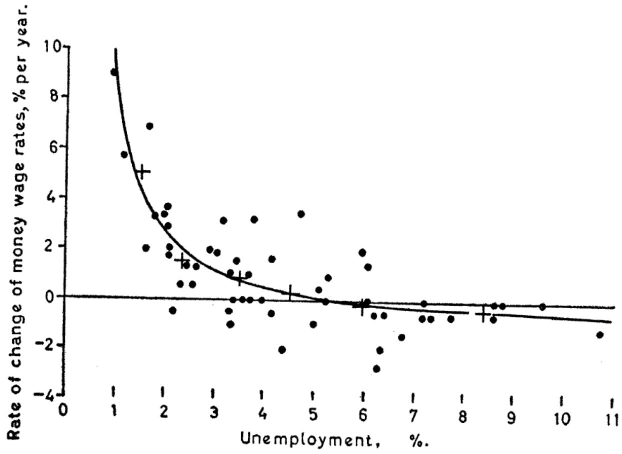
where θ is a coefficient, β is a discount factor, π_t is inflation, $\mathbb{E}_t\pi_{t+1}$ is expected inflation, u_t is the unemployment rate, and u^N is the "natural rate" of unemployment. θ and β are structural parameters.

Particularly before Rational Expectations, macroeconomists didn't know how to treat expectations seriously; and indeed, many models were static and so had no role for expectations of what was going to happen in the future. Suppose an econometrician estimated the following regression:

$$\pi_t = \xi(u_t - u^N) + \varepsilon_t.$$

As in the above example, this regression is misspecified relative to the theory – the error term includes expected future inflation. But suppose that in historical data expected inflation was pretty stable. This would mean there wouldn't be much bias in the coefficient estimate, and we would expect that an estimate of ξ would be close to the true θ. Suppose that the true $\theta < 0$: there is a negative relationships between inflation and unemployment. One would be tempted to

Figure 10.1: A.W. Phillips' Graph



conclude that raising inflation would lead to a reduction in unemployment. So the econometrician goes to the policymaker and says "Let's raise inflation and this will result in lower unemployment!" But will it?

It will, but only to the extent to which higher inflation doesn't get incorporated into higher inflation expectations. If people are paying attention, they will expected more inflation – $\mathbb{E}_t\pi_{t+1}$ will rise, which means u_t won't fall by as much as the simple regression would have predicted. Again, using past correlations to predict the effects of a policy change may very well be misleading.

As an aside, the modern incarnation of the Phillips Curve is usually traced to a late 1958 study by the LSE's A.W. Phillips. Phillips showed that low unemployment was associated wth high inflation, presumably because tight labour markets stimulated wage inflation. A 1960 study by Solow and Samuelson replicated these findings for the US.

Interestingly, if one reads Friedman's 1967 AEA Presidential Address, there are references to agents forming expectations of future inflation values:

> [...]At any moment of time, there is some level of unemployment which has the property that it is consistent with equilibrium in the structure of real wage rates. At that level of unemployment, real wage rates are tending on the average to rise at a "normal" secular rate[...]
>
> [...]A lower level of unemployment is an indication that there is an excess demand for labor that will produce upward pressure on real wage rates. A higher level of unemployment is an indication that there is an excess supply of labor that will produce downward pressure on real wage rates.[...]
>
> [...]The "natural rate of unemployment" in other words, is the level that would be ground out by the Walrasian system of general equilibrium equations, provided there

Figure 10.2: Solow and Samuelson's Description of the Phillips Curve

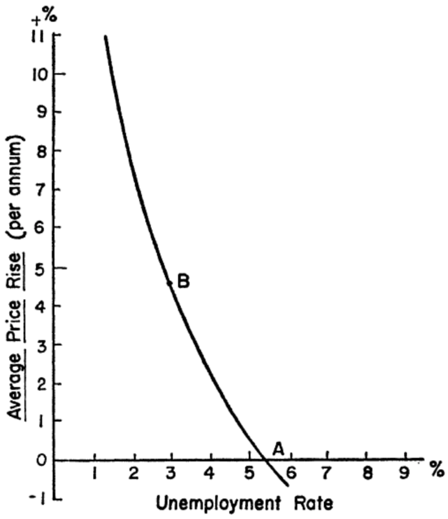


Figure 2

Modified Phillips Curve for U.S.
This shows the menu of choice between different degrees of unemployment and price stability, as roughly estimated from last twenty-five years of American data.

is imbedded in them the actual structural characteristics of the labor and commodity markets, including market imperfections[...]

[...]You will recognise the close similarity between this statement and the celebrated Phillips Curve. The similarity is not coincidental. Phillips' analysis of the relation between unemployment and wage change is deservedly celebrated as an important and original contribution. But, unfortunately, it contains a basic defect—the failure to distinguish between nominal wages and real wages.[...]

[...]Implicitly, Phillips wrote his article for a world in which everyone anticipated that nominal prices would be stable and in which that anticipation remained unshaken and immutable whatever happened to actual prices and wages. Suppose, by contrast, that everyone anticipates that prices will rise at a rate of more than 75 percent a year[...]

[...]Then wages must rise at that rate simply to keep real wages unchanged. An excess supply of labor will be reflected in a less rapid rise in nominal wages than in anticipated prices, not in an absolute decline in wages.[...]

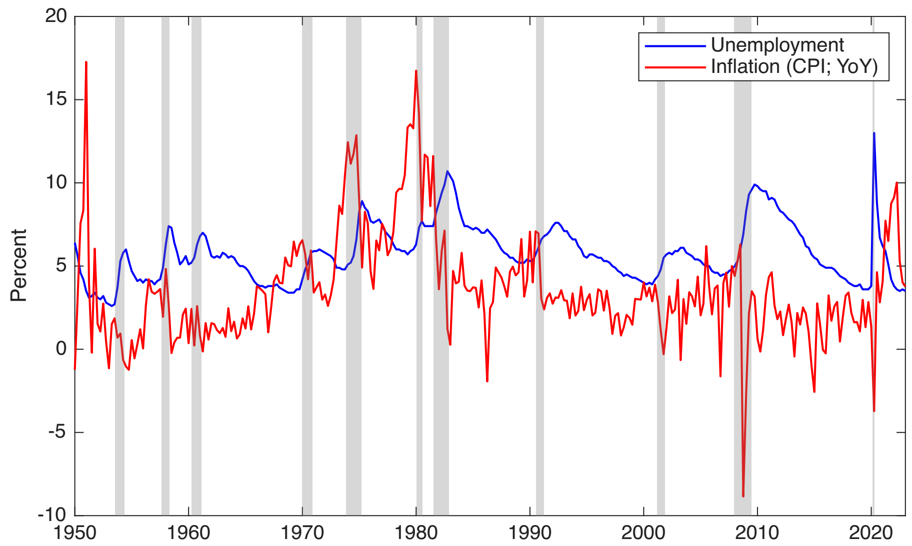
[...]Restate Phillips' analysis in terms of the rate of change of real wages—and even more precisely, anticipated real wages—and it all falls into place.[...]

[...]Income and spending will start to rise. To begin with, much or most of the rise in income will take the form of an increase in output and employment rather than in prices ..... Employees will start to reckon on rising prices of the things they buy and to demand higher nominal wages for the future.[...]

> [...]In order to keep unemployment at its target level [below the natural rate], the monetary authority would have to raise monetary growth still more ... the "market" rate can be kept below the "natural" rate only by inflation. And ... only by accelerating inflation.[...]

Essentially, Friedman predicted the Phillips Curve relationship would collapse. Turns out he was right:

Figure 10.3: US Unemployment and Inflation (1950-2023)



Friedman didn't use equations in his AEA address, but a rough model of his ideas were much like what we have in equation (10.1):

$$\pi_t = \pi_t^e - \theta(u_t - u^N).$$

Friedman pointed out that if policymakers tried to exploit an apparent Phillips Curve tradeoff, then the public would get used to high inflation and come to expect it: π_t^e would drift up and the trade-off between inflation and output would worsen. In the long-run, you can't fool the public ($\pi_t^e \approx \pi_t$), so you can't keep unemployment away from its "natural rate" $u_t \approx u^N$.

10.2.3 Is econometrics useful?

The conclusion of the Lucas Critique is that we need to take economic theory seriously – correlations (or regression coefficients) estimated in the data may not be policy-invariant, and therefore may not be useful in thinking about "counterfactuals" where we think of what would happen under alternative policy regimes.

Some people (incorrectly) interpret the Lucas Critique as saying we shouldn't do econometrics at all in macro. This is too strong. The Lucas Critique tells us that we need to take theory seriously when doing econometrics; and when we do econometrics without theory (e.g. reduced form econometrics), be honest and open about the potential misgivings. In both of the examples we have above, we actually have regression specifications implied by the theory – it's just that in the regressions we considered running, there was an omitted variable. "Theory" doesn't tell us values of structural parameters like β or θ – that's what econometrics is for. But theory might tell us what kind of econometric models to run, what kind of restrictions we can impose, etc. Then once we have good estimates of the structural parameters, we can use the model to to consider the effects of different policies.

It is actually here where the implications of rational expectations can be useful. Consider the two period consumption model (this time, make it stochastic so that the point is clearer). The theory tells us to run a regression like:

$$C_t = \alpha_1 Y_t + \alpha_2 \mathbb{E}_t Y_{t+1} + \varepsilon_t.$$

But the problem here is that we don't necessarily observe $\mathbb{E}_t Y_{t+1}$. Rational Expectations tells us how to get around this, however. In particular, Rational Expectations tells us that $\mathbb{E}_t Y_{t+1} = Y_{t+1} + u_{t+1}$, where u_{t+1} is i) mean zero, and ii) uncorrelated with anything known at date t or earlier. So Rational Expectations tells us that we can run the following regression:

$$C_t = \alpha_1 Y_t + \alpha_2 Y_{t+1} + v_t.$$

Now v_t is a composite error term, equal to $\varepsilon_t + \alpha u_{t+1}$. Y_{t+1} is correlated with u_{t+1}, so OLS won't work here. But Rational Expectations tells us that we can instrument for Y_{t+1} with anything known at date t or earlier – that the forecast error, u_{t+1}, is uncorrelated with anything dated t or earlier, making anything dated t or earlier valid instruments. We could do a similar exercise for the Phillips Curve equation, including realised future inflation on the RHS and instrumenting for it with something known at time t or earlier. In other words, taking Rational Expectations seriously often gives us a "theory of the error term" in regression models and therefore guides us on how to deal with that error term.

10.3 Does money matter? Evidence and stylised facts

What are the basic empirical regularities that monetary economics must explain? Monetary economics focuses on the behaviour of prices, monetary aggregates, nominal and real interest rates, and output. So a useful starting point is to summarise briefly what macroeconomic data tells us about the relationships amongst these variables.

10.3.1 Long-run relationships

A nice summary of long-run monetary relationships is provided by McCandless and Weber (1995). They examined data by covering a 30-year period from 110 countries using several definitions of money. Based on their analysis, two primary conclusions emerge. The first is that the correlation between inflation and the growth rate of the money supply is almost 1, varying between 0.92 and 0.96. This strong positive relationship between inflation and money is consistent with many other studies based on smaller samples of countries and different time periods. This correlation is normally taken to support one of the basic tenets of the quantity theory of money: a change in the growth rate of money induces "an equal change in the rate of price inflation" (Lucas, 1980). Using US data from 1955 and 1975, Lucas plotted annual inflation against the annual growth rates of money. While the scatter plot suggests only a loose but positive relationship between inflation and money growth, a much stronger relationship emerged when Lucas filtered the data to remove short-run volatility.

The high correlation between inflation and money growth does not, however, have any implication for causality. If countries followed policies under which money supply growth rates were exogenously determined, then the correlation could be taken as evidence that money growth causes inflation, with an almost one-to-one relationship between them. An alternative possibility, equally consistent with the high correlation, is that other factors generate inflation, and central banks allow the growth rate of money to adjust. But in any case, a sensible model of monetary economics would imply neutrality in the long-run.

McCandless and Weber's second general conclusion is that there is no correlation between either inflation or money growth and the growth rate of real output. Thus, there are countries with low output growth and high money growth and inflation – and countries with every other combination as well. This conclusion is not as robust as the money growth-inflation one; McCandless and Weber reported a positive correlation between real growth and money growth, but not inflation, for a subsample of OECD countries. Barro (1998; 2013) reported a negative correlation between inflation and growth in a cross-country sample. Bullard and Keating (1995) examined post-WWII data from 58 countries, concluding for the sample as a whole that the evidence that permanent shifts in inflation produce permanent effects on the level of output is weak, with some evidence of positive effects of inflation on output amongst low-inflation countries and zero or negative effects for higher-inflation countries.

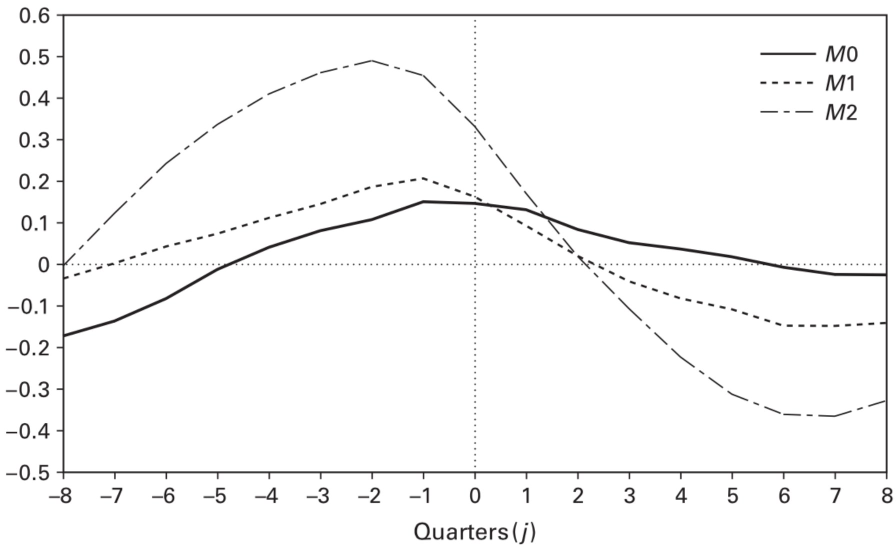
Bullard (1999) surveyed much of the existing empirical work on the long-run relationship between money growth and real output. His main finding was that while shocks to the level of the money supply do not appear to have long-run effects on real output, this was not the case with respect to shocks to money growth. Despite the diversity of empirical findings concerning the long-run relationship between inflation and real growth, and other measures of real economic activity such as unemployment, the general consensus was well summarised by the proposition, "[...]that there is no long-run tradeoff between the rate of inflation and the rate of unemployment" (Taylor, 1996).

10.3.2 Short-run relationships

The long-run empirical regularities of monetary economics are important for gauging how well the steady state properties of a theoretical model match the data. Much of our interest in monetary economics, however arises because of a need to understand how monetary phenomena in general and monetary policy in particular affect the behaviour of the macroeconomy over time periods of months or quarters. short-run dynamic relationships between money, inflation, and output reflect both the way in which private agents respond to economic disturbances and the way in which the monetary policy authority responds to those same disturbances.

Some evidence on short-run correlations for the US are provided in Figures 10.4 and 10.5. The figures show correlations between the detrended[2] log of real GDP and three different monetary aggregates, each also in detrended log form. Data are quarterly from 1967 Q1 to 2008 Q2, and the figures plot, for the entire sample and for the subperiod 1984 Q1 to 2008 Q2, the correlation between real GDP, Y_t, and a monetary aggregate, M_t. The three aggregates are the monetary base (sometimes referred to as M_0 or M^B), $M1$, and $M2$.

Figure 10.4: Dynamic Correlations for Y_t and M_{t+j} (1967 Q1 - 2008 Q2)

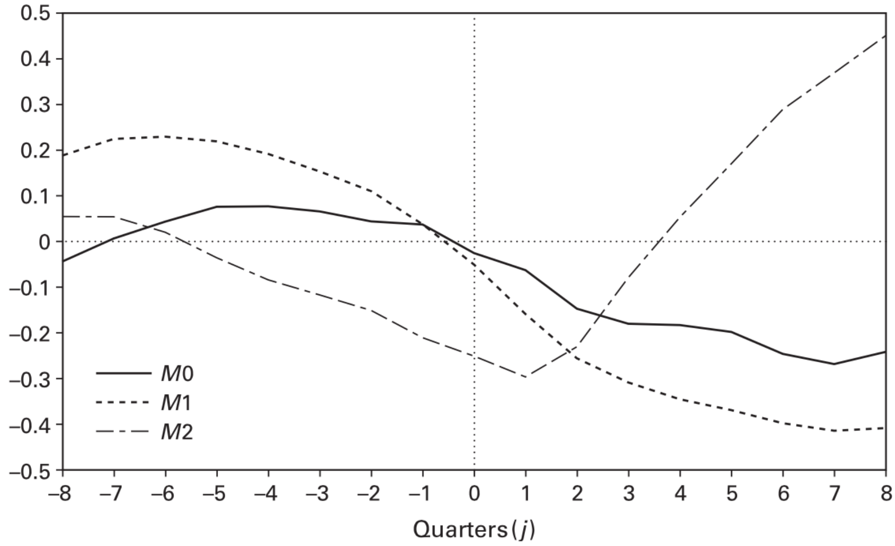


Source: Walsh (2010).

As Figure 10.4 shows, the correlations with real output change substantially as one moves from M_0 to $M2$. The narrow measure M_0 is positively correlated with real GDP at both leads and lags over the entire period, but future M_0 is negatively correlated with real GDP in the period since 1984. $M1$ and $M2$ are positively correlated at lags but negatively correlated at leads over the full sample. In other words, high GDP (relative to trend) tends to be preceded by high values of $M1$ and $M2$ but followed by low values. The positive correlation between Y_t and M_{t+j} for $j < 0$

2. Trends are estimated using a HP filter.

Figure 10.5: Dynamic Correlations for Y_t and M_{t+j} (1984 Q1 - 2008 Q2)



Source: Walsh (2010).

indicates that movements in money lead movements in output. This timing pattern played an important role in Friedman and Schwartz's classic and highly influential *A Monetary History of the United States, 1867-1960*. The larger correlations between GDP and $M2$ arise in part from the endogenous nature of an aggregate such as $M2$, depending as it does on banking sector behaviour as well as on that of the nonbank private sector.[3] However, these patterns for $M2$ are reversed in the later period, though $M1$ still leads GDP.

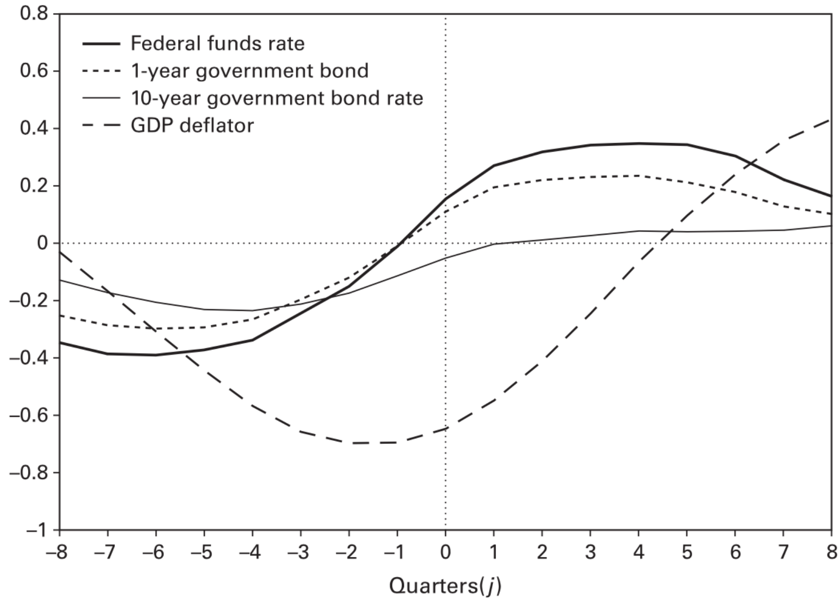
Figures 10.6 and 10.7 show the cross correlations between detrended real GDP and several interest rates and between detrended real GDP and the detrended GDP deflator. The interest rates range from the Federal Funds Rate, to the 1-year and 10-year rates on government bonds. The three interest rate series display similar correlations with real output, although the correlations become smaller for the longer term rates.

For the entire sample period (Figure 10.6), low interest rates tend to lead output, and a rise in output tends to be followed by higher interest rates. This pattern is less pronounced in the later period (Figure 10.7), and interest rates appear to rise prior to an increase in detrended GDP.

In contrast, the GDP deflator tends to be below trend when output is above trend, but increases in real output tend to be followed by increases in prices, though this effect is absent in the more recent period. Kydland and Prescott (1990) argued that the negative contemporaneous correlation between output and price series suggests that supply shocks, not demand shocks, must be responsible for business cycle fluctuations. Aggregate supply shocks would cause prices to be countercyclical, whereas demand shocks would be expected to make prices procyclical. However, if prices were sticky, a demand shock would initially raise output above trend, and prices would

3. See King and Plosser (1984) and Coleman (1996) for more.

Figure 10.6: DYNAMIC CORRELATIONS OF OUTPUT, PRICES, AND INTEREST RATES (1967 Q1 - 2008 Q2)

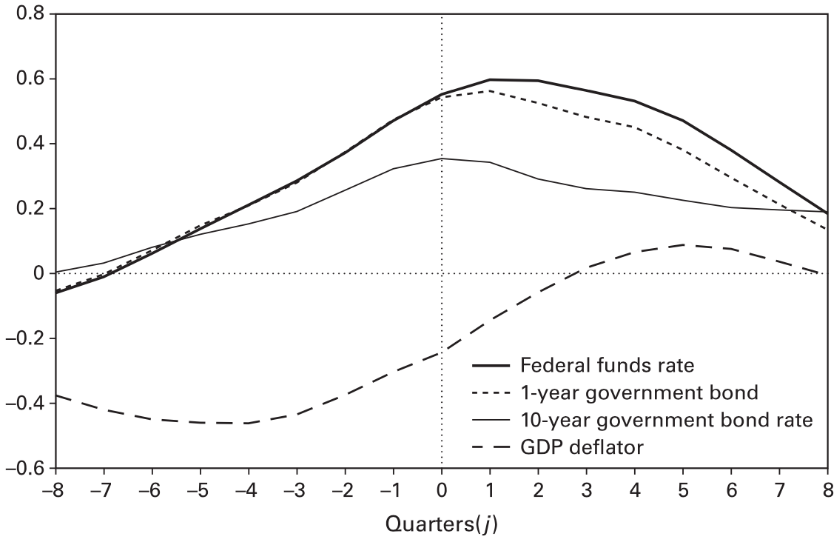


Source: Walsh (2010).

respond very little. If prices did eventually rise while output eventually returned to trend, prices could be rising as output was falling, producing a negative unconditional correlation between the two even though it was demand shocks generating the fluctuations (Ball and Mankiw, 1994).

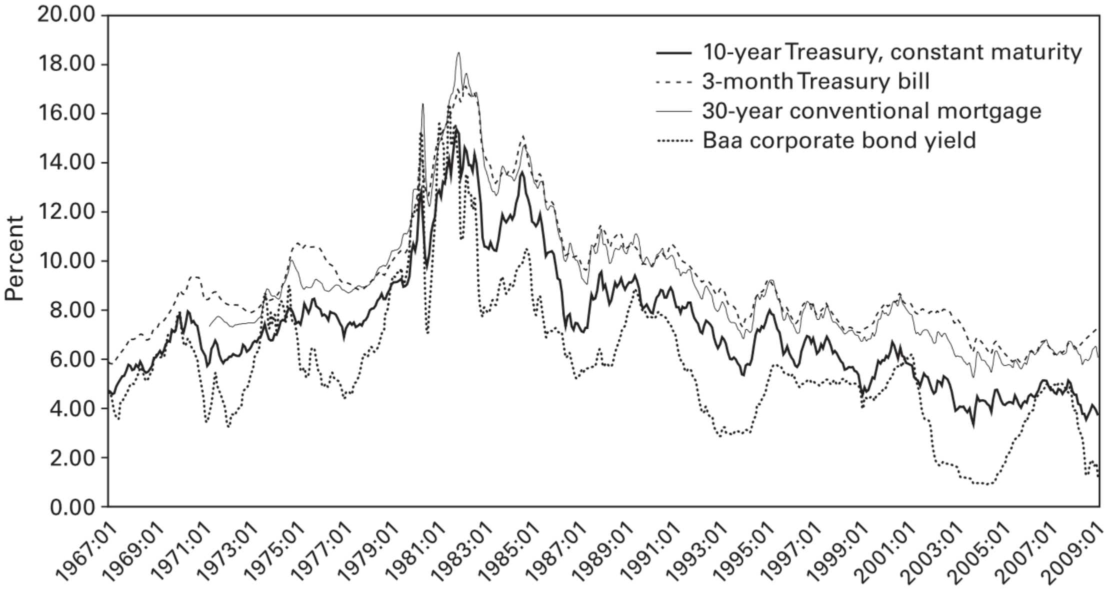
Most models used to address issues in monetary theory and policy contain only a single interest rate. Generally, this is interpreted as a short term rate of interest and is often viewed as an overnight market interest rate that the central bank can control. The assumption of a single interest rate is a useful simplification if all interest rates tend to move together. Figure 10.8 shows several longer term market rates of interest for the US. As the figure suggests, interest rates do tend to display similar behaviour, although the 3-month T-Bill rate, the shortest maturity shown, is more volatile than the other rates. There are periods, however, when rates at different maturities and riskiness move in opposite directions. For example, during 2008, the rate on corporate bonds rose while the rates on government debt, both at 3-month and 10-year maturities, were falling.

Figure 10.7: Dynamic Correlations of Output, Prices, and Interest Rates (1984 Q1 - 2008 Q2)



Source: Walsh (2010).

Figure 10.8: Interest Rates (1967:01 - 2008:09)



Source: Walsh (2010).

10.3.3 Estimating the effect of money on output

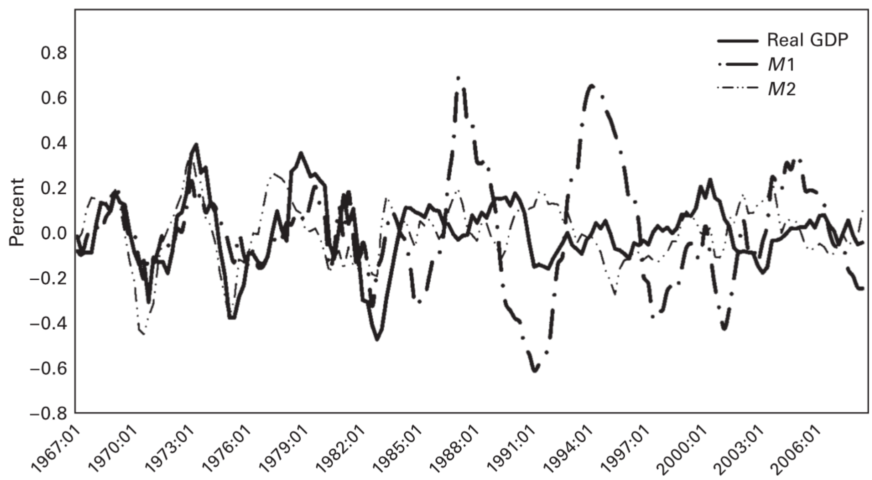
The tools that have been employed to estimate the impact of monetary have evolved over time as the result of developments in time series econometrics and changes in the specific questions posed by theoretical models. The literature of the empirical evidence on the relationship between monetary policy and US macroeconomic behaviour has focused on whether monetary policy disturbances actually have played an important role in US economic fluctuations. Equally important, the empirical evidence is useful in judging whether the predictions of different theories about the effects of monetary policy are consistent with the evidence. A key paper in this literature is Christiano, Eichenbaum, and Evans (1999).

10.3.4 The evidence of Friedman and Schwartz

Friedman and Schwartz's (1963) study of the relationship between money and business cycles still represents probably the most influential empirical evidence that money does matter for business cycle fluctuations. Their evidence, based on almost 100 years of data from the US, relies heavily on patterns timing; systematic evidence that money growth rate changes lead changes in real economic activity is taken to support a causal interpretation in which money causes output fluctuations.

The nature of this evidence is apparent in Figure 10.9, which shoes two detrended money supply measures and real GDP. The monetary aggregates in the figure, $M1$ and $M2$, are quarterly observations on the deviations of the actual series from trend. The sample period is 1967:1-2008:2, so it's after the Friedman and Schwartz period of study. The figure reveals slowdowns in money leading most business cycle downturns through the early 1980's. However, the pattern is not so apparent after 1982.

Figure 10.9: Detrended Money and Real GDP (1967 Q1 - 2008 Q2)



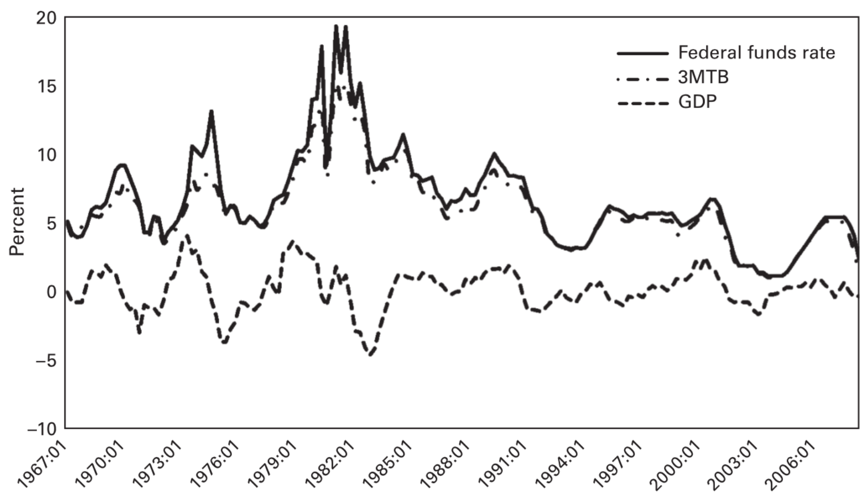
Source: Walsh (2010).

While it is suggestive, evidence based on timing patterns and simple correlations may not indicate the true causal role of money. Since the Fed and the banking sector respond to economic developments, movements in the monetary aggregates are not exogenous, and the correlation patterns need no reflect any causal effect of monetary policy on economic activity. If, for example, the central bank is implementing monetary policy by controlling the value of some short-term market interest rate, the nominal stock of money will be affected both by policy actions that change interest rates and by developments in the economy that are not related to policy actions. An economic expansion may lead banks to expand lending in ways that produce an increase in the stock of money, even if the central bank has not changed its policy. If the money stock is used to measure monetary policy, the relationship observed in the data between money and output may reflect the impact of output on money, nit the impact of money and monetary policy on output.

10.3.5 Reverse causation argument

Tobin (1970) was the first to model formally the idea that the positive correlation between money and output – the correlation that Friedman and Schwartz interpreted as providing evidence that money caused output movements – could in fact reflect just the opposite – output might be causing money. A more modern treatment of what is known as the reverse causation argument was provided by King and Plosser (1984) and Coleman (1996).

Figure 10.10: Interest Rates and Detrended Real GDP (1967 Q1 - 2008 Q2)



Source: Walsh (2010).

King and Plosser deduced that the correlation between broad aggregates such as $M1$ and $M2$ and output arises from the endogenous response of the banking sector economic disturbances that are not the result of monetary policy actions. The endogeneity problem is likely to be particularly severe if the monetary authority has employed a short-term interest rate as its main policy instrument, and this has generally been the case in the US. Changes in the money stock will then be endogenous and cannot be interpreted as representing policy actions. Figure 10.10 shows the behaviour of two short-term nominal interest rates, the 3-month T-Bill rate (3MTB) and the Federal Funds Rate, together with detrended GDP. Like Figure 10.9, Figure 10.10 provides some support for the notion that monetary policy actions have contributed to US business cycles. Interest rates have typically increased prior to economic downturns. But whether this is evidence that monetary policy has caused or contributed to cyclical fluctuations cannot be inferred from the figure; the movements in interest rates may simply reflect the Fed's response to the state of the economy.

10.3.6 Technical aside: Granger causality

One of the goals of economic analysis is to judge whether there is a causal relationship between economic variables. Generally, it is hard to detect the causal relationship from data, and we have to rely on economic theory. Recall our discussion in the "Primer to DSGE Models" section – one advantage of time series analysis is that it does not have to rely on economic theory. So what if there was a way to filter out potential causation flows between variables from the data? Granger (1969) introduced the concept of Granger causality based on forecasting techniques in time series econometrics. A variable X is said to Granger-cause Y if and only if lagged values of X have marginal predictive content in a forecasting equation for Y. In other words, having some information about the future of Y is not enough for Granger causality, and that past observations of X should have more information about the future of Y than the past Y observations.

Consider the simple bivariate VAR(2) model:

$$\begin{aligned}
Y_{1,t} &= c_1 + \phi_{11}^{(1)} Y_{1,t-1} + \phi_{12}^{(1)} Y_{2,t-1} + \phi_{11}^{(2)} Y_{1,t-2} + \phi_{12}^{(2)} Y_{2,t-2} + \varepsilon_{1,t} \\
Y_{2,t} &= c_2 + \phi_{21}^{(1)} Y_{1,t-1} + \phi_{22}^{(1)} Y_{2,t-1} + \phi_{21}^{(2)} Y_{1,t-2} + \phi_{22}^{(2)} Y_{2,t-2} + \varepsilon_{2,t}.
\end{aligned}$$

Y_2 does not Granger-cause Y_1 is analogous to saying that the coefficients of Y_2 in the Y_1 equation are all 0. In other words, $\phi_{12}^{(1)} = \phi_{12}^{(2)} = 0$. We can test for the Granger causality running from Y_2 to Y_1 by F-testing the null hypothesis that $\phi_{12}^{(1)} = \phi_{12}^{(2)} = 0$.

10.3.7 Model-based approach

In an important contribution, Sims (1972) introduced the notion of Granger causality into the debate over the real effects of money. Sims' original work used log levels of US nominal GNP and money (both $M1$ and the monetary base). He found evidence that money Granger-caused GNP. That is, the behaviour of money helped to predict future GNP. However, using the index of industrial protection to measure real output, Sims (1980) found that the fraction of output variation explained by money was greatly reduced when a nominal interest rate was added to the regression equation. Stock and Watson (1989) provided a systematic treatment of the trend specification in testing whether money Granger-causes real output. They concluded that money does help to predict future output even when prices and an interest rate are included.

A large literature has examined the value of monetary indicators in forecasting output. One interpretation of Sims' finding was that including an interest rate reduces the apparent role of money because, at least in the US, a short-term interest rate than the money supply provides a better measure of monetary policy actions (we will cover this soon).

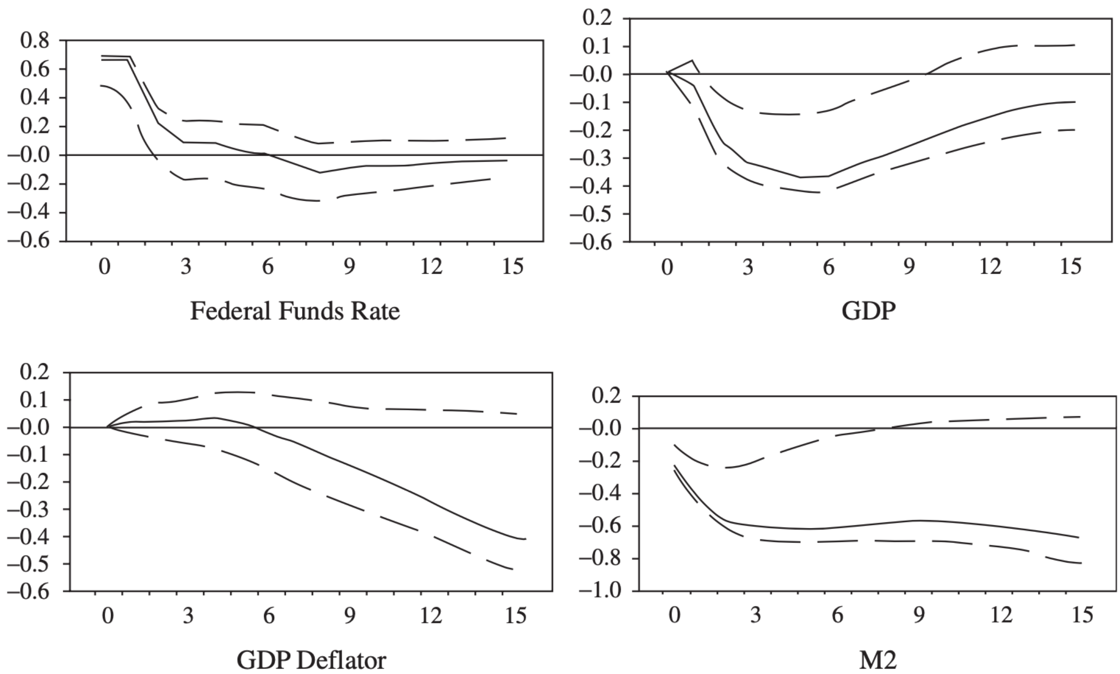
As alluded to earlier, the seminal papers to understand effects of monetary policy shocks on output is Christiano, Eichenbaum, and Evans (1999) (CEE) and Stock and Watson (2001).[4] Figure 10.11 shows the dynamic responses of the Federal Funds Rate, log GDP, log GDP deflator, and the money supply ($M2$) to an exogenous tightening of monetary policy from the CEE paper. Note that the path of the funds rate itself, depicted in the top left graph, shows an initial increase of about 75 basis points, followed by a gradual return to its original level. In response to that tightening of monetary policy, GDP declines with a characteristic hump-shaped pattern. It reaches a trough after five quarters at a level about 50 basis points below its original level, and then it slowly reverts back to its original level. That estimated response of GDP can be viewed as evidence of sizeable and persistent real effects of monetary policy shocks. On the other hand, the GDP deflator displays a flat response for over a year, after which it declines. That estimated sluggish response of prices to the policy tightening is generally interpreted as evidence of substantial price rigidities (more on this soon). Finally, note that $M2$ displays a persistent decline the face of the rise in the Federal Funds Rate, suggesting that the Fed needs to reduce the amount of money in circulation in order to bring about the increase in the nominal rate. The observed negative co-movement is between money supply and nominal interest rates is known as the "liquidity effect".

Also notice, however, that a contractionary monetary policy shock seems to lead to an initial small increase in the GDP deflator. This is referred to by macroeconomists as a "price puzzle". The effect is small and temporary (and barely statistically significant) but still puzzling. The most commonly accept explanation for the price puzzle is that it reflects the fact that the variables included in the VAR do not span the full information set available to the Fed. Suppose the Fed tends to raise the Funds Rate whenever it forecasts that inflation might rise in the future. To the extend that the Fed is unable to offset the factors that led it to forecast higher inflation, or to the extend that the Fed acts too late to prevent inflation from rising, the increase in the Funds Rate will be followed by a rise in prices. Sims (1992), using similar VARs as CEE, also showed that this price puzzle occurs for monetary policy shocks for France, Germany, Japan, and the UK.

One solution to resolving this puzzle would then be to include variables like commodity prices or other asset prices in the VAR (Sims, 1992; Chari, Christiano, and Eichenbaum, 1995; and Bernanke and Mihov, 1998). An alternative interpretation of the price puzzle is provided by Barth and Ramey (2001). They argued that contractionary monetary policy operates on aggregate supply as well as aggregate demand. For example, an increase in interest rates raises the cost of holding inventories and thus acts as a positive cost push shock. This negative supply effect raises prices and lowers output. Such an effect is called the cost channel of monetary policy.

4. The contemporary literature is *extremely vast*. There has been a lot of recent work done by Gertler and Karadi, Òscar Jordà, Miranda-Agrippino and Ricco, Nakamura and Steinsson, and – of course – more recent work by Stock and Watson.

Figure 10.11: Estimated Dynamic Response to a Monetary Policy Shock (Quarterly)



Source: Christiano, Eichenbaum, and Evans (1999).

10.3.8 Criticisms of the VAR approach

First, some of the impulse responses do not accord with most economists' priors. In particular, the price puzzle – the finding that a contractionary policy shock, as measured by a Federal Funds Rate shock, tends to be followed by a rise in the price level – is troublesome. As noted earlier, the price puzzle can be solved by including oil prices or commodity prices in the VAR system, and the generally accepted interpretation is that lacking these inflation-sensitive prices, a standard VAR misses important information that is available to policymakers. A related but more general point is that many of the VAR models used to assess monetary policy fail to incorporate forward-looking variables. Central banks look at a lot of information in setting policy. But because policy is likely to respond to forecasts of future economic conditions, VARs may attribute the subsequent movements input and inflation to the policy action.

At best, the VAR approach identifies only the effects of monetary policy shocks, shifts in policy unrelated to the endogenous response of policy to developments in the economy. Yet most, if not all, of what one thinks of in terms of policy and policy design represents the endogenous response of policy to the economy, and "most variation in monetary policy instruments is accounted for by responses of policy to the state of the economy, not by random disturbances to policy" (Sims and Zha, 1998). So it is unfortunate that primary empirical tool – VAR analysis – used to assess the impact of monetary policy is uninformative about the role played by policy rules. If policy is completely characterised as a feedback rule on the economy, so that there are no exogenous policy shocks, then the VAR methodology would conclude that monetary policy doesn't matter. Yet while monetary policy is not causing output movements in this example, it does not follow that policy is unimportant; the response of the economy to non-policy shocks may depend importantly on the way monetary policy endogenous adjusts. This broadly echoes our discussion of the Lucas Critique and the role of econometrics in macroeconomic theory.

10.4 Money in the utility function

Putting money into a general equilibrium model is not easy. The main issue being is that money is a dominated asset – it does not have a return, and hence it has no value in equilibrium. We need to specify a role for money so that agents wish to hold positive quantities in equilibrium. There are broadly three approaches to integrating money into general equilibrium models:

- Money in the utility (MIU) function (Sidrauski, 1967);
- Transaction costs:
  - Shoe-leather costs (Baumol, 1952; Tobin, 1956);
  - Cash-in-advance (CIA) constraint (Clower, 1967; Lucas, 1982; Svensson, 1985); and
  - Transaction technologies (e.g. Schmitt-Grohé and Uribe (2010)).
- Search and match (Kiyotaki and Wright, 1989; Williamson and Wright, 2010).

All of the above methods – with the exception of search and match – are fairly ad-hoc and aren't microfounded.

In this section we will examine the MIU approach. The idea is that money provides some service to the economy and that the benefits of that service can be expressed in the utility function. If one assumes that having more real money balances means that one will be able to reduce the time and energy spent making transactions, for example, one might include real balances in the utility function as a way of representing these utility gains. We will use a utility function in which increased holdings of real balances directly increases welfare. Utility that individuals wish to maximise is still the present value of an infinite sequence of additively separable subutilities. The utility function of an individual in period t is:

$$u\left(c_t, \frac{M_t}{P_t}\right) = u\left(c_t, m_t\right),$$

where the only important change is that we now add real balances of the individual, M_t/P_t, as a variable, and to keep to intuition simple we abstract from labour supply decisions of the household. The rationale for adding real balances to the utility function is the presumption that additional real balances reduce the cost of making transactions or reduce search (since they solve the noncoincidence-of-wants problem that arises in barter trade). One of the benefits of putting money in the utility function is that if there are other assets that individuals can hold, capital, for instance, the model will produce a real rate of return for money that is less than that of the other assets.

The benefit does not come without costs. As noted, money doesn't do anything, and so there is no clear use for it. Just keeping the money in your possession creates the utility. In economies with just one good in which all agents are identical, no trades ever take place and money really is not ever used for anything. Nevertheless, as a rough approximation of the gains from using money, and in particular, for giving money value when there are interest-earning assets available, MIU functions models are useful.

10.4.1 A simple Sidrauski MIU model

A unit mass of identical households each choose sequences of $\{c_t, M_{t+1}, k_{t+1}, B_{t+1}\}$ to maximise the infinite horizon discounted utility function:

$$\max_{\{c_t, M_{t+1}, k_{t+1}, B_{t+1}\}} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s u\left(c_{t+s}, m_{t+s}\right),$$

subject to the sequence of period t budget constraints:

$$P_t c_t + P_t \underbrace{[k_{t+1} - (1-\delta)k_t]}_{I_t} + B_{t+1} + M_{t+1} = P_t y_t + P_t \tau_t + R_t B_t + M_t,$$

where τ_t are lump sum transfers of money from the monetary authority to the household in period t, and $R_t = 1 + i_t$ is the gross nominal interest rate. Dividing through by P_t, we can rewrite the budget constraint in real terms as:[5]

$$c_t + k_{t+1} - (1-\delta)k_t + b_{t+1} + m_{t+1} = y_t + \tau_t + \frac{R_{t-1} b_t + m_t}{\pi_t} \tag{10.2}$$

where gross inflation π_t is defined as:

$$\pi_t = \frac{P_t}{P_{t-1}}.$$

Note that here we use the start of period timing convention for the capital stock, bond holdings, and money balances. If you wish, you can write everything in end of period notation – you'll get the same results either way.

Instead of proceeding with a Lagrange, we can use dynamic programming to solve. So dust off your Bellman equation techniques! First, define ω_t as a composition of the choice variables for the household, and use what we know about the economy's aggregate resources and its production

5. This was because $B_{t+1} = P_t b_{t+1}$.

technology to write:

$$\omega_t \equiv f(k_t) + \tau_t + \frac{R_{t-1}b_t + m_t}{\pi_t} + (1-\delta)k_t = c_t + k_{t+1} + b_{t+1} + m_{t+1}. \tag{10.3}$$

The household's problem is to choose paths for c_t, k_{t+1}, b_{t+1}, and m_{t+1} to maximise utility subject to its period budget constraint. Then we can write the Bellman equation as:

$$V(\omega_t) = \max_{\{c_{t+1},k_{t+1},m_{t+1},b_{t+1}\}} \left\{ u(c_t, m_t) + \beta V(\omega_{t+1}) \right\}, \tag{10.4}$$

subject to (10.3) and:

$$\omega_{t+1} = f(k_{t+1}) + \tau_{t+1} + \frac{R_t b_{t+1} + m_{t+1}}{\pi_{t+1}} + (1-\delta)k_{t+1}.$$

Using (10.3), we can write:

$$k_{t+1} = \omega_{t+1} - c_t - m_{t+1} - b_{t+1},$$

and then we can write (10.4) as:

$$V(\omega_t) = \max_{c_t,b_{t+1},m_{t+1}} \left\{ u(c_t, m_t) + \beta V \begin{bmatrix} f(\omega_{t+1} - c_t - m_{t+1} - b_{t+1}) \\ +\tau_{t+1} + \frac{R_t b_{t+1} + m_{t+1}}{\pi_{t+1}} \\ +(1-\delta)(\omega_{t+1} - c_t - m_{t+1} - b_{t+1}) \end{bmatrix} \right\}.$$

So the maximisation problem is now an unconstrained problem over the choices of c_t, b_{t+1}, and m_{t+1}. The FOCs are:

$$\frac{\partial V(\omega_t)}{\partial c_t} = u_c(c_t, m_t) - \beta V_\omega(\omega_{t+1}) \left[f_k(k_{t+1}) + (1-\delta) \right] = 0, \tag{10.5}$$

$$\frac{\partial V(\omega_t)}{\partial b_{t+1}} = \beta V_\omega(\omega_{t+1}) \left[\frac{R_t}{\pi_{t+1}} - f_k(k_{t+1}) - (1-\delta) \right] = 0, \tag{10.6}$$

$$\frac{\partial V(\omega_t)}{\partial m_{t+1}} = u_m(c_t, m_t) - \beta V_\omega(\omega_{t+1}) \left[f(k_{t+1}) + (1-\delta) + \frac{1}{\pi_{t+1}} \right] = 0, \tag{10.7}$$

together with the transversality conditions:

$$\lim_{t\to\infty} \beta^t \lambda_t x_t = 0, \;\; x = \{k, b, m\},$$

where λ_t is the marginal utility of period t consumption. The envelope theorem implies:

$$V_\omega(\omega_t) = \beta V_\omega(\omega_{t+1}) \left[f_k(k_{t+1}) + (1-\delta) \right]. \tag{10.8}$$

Combine the envelope condition (10.8) with (10.5) to get:

$$\lambda_t = u_c(c_t, m_t) = V_\omega(\omega_t). \tag{10.9}$$

The FOCs have straightforward interpretations. Since initial resources ω_t must be divided between consumption, capital, bonds, and money balances, each use must yield the same marginal benefit at an optimum allocation. Using (10.5) and (10.9), we can write (10.7) as:

$$u_m(c_t, m_t) + \beta \frac{u_c(c_{t+1}, m_{t+1})}{\pi_{t+1}} = u_c(c_t, m_t), \tag{10.10}$$

which states that the marginal benefit of adding to money holdings at time t must equal the marginal utility of consumption at time t. The marginal benefit of additional money holdings has two components. First, money directly yields utility u_m. Second, real money balances at time t adds $1/\pi_{t+1}$ to real resources at time $t+1$. This addition to ω_{t+1} is worth $V_\omega(\omega_{t+1})$ at $t+1$, or $\beta V_\omega(\omega_{t+1})$ at time t. Thus, the total marginal benefit of money at time t is $u_m(c_t, m_t) + \beta V_\omega(\omega_{t+1})/(1+\pi_{t+1})$. Equation (10.10) is then obtained by noting that $V_\omega(\omega_{t+1}) = u_c(c_{t+1}, m_{t+1})$.

We can derive the value of money as an asset by rewriting (10.10) as:

$$\begin{aligned}
\frac{u_c(c_t, m_t)}{P_t} &= \frac{u_m(c_t, m_t)}{P_t} + \beta \frac{1}{P_t} \frac{u_c(c_{t+1}, m_{t+1})}{\pi_{t+1}}, \\
\Leftrightarrow \underbrace{\frac{u_c(c_t, m_t)}{P_t}}_{\text{Price}} &= \underbrace{\frac{u_m(c_t, m_t)}{P_t}}_{\text{Dividend}} + \underbrace{\beta \frac{u_c(c_{t+1}, m_{t+1})}{P_{t+1}}}_{\text{Price tomorrow}}.
\end{aligned}$$

Then, roll forward to get:

$$\begin{aligned}
\frac{u_c(c_t, m_t)}{P_t} &= \frac{u_m(c_t, m_t)}{P_t} + \beta \left(\frac{u_m(c_{t+1}, m_{t+1})}{P_{t+1}} + \beta \frac{u_c(c_{t+2}, m_{t+2})}{P_{t+2}} \right) \\
&= \frac{u_m(c_t, m_t)}{P_t} + \beta \frac{u_m(c_{t+1}, m_{t+1})}{P_{t+1}} + \beta^2 \frac{u_c(c_{t+2}, m_{t+2})}{P_{t+2}} + ...,
\end{aligned}$$

Hence, the value of money (in terms of utils) can be written as:

$$\frac{u_c(c_t, m_t)}{P_t} = \sum_{i=0}^{\infty} \beta^i \frac{u_m(c_{t+i}, m_{t+i})}{P_{t+i}}. \tag{10.11}$$

10.4.2 Opportunity cost of holding money and the Fisher relation

Rewrite (10.6) as:

$$\begin{aligned}
f_k(k_{t+1}) &= \frac{R_t}{\pi_{t+1}} - 1 + \delta \\
\Leftrightarrow 1 + r_t &= f_k(k_{t+1}) + 1 - \delta,
\end{aligned}$$

where $1 + r_t$ is the [gross] real interest rate.[6] Then combine this with (10.5) and (10.10) to get the opportunity cost of holding money:

$$
\begin{aligned}
1 &= \frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} + \frac{\beta}{\pi_{t+1}} \frac{u_c(c_{t+1}, m_{t+1})}{u_c(c_t, m_t)} \\
\frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} &= 1 - \frac{1}{\underbrace{[f_k(k_{t+1}) + 1 - \delta]}_{1+r_t}\pi_{t+1}} \\
\frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} &= \frac{i_t}{R_t} \equiv \Upsilon_t, \qquad (10.12)
\end{aligned}
$$

where (10.5) implied:

$$\beta \frac{u_c(c_{t+1}, m_{t+1})}{u_c(c_t, m_t)} = \frac{1}{1 + r_t}.$$

Note, from (10.12) can be written as:

$$
\begin{aligned}
\frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} &= \frac{P_t i_t}{P_t(1 + i_t)} \\
\Leftrightarrow \frac{u_m(c_t, m_t)}{u_c(c_t, m_t)} &= \frac{P_m(t)}{P_c(t)} \\
\Leftrightarrow \text{MRS} &= \text{MRT}.
\end{aligned}
$$

Equation (10.12) also makes use of (10.6), which links the nominal return on bonds, inflation, and the real interest rate (or, alternatively, the real return on capital). This latter equation can be written as:

$$
\begin{aligned}
R_t &= [f_k(k_{t+1}) + 1 - \delta]\, \pi_{t+1} \\
&= (1 + r_t)\pi_{t+1}.
\end{aligned}
$$

This relationship between the real and nominal rates of interest is called the Fisher equation, after Irving Fisher (1896). It expresses the gross nominal rate of interest as equal to the gross real return on capital times 1 plus the expected rate of inflation. If one notes that $(1 + x)(1 + y) \approx 1 + x + y$ when x and y are small, we can write the Fisher relationship as:

$$i_t = r_t + \hat{\pi}_{t+1}, \qquad (10.13)$$

where here $\hat{\pi}_{t+1}$ is the net inflation rate.

To interpret (10.12), consider a very simple choice problem in which the agent must pick x and z to maximise $u(x, z)$ subject to a budget constraint of the form $x + pz = y$, where p is the relative price of z. The FOC conditions imply:

$$\frac{u_z}{u_x} = p,$$

6. Notice that this also commonly written as the gross return on capital, R_t^k, in many DSGE models.

in words, the marginal rate of substitution between z and x equals the relative price of z in terms of x. Comparing this to (10.12) shows that Υ can be interpreted as the relative price of real money balances in terms of the consumption good. The marginal rate of substitution between money and consumption is set equal to the price, or opportunity cost, of holding money. The opportunity cost of holding money is directly related to the nominal rate of interest. The household could hold one unit less of money, purchasing instead a bond yielding a nominal return of R (gross) or i (net); the real value of this payment is i/π, and since it is received in period $t+1$, its present value is:

$$\frac{\bar{i}}{(1+\bar{r})\bar{\pi}} = \frac{\bar{i}}{1+\bar{i}} = \frac{\bar{i}}{\bar{R}}.$$

Since money is assumed to pay no rate of interest, the opportunity cost of holding money is affected both by the real return on capital and the rate of inflation. If the price level is constant (so $\pi = 1$), then the foregone earnings from holding money rather than capital are determined by the real return to capital. If the price level is rising ($\pi > 1$), the real value of money in terms of consumption declines, and this adds to the opportunity cost of holding money.

In deriving the FOCs for the household's problem, it could have been equivalently assumed that the household leased its capital to firms, receiving a rental rate of r^k, and sold its labour services at a wage rate of w. Household income would then be $r^k k + w$. With competitive firms hiring capital and labour in perfectly competitive markets under constant returns to scale, $r^k = f'(k)$ and $w = f(k) - kf'(k)$, so household income would be (using Euler's Theorem):

$$r^k k + w = f_k(k)k + [f(k) - kf_k(k)] = f(k),$$

as in (10.3).[7]

10.4.3 Steady state equilibrium and money neutrality

Consider the properties of this economy when it is in a steady-state equilibrium and the nominal supply of money growing at the rate θ. We use our usual notation to denote steady state variables with a bar (e.g. $\bar{x}$). The steady state values of consumption, the capital stock, real money balances, inflation, and the nominal interest rate must satisfy the FOCs for the household's decision problem given by (10.5)-(10.7), the economy wide budget constraint, and the specification of the exogenous growth rate of M. Note that with real money balances constant in the steady state, it must be that the prices are growing at the same rate as the nominal stock of money, $\bar{\pi} = 1 + \theta$. Use (10.9) to eliminate $V_\omega(\bar{\omega})$, the FOCs can then be written as:

$$0 = u_c(\bar{c}, \bar{m}) - \beta u_c(\bar{c}, \bar{m}) \left[f_k(\bar{k}) + 1 - \delta \right], \tag{10.14}$$

$$0 = \frac{1+\bar{i}}{1+\theta} - f_k(\bar{k}) - 1 + \delta, \tag{10.15}$$

7. For this approach, see McCandless (2008).

$$0 = u_m(\bar{c}, \bar{m}) - \beta u_c(\bar{c}, \bar{m}) \left[f(\bar{k}) + 1 - \delta \right] + \beta \frac{u_c(\bar{c}, \bar{m})}{1 + \theta}, \tag{10.16}$$

and the economy wide resource constraint is:

$$f(\bar{k}) + \bar{\tau} + (1 - \delta)\bar{k} + \frac{\bar{m}}{1 + \theta} = \bar{c} + \bar{k} + \bar{m}, \tag{10.17}$$

where $\bar{b} = 0$.

Equation (10.15) is the steady state form of the Fisher relation, linking real and nominal interest rates. This can be seen by noting that the real return on capital (net of depreciation) is:

$$\bar{r} \equiv f_k(\bar{k}) - \delta,$$

so (10.15) can be written as:

$$1 + \bar{i} = (1 + \bar{r})\bar{\pi} = (1 + \bar{r})(1 + \theta).$$

Notice that in (10.14)-(10.17) money appears only in the form of real money balances. Thus, any change in the nominal quantity of money that is matched by a proportional change in the price level – leaving $\bar{m}$ unchanged – has no effect on the economy's real equilibrium. This is described by saying that the model exhibits neutrality of money. If prices do not adjust immediately in response to a change in M, then a model might display non-neutrality with respect to changes in M in the short-run but still exhibit monetary neutrality in the long-run, once all prices have adjusted.

Dividing (10.14) by $u_c(\bar{c}, \bar{m})$ yields:

$$\begin{aligned} 0 &= 1 - \beta \left[f_k(\bar{k}) + 1 - \delta \right] \\ \Leftrightarrow f_k(\bar{k}) &= \frac{1}{\beta} - 1 + \delta. \end{aligned} \tag{10.18}$$

This equation defines the steady state capital labour ratio $\bar{k}$ as a function of β and δ. If the production function is Cobb-Douglas, say $f(k) = k^\alpha$ for $\alpha \in (0, 1]$, then $f_k(k) = \alpha k^{\alpha-1}$ and:

$$\bar{k} = \left[\frac{\alpha\beta}{1 + \beta(\delta - 1)} \right]^{\frac{1}{1-\alpha}}. \tag{10.19}$$

What is particularly relevant here is the implication that the steady state capital labour ratio is independent of i) all parameters of the utility function other than the subjective discount rate β, and ii) the steady state rate of inflation $\bar{\pi}$. In fact, $\bar{k}$ depends only on the production function, the depreciation rate, and the discount rate. It is independent of the rate of inflation and the growth of money.

Because changes in the nominal quantity of money are engineered in this model by making lump sum transfers to the public, the real value of these transfers must equal:

$$\frac{M_{t+1} - M_t}{P_t} = \frac{\theta M_t}{P_t} \\ = \frac{\theta m_t}{\pi_t}.$$

Hence, steady state transfers are given by:

$$\bar{\tau} = \frac{\theta \bar{m}}{\bar{\pi}} = \frac{\theta \bar{m}}{1+\theta},$$

and the budget constraint (10.17) reduces to:

$$\bar{c} = f(\bar{k}) - \delta \bar{k}. \tag{10.20}$$

The steady state level of consumption is equal to output minus replacement investment and is completely determined once the level of steady state capital is known. Assuming that $f(k) = k^\alpha$, then $\bar{k}$ is given by (10.19) and so

$$\bar{c} = \left[\frac{\alpha\beta}{1+\beta(\delta-1)}\right]^{\frac{\alpha}{1-\alpha}} - \delta\left[\frac{\alpha\beta}{1+\beta(\delta-1)}\right]^{\frac{1}{1-\alpha}}.$$

Steady state consumption per capital depends on the parameters of the production function α, the rate of depreciation δ, and the subjective rate of time discount β.

The Sidrauski MIU model exhibits a property called the superneutrality of money; the steady state values of the capital stock, consumption, and output are all independent of the rate of growth of the nominal money stock. That is, not only is money neutral, so that proportional changes in the level of nominal money balances and prices have no real effects, but changes in the rate of growth of nominal money also have no effect on the steady state capital stock nor on output and consumption. Since the real rate of interest is equal to the marginal product of capital, it also is invariant across steady states that differ only in their rates of money growth. Thus, the Sidrauski MIU model possesses the properties of both neutrality and superneutrality.

10.4.4 The demand for money

Returning to the opportunity cost of money (10.12), this equation characterises the demand for real money balances as a function of the nominal rate of interest and real consumption. For example, suppose that the utility function in consumption and real balances is of the constant elasticity of substitution (CES) form:

$$u(c_t, m_t) = \left[a c_t^{1-b} + (1-a) m_t^{1-b}\right]^{\frac{1}{1-b}},$$

with $a \in (0,1)$ and $b > 0, b \neq 1$. Then:

$$\frac{u_m}{u_c} = \left(\frac{1-a}{a}\right)\left(\frac{c_t}{m_t}\right)^b,$$

and (10.12) can be written as (in the limit as $b \to \infty$):

$$m_t = \left(\frac{1-a}{a}\right)^{\frac{1}{b}} \left(\frac{i_t}{R_t}\right)^{-\frac{1}{b}} c_t. \tag{10.21}$$

In terms of the more common log specification used to model empirical money demand equations, we have:

$$\ln \frac{M_t}{P_t} = \frac{1}{b} \ln\left(\frac{1-a}{a}\right) + \ln c_t - \frac{1}{b} \ln \frac{i_t}{R_t}, \tag{10.22}$$

which gives the real demand for money as a negative function of the nominal rate of interest and a positive function of consumption. The consumption (income) elasticity of money demand is equal to 1 in this specification. The elasticity of money demand with respect to the opportunity cost variable $\Upsilon_t = i_t/R_t$ is $1/b$. For simplicity, this will often be referred to as the interest elasticity of demand.

For $b = 1$, the CES specification becomes $u(c_t, m_t) = c_t^a m_t^{1-a}$. Note from (10.22) that in this case, the consumption (income) elasticity of money demand and the elasticity with respect to the opportunity cost measure Υ_t are both equal to 1.

While the parameter b governs the interest elasticity of demand, the steady state level of money holdings depends on the value of a. From (10.21), the ratio of real money balances to consumption in the steady state will be:

$$\begin{aligned}
\frac{\bar{m}}{\bar{c}} &= \left(\frac{1-a}{a}\right)^{\frac{1}{b}} \left(\frac{\bar{i}}{\bar{R}}\right)^{-\frac{1}{b}} \\
&= \left(\frac{1-a}{a}\right)^{\frac{1}{b}} \left(\frac{\bar{r} + \bar{\pi}}{(1+\bar{r})\bar{\pi}}\right)^{-\frac{1}{b}} \\
&= \left(\frac{1-a}{a}\right)^{\frac{1}{b}} \left(\frac{\beta^{-1} - 1 + \bar{\pi}}{\beta^{-1}\bar{\pi}}\right)^{-\frac{1}{b}} \\
\therefore \frac{\bar{m}}{\bar{c}} &= \left(\frac{1-a}{a}\right)^{\frac{1}{b}} \left[\frac{1 + \beta(\bar{\pi} - 1)}{\bar{\pi}}\right]^{-\frac{1}{b}}.
\end{aligned} \tag{10.23}$$

The ratio of $\bar{m}$ to $\bar{c}$ is decreasing in a; an increase in a reduces the weight given to real money balances in the utility function and results in smaller holdings of money (relative to consumption) in the steady state. Increases in inflation also reduce the ratio of money holdings to consumption by increasing the opportunity cost of holding money!

10.4.5 The welfare cost of inflation

Because money holdings yield direct utility and higher inflation reduces real money balances, inflation generates a welfare loss. This raises two questions: How large is the welfare cost of inflation? Is there an optimal rate of inflation that maximises the steady state welfare of the representative household?

The second question – the optimal rate of inflation – was originally addressed by Bailey (1956) and Friedman (1969). Their basic intuition was the following. The private opportunity cost of holding money depends on the nominal rate of interest. The social marginal cost of producing money, that is, running the printing presses, is essentially zero. The wedge that arises between the private marginal cost and the social marginal cost when the nominal rate of interest is positive generates an inefficiency. This inefficiency would be eliminated if the private opportunity cost were also equal to zero, and this will be the case if the nominal rate of interest equals zero.

But $i = 0$ requires that $\pi = -r/(1+r) \approx -r$. So the optimal rate of inflation is a rate of deflation approximately equal to the real return on capital.

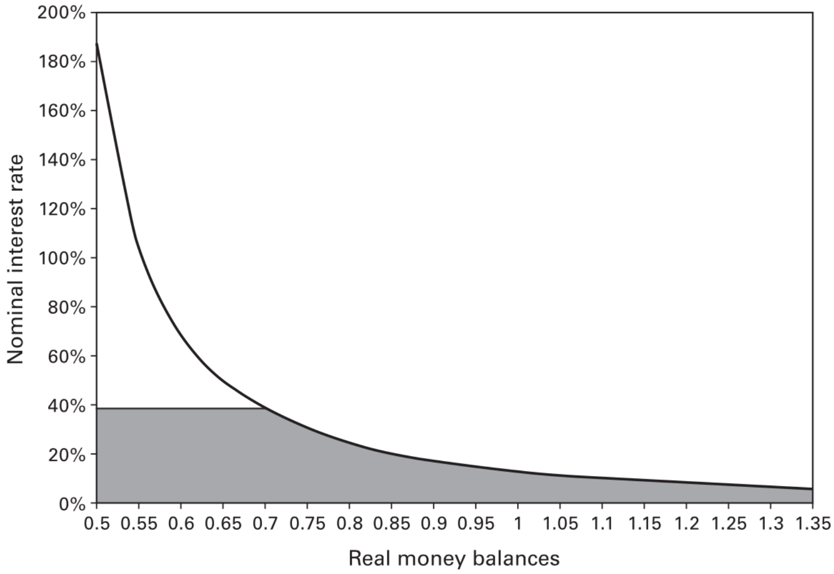
In the steady state, real money balances are directly related to the inflation rate, so the optimal rate of inflation is also frequently discussed under the heading of the optimal quantity of money (Friedman, 1969). With utility depending direct on m, one can think of the government choosing its policy instrument θ (and therefore π) to achieve the steady state optimal value of m.

The major criticism of this result is due to Phelps (1973), who pointed out that money growth generates revenue for the government – the inflation tax. The implicit assumption so far has been that variations in money growth are engineered via lump sum transfers. Any effects on government revenue can be offset by a suitable adjustment in these lump sum transfers. But if governments only have distortionary taxes available for financing expenditures, then reducing inflation tax revenues to achieve the Friedman rule of a zero nominal interest rate requires that the lost revenue be replaced through increases in other distortionary taxes. Reducing the nominal rate of interest to zero would increase the inefficiencies generated by the higher level of other taxes that would be needed to replace the lost inflation tax revenues. To minimise the total distortions associated with raising a given amount of revenue, it may be optimal to rely on the inflation tax to some degree. Other work on this issue is by Chari, Christiano, and Kehoe (1991, 1996) and Correia and Teles (1996).

As for the first question – how large is the welfare cost of inflation? Beginning with Bailey (1956), this welfare cost has been calculated from the area under the money demand curve (showing money demand as a function of the nominal interest rate) because this provides a measure of the consumer surplus lost as a result of having a positive nominal rate of interest. Figure 10.12 is based on the money demand function given by 10.21 with $a = 0.9$ and Chari, Kehoe, and McGrattan (2000)'s implied value for b of 2.56. At a nominal interest rate of i^*, the deadweight loss is measured by the shaded area under the money demand curve.

Lucas (1994) provided estimates of the welfare costs of inflation by starting with the following

Figure 10.12: Welfare Costs of Inflation as Measured by the Area under the Demand Curve



Source: Walsh (2010).

specification of the instantaneous utility function:

$$u(\bar{c}, \bar{m}) = \frac{1}{1-\sigma}\left\{\left[\bar{c}\varphi\left(\frac{\bar{m}}{\bar{c}}\right)\right]^{1-\sigma} - 1\right\}.$$

With this utility function (10.12) becomes:

$$\frac{u_m}{u_c} = \frac{\varphi'(\bar{x})}{\varphi(\bar{x}) - \bar{x}\varphi'(\bar{x})} = \frac{\bar{i}}{\bar{R}} = \bar{\Upsilon}, \tag{10.24}$$

where $\bar{x} \equiv \bar{m}/\bar{c}$. Normalising so that the steady state consumption equals 1, $u(1, \bar{m})$ will be maximised when $\bar{\Upsilon} = 0$, implying that the optimal x is defined by $\varphi'(\bar{m}^*) = 0$. Lucas proposed to measure the costs of inflation by the percentage increase in steady state consumption necessary to make the household indifferent between a nominal interest rate of i and a nominal rate of 0. If this cost is denoted $w(\bar{\Upsilon})$ it is defined by:

$$u\left(1 + w(\bar{\Upsilon}), m(\bar{\Upsilon})\right) \equiv u(1, \bar{m}^*),$$

where $m(\bar{\Upsilon})$ denotes the solution of (10.24) for real money balances evaluated at steady state consumption $\bar{c} = 1$.

Suppose, following Lucas, that $\varphi(\bar{m}) = (1 + B\bar{m}^{-1})^{-1}$, where B is a positive constant. Solving (10.24), one obtains $m(\bar{i}) = B^{0.5}\bar{\Upsilon}^{-0.5}$. Note that $\varphi' = 0$ requires that $\bar{m}^* = \infty$. But $\varphi(\infty) = 1$

and $u(1, \infty) = 0$, so $w(\bar{\Upsilon})$ is the solution to $u(1 + w(\bar{\Upsilon}), B^{0.5}\bar{\Upsilon}^{-0.5}) = u(1, \infty) = 0$. Using the definition of the utility function, one obtains $1 + w(\bar{\Upsilon}) = 1 + \sqrt{B\bar{\Upsilon}}$, or:

$$w(\bar{\Upsilon}) = \sqrt{B\bar{\Upsilon}}.$$

Based on US annual data from 1900 to 1985, Lucas reported an estimate of 0.0018 for B. Hence, the welfare loss arising from a nominal interest rate of 10 percent would be $\sqrt{(0.0018)(0.1/1.1)} = 0.0013$, or just over 1 percent of aggregate consumption.

10.4.6 Breaking superneutrality and model dynamics

Suppose that labour was endogenously determined, so that the household's utility function is:

$$u(c_t, m_t, l_t) = u(c_t, m_t, 1 - h_t),$$

and that we get a new optimality condition:

$$\frac{u_l(c_t, m_t, l_t)}{u_c(c_t, m_t, l_t)} = f_h(k_t, h_t).$$

This states that an optimum, the marginal rate of substitution between consumption and leisure must equal the marginal product of labour.

The full details are in McCandless (2008) and Walsh (2010), which we won't delve into now, but what we find is that so long as the household's preferences are separable, superneutrality holds – changes in the steady state rate of inflation will alter nominal interest rates and the demand for real money balances, but different inflation rates have no effect on the steady state values of the capital stock, labour supply, or consumption.

If utility is not separable, so that either u_l or u_c (or both) depend on m, then money is not superneutral. Variations in average inflation that affect the opportunity cost of holding money will affect the steady state level of m. Different levels of $\bar{m}$ will change the value of $\bar{h}$. In other words, the steady state effect of money growth on real variables will depend on "strange" cross elasticities: u_{cm} and u_{lm}.

But does it make sense for the effects of monetary growth be channelled through effects of m on labour supply? What's even more troubling for us that the model's dynamics have unpalatable implications. For example, Walsh (2010) demonstrates that a positive money shock increases the nominal rate of interest; if there is persistence in the process for money growth, money growth rate shocks increase expected inflation and raise the nominal interest rate, while the real quantity of money actually falls.

This doesn't quite line up with the empirical findings we discussed earlier in this section, and so our search for a model which accommodates money continues.

10.5 Cash in advance constraints

A direct approach to generating a role for money proposed by Clower (1967) and developed by Lucas (1980; 1982), Stockman (1981), Svensson (1985), Lucas and Stokey (1987), and Cooley and Hansen (1989), captures the role of money as a medium of exchange by requiring explicitly that money be used to purchase goods.

Timing assumptions are important in cash in advance (CIA) models. In Lucas (1982), agents are able to allocate their portfolios between cash and other assets at the start if each period, after observing any current shocks but prior to purchasing goods. This timing is often described by saying that the asset market opens first and then the goods market opens. If there is a positive opportunity cost of holding money and the asset market opens first, agents will only hold an amount of money that is sufficient to finance their desired level of consumption. In Svensson (1985), the goods market opens first. this implies that agents have available for spending only the cash carried over from the previous period, and so cash balances must be chosen before agents know how much spending they will wish to undertake. For example, if uncertainty is resolved after money balances are chosen, an agent may find that he is holding cash balances that are too low to finance his desired spending level. Or he may be left with more cash than he needs, thereby forgoing interest income.

To understand the structure of CIA models, this section reviews a simplified version of a model due to Svensson (1985). Although, the alternative timing used by Lucas (1982) is also briefly discussed. After the model and its equilibrium conditions are set out, the steady state is examined and the welfare costs of inflation a CIA model are discussed.

10.5.1 A simple Svensson CIA model

Consider the following representative agent model. The agent's objective is to chose a path for consumption and asset holdings to maximise:

$$\sum_{t=0}^{\infty} \beta^t u(c_t), \ \beta \in (0, 1),$$

where $u(\cdot)$ is bounded, continuously differentiable strictly increasing, and strictly concave, and the maximisation is subject to a sequence of CIA and budget constraints. The agent enters the period with money holdings M_t and receives a lump-sum transfer T_t (in nominal terms). If goods markets open first, the CIA constraint takes the form:

$$P_t c_t \leq M_t + T_t,$$

where c_t is real consumption, P_t is the aggregate price level, and T_t are nominal lump sum transfers in period t. In real terms this is:

$$c_t \leq \frac{M_t}{P_t} + \frac{T_t}{P_t} = \frac{m_t}{\pi_t} + \tau_t, \tag{10.25}$$

where $m_t = M_t/P_{t-1}$, $\pi_t = P_t/P_{t-1}$ is the gross inflation rate, and $\tau_t = T_t/P_t$. Note the timing: M_t refers to nominal money balances chosen by the agent in period $t-1$ and carried into period t. The real value of these balances is determined by the period t price level P_t. Since we have assumed away any uncertainty, the agent knew P_t at the time M_t was chosen. This specification of the CIA constraint assumes that income from production during period t will not be available for consumption purchases until period $t+1$.

The budget constraint, nominal terms, is:

$$P_t\omega_t \equiv P_t f(k_t) + (1-\delta)P_t k_t + M_t + T_t + R_{t-1}B_t \geq P_t c_t + P_t k_{t+1} + M_{t+1} + B_{t+1},$$

where ω_t is the agent's time t real resources, consisting of income generated during period t, $f(k_t)$, the undepreciated capital stock $(1-\delta)k_t$, money holdings, the transfer from the government, and gross nominal interest earnings on the agent's $t-1$ holdings of nominal one-period bonds, B_t. Physical capital depreciates at the rate δ. These resources are used to purchase consumption, capital, bonds, and nominal money holdings that are then carried into period $t+1$. Dividing through by the time t price level, the budget constraint can be rewritten in real terms as:

$$\omega_t \equiv f(k_t) + (1-\delta)k_t + \tau_t + \frac{m_t + R_{t-1}b_t}{\pi_t} \geq c_t + m_{t+1} + b_{t+1} + k_{t+1}. \tag{10.26}$$

Note that real resources available to the representative agent in period $t+1$ are given by:

$$\omega_{t+1} = f(k_{t+1}) + (1-\delta)k_{t+1} + \tau_{t+1} + \frac{m_{t+1} + R_t b_{t+1}}{\pi_{t+1}}. \tag{10.27}$$

The period t gross nominal interest rate R_t can be divided by π_{t+1} to yield the gross real rate of return from period t to $t+1$, and can be denoted by:

$$1 + r_t = \frac{R_t}{\pi_{t+1}} = \frac{1+i_t}{\pi_{t+1}}. \tag{10.28}$$

With this notation (10.27) can be written as:

$$\omega_{t+1} = f(k_{t+1}) + (1-\delta)k_{t+1} + \tau_{t+1} + (1+r_t)a_{t+1} - \left(\frac{i_t}{\pi_{t+1}}\right) m_{t+1}, \tag{10.29}$$

where $a_{t+1} \equiv m_{t+1} + b_{t+1}$ is the agent's holding of nominal financial assets (money and bonds). Writing it this way shows that there is a cost to holding money when the nominal interest rate is positive. This cost is $i_t/(1+\pi_{t+1})$; since this is the cost in terms of period $t+1$ real resources, the discounted cost at time t of holding an additional unit of money is:

$$\frac{i_t}{(1+r_t)\pi_{t+1}} = \frac{i_t}{R_t} = \Upsilon_t.$$

This is the same expression for the opportunity cost of money in the MIU model.

10.5.2 Lucas' alternate timing convention

Equation (10.25) is based on the timing convention that goods markets open before asset markets. The model of Lucas (1982) assumed the reverse, and individuals can engage in asset transactions at the start of each period before the goods market has opened. In the present model, this would mean that the agent enters period t with financial wealth that can be used to purchase nominal bonds B_{t+1} or carried as cash into the goods market to purchase consumption goods. The CIA constraint would then take the form:

$$c_t \leq \frac{m_t}{\pi_t} + \tau_t - b_{t+1}. \tag{10.30}$$

In this case, the household is able to adjust its portfolio between money and bonds before entering the goods market to purchase consumption goods.

To understand the implications of this alternative timing, suppose there is a positive opportunity cost of holding money. Then, if the asset market opens first, the agent will only hold an amount of money that is just sufficient to finance the desired level of consumption. Since the opportunity cost of holding m is positive whenever the nominal interest rate is greater than zero, (10.30) will always hold with equality as long as the nominal rate of interest is positive. When uncertainty is introduced, the CIA constraint may not bind when (10.25) is used and the goods market opens before the asset market. For example, if period t's income is uncertain and is realised after M_{t-1} has been chosen, a bad income realisation may cause the agent to reduce consumption to a point where the CIA constraint is no longer binding. Or a disturbance that causes an unexpected price decline might, by increasing the real value of the agent's money holdings, result in a nonbinding constraint. Since a nonstochastic environment holds in this section, the CIA constraint will bind under either timing assumption if the opportunity cost of holding money is positive.

10.5.3 Equilibrium

The choice variables at time t are c_t, m_{t+1}, b_{t+1}, and k_{t+1}. An individual agent's state at time t can be characterised by her resources ω_t and her real cash holdings m_t; both are relevant because the consumption choice is constrained by the agent's resources and by cash holdings. To analyse the agent's decision problem, one can define the value function:

$$V(\omega_t, m_t) = \max_{c_t, k_{t+1}, b_{t+1}, m_{t+1}} \left\{ u(c_t) + \beta V(\omega_{t+1}, m_{t+1}) \right\}, \tag{10.31}$$

subject to (10.26), (10.25), and (10.29):

$$\omega_t \geq c_t + m_{t+1} + b_{t+1} + k_{t+1},$$
$$c_t = \frac{m_t}{\pi_t} + \tau_t,$$
$$\omega_{t+1} = f(k_{t+1}) + (1-\delta)k_{t+1} + \tau_{t+1} + (1+r_t)a_{t+1} - \left(\frac{i_t}{\pi_{t+1}}\right) m_{t+1},$$

Using the expression for ω_{t+1} and letting λ_t and μ_t denote the Lagrangian/Kuhn-Tucker multipliers for the budget constraint and CIA constraint, respectively, the FOCs take the form:

$$\frac{\partial V(\omega_t, m_t)}{\partial c_t} = u'(c_t) - \lambda_t - \mu_t = 0, \tag{10.32}$$
$$\frac{\partial V(\omega_t, m_t)}{\partial k_{t+1}} = \beta(f'(k_{t+1}) + 1 - \delta)V_\omega(\omega_{t+1}, m_{t+1}) - \lambda_t = 0, \tag{10.33}$$
$$\frac{\partial V(\omega_t, m_t)}{\partial m_{t+1}} = \beta\left(1 + r_t - \frac{i_t}{\pi_{t+1}}\right) V_\omega(\omega_{t+1}, m_{t+1}) + \beta V_m(\omega_{t+1}, m_{t+1}) - \lambda_t = 0, \tag{10.34}$$
$$\frac{\partial V(\omega_t, m_t)}{\partial b_{t+1}} = \beta\left(1 + r_t\right) V_\omega(\omega_{t+1}, m_{t+1}) - \lambda_t = 0. \tag{10.35}$$

By the envelope theorem:

$$\frac{\partial V(\omega_t, m_t)}{\partial \omega_t} = \lambda_t, \tag{10.36}$$
$$\frac{\partial V(\omega_t, m_t)}{\partial m_t} = \left(\frac{1}{\pi_t}\right)\mu_t. \tag{10.37}$$

From (10.36), λ_t is equal to the marginal utility of wealth. According to (10.32), the marginal utility of consumption exceeds the marginal utility of wealth by the value of liquidity services, μ_t. The individual must hold money in order to purchase consumption, so the "cost", to which the marginal utility of consumption is set equal, is the marginal utility of wealth plus the cost of the liquidity services needed to finance the transaction.

In terms of λ, (10.35) becomes:

$$\lambda_t = \beta(1 + r_t)\lambda_{t+1}, \tag{10.38}$$

which is nothing but the familiar consumption Euler equation! Along the optimal path, the marginal cost (in terms of today's utility) from reducing wealth slightly, λ_t, must equal the utility value of carrying that wealth forward one period, earning a gross real return $1+r_t$, where tomorrow's utility is discounted back to today at the rate β; that is, $\lambda_t = \beta(1 + r_t)\lambda_{t+1}$ along the optimal path.

Using (10.36) and (10.37), the FOC (10.34) can be expressed as:

$$\lambda_t = \beta \left(\frac{\lambda_{t+1} + \mu_{t+1}}{\pi_{t+1}} \right).$$

This equation can be interpreted as an asset pricing equation for money. The price of a unit of money in terms of goods is just $1/P_t$ at time t; its value in utils is λ_t/P_t. Now, by dividing the above equation by P_t, it can be rewritten as:

$$\frac{\lambda_t}{P_t} = \beta \left(\frac{\lambda_{t+1}}{P_{t+1}} + \frac{\mu_{t+1}}{P_{t+1}} \right).$$

Solving this equation forward implies that:

$$\frac{\lambda_t}{P_t} = \sum_{i=1}^{\infty} \beta^i \left(\frac{\mu_{t+i}}{P_{t+i}} \right).$$

From (10.37), μ_{t+i}/P_{t+i} is equal to $V_m(\omega_{t+i}, m_{t+i})/P_{t+i-1}$. This last expression though is just the partial of the value function with respect to time $t+i1$ nominal money balances:

$$\begin{aligned}
\frac{\partial V(\omega_{t+i}, m_{t+i})}{\partial M_{t+i}} &= V_m(\omega_{t+i}, m_{t+i}) \left(\frac{\partial m_{t+i}}{\partial M_{t+i}} \right) \\
&= \frac{V_m(\omega_{t+i}, m_{t+i})}{P_{t+i-1}} \\
&= \left(\frac{\mu_{t+i}}{P_{t+i}} \right).
\end{aligned}$$

This means that we can write:

$$\frac{\lambda_t}{P_t} = \sum_{i=1}^{\infty} \beta^i \frac{\partial V(\omega_{t+i}, m_{t+i})}{\partial M_{t+i}}. \tag{10.39}$$

In words, the current value of money in terms of utility is equal periods. This is an interesting result; it says that money is just like any other asset in the sense that its value can be thought of as equal to the present discounted value of the stream of returns generated by the asset. In the case of money, these returns take the form of liquidity services. If the CIA constraint were not binding, these liquidity services would have no value ($\mu = V_m = 0$) and nor would money. But if the constraint is binding, then money has value because it yields valued liquidity services.

The result that the value of money, λ/P, satisfies an asset pricing relationship is not unique to the CIA approach. For example, a similar relationship is implied by the MIU approach. The model employed in the analysis of the MIU approach implied that:

$$\frac{\lambda_t}{P_t} = \beta \left(\frac{\lambda_{t+1}}{P_{t+1}} \right) + \frac{u_m(c_t, m_t)}{P_t},$$

which can be solved forward to yield:

$$\frac{\lambda_t}{P_t} = \sum_{i=0}^{\infty} \beta^i \left[\frac{u_m(c_{t+i}, m_{t+i})}{P_{t+i}} \right].$$

Here, the marginal utility of money u_m plays a role exactly analogous to that played by the Lagrangian on the CIA constraint μ. The one difference is that in the MIU approach, m_t yields utility at time t, whereas in the CIA approach, the value of money accumulated at time t is measured by μ_{t+1} because the cash cannot be used to purchase consumption goods until period $t+1$.

An expression for nominal rate of interest can be obtained by using our results for λ_t to obtain:

$$\begin{aligned} \lambda_t &= \beta(1+r_t)\lambda_{t+1} \\ \lambda_t &= \beta \left[\frac{\lambda_{t+1} + \mu_{t+1}}{\pi_{t+1}} \right] \\ \Leftrightarrow (1+r_t)\pi_{t+1}\lambda_{t+1} &= \lambda_{t+1} + \mu_{t+1}. \end{aligned}$$

Since $1 + i_t = (1 + r_t)\pi_{t+1}$, the nominal interest rate is given by:

$$i_t = \left(\frac{\lambda_{t+1} + \mu_{t+1}}{\lambda_{t+1}} \right) - 1 = \frac{\mu_{t+1}}{\lambda_{t+1}}. \tag{10.40}$$

Thus, the nominal rate of interest is positive if and only if money yields liquidity services ($\mu_{t+1} > 0$). In particular, if the nominal interest rate is positive, the CIA constraint is binding ($\mu > 0$).

10.5.4 The steady state

In the steady state, (10.38) implies that:

$$1 + \bar{r} = \frac{1}{\beta},$$

and:

$$1 + i = \frac{\bar{\pi}}{\beta}$$

In addition, (10.33) gives the steady state capital stock as the solution to:

$$f_k(\bar{k}) = \bar{r} + \delta = \frac{1}{\beta} - 1 + \delta.$$

So this CIA model, like the Sidrauski MIU model, exhibits superneutrality. The steady state capital stock depends only on the time preferences parameter β, the rate of depreciation δ, and the production function. It is independent of the rate of inflation. Since steady state consumption is equal to $f(\bar{k}) - \delta\bar{k}$, it too is independent of the rate of inflation.

It has been shown that the marginal utility of consumption could be written as the marginality of wealth (λ) times 1 plus the nominal rate of interest, reflecting the opportunity cost of holding the money required to purchase goods for consumption. Using (10.40), the ratio of the liquidity value of money, measured by the Lagrangian multiplier μ, to the marginal utility of consumption is:

$$\frac{\mu_t}{u_c(c_t)} = \frac{\mu_t}{\lambda_t(1+i_t)} = \frac{i_t}{R_t}.$$

This expression is exactly parallel to the result in the MIU framework, where the ratio of the marginal utility of money to the marginal utility of consumption was equal to the nominal interest rate divided by 1 plus the nominal rate, that is, the relative price of money in terms of consumption.

With the CIA constraint binding, real consumption is equal to real money balances. In the steady state, constant consumption implies that the stock of nominal money balances and the price level must be changing at the same rate. Define θ as the growth rate of the nominal quantity of money (so that $T_t = \theta M_t$); then:

$$\bar{\pi} = 1 + \theta.$$

The steady state inflation rate is, as usual, determined by the rate of growth of the nominal money stock.

One difference between the CIA model and the MIU model is that with $\bar{c}$ independent of inflation and the CIA constraint binding, the fact that $\bar{c} = \bar{m}$ in the CIA model implies that the steady state money holdings are also independent of inflation.

10.5.5 Welfare costs of inflation and model dynamics

The CIA model, because it is based explicitly on behavioural relationships consistent with utility maximisation, can be used to assess the welfare costs of inflation and to determine the optimal rate of inflation. The MIU approach had very strong implications for the optimal inflation rate. Steady state utility of the representative household was maximised when the nominal rate of interest equalled zero. It has already been suggested that this conclusion continues to hold when money produces transaction services.

In the basic CIA model, however, there is no optimal rate of inflation that maximises the steady state welfare of the representative household. The reason follows directly from the specification of utility as a function only of consumption and the result that consumption is independent of the rate of inflation (superneutrality). Steady state welfare is equal to:

$$\sum_{t=0}^{\infty} \beta^t u(\bar{c}) = \frac{u(\bar{c})}{1-\beta},$$

and is invariant to the inflation rate. Comparing across steady states, any inflation rate is as good as any other!

This finding is not robust to modifications in the basic CIA model. In particular, once the model is extended to incorporate a labour-leisure choice, consumption will no longer be independent of the inflation rate, and there will be a well defined optimal rate of inflation. Because leisure can be "purchased" without the use of money (i.e. leisure is not subject to the CIA constraint), variations in the rate of inflation will affect the marginal rate of substitution between consumption and leisure. With different inflation rates leading to different levels of steady state consumption and leisure, steady state utility will be a function of inflation. This type of substitution plays an important role in the model of Cooley and Hansen (1989).

In other words, in the Cooley and Hansen (1989) model (which is essentially an RBC model with CIA constraints), including leisure breaks superneutrality. An increase in π will lead to an increase in the relative price of consumption verses leisure, causing a substitution and eventually a decline in employment and consumption. This effect may be ambiguous as in the MIU model, but if the cross elasticity $u_{cm} > 0$, which implies that money and consumption are complements, then an increase in π will lead to a decline in m, a decline in u_c, and a reduction in the supply of labour.

Furthermore, like the MIU model, the results of the Cooley and Hansen model suggest that an increase in money supply lead to an increase in the nominal interest rate (the other effects are qualitatively much like the MIU model, but greater in magnitude).

10.6 A classical monetary model

In this section, we move toward a full-scale DSGE model by introducing a simple model of a classical monetary economy, featuring perfect competition and fully flexible prices in all markets. Many of the predictions from this simple monetary model will not align with the empirical evidence we reviewed in an early section. But nevertheless, the analysis of the simple monetary model provides a benchmark that will be useful later on. To remotivate why we're doing this, the following quote from Lucas is fitting:

> "Nominal variables - the quantity of money, the general price level, and nominal rates of interest - play no role in the Kydland-Prescott model [...] One consequence of this omission is that these theories cannot shed light on the problem of inflation or on the observed associations between movements in money and prices and real economic activity. [...] What I would like to do next, then, is to introduce money into a neoclassical dynamic framework in such a way as to restate in modern terms the quantity theory of money, inflation, and interest." – Lucas (1987)

For this section we will adopt the notation of Galí (2015) – so I'm going to be switching to end of period notation. Also, be careful with variables that are denoted as log levels and log deviations from steady state! Without further ado, let's get started.

10.6.1 Households

The representative household seeks to maximise the objective function:

$$\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s u(C_{t+s}, N_{t+s}; Z_{t+s}),$$

where C_t is consumption of the single good, and N_t denotes hours of work or employment, Z_t is a preference shock, and this maximisation problem is subject to the following flow budget constraints:

$$P_t C_t + Q_t B_t \leq B_{t-1} + W_t N_t - D_t,$$

where P_t is the price of the consumption good, W_t denotes the nominal wage, B_t are holdings of one-period nominally riskless discount bonds purchased in period t and maturing in $t+1$, Q_t is the bond price,[8] and D_t represents dividends to the household expressed in nominal terms. We also assume that the household is subject to a solvency constraint that prevents it from engaging in Ponzi type schemes:

$$\lim_{T \to \infty} \mathbb{E}_t \Lambda_{t,T} \frac{B_T}{P_T} \geq 0, \ \forall t,$$

where $\Lambda_{t,T}$ is the household stochastic discount factor (SDF):

$$\Lambda_{t,T} = \beta^{T-t} \frac{u_{c,T}}{u_{c,t}}.$$

The optimality conditions implied by the household maximisation problem are given by:

$$-\frac{u_{n,t}}{u_{c,t}} = \frac{W_t}{P_t}, \tag{10.41}$$

$$Q_t = \beta \mathbb{E}_t \frac{u_{c,t+1}}{u_{c,t}} \frac{P_t}{P_{t+1}}. \tag{10.42}$$

If we assume CRRA utility,

$$u(C_t, N_t; Z_t) = \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} \right) Z_t,$$

where σ is coefficient of relative risk aversion,[9] and we can assume that $z_t \equiv \ln Z_t$ follows an exogenous AR(1) process:

$$z_t = \rho_z z_{t-1} + \varepsilon_t^z.$$

8. The yield on the one period bond is defined by $Q_t \equiv (1 + \text{yield})^{-1}$. Note that $i_t \equiv -\ln Q_t = \ln(1 + \text{yield}_t) \approx \text{yield}_t$, where the latter approximation will be accurate as long as the nominal yield is "small".

9. Remember that if $\sigma = 1$ then the consumption subutility collapses to log preferences.

Then, the optimality conditions become:

$$\frac{W_t}{P_t} = C_t^{\sigma} N_t^{\varphi}, \tag{10.43}$$

$$Q_t = \beta \mathbb{E}_t \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma} \frac{Z_{t+1}}{Z_t} \frac{P_t}{P_{t+1}}. \tag{10.44}$$

Note, for future reference, (10.43), which is our familiar intratemporal Euler equation (or labour supply schedule) can be written in log-linear form as:

$$w_t - p_t = \sigma c_t + \varphi n_t, \tag{10.45}$$

where lower case variables denote the natural logs of the corresponding variable (e.g., $w_t \equiv \ln W_t$).

We can log-linearise (10.44) by first taking logs, and use $\ln Q_t = q_t = -i_t$ and $\ln \beta = -\rho$:

$$\begin{aligned} \ln Q_t &= \mathbb{E}_t \left[\ln \beta - \sigma \left(\ln C_{t+1} - \ln C_t \right) + \ln Z_{t+1} - \ln Z_t + \ln P_t - \ln P_{t+1} \right] \\ q_t &= \mathbb{E}_t \left[-\rho - \sigma (c_{t+1} - c_t) + z_{t+1} - z_t + p_t - p_{t+1} \right] \\ -i_t &= \mathbb{E}_t \left[-\rho - \sigma (c_{t+1} - c_t) + \Delta z_{t+1} - \pi_{t+1} \right] \\ 0 &= \mathbb{E}_t \left[i_t - \rho - \sigma \Delta c_{t+1} - \Delta z_{t+1} - \pi_{t+1} \right] \\ 1 &= \mathbb{E}_t \exp \left\{ i_t - \rho - \sigma \Delta c_{t+1} - \Delta z_{t+1} - \pi_{t+1} \right\}, \end{aligned} \tag{10.46}$$

where ρ is the household's discount rate, $\rho \equiv -\ln \beta \approx \beta^{-1} - 1$, and $\pi_t = p_t - p_{t-1}$ is the period t inflation rate. Then in the deterministic steady state, suppose that the net nominal interest rate is given by:

$$i = \rho + \pi + \sigma \gamma, \tag{10.47}$$

where γ is the trend growth rate of consumption. Take a first-order Taylor expansion of the exponent about the deterministic steady state to get:

$$\begin{aligned} \exp \left\{ i_t - \rho - \sigma \Delta c_{t+1} - \Delta z_{t+1} - \pi_{t+1} \right\} &\approx 1 + (i_t - i) - \sigma (\Delta c_{t+1} - \gamma) - (\pi_{t+1} - \pi) + (\Delta z_{t+1} - 0) \\ &= 1 + i_t - \rho - \sigma \Delta c_{t+1} - \pi_{t+1} + \Delta z_{t+1}, \end{aligned}$$

by using (10.47). Substitute the above into (10.46), do a bit of rearranging,[10] and you can get the log-linearised expression of the consumption Euler equation about the deterministic steady state (10.47):

$$c_t = \mathbb{E}_t c_{t+1} - \frac{1}{\sigma} (\hat{i}_t - \mathbb{E}_t \pi_{t+1} - \rho) + \frac{1}{\sigma} (1 - \rho_z) z_t, \tag{10.48}$$

Notice that $\hat{i}_t$ corresponds to the log of the gross yield on the one period bond; henceforth, it is referred to as the nominal interest rate.

One more thing we need to do with households is to specify a money demand equation in this

10. Use the fact that $\mathbb{E}_t z_{t+1} = \rho_z z_t$ – a common trick that we will use over and over again.

model. In log-linear form, let's say that the demand for real money balances is given by:

$$m_t - p_t = c_t - \eta i_t, \tag{10.49}$$

where $\eta \geq 0$ denotes the interest semi-elasticity of money demand.[11]

10.6.2 Firms

The supply side of the economy is very simple. A representative firm is assumed whose technology is described by a production function given by:

$$Y_t = A_t N_t^{1-\alpha},$$

where A_t represents the level of technology, and $a_t \equiv \ln A_t$ evolves exogenously according to some AR(1) stochastic process:

$$a_t = \rho_a a_{t-1} + \varepsilon_t^a,$$

where $\rho_a \in [0, 1]$.

Each period the firm maximises profits:

$$P_t Y_t - W_t N_t,$$

subject to the production technology, taking prices and wages as given. The firm's maximisation problem yields the optimality condition:

$$\frac{W_t}{P_t} = (1-\alpha) A_t N_t^{-\alpha}. \tag{10.50}$$

In words, the firm hires labour up to the point where its marginal product equals the real wage. Equivalently, the price of output must equal the marginal cost:

$$P_t = \frac{W_t}{(1-\alpha) A_t N_t^{-\alpha}}.$$

In log-linear terms (10.50) is:

$$w_t - p_t = a_t - \alpha n_t + \ln(1-\alpha). \tag{10.51}$$

This equation can be interpreted as a labour demand schedule, mapping the real wage into the quantity of labour demanded, given the level of technology.

11. We explored motivations for holding real money balances previously in this chapter (the MIU and CIA models). But here we're being a bit ad-hoc with (10.49).

10.6.3 Equilibrium

The baseline model abstracts from sources of goods demand other than consumption (like investment, government purchases, or net exports). Accordingly, the goods market clearing condition is given by:

$$y_t = c_t, \tag{10.52}$$

that is, all output must be consumed.

By combining the optimality conditions of households and firms with the goods market clearing condition and the log-linear aggregate production relationship:

$$y_t = a_t + (1-\alpha)n_t, \tag{10.53}$$

one can determine the equilibrium levels of employment and output:

$$n_t = \psi_{na}a_t + \psi_n \tag{10.54}$$

$$y_t = \psi_{ya}a_t + \psi_y, \tag{10.55}$$

where:

$$\psi_{na} = \frac{1-\sigma}{\sigma(1-\alpha)+\varphi+\alpha}, \quad \psi_n = \frac{\ln(1-\alpha)}{\sigma(1-\alpha)+\varphi+\alpha},$$
$$\psi_{ya} = \frac{1+\varphi}{\sigma(1-\alpha)+\varphi+\alpha}, \quad \psi_y = (1-\alpha)\psi_n.$$

Furthermore, given the equilibrium process for output, (10.48) can be used to determine the implied real interest rate $r_t \equiv i_t - \mathbb{E}_t\pi_{t+1}$, as:

$$\begin{aligned} r_t &= \rho + (1-\rho_z)z_t + \sigma\mathbb{E}_t\Delta y_{t+1} \\ &= \rho + (1-\rho_z)z_t - \sigma(1-\rho_a)\psi_{ya}a_t. \end{aligned} \tag{10.56}$$

Finally, the equilibrium real wage $\omega_t \equiv w_t - p_t$, is given by:

$$\begin{aligned} \hat{\omega}_t &= a_t - \alpha n_t + \ln(1-\alpha) \\ &= \psi_{\omega a}a_t + \psi_\omega, \end{aligned} \tag{10.57}$$

where

$$\psi_{\omega a} = \frac{\sigma+\varphi}{\sigma(1-\alpha)+\varphi+\alpha}, \quad \psi_\omega = \frac{(\sigma(1-\alpha)+\varphi)\ln(1-\alpha)}{\sigma(1-\alpha)+\varphi+\alpha}.$$

Notice that the equilibrium levels of employment, output, and the real interest rate are determined independently of monetary policy. In other words, monetary policy is neutral with respect to those real variables (i.e. money is neutral in the short-run). In the simple model, output, employment, and the real wage fluctuate in response to variations in technology. In particular, output always rises in the face of a productivity increase, with the size of the increase being given by $\psi_{ya} > 0$. The same is true for the real wage. On the other hand, the sign of the employment is ambiguous, depending on whether σ (which measures the strength of the income effect on labour supply) is larger or small than 1. When $\sigma < 1$, the substitution effect on labour supply resulting from a higher real wage dominates the negative effect caused by a smaller marginal utility of consumption, leading to an increase in employment. The converse is true whenever $\sigma > 1$. When the utility of consumption is logarithmic ($\sigma = 1$), employment remains unchanged in the face of technology variations, for substitution and income effects cancel one another. Finally, and under the assumption on the process of technology, the real interest rate goes down in response to a positive technology shock.

What about nominal variables, like inflation or the nominal interest rate? Not surprisingly, and in contrast with real variables, their policy. To illustrate how nominal variables are influenced by the way monetary policy is conducted, their equilibrium behaviour under alternative monetary policy rules will be considered next.

10.6.4 Monetary policy and price level determination

Let us start by examining the implications of some interest rate rules. We will need these rules in order to close our model. Rules that involve monetary aggregates will be introduced later. Recall the Fisher relation:

$$i_t = \mathbb{E}_t \pi_{t+1} + r_t, \tag{10.58}$$

which implies that the nominal rate adjusts one-for-one with expected inflation, given a real interest rate that is determined exclusively by real factors. Equation (10.56) implies that in the steady state without growth $r = \rho$, that is, the real interest rate is equal corresponds to the household's discount rate. Thus it follows from the Fisher relation that in the deterministic steady state:

$$i = \rho + \pi.$$

10.6.5 An exogenous path for the nominal interest rate

Let us first consider the case of a monetary policy that implies an exogenous path for the nominal interest rate. A particular case of this rule corresponds to a constant interest rate, that is, $i_t = i$ for all t. For concreteness, let us assume the rule:

$$i_t = i + v_t, \tag{10.59}$$

where v_t is assumed to follow an exogenous AR(1) process:

$$v_t = \rho_v v_{t-1} + \varepsilon_t^v.$$

The stochastic component ε_t^v is commonly referred to as a "monetary policy shock". We've referred to this in previous chapters, but it is essentially a purely random, unanticipated shock to the nominal interest rate. It could even be interpreted as a mistake or error in the conduct of monetary policy.

A particular case of this simple rule is $v_t = 0, i_t = i, \forall t$. Note that $\pi = i - \rho$ is the steady state inflation (or the implicit long-run inflation target) associated with the rule above. So from the Fisher relation, putting everything together we have:

$$\begin{aligned} i_t &= \mathbb{E}_t \pi_{t+1} + r_t \\ i + v_t &= \mathbb{E}_t \pi_{t+1} + r_t \\ \rho + \pi + v_t &= \mathbb{E}_t \pi_{t+1} + r_t \\ \mathbb{E}_t \pi_{t+1} &= \pi + v_t - \hat{r}_t, \end{aligned}$$

where $\hat{r}_t = r_t - \rho$. While this pins down expected inflation $\mathbb{E}_t \pi_{t+1}$ because $\hat{r}_t$ is a function of α_t, z_t, and deep parameters, it does not pin down actual inflation, π_t. Why? Because any process which satisfies the following:

$$\begin{aligned} \pi_t &= \pi + v_{t-1} - \hat{r}_{t-1} + \xi_t, \\ \Leftrightarrow p_t &= \pi + p_{t-1} + v_{t-1} - \hat{r}_{t-1} + \xi_t, \end{aligned}$$

where $\mathbb{E}_{t-1} \xi_t = 0, \forall t$ is consistent with the equilibrium equations. Shocks such as ξ_t are referred to in the literature as *sunspot shocks*. An equilibrium in which such non-fundamental factors may cause fluctuations in one or more variables is referred to as an indeterminate equilibrium. The example above shows how an exogenous nominal interest rate leads to price level indeterminacy.

Notice also that the nominal wage is also indeterminate in this model, since it is equal to the real wage, ω_t, plus p_t which is indeterminate.

10.6.6 A simple interest rate rule

Suppose that the central bank adjusts the nominal interest rate in response to deviations of inflation from a target, $\bar{\pi}$, according to the interest rate rule:

$$i_t = \rho + \pi + \phi_\pi (\pi_t - \pi) + v_t, \tag{10.60}$$

where $\phi_\pi \geq 0$ is a coefficient determining the strength of the endogenous response of monetary policy, and v_t are monetary policy shocks as before. Combining this rule with the Fisher relation, we have:

$$\phi_\pi \hat{\pi}_t = \mathbb{E}_t \hat{\pi}_{t+1} + \hat{r}_t - v_t, \tag{10.61}$$

where $\hat{\pi}_t = \pi_t - \pi$. If $\phi_\pi > 1$, the previous difference equation has only one nonexplosive solution. that solution can be obtained by solving (10.61) forward which yields:

$$\hat{\pi}_t = \mathbb{E}_t \sum_{k=0}^{\infty} \phi_\pi^{-(k+1)} \left[\hat{r}_{t+k} - v_{t+k} \right].$$

So inflation (and, hence, the price level) is determined by the path of the real interest rate, which in turn is a function of exogenous real forces, as shown in (10.56). Given the processes for preferences and monetary policy, inflation can be written as:

$$\pi_t = \pi - \frac{\sigma(1-\rho_a)\psi_{ya}}{\phi_\pi - \rho_a} a_t + \frac{1-\rho_z}{\phi_\pi - \rho_z} z_t - \frac{1}{\phi_\pi - \rho_v} v_t. \tag{10.62}$$

Note that a central bank's choice of ϕ_π can influence the degree of inflation volatility. The larger that coefficient is, the smaller will be the impact of real shocks on inflation. Monetary policy shocks, v_t, are seen to generate "unnecessary" fluctuations in inflation. Given the equilibrium path for inflation, the price, and nominal wage levels are uniquely determined by the identities $p_t = p_{t-1} + \pi_t$ and $w_t = \omega_t - p_t$.

Also, a monetary policy tightening (positive v_t) reduces inflation and nominal interest rates (if $\rho_v > 0$), since $\partial i_t / \partial \varepsilon_t^v = -\rho_b/(\phi_\pi - \rho_v)$. Assuming that money demand is given by (10.49), the contemporaneous response of the money supply to the tightening is,

$$\frac{\partial m_t}{\partial \varepsilon_t^v} = \frac{\eta \rho_v - 1}{\phi_\pi - \rho_v},$$

which can be negative or positive depending on the value of $\eta\rho_v$. If $\eta\rho_v > 1$, then there will be a liquidity effect in response to a monetary policy tightening.

On the other hand, if $\phi_\pi < 1$, the forward solution of (10.61) does not converge. Instead, the stationary solution to (10.61) takes the form:

$$\pi_t = (1-\phi_\pi)\pi + \phi_\pi \pi_{t-1} - \hat{r}_{t-1} + v_{t-1} + \xi_t, \tag{10.63}$$

where $\{\xi_t\}$ are, again, a sequence of shocks that are related or unrelated to fundamentals, satisfying $\mathbb{E}_{t-1}\xi_t = 0, \forall t$.[12]

Accordingly, any process for π_t satisfying (10.63) is consistent with equilibrium, while remaining in a neighbourhood of the steady state (for sufficiently small shocks). So, as in the case of an exogenous nominal rate, the price level (and, hence, inflation) are not determined uniquely when the interest rate rule implies a weak response of the nominal rate to deviations of inflation from target.

More specifically, the condition for a determinate price level, $\phi_\pi > 1$, requires that the central

12. We could write

$$\xi_t = \tau_a \varepsilon_t^a + \tau_z \varepsilon_t^z + \tau_v \varepsilon_t^v + \xi_t^*,$$

where τ_a, τ_z, and τ_v can take any value (they are indeterminate) and ξ_t^* is a "pure sunspot" shock.

bank adjusts nominal interest rates more than one-for-one in response to any change in inflation, a property known as the Taylor principle. The previous result can be viewed as a particular instance of the need to satisfy the Taylor principle in order for an interest rate rule to bring about a determinate equilibrium.

10.6.7 An exogenous path for the money supply

What about a money supply rule? To eliminate i_t from our model and close it, we need to specify money demand (which we did in (10.49)):

$$m_t = p_t + y_t - \eta i_t,$$

and combine it with the Fisher relation to get:

$$p_t = \left(\frac{\eta}{1+\eta}\right)\mathbb{E}_t p_{t+1} + \left(\frac{1}{1+\eta}\right)m_t + u_t, \tag{10.64}$$

where:

$$u_t = \frac{\eta r_t - y_t}{1+\eta},$$

which evolves independently of m_t.

Assuming that $\eta > 0$ and solving (10.64) forward, we get:

$$\begin{aligned}
p_t &= \frac{1}{1+\eta}\mathbb{E}_t\sum_{i=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^i m_{t+i} + \mathbb{E}_t\sum_{i=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^i u_{t+i} \\
&= \left(1-\frac{\eta}{1+\eta}\right)\mathbb{E}_t\sum_{i=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^i m_{t+i} + \mathbb{E}_t\sum_{i=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^i u_{t+i} \\
&= \mathbb{E}_t\sum_{i=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^i m_{t+i} - \mathbb{E}_t\sum_{i=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^{i+1} m_{t+i} + \mathbb{E}_t\sum_{i=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^i u_{t+i} \\
&= m_t + \mathbb{E}_t\sum_{i=1}^{\infty}\left(\frac{\eta}{1+\eta}\right)^i \Delta m_{t+i} + \mathbb{E}_t\sum_{i=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^i u_{t+i}.
\end{aligned} \tag{10.65}$$

Thus, when the monetary policy rule takes the form of an exogenous path for the money supply, the equilibrium price level is always determined uniquely.

Given our forward iterated solution, the money demand equation (10.49) can be used to solve for the nominal interest rate:

$$\begin{aligned}
i_t &= \frac{1}{\eta}\left[y_t - (m_t - p_t)\right] \\
&= \frac{1}{\eta}\mathbb{E}_t\sum_{i=1}^{\infty}\left(\frac{\eta}{1+\eta}\right)^i \Delta m_{t+i} + \frac{1}{\eta}\left[y_t + \mathbb{E}_t\sum_{i=0}^{\infty}\left(\frac{\eta}{1+\eta}\right)^i u_{t+i}\right].
\end{aligned} \tag{10.66}$$

Consider the case of an AR(1) process for the money supply:

$$\Delta m_t = \rho_m \Delta m_{t-1} + \varepsilon_t^m,$$

then we have:

$$\mathbb{E}_t \Delta m_{t+i} = \rho_m^i \Delta m_t.$$

This would yield a solution for the price level as:

$$p_t = m_t + \frac{\eta \rho_m}{1 + \eta(1 - \rho_m)} \Delta m_t + \mathbb{E}_t \sum_{i=0}^{\infty} \left(\frac{\eta}{1+\eta} \right)^i u_{t+i},$$

which would imply that the price level response more than one-to-one with respect to an increase in the money supply.

The nominal interest rate is in turn given by:

$$i_t = \frac{\rho_m}{1 + \eta(1 - \rho_m)} \Delta m_t + \frac{1}{\eta} \left[y_t + \mathbb{E}_t \sum_{i=0}^{\infty} \left(\frac{\eta}{1+\eta} \right)^i u_{t+i} \right].$$

That is, in response to an expansion of the money supply, and as long as $\rho_m > 0$, the nominal interest rate is predicted to go up. In other words, the model implies the absence of the liquidity effect, in stark contrast with the evidence discussed earlier in the chapter.

10.7 Comments and key readings

This was quite a long and in depth chapter, where we motivated some of the deficiencies of an RBC model, as well as the empirical facts a model with nominal variables should achieve.

We started by looking at the long-run relationship between inflation and the growth of money, where for additional reading refer to McCandless and Weber (1995) and Lucas (1980). Then we looked at the relationship between inflation and growth, where additional recommended readings are Barro (1995, 1996) and Bullard (1999).

Then we moved onto short term responses of output to monetary policy: Key papers are Leeper, Sims, and Zha (1996) and Christiano, Eichenbaum, and Evans (1999), which focus on the role of identified VARs in estimating the effects of monetary policy, and King and Watson (1996), where the focus is on using empirical evidence to distinguish among competing business cycle models. King and Plosser (1984) examined the reverse causation argument – that changes in output lead changes in money. Coleman (1996) confirmed King and Plosser's results and shows that money should be highly correlated with lagged output than future output. Sims (1972, 1980) introduced Granger causality – that is the notion that changes in money Granger-caused changes in GNP.

Critically, we learnt about the price puzzle – that monetary policy shocks lead to an initial small increase in the price level before they have a prolonged decline – and we also discovered the liquidity effect (an increase in money should be followed by a decline in interest rates). Key papers were Chari, Christiano, and Eichenbaum (1995), Christiano, Eichenbaum, and Evans (1999), Sims (1992), and Stock and Watson (2001).

We then looked at three models which essentially just added money to an RBC model: the MIU model by Sidrauski (1967), a model with CIA constraints (Lucas, 1980; Stockman, 1981; Svensson, 1985), and a classical monetary model with some interest rules (as in Galí (2015) – where we also explored some price level determinacy problems). None of these models seem to say much about either the pricing puzzle, nor – more critically – about the liquidity effect. So, we now move onto the New Keynesian model.

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11 The New Keynesian DSGE Model

11.1 Introduction

In the previous sections, we discussed critiques during the 1970s of Keynesian ideas from economists who favoured the use of rational expectations as a modelling device. Recall that following the Lucas Critique and the RBC boom, Keynesian economists had to head back to the drawing board to derive a model in which parameters were independent of shocks, and in which the behaviour of agents were fully rational and microfounded. Furthermore, these models needed to be dynamic. This was in stark contrast to past Keynesian static models such as in the Mundell-Fleming based IS-LM-AD-AS-BoP[1] framework, in which agents' behaviour was mostly ad-hoc for the purposes of fitting macro data.

Many different mechanisms were invoked, but most common was sticky prices. If prices didn't jump in line with money, then central banks could control real money supply and hence real interest rates. In a nutshell, the core focus of the Keynesian school of economics was that policy could affect real economic variables. So in the 1980's, after throwing out their static models, "New Keynesians" brought about their models showing some key theoretical points. See for example *New Keynesian Economics* by Mankiw and Romer (1991), "Monopolistic Competition and the Effects of Aggregate Demand" by Blanchard and Kiyotaki (1987), and Akerlof and Yellen's idea of bounded rationality of firms leading to sticky price dynamics. Then, in the 1990's, the New Keynesian school of economics had an important breakthrough: they developed the New Keynesian Phillips Curve (NKPC) (Roberts, 1995). While it looked a lot like the old Phillips Curve, it featured future expected inflation based on rational expectations. An immense amount of research and literature surrounding New Keynesian economics and the NKPC boomed. The New Keynesian DSGE revolution flourished, essentially kicking the RBC and neoclassical economists to the curb when it came to mainstream macroeconomic theory.

In this section we derive the canonical New Keynesian (NK) model. For more information on the derivation, see textbooks by Galí, Walsh, and Woodford, as well as the seminal paper "The Science of Monetary Policy: A New Keynesian Perspective" by Clarida, Galí, and Gertler (1999). For the sake of sanity, we will initially omit capital and investment in this model (doing so will allow us to derive nice analytical solutions).

The NK model takes an RBC model as its backbone and adds some nominal rigidities – here, we will add sticky prices – which allows shocks to affect the real economy. Namely, it will allow monetary policy shocks to affect real variables. This is different to the simple monetary models or RBC models we looked at before where money was neutral. To get price stickiness in the model, we will assume that some firms are price-setters. As such, we need to move away from perfect competition toward monopolistic competition, where we have a continuum of firms all of which produce a slightly differentiated product and a downward sloping demand curve. To keep things conceptual and digestible, we will split production into two sectors: final goods which

1. Investment-Saving-Liquidity-Money-Aggregate Demand-Aggregate Supply-Balance of Payments.

are produced by perfectly competitive firms (or a single representative firm), and intermediate goods which are produced by the monopolistically competitive firms. So the intermediate firms produce their differentiated output which is then aggregated and combined into a final good for consumption.

All the key mechanisms of the canonical NK model happen with intermediate firms. There are two main ways to model the market power and price stickiness that these monopolistically competitive firms induce: Calvo pricing and Rotemberg pricing. Up to a first order Taylor approximation, these two pricing schemes produce identical results (they result in the same NKPC). We will see that with either Calvo or Rotemberg pricing, we get nice aggregation of the behaviour of the intermediate firms, and it will allow us to derive the NKPC. Without any further ado, let's begin.

11.2 Motivation: Nominal rigidities and the Phillips Curve

We've gone over non-neutrality of money in the previous section, so there's a couple more pieces to go over to motivate our New Keynesian model: nominal rigidities ("sticky prices") and going over the old-school Keynesian favourite: the Phillips Curve.

11.2.1 Evidence of nominal rigidities

Most attempts to uncover evidence on the existence and importance of price rigidities have generally relied on the analysis of micro data, that is, data on the prices of individual goods and services. In an early survey of that research, Taylor (1999) concludes that of price adjustment being about one year. In addition, he points to the very limited evidence of synchronisation of price adjustments, thus providing some justification for the assumption of staggered price setting commonly found in the New Keynesian model. The study of Bils and Klenow (2004), based on the analysis of the average frequencies of price changes for 350 product categories underlying the US CPI called into question that conventional wisdom by uncovering a median duration of prices between 4 and 6 months. Nevertheless, more recent evidence by Nakamura and Steinsson (2008), using data on the individual prices underlying the US CPI and excluding price changes associated with sales, has led to a reconsideration of the Bils-Klenow evidence, with an upward adjustment of the estimated median duration to a range between 8 and 11 months. Dhyne et al. (2006) found similar evidence for the Euro area. It is worth mentioning that, in addition to evidence of substantial price rigidities, most studies find a large amount of heterogeneity in price duration across sectors/types of goods.

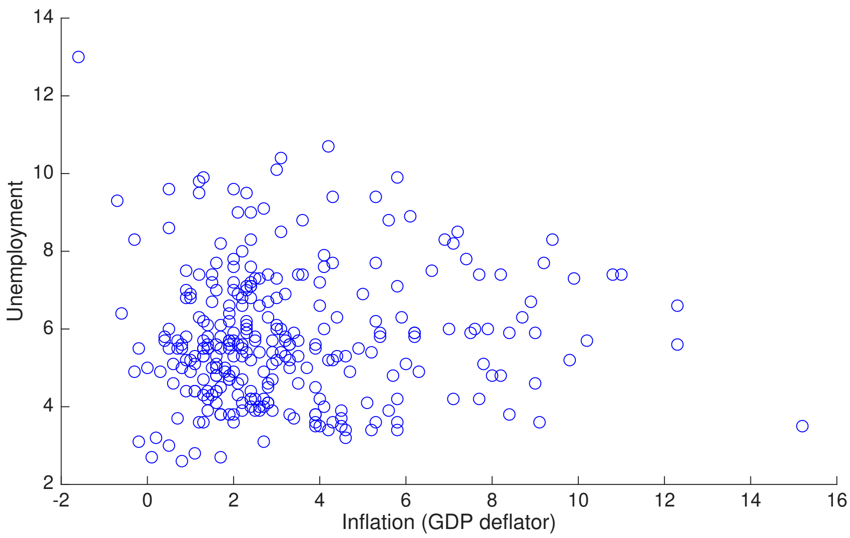
The literature also contains several studies using micro data that provides analogous evidence on nominal rigidities for wages. Taylor (1999) contains an early survey of that literature and suggests an estimate of the average frequency of wage changes of about one year, the same as for prices. A significant branch of the literature on wage rigidities has focused on the possible existence of asymmetries that make wage cuts very rare or unlikely. Bewley (1999) detailed study of firms' wage policies finds ample evidence of downward nominal wage rigidities.

11.2.2 The Phillips Curve (again)

Recall the discussion we had about the Phillips Curve when we explored the Lucas Critique. In a nutshell, to allow for a realistic model of monetary policy, we needed a framework in which prices didn't simply follow the money supply and nominal interest rates don't just move together one-for-one. In this kind of Keynesian model, prices are sticky, so real interest rates can be influenced by the central bank. Real interest rates can affect the performance of the economy, which in turn influences inflation via Phillips Curve relationship (see Figures 10.1 and 10.2).

Recall however that the Phillips Curve has failed empirically:

Figure 11.1: US Inflation and Unemployment (1950-2023)



We discussed briefly some potential reasons for why the Phillips Curve failed – which essentially followed the gist of Lucas' critique of macroeconomics at the time, and followed the speech given by Friedman in his 1967 AEA address – in the long-run, you can't fool the public ($\pi_t^e \approx \pi_t$) so you can't keep unemployment away from its "natural rate" ($U_t \approx U^*$).

Friedman thought that inflation expectations were determined adaptively. For instance, people use last year's inflation rate as a guide to what to expect this year (a rule of thumb approach). If we set $\pi_t^e = \pi_{t-1}$ then the expectations augmented Phillips Curve,

$$\pi_t = \pi_t^e - \gamma(U_t - U^*),$$

becomes:

$$\pi_t = \pi_{t-1} - \gamma(U_t - U^*). \tag{11.1}$$

This relates the change in inflation to the gap between unemployment and its natural rate. When unemployment is below its natural rate, inflation will be increasing; when it is above it, it will be decreasing. Unemployment below the natural rate implies an accelerating price level. The relationship (11.1) is known as the accelerationist Phillips Curve.

In practice, there are a few complications. Inflation expectations are likely to be better captured by a weighted average of past inflation rates rather than just a single lag, implying:

$$\pi_t = \sum_{i=1}^{N} \beta_i \pi_{t-i} - \gamma(U_t - U^*),$$

where $\sum_{i=1}^{N} \beta_i = 1$.

Further, we don't know what the natural rate is, but we can estimate it from:

$$\pi_t = \alpha - \gamma U_t + \sum_{i=1}^{N} \beta_i \pi_{t-i},$$

where if we set:

$$\begin{aligned} \alpha - \gamma U^* &= 0 \\ \implies U^* &= \frac{\alpha}{\gamma}, \end{aligned}$$

which is known as the NAIRU (Non Accelerating Inflation Rate of Unemployment).

But this kind of estimation was precisely in the firing line of the Lucas Critique – which implied that econometric NAIRU estimates were not useful. Furthermore, consider the expectations-augmented Phillips Curve again:

$$\pi_t = \pi_t^e - \gamma(U_t - U^*).$$

We can only have $U_t \neq U^*$ when there is unexpected inflation so $\pi_t \neq \pi_t^e$. If expectations are rational, then these must be random and unpredictable based on publicly available information, so there's no room for systematic predictable (Keynesian) stabilisation.

The Rational Expectations school pioneered a different approach with models based on individual agents pursuing optimising behaviour, and over time many advocates of Rational Expectations came to believe that monetary policy had little to do with business cycles. Many turned to RBC theory. But – as we now well know – the RBC school hit a brick wall, and Keynesians came back with a new model...

In what follows, and since originally writing these notes, I try and stick to the notation of Galí (2015) for things such as symbols for parameters. Also now, more than ever, it's very important to keep track of variables that are levels, log-levels, and log deviations from steady state; and I will try to be as consistent as possible. Being very clear now will save us a lot of sanity as we go forward.[2]

2. Generally, it's also good to develop these habits as you begin to write research papers during your PhD.

11.3 Households

We assume a representative agent household[3] that consumes supplies labour, accumulates bonds, holds shares in firms, and accumulates money. It gets utility from holding real money balances and disutility from working. Its problem is:

$$\max_{\{C_t,N_t,B_t,M_t\}} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left[\frac{C_{t+s}^{1-\sigma}}{1-\sigma} - \varphi_0 \frac{N_{t+s}^{1+\varphi}}{1+\varphi} + \zeta_M \ln\left(\frac{M_{t+s}}{P_{t+s}}\right) \right] Z_{t+s}, \tag{11.2}$$

where we assume that the household has the familiar CRRA preferences with σ being the Arrow-Pratt coefficient of relative risk aversion, and φ is the inverse-Frisch elasticity of labour supply.[4] As before, Z_t is a "preference shock", and it follows a stationary AR(1) process:

$$z_t \equiv \ln Z_t = \rho_z z_{t-1} + \varepsilon_t^z,$$

with $\varepsilon_t^z \overset{\text{IID}}{\sim} \mathcal{N}(0, \sigma_z^2)$.

We've been hand wavy and assumed that utility from holding real money balances is logarithmic. So long as real money balances are additively separable, they don't affect our results. In fact, if we were to assume that the central bank targets the interest rate rather than money supply, then we could ignore real money in utility altogether (which we will do later on). The nominal flow budget constraint of the household is:

$$P_t C_t + B_t + M_t + P_t T_t \leq W_t N_t + M_{t-1} + D_t + R_{t-1} B_{t-1}, \tag{11.3}$$

where money is the numeraire, P_t is the price goods in terms of money, B_{t-1} is the stock of nominal bonds a household enters the period with,[5] and they pay out a gross interest rate of $R_{t-1} = 1 + i_{t-1}$. Households enter period t with nominal money balances of M_{t-1}, earn a nominal wage of W_t on the labour they supply, earn nominal profits D_t remitted to them by firms, and T_t is a lump sum tax paid to the government. Using our timing trick from when we solved the RBC model, the Lagrangian for the household is:

$$\begin{aligned}
\mathcal{L} = {} & \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left[\frac{C_{t+s}^{1-\sigma}}{1-\sigma} - \varphi_0 \frac{N_{t+s}^{1+\varphi}}{1+\varphi} + \zeta_M \ln\left(\frac{M_{t+s}}{P_{t+s}}\right) \right] Z_{t+s} \\
& + \lambda_t \left(W_t N_t + D_t - P_t T_t + R_{t-1} B_{t-1} - P_t C_t - B_t - M_t + M_{t-1} \right) \\
& + \beta \mathbb{E}_t \left[W_{t+1} N_{t+1} + D_{t+1} - P_{t+1} T_{t+1} + R_t B_t - P_{t+1} C_{t+1} - B_{t+1} - M_{t+1} + M_t \right].
\end{aligned}$$

3. So we are working with a representative agent NK (RANK) model, as opposed to the newer heterogeneous-agent NK (HANK) models.

4. $\sigma \geq 0$ and $\varphi \geq 0$ control the curvature of the utility function from consumption and disutility from labour, respectively.

5. Note that I am using end of period notation.

The FOCs – which look very familiar – are:

$$\frac{\partial \mathcal{L}}{\partial C_t} = C_t^{-\sigma} Z_t - \lambda_t P_t = 0, \tag{11.4}$$

$$\frac{\partial \mathcal{L}}{\partial N_t} = -\varphi_0 N_t^{\varphi} Z_t + \lambda_t W_t = 0, \tag{11.5}$$

$$\frac{\partial \mathcal{L}}{\partial B_t} = -\lambda_t + \beta \mathbb{E}_t \lambda_{t+1} R_t = 0, \tag{11.6}$$

$$\frac{\partial \mathcal{L}}{\partial M_t} = \zeta_M \frac{1}{M_t} Z_t - \lambda_t + \beta \mathbb{E}_t \lambda_{t+1} = 0. \tag{11.7}$$

From (11.6) we know that $\lambda_t = \beta \mathbb{E}_t \lambda_{t+1} R_t$, so we can write the FOCs as:

$$C_t^{-\sigma} Z_t = P_t \beta \mathbb{E}_t \lambda_{t+1} R_t, \tag{11.8}$$

$$\varphi_0 N_t^{\varphi} Z_t = \lambda_t W_t, \tag{11.9}$$

$$\zeta_M \frac{1}{M_t} Z_t = \lambda_t - \beta \mathbb{E}_t \lambda_{t+1}. \tag{11.10}$$

Then, from (11.8), like we do in the RBC models, we have:

$$\frac{C_t^{-\sigma}}{P_t} Z_t = \beta \mathbb{E}_t \lambda_{t+1} R_t = \lambda_t,$$

so we can roll one period ahead to get:

$$\frac{C_{t+1}^{-\sigma}}{P_{t+1}} Z_{t+1} = \lambda_{t+1},$$

and combining (11.8) and (11.9) we can get rid of λ_t from our FOCs:

$$\varphi_0 N_t^{\varphi} = \frac{C_t^{-\sigma}}{P_t} W_t, \tag{11.11}$$

$$C_t^{-\sigma} = \frac{P_t}{Z_t} \beta \mathbb{E}_t \frac{C_{t+1}^{-\sigma}}{P_{t+1}} Z_{t+1} R_t = \beta \mathbb{E}_t C_{t+1}^{-\sigma} \frac{P_t}{P_{t+1}} \frac{Z_{t+1}}{Z_t} R_t, \tag{11.12}$$

and for our third FOC:

$$\begin{aligned}
\zeta_M \frac{1}{M_t} Z_t &= \lambda_t - \beta \mathbb{E}_t \lambda_{t+1} \\
&= \frac{C_t^{-\sigma}}{P_t} Z_t - \beta \mathbb{E}_t \frac{C_{t+1}^{-\sigma}}{P_{t+1}} Z_{t+1} \\
\zeta_M \frac{P_t}{M_t} &= C_t^{-\sigma} Z_t - \beta \mathbb{E}_t \frac{P_t C_{t+1}^{-\sigma}}{P_{t+1}} Z_{t+1},
\end{aligned}$$

and since $\frac{C_t^{-\sigma}}{R_t} Z_t = \beta \mathbb{E}_t \frac{P_t C_{t+1}^{-\sigma}}{P_{t+1}} Z_{t+1}$:

$$\begin{aligned}
\zeta_M \frac{P_t}{M_t} &= C_t^{-\sigma} Z_t - \frac{C_t^{-\sigma}}{R_t} Z_t \\
&= \frac{C_t^{-\sigma} R_t Z_t - C_t^{-\sigma} Z_t}{R_t} \\
&= \frac{C_t^{-\sigma} Z_t + i_t C_t^{-\sigma} Z_t - C_t^{-\sigma} Z_t}{R_t} \\
\therefore \zeta_M \left(\frac{M_t}{P_t} \right)^{-1} &= \frac{i_t C_t^{-\sigma}}{R_t} Z_t. \qquad (11.13)
\end{aligned}$$

These FOCs characterise the optimising behaviour of households in the canonical NK model.

11.4 Firms and production

As previously mentioned, we split production into two sectors. There is a representative perfectly competitive final good firm which aggregates intermediate inputs according to a CES technology. But these intermediate goods are imperfect substitutes, which causes the demand for these goods to be downward sloping. Hence, intermediate firms have a degree of market power. Intermediate firms are large in number (we assume a continuum of them), and so they behave as in monopolistic competition. They control their price, but treat other prices as given. These firms produce output using labour and are subject to an aggregate productivity shock. They are not freely able to adjust prices each period, however.

11.4.1 Final goods producer

The final output good is a CES aggregate, utilising the Dixit-Stiglitz aggregator, of a continuum of intermediate goods:

$$Y_t = \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1}}, \quad \epsilon > 0, \qquad (11.14)$$

so final good firms maximise their profits by selecting how much of each intermediate good to purchase, and so their problem is:

$$\max_{Y_t(j)} \left\{ P_t Y_t - \int_0^1 P_t Y_t(j) dj \right\}.$$

The FOC for a typical intermediate good j is:

$$\begin{aligned}
0 &= P_t \frac{\epsilon}{\epsilon - 1} \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{\epsilon}{\epsilon-1} - 1} \left(\frac{\epsilon - 1}{\epsilon} \right) Y_t(j)^{\frac{\epsilon-1}{\epsilon} - 1} - P_t(j) \\
P_t(j) &= P_t \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj \right)^{\frac{1}{\epsilon-1}} Y_t(j)^{-\frac{1}{\epsilon}}
\end{aligned}$$

$$\begin{aligned}
\frac{P_t(j)}{P_t} &= \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj\right)^{\frac{1}{\epsilon-1}} Y_t(j)^{-\frac{1}{\epsilon}} \\
\left[\frac{P_t(j)}{P_t}\right]^{-\epsilon} &= \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj\right)^{-\frac{\epsilon}{\epsilon-1}} Y_t(j) \\
Y_t(j) &= \left[\frac{P_t(j)}{P_t}\right]^{-\epsilon} \left(\int_0^1 Y_t(j)^{\frac{\epsilon-1}{\epsilon}} dj\right)^{\frac{\epsilon}{\epsilon-1}} \\
\implies Y_t(j) &= \left[\frac{P_t(j)}{P_t}\right]^{-\epsilon} Y_t. \qquad (11.15)
\end{aligned}$$

The relative demand for intermediate good j is dependent of j's relative price, with ϵ the price elasticity of demand, and is proportional to aggregate output, Y_t. So, for example, demand for j scales with aggregate economy size.

From Blanchard and Kiyotaki (1987), we can derive a price index for the aggregate economy:

$$P_t Y_t \equiv \int_0^1 P_t(j) Y_t(j) dj.$$

Then, plugging in the demand for good j from (11.15) we have:

$$\begin{aligned}
P_t Y_t &= \int_0^1 P_t(j) \left[\frac{P_t(j)}{P_t}\right]^{-\epsilon} Y_t dj \\
&= \int_0^1 P_t(j) P_t(j)^{-\epsilon} P_t^{\epsilon} Y_t dj \\
&= P_t^{\epsilon} Y_t \int_0^1 P_t(j)^{1-\epsilon} dj \\
P_t^{1-\epsilon} &= \int_0^1 P_t(j)^{1-\epsilon} dj \\
\implies P_t &= \left(\int_0^1 P_t(j)^{1-\epsilon} dj\right)^{\frac{1}{1-\epsilon}}. \qquad (11.16)
\end{aligned}$$

11.4.2 Intermediate producers

A typical intermediate firm produces output according a constant returns to scale technology in labour, with a common productivity shock, A_t:

$$Y_t(j) = A_t N_t(j)^{1-\alpha}. \qquad (11.17)$$

Intermediate firms pay a common wage. They are not freely able to adjust price so as to maximise profit each period, but will always act to minimise cost. The cost minimisation problem is to minimise total cost subject to the constraint producing enough to meet demand (again, see Blanchard and Kiyotaki (1987) for the derivation of this problem):

$$\min_{N_t(j)} \; W_t N_t(j),$$

subject to

$$A_t N_t(j)^{1-\alpha} \geq Y_t(j) = \left[\frac{P_t(j)}{P_t}\right]^{-\epsilon} Y_t.$$

The Lagrangian for an intermediate firm j's problem is:

$$\mathcal{L} = W_t N_t(j) - \Psi_t(j)\left(A_t N_t(j)^{1-\alpha} - \left[\frac{P_t(j)}{P_t}\right]^{-\epsilon} Y_t\right),$$

where $\Psi_t(j)$ is the Lagrangian multiplier for firm j. The FOC is:

$$\frac{\partial \mathcal{L}}{\partial N_t(j)} = W_t - (1-\alpha)\Psi_t(j) A_t N_t(j)^{-\alpha} = 0,$$

which then implies:

$$\Psi_t(j) = \frac{W_t}{(1-\alpha)A_t} N_t(j)^{\alpha} = \frac{W_t}{(1-\alpha)A_t}\left[\frac{Y_t(j)}{A_t}\right]^{\frac{\alpha}{1-\alpha}}. \tag{11.18}$$

Notice that when $\alpha = 0$ neither W_t nor A_t are firm j specific, so in fact we can write $\Psi_t(j)$ as simply Ψ_t.[6] Now, what is the economic interpretation of $\Psi_t(j)$? It is an intermediate firm's nominal marginal cost – how much costs change if you are forced to produce an extra unit of output.

For the case where $\alpha \neq 0$, define the economy-wide average marginal cost as

$$\Psi_t = \frac{W}{(1-\alpha)A_t}\left(\frac{Y_t}{A_t}\right)^{\frac{\alpha}{1-\alpha}}. \tag{11.19}$$

Then we can write:

$$\Psi_t(j) = \Psi_t \left[\frac{Y_t}{Y_t(j)}\right]^{\frac{-\alpha}{1-\alpha}} = \Psi_t \left[\frac{P_t(j)}{P_t}\right]^{\frac{\epsilon\alpha}{\alpha-1}}, \tag{11.20}$$

which shows that firms with high prices have low marginal costs.

Now, formulate the intermediate firm's real flow profit as:

$$\frac{D_t(j)}{P_t} = \frac{P_t(j)}{P_t} Y_t(j) - \frac{W_t}{P_t} N_t(j),$$

6. In other words, if $\alpha = 0$, then you can just skip ahead to the firm's pricing problem under Calvo's sticky prices assumption.

and substitute in the nominal wage from (11.18):

$$\begin{aligned}\frac{D_t(j)}{P_t} &= \frac{P_t(j)}{P_t}Y_t(j) - (1-\alpha)\frac{\Psi_t(j)A_t}{P_t}N_t(j)^{1-\alpha} \\ &= \frac{P_t(j)}{P_t}Y_t(j) - MC_t Y_t(j), \end{aligned} \tag{11.21}$$

where $MC_t(j) = \frac{\Psi_t(j)}{P_t}$ is the real marginal cost for an intermediate firm. Now, buckle up because this is where the fun begins...

11.4.3 Monopolistic competition with Calvo pricing

Firms are not freely able to adjust price each period. In particular, each period there is a fixed probability $1-\theta$ that a firm can adjust its price. This means that firms are unable to change their price with probability θ each period. Since we assume a unit mass of intermediate firms, the probability that a firm is stuck with its price in any given period is θ, θ^2 for two periods, and so on. Since there is a chance that a firm will get stuck with its price for multiple periods, the pricing problem becomes dynamic. After all, suppose if you were an intermediate firm with the chance to change your price:[7] You would want to optimise your pricing decision taking into account all future periods where you are potentially unable to price your output optimally. Thus, firms will need to be smart and discount future profits by some discount factor that is dynamic: Firms will discount s periods into the future by:

$$\Lambda_{t,t+s}\theta^s,$$

where

$$\Lambda_{t,t+s} = \beta^s \frac{u_{C,t+s}}{u_{C,t}},$$

is the household stochastic discount factor.[8] Note that discounting is by both the usual stochastic discount factor as well as by the probability that a price chosen in period t will still be in use in period $t+s$. The dynamic problem of an updating firm can be written as

$$\max_{P_t(j)} \mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_{t,t+s}\theta^s \left(\frac{P_t(j)}{P_{t+s}} \underbrace{\left[\frac{P_t(j)}{P_{t+s}}\right]^{-\epsilon} Y_{t+s}}_{Y_{t+s}(j)} - MC_{t+s}(j) \underbrace{\left[\frac{P_t(j)}{P_{t+s}}\right]^{-\epsilon} Y_{t+s}}_{Y_{t+s}(j)} \right), \tag{11.22}$$

where we assume that output will equal demand. Expanding the terms, we get

$$\max_{P_t(j)} \mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_{t,t+s}\theta^s \left(P_t(j)^{1-\epsilon} P_{t+s}^{\epsilon-1} Y_{t+s} - MC_{t+s}(j) P_t(j)^{-\epsilon} P_{t+s}^{\epsilon} Y_{t+s} \right)$$

7. In other words, you get a visit from the "Calvo fairy", giving you the chance to change your price.
8. Remember, ultimately, the firms are owned by the households. Thus, future profits and dividends are discounted using the household's marginal utility of consumption and discount factor.

$$\Leftrightarrow \max_{P_t(j)} \mathbb{E}_t \sum_{s=0}^{\infty} \left(\Lambda_{t,t+s}\theta^s P_t(j)^{1-\epsilon} P_{t+s}^{\epsilon-1} Y_{t+s} - \Lambda_{t,t+s}\theta^s MC_{t+s}(j)(j) P_t(j)^{-\epsilon} P_{t+s}^{\epsilon} Y_{t+s} \right),$$

and so the FOC is:

$$\begin{aligned} 0 &= \mathbb{E}_t \sum_{s=0}^{\infty} \left[(1-\epsilon)\Lambda_{t,t+s}\theta^s P_t(j)^{-\epsilon} P_{t+s}^{\epsilon-1} Y_{t+s} + \epsilon \Lambda_{t,t+s}\theta^s MC_{t+s} P_t(j)^{-\epsilon-1} P_{t+s}^{\epsilon} Y_{t+s} \right] \\ &= (1-\epsilon) P_t(j)^{-\epsilon} \mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_{t,t+s}\theta^s P_{t+s}^{\epsilon-1} Y_{t+s} + \epsilon P_t(j)^{-\epsilon-1} \mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_{t,t+s}\theta^s MC_{t+s}(j) P_{t+s}^{\epsilon} Y_{t+s}. \end{aligned}$$

Then, move the first summation to the LHS to get:

$$\begin{aligned} (\epsilon-1) P_t(j)^{-\epsilon} \mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_{t,t+s}\theta^s P_{t+s}^{\epsilon-1} Y_{t+s} &= \epsilon P_t(j)^{-\epsilon-1} \mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_{t,t+s}\theta^s MC_{t+s}(j) P_{t+s}^{\epsilon} Y_{t+s} \\ P_t(j) &= \frac{\epsilon}{\epsilon-1} \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_{t,t+s}\theta^s MC_{t+s}(j) P_{t+s}^{\epsilon} Y_{t+s}}{\mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_{t,t+s}\theta^s P_{t+s}^{\epsilon-1} Y_{t+s}} \\ P_t(j) &= \frac{\epsilon}{\epsilon-1} \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \frac{u_{C,t+s}}{u_{C,t}} \theta^s MC_{t+s}(j) P_{t+s}^{\epsilon} Y_{t+s}}{\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \frac{u_{C,t+s}}{u_{C,t}} \theta^s P_{t+s}^{\epsilon-1} Y_{t+s}} \\ P_t(j) &= \frac{\epsilon}{\epsilon-1} \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s u_{C,t+s} \theta^s MC_{t+s}(j) P_{t+s}^{\epsilon} Y_{t+s}}{\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s u_{C,t+s} \theta^s P_{t+s}^{\epsilon-1} Y_{t+s}}. \end{aligned}$$

Now, because we decided to inflict some pain on ourselves and didn't assume a CRTS production technology, we need to get rid of the heterogeneity in marginal costs. Use (11.20) (adjust appropriately to get MC_t) to write the above expression as:

$$\begin{aligned} P_t(j) &= \frac{\epsilon}{\epsilon-1} \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s u_{C,t+s} \theta^s MC_t \left[\frac{P_t(j)}{P_{t+s}} \right]^{\frac{\epsilon\alpha}{\alpha-1}} P_{t+s}^{\epsilon} Y_{t+s}}{\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s u_{C,t+s} \theta^s P_{t+s}^{\epsilon-1} Y_{t+s}} \\ P_t(j)^{1+b} &= \frac{\epsilon}{\epsilon-1} \frac{\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s u_{C,t+s} \theta^s MC_t P_{t+s}^{\epsilon+\frac{\epsilon\alpha}{1-\alpha}} Y_{t+s}}{\mathbb{E}_t \sum_{s=0}^{\infty} \beta^s u_{C,t+s} \theta^s P_{t+s}^{\epsilon-1} Y_{t+s}}, \end{aligned}$$

where $b = \frac{\epsilon\alpha}{1-\alpha}$. It's worth noting that none of the variables on the RHS of the above equation depend on j. This means that any firm able to update their prices will update their prices to the same optimal price, say, $P_t(j) = P_t^*$. We can write P_t^* compactly as:[9]

$$(P_t^*)^{1+b} = \mathcal{M} \frac{X_{1,t}}{X_{2,t}}, \tag{11.23}$$

9. This expression looks gross because of the assumption of DRTS of the firm's production function. If we simply assume $\alpha = 0$, everything will look at lot nicer! Additionally, if we had production subsidies for an efficient steady state, they would show up in the RHS of (11.23) as $(1-\tau^s)$. More on this later.

where $\mathcal{M} = \frac{\epsilon}{\epsilon-1}$ is the optimal markup charged by monopolistically competitive firms, and our auxiliary variables $X_{1,t}$ and $X_{2,t}$ are:

$$X_{1,t} = u_{C,t} MC_t P_t^{\epsilon+b} Y_t + \theta\beta\mathbb{E}_t X_{1,t+1}, \tag{11.24}$$

$$X_{2,t} = u_{C,t} P_t^{\epsilon-1} Y_t + \theta\beta\mathbb{E}_t X_{2,t+1}. \tag{11.25}$$

Notice that the second terms of our auxiliary variables are equal to 0 when $\theta = 0$, i.e., if all firms are able to change their prices freely, then prices are flexible which means that $MC_t P_t = \Psi_t$ and $\Psi_t = \mathcal{M}^{-1}$. This is important to note down.

11.5 Equilibrium and aggregation

To close the model, we need to specify an exogenous process for our technology shocks A_t, some kind of monetary policy rule to determine M_t, and a fiscal rule to determine T_t. As in the previous sections, let the aggregate productivity term follow an AR(1) process such as:

$$a_t \equiv \ln A_t = \rho_a a_{t-1} + \varepsilon_t^a. \tag{11.26}$$

Then, for money, let's suppose that the nominal money supply also follows an AR(1) process in the growth rate:

$$\Delta m_t \equiv \Delta \ln M_t = (1 - \rho_m)\pi + \rho_m \Delta m_{t-1} + \varepsilon_t^m, \tag{11.27}$$

where $\Delta m_t \equiv m_t - m_{t-1}$. Writing the growth of money in this way implies that the mean growth rate of money is equal to the steady state net inflation rate π, as we want real money balances to be stationary so M_t and P_t grow at the same rate in the steady state. For both the law of motion of technology and nominal money, I assume that they contain white noise shock terms such that $\varepsilon_t^a \overset{\text{IID}}{\sim} \mathcal{N}(0, \sigma_a^2)$ and $\varepsilon_t^m \overset{\text{IID}}{\sim} \mathcal{N}(0, \sigma_m^2)$.

In this economy, the government prints money, so it earns seignorage. Right now, we assume that the government does not consume, and that it does not take part in bond markets. The nominal government budget constraint is:

$$0 \leq P_t T_t + M_t - M_{t-1}.$$

In words, the change in the nominal money supply, $M_t - M_{t-1}$ is nominal revenue for the government. Since it does no spending, at equality lump sum taxes must satisfy:

$$T_t = -\frac{M_t - M_{t-1}}{P_t}.$$

So if money growth is positive, e.g. $M_t > M_{t-1}$, then lump sum taxes will be negative – the government will be rebating its seignorage revenue to the households via lump sum transfers.

In equilibrium, bond-holding is always zero in all periods: $B_t = 0$. Using this, plus the relationship between the lump sum tax and money growth derived above, the household budget constrain can be written in real terms:

$$
\begin{aligned}
P_tC_t + B_t + M_t + P_tT_t &\leq W_tN_t + M_{t-1} + D_t + R_{t-1}B_{t-1} \\
\Leftrightarrow P_tC_t + M_t - M_{t-1} + P_t\left(-\frac{M_t - M_{t-1}}{P_t}\right) &\leq W_tN_t + D_t \\
\Leftrightarrow C_t &= \frac{W_tN_t}{P_t} + \frac{D_t}{P_t}.
\end{aligned}
$$

Real dividends received by the household are just the sum of real profits from intermediate goods firms (since the final good firm is competitive and earns no economic profit):

$$
\begin{aligned}
\frac{D_t}{P_t} &= \int_0^1 \left[\frac{P_t(j)}{P_t}Y_t(j) - \frac{W_t}{P_t}N_t(j)\right] dj \\
&= \int_0^1 \frac{P_t(j)}{P_t}Y_t(j)dj - \frac{W_t}{P_t}\int_0^1 N_t(j)dj \\
&= \int_0^1 \frac{P_t(j)}{P_t}Y_t(j)dj - \frac{W_t}{P_t}N_t.
\end{aligned}
$$

So, the household budget constraint becomes:

$$
\begin{aligned}
C_t &= \frac{W_tN_t}{P_t} + \int_0^1 \frac{P_t(j)}{P_t}Y_t(j)dj - \frac{W_tN_t}{P_t} \\
\implies C_t &= \int_0^1 \frac{P_t(j)}{P_t}Y_t(j)dj,
\end{aligned}
$$

and since:

$$
Y_t(j) = \left[\frac{P_t(j)}{P_t}\right]^{-\epsilon} Y_t,
$$

we have:

$$
\begin{aligned}
C_t &= \int_0^1 \frac{P_t(j)}{P_t}\left[\frac{P_t(j)}{P_t}\right]^{-\epsilon} Y_t dj \\
&= \int_0^1 P_t(j)^{1-\epsilon}P_t^{\epsilon-1}Y_t dj \\
&= P_t^{\epsilon-1}Y_t\int_0^1 P_t(j)^{1-\epsilon}dj,
\end{aligned}
$$

but $\int_0^1 P_t(j)^{1-\epsilon}dj = P_t^{1-\epsilon}$ from (11.16), so the P_t terms drop out and we have the market clearing condition:

$$
C_t = Y_t \tag{11.28}
$$

Now, we need to solve for Y_t. But we first need to get aggregate labour demand by firms, N_t:

(11.15) and (11.17) give:

$$N_t(j) = \left[\frac{Y_t(j)}{A_t}\right]^{\frac{1}{1-\alpha}}.$$

Aggregate this across firms to then get

$$N_t \equiv \int_0^1 N_t(j)dj = \int_0^1 \left[\frac{Y_t(j)}{A_t}\right]^{\frac{1}{1-\alpha}} dj,$$

then substitute in for $Y_t(j)$ using the demand for intermediate goods:

$$N_t = \int_0^1 \left\{ \frac{\left[\frac{P_t(j)}{P_t}\right]^{-\epsilon} Y_t}{A_t} \right\}^{\frac{1}{1-\alpha}} dj. \tag{11.29}$$

Now, do a bit of algebra and solve for Y_t:

$$\begin{aligned} Y_t &= \frac{A_t N_t^{1-\alpha}}{\underbrace{\left\{\int_0^1 \left[\frac{P_t(j)}{P_t}\right]^{-\frac{\epsilon}{1-\alpha}} dj\right\}^{1-\alpha}}_{V_t^P}} \\ &= \frac{A_t N_t^{1-\alpha}}{V_t^P}. \end{aligned} \tag{11.30}$$

The new variable we have defined, V_t^P, is a measure of price dispersion. If there were no pricing frictions, all firms would charge the same price, and $V_t^P = 1$. If prices are different, one can show that this expression is bound from below by unity. Since $V_t^P \geq 1$, price dispersion leads to lower output. The economy produces less than it otherwise would given A_t and aggregate labour input if prices are disperse. This is the gist for why price stability is a good thing.

Our full set of equilibrium conditions are:

$$\begin{aligned} C_t^{-\sigma} &= \beta \mathbb{E}_t \frac{C_{t+1}^{-\sigma} R_t P_t Z_{t+1}}{P_{t+1} Z_t}, \\ \varphi_0 N_t^{\varphi} &= C_t^{-\sigma} \frac{W_t}{P_t}, \\ \frac{M_t}{P_t} &= \zeta_M \frac{R_t}{i_t Z_t} C_t^{\sigma}, \\ MC_t &= \frac{W_t/P_t}{(1-\alpha)A_t (Y_t/A_t)^{\frac{-\alpha}{1-\alpha}}}, \\ C_t &= Y_t, \\ Y_t &= \frac{A_t N_t^{1-\alpha}}{V_t^P}, \end{aligned}$$

$$
\begin{aligned}
V_t^P &= \left\{ \int_0^1 \left[\frac{P_t(j)}{P_t} \right]^{-\frac{\epsilon}{1-\alpha}} dj \right\}^{1-\alpha}, \\
P_t^{1-\epsilon} &= \int_0^1 P_t(j)^{1-\epsilon} dj, \\
(P_t^*)^{1+\frac{\epsilon\alpha}{1-\alpha}} &= \mathcal{M} \frac{X_{1,t}}{X_{2,t}}, \\
X_{1,t} &= C_t^{-\sigma} Z_t MC_t P_t^{\epsilon + \frac{\epsilon\alpha}{1-\alpha}} Y_t + \theta\beta \mathbb{E}_t X_{1,t+1}, \\
X_{2,t} &= C_t^{-\sigma} Z_t P_t^{\epsilon-1} Y_t + \theta\beta \mathbb{E}_t X_{2,t+2}, \\
\ln Z_t &= \rho_z \ln Z_{t-1} + \varepsilon_t^z, \\
\ln A_t &= \rho_a \ln A_{t-1} + \varepsilon_t^a, \\
\Delta m_t &= (1-\rho_m)\pi + \rho_m \Delta m_{t-1} + \varepsilon_t^m, \\
\Delta m_t &= m_t - m_{t-1}.
\end{aligned}
$$

This is 15 equations in 15 aggregate variables. But there are three issues with the way we have written up this system of equations: 1) we have heterogeneity (j shows up); 2) the price level shows up and it isn't stationary; and, 3) nominal money growth shows up and it isn't stationary. So we will rewrite these conditions using Calvo pricing to get rid of the j terms, using inflation instead of price levels. We also need to ensure that trending variables are detrended – for example, we will need to ensure that M_t is divided through by another trending variable such as P_t.

11.5.1 Re-writing the equilibrium conditions

Begin by rewriting gross inflation as $\Pi_t = 1 + \pi_t = \frac{P_t}{P_{t-1}}$. The consumption Euler equation can be re-written as:

$$
C_t^{-\sigma} = \beta \mathbb{E}_t C_{t+1}^{-\sigma} R_t \Pi_{t+1} \frac{Z_{t+1}}{Z_t}.
$$

The demand for money equation is already written in terms of real money balances, M_t/P_t, which is stationary so it's fine:

$$
\frac{M_t}{P_t} = \zeta_M \frac{R_t}{i_t Z_t} C_t^{\sigma}.
$$

Now we need to get rid of the j terms in the price level and price dispersion expansions. The expression for the price level is:

$$
P_t^{1-\epsilon} = \int_0^1 P_t(j)^{1-\epsilon} dj.
$$

Now, recall that a fraction $1-\theta$ of these firms will update their price – after a visit from the Calvo fairy – to the same optimal price, P_t^*. The other fraction θ will charge the price they charged in the previous period. Since it doesn't matter how we "order" these firms along the unit interval, this means we can break up the integral on the RHS above as:

$$\begin{aligned} P_t^{1-\epsilon} &= \int_0^{1-\theta} \left(P_t^*\right)^{1-\epsilon} dj + \int_{1-\theta}^1 P_{t-1}(j)^{1-\epsilon} dj \\ \Leftrightarrow P_t^{1-\epsilon} &= (1-\theta)\left(P_t^*\right)^{1-\epsilon} + \int_{1-\theta}^1 P_{t-1}(j)^{1-\epsilon} dj. \end{aligned}$$

Now, watch the Calvo magic. Because the firms who get to update are randomly chosen, and because there are a large number of firms, the integral of individual prices over some subset of the unit interval will simply be proportional to the integral over the entire unit interval, where the proportion is equal to the subset of the unit interval over which the integral is taken. This means:

$$\int_{1-\theta}^1 P_t(j)^{1-\epsilon} dj = \theta \int_0^1 P_{t-1}(j)^{1-\epsilon} dj = \theta P_{t-1}^{1-\epsilon}.$$

Therefore we have

$$P_t^{1-\epsilon} = (1-\theta)\left(P_t^*\right)^{1-\epsilon} + \theta P_{t-1}^{1-\epsilon}.$$

Tada! We've gotten rid of the heterogeneity. The Calvo assumption allows us to integrate out the heterogeneity and not worry about keeping track of what each firm is doing from the perspective of looking at the behaviour of aggregates. Now, we want to write things in terms of inflation, so divide both sides by $P_{t-1}^{1-\epsilon}$, and define $\Pi_t^* = 1 + \pi_t^* = \frac{P_t^*}{P_{t-1}}$ as "optimal price inflation":

$$\begin{aligned} \frac{P_t^{1-\epsilon}}{P_{t-1}^{1-\epsilon}} &= (1-\theta)\frac{\left(P_t^*\right)^{1-\epsilon}}{P_{t-1}^{1-\epsilon}} + \theta \frac{P_{t-1}^{1-\epsilon}}{P_{t-1}^{1-\epsilon}} \\ \Leftrightarrow \Pi_t^{1-\epsilon} &= (1-\theta)(\Pi_t^*)^{1-\epsilon} + \theta. \end{aligned} \tag{11.31}$$

Now, look at the price dispersion term. Notice we can use the same Calvo trick we used above here:

$$\begin{aligned} (V_t^P)^{\frac{1}{1-\alpha}} &= \int_0^1 \left[\frac{P_t(j)}{P_t}\right]^{-\frac{\epsilon}{1-\alpha}} dj \\ &= \int_0^{1-\theta} \left[\frac{P_t(j)}{P_t}\right]^{-\frac{\epsilon}{1-\alpha}} dj + \int_{1-\theta}^1 \left[\frac{P_{t-1}(j)}{P_t}\right]^{-\frac{\epsilon}{1-\alpha}} dj \\ &= \int_0^{1-\theta} \left(\frac{P_t^*}{P_t}\right)^{-\epsilon} dj + \int_{1-\theta}^1 \left[\frac{P_{t-1}(j)}{P_t}\right]^{-\frac{\epsilon}{1-\alpha}} dj \\ &= \int_0^{1-\theta} \left(\frac{P_t^*}{P_{t-1}}\right)^{-\frac{\epsilon}{1-\alpha}} \left(\frac{P_{t-1}}{P_t}\right)^{-\frac{\epsilon}{1-\alpha}} dj + \int_{1-\theta}^1 \left[\frac{P_{t-1}(j)}{P_{t-1}}\right]^{-\frac{\epsilon}{1-\alpha}} \left(\frac{P_{t-1}}{P_t}\right)^{-\frac{\epsilon}{1-\alpha}} dj \\ &= (1-\theta)\left(\frac{P_t^*}{P_{t-1}}\right)^{-\frac{\epsilon}{1-\alpha}} \left(\frac{P_t}{P_{t-1}}\right)^{\frac{\epsilon}{1-\alpha}} + \left(\frac{P_t}{P_{t-1}}\right)^{\frac{\epsilon}{1-\alpha}} \int_{1-\theta}^1 \left[\frac{P_{t-1}(j)}{P_{t-1}}\right]^{-\frac{\epsilon}{1-\alpha}} dj \\ &= (1-\theta)(\Pi_t^*)^{-\frac{\epsilon}{1-\alpha}} \Pi_t^{\frac{\epsilon}{1-\alpha}} + \Pi_t^{\frac{\epsilon}{1-\alpha}} \int_{1-\theta}^1 \left[\frac{P_{t-1}(j)}{P_{t-1}}\right]^{-\frac{\epsilon}{1-\alpha}} dj, \end{aligned}$$

and use the Calvo trick, and the definition of price dispersion, on the last term of the RHS to get:

$$(V_t^P)^{\frac{1}{1-\alpha}} = (1-\theta)(\Pi_t^*)^{-\frac{\epsilon}{1-\alpha}}\Pi_t^{\frac{\epsilon}{1-\alpha}} + \theta\Pi_t^{\frac{\epsilon}{1-\alpha}}(V_{t-1}^P)^{\frac{1}{1-\alpha}}.$$

Now, adjust the reset price expression. First, divide the following auxiliary variables by P_t raised to the relevant powers:

$$\begin{aligned}
\frac{X_{1,t}}{P_t^{\epsilon+b}} &= \frac{C_t^{-\sigma}Z_t MC_t P_t^{\epsilon+b}Y_t}{P_t^{\epsilon+b}} + \theta\beta\frac{\mathbb{E}_t X_{1,t+1}}{P_t^{\epsilon+b}}, \\
\frac{X_{2,t}}{P_t^{\epsilon-1}} &= \frac{C_t^{-\sigma}Z_t P_t^{\epsilon-1}Y_t}{P_t^{\epsilon-1}} + \theta\beta\frac{\mathbb{E}_t X_{2,t+1}}{P_t^{\epsilon-1}}.
\end{aligned}$$

Multiplying and dividing the second terms on the RHS by P_{t+1} to the right powers yields:

$$\begin{aligned}
\frac{X_{1,t}}{P_t^{\epsilon+b}} &= C_t^{-\sigma}Z_t MC_t Y_t + \theta\beta\mathbb{E}_t\frac{X_{1,t+1}}{P_{t+1}^{\epsilon+b}}\left(\frac{P_{t+1}}{P_t}\right)^{\epsilon+b} \\
&= C_t^{-\sigma}Z_t MC_t Y_t + \theta\beta\mathbb{E}_t\Pi_{t+1}^{\epsilon+b}\frac{X_{1,t+1}}{P_{t+1}^{\epsilon+b}}, \\
\frac{X_{2,t}}{P_t^{\epsilon-1}} &= C_t^{-\sigma}Z_t Y_t + \theta\beta\mathbb{E}_t\frac{X_{2,t+1}}{P_{t+1}^{\epsilon-1}}\left(\frac{P_{t+1}}{P_t}\right)^{\epsilon-1} \\
&= C_t^{-\sigma}Z_t Y_t + \theta\beta\mathbb{E}_t\Pi_{t+1}^{\epsilon-1}\frac{X_{2,t+1}}{P_{t+1}^{\epsilon-1}}.
\end{aligned}$$

So the reset price expression can now be written as:

$$(P_t^*)^{1+b} = \mathcal{M}\frac{X_{1,t}/P_t^{\epsilon+b}}{X_{2,t}/P_t^{\epsilon-1}}P_t^{1+b},$$

and by dividing both sides by P_{t-1}^{1+b} we can write this in terms of inflation:

$$(\Pi_t^*)^{1+b} = \mathcal{M}\frac{X_{1,t}/P_t^{\epsilon+b}}{X_{2,t}/P_t^{\epsilon-1}}\Pi_t^{1+b}.$$

The process for real money balances can be converted into real terms quite easily:

$$\Delta m_t = m_t - m_{t-1},$$

and then do some add and subtraction:

$$\begin{aligned}
\Delta m_t &= m_t - m_{t-1} + p_t - p_t + p_{t-1} - p_{t-1} \\
&= m_t - p_t - (m_{t-1} - p_{t-1}) + p_t - p_{t-1} \\
&= \Delta(m_t - p_t) + \pi_t.
\end{aligned}$$

If you really want to, you could make some new auxiliary variables to easily track the real variables from their nominal counterparts.[10] So we can write the process for money growth in terms of real balance growth as:

$$\Delta(m_t - p_t) + \pi_t = (1-\rho_m)\pi + \rho_m \Delta(m_{t-1} - p_{t-1}) + \rho_m \pi_{t-1} + \varepsilon_t^m.$$

The full set of rewritten equilibrium conditions is:

$$\begin{aligned}
C_t^{-\sigma} &= \beta \mathbb{E}_t C_{t+1}^{-\sigma} \Pi_{t+1} R_t \frac{Z_{t+1}}{Z_t}, \\
\varphi_0 N_t^{\varphi} &= C_t^{-\sigma} \frac{W_t}{P_t}, \\
\frac{M_t}{P_t} &= \zeta_M \frac{R_t}{i_t Z_t} C_t^{\sigma}, \\
MC_t &= \frac{W_t/P_t}{(1-\alpha) A_t (Y_t/A_t)^{\frac{-\alpha}{1-\alpha}}}, \\
C_t &= Y_t, \\
Y_t &= \frac{A_t N_t^{1-\alpha}}{V_t^P}, \\
(V_t^P)^{\frac{1}{1-\alpha}} &= (1-\theta)(\Pi_t^*)^{-\frac{\epsilon}{1-\alpha}} \Pi_t^{\frac{\epsilon}{1-\alpha}} + \theta \Pi_t^{\frac{\epsilon}{1-\alpha}} (V_{t-1}^P)^{\frac{1}{1-\alpha}}, \\
\Pi_t^{1-\epsilon} &= (1-\theta)(\Pi_t^*)^{1-\epsilon} + \theta, \\
(\Pi_t^*)^{1+b} &= \mathcal{M} \frac{X_{1,t}/P_t^{\epsilon+b}}{X_{2,t}/P_t^{\epsilon-1}} \Pi_t^{1+b}, \\
\frac{X_{1,t}}{P_t^{\epsilon+b}} &= C_t^{-\sigma} MC_t Y_t + \theta\beta \mathbb{E}_t \Pi_{t+1}^{\epsilon+b} \frac{X_{1,t+1}}{P_{t+1}^{\epsilon+b}}, \\
\frac{X_{2,t}}{P_t^{\epsilon-1}} &= C_t^{-\sigma} Y_t + \theta\beta \mathbb{E}_t \Pi_{t+1}^{\epsilon-1} \frac{X_{2,t+1}}{P_{t+1}^{\epsilon-1}}, \\
z_t &= \rho_z z_{t-1} + \varepsilon_t^z, \\
a_t &= \rho_a a_{t-1} + \varepsilon_t^a, \\
\Delta m_t &= \Delta(m_t - p_t) + \pi_t, \\
\Delta(m_t - p_t) + \pi_t &= (1-\rho_m)\pi + \rho_m \Delta(m_{t-1} - p_{t-1}) + \rho_m \pi_{t-1} + \varepsilon_t^m.
\end{aligned}$$

11.6 The steady state

We now solve for the non-stochastic steady state of the model. We have $A = 1$, and since output and consumption are always equal, it must be that $Y = C$. Steady state inflation is equal to the exogenous target, π, which we will assume to be 0. In other words, in what follows, we will assume that there is no trend inflation in the steady state.

10. For example, you could define, say, $M_t^r = M_t/P_t$ or $\tilde{X}_{1,t} = X_{1,t}/P_t^{\epsilon+b}$.

Next, from the consumption Euler equation, we have:

$$\begin{aligned} C^{-\sigma} &= \beta C^{-\sigma}\Pi R \frac{Z}{Z} \\ \implies R &= \frac{1+\pi}{\beta} \\ \Leftrightarrow 1+i &= \frac{1+\pi}{\beta} \\ \implies i &= \rho + \pi, \end{aligned} \tag{11.32}$$

where

$$\beta = \frac{1}{1+\rho}.$$

(11.32) is the familiar Fisher equation, and ρ in the expression for β is the discount rate (whereas β is the discount factor), and is also referred to as the net real interest rate.

From the price evolution equation, we can derive the steady state expression for reset price inflation:

$$\begin{aligned} \Pi^{1-\epsilon} &= (1-\theta)(\Pi^*)^{1-\epsilon} + \theta \\ \frac{\Pi^{1-\epsilon} - \theta}{1-\theta} &= (\Pi^*)^{1-\epsilon} \\ \implies \Pi^* &= \left(\frac{\Pi^{1-\epsilon} - \theta}{1-\theta}\right)^{\frac{1}{1-\epsilon}}. \end{aligned} \tag{11.33}$$

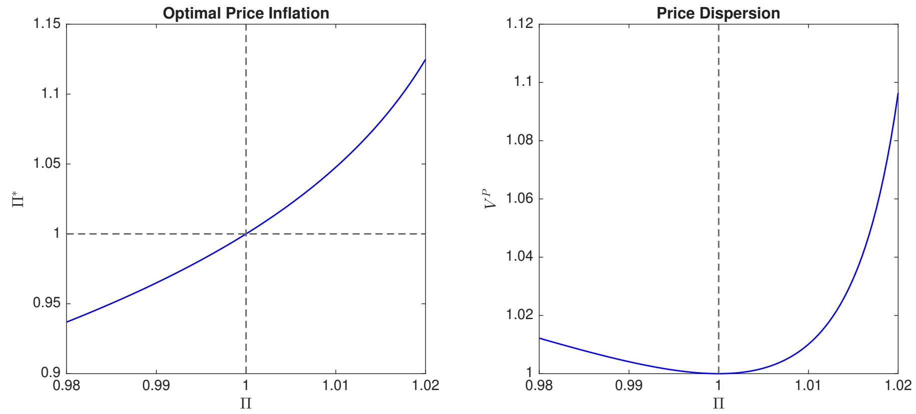
If $\Pi = 1$, then $\Pi^* = \Pi$, since the RHS of the above expression is equal to 1. If $\Pi > 1 \implies \Pi^* > \Pi$, and if $\Pi < 1 \implies \Pi^* < \Pi$. With this in hand, we can solve for steady state price dispersion:

$$\begin{aligned} (V^P)^{\frac{1}{1-\alpha}} &= (1-\theta)(\Pi^*)^{-\frac{\epsilon}{1-\alpha}}\Pi^{\frac{\epsilon}{1-\alpha}} + \theta\Pi^{\frac{\epsilon}{1-\alpha}}(V^P)^{\frac{1}{1-\alpha}} \\ \left(1 - \theta\Pi^{\frac{\epsilon}{1-\alpha}}\right)(V^P)^{\frac{1}{1-\alpha}} &= \frac{(1-\theta)(\Pi)^{\frac{\epsilon}{1-\alpha}}}{(\Pi^*)^{\frac{\epsilon}{1-\alpha}}}. \end{aligned} \tag{11.34}$$

If $\Pi = 1$, then $V^P = 1$. If $\Pi \neq 1$, then $V^P > 1$. Figure 11.2 contains two plots. The left plot maps steady state inflation (Π) and steady state optimal price inflation (Π^*). We can see that steady state reset price inflation is less than steady state inflation for negative steady state inflation, and greater than steady state inflation for positive steady state inflation.[11] The right plot maps steady state inflation against steady state price dispersion. We see that steady state price dispersion is equal to unity for when steady state inflation is equal to zero, and is rising for any value of steady state inflation not equal to zero.

11. This is an awful sentence. Essentially we are confirming what we found in the equation for the steady set reset price inflation.

Figure 11.2: Steady State Optimal Inflation and Price Dispersion



$\epsilon = 10, \theta = 0.75, \alpha = 0$

Now, we can solve for the steady state ratio of $(X_1/P^{\epsilon+b})/(X_2/P^{\epsilon-1})$:

$$\frac{X_1/P^{\epsilon+b}}{X_2/P^{\epsilon-1}} = \mathcal{M}^{-1}\left(\frac{\Pi^*}{\Pi}\right)^{1+b}. \tag{11.35}$$

Then take the equation for the auxiliary variable $X_{1,t}/P_t^{\epsilon+b}$, and do some rearranging:

$$\begin{aligned}\frac{X_1}{P^{\epsilon+b}}(1-\theta\beta\Pi^{\epsilon+b}) &= MC\frac{Y}{C} \\ \frac{X_1}{P^{\epsilon+b}} &= MC\frac{Y}{C}(1-\theta\beta\Pi^{\epsilon+b})^{-1},\end{aligned}$$

and do the same for $X_{2,t}/P_t^{\epsilon-1}$:

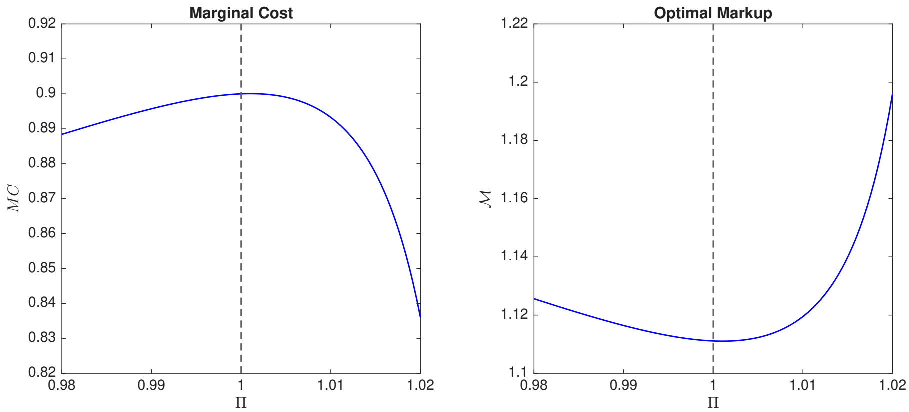
$$\frac{X_2}{P^{\epsilon-1}} = \frac{Y}{C}(1-\theta\beta\Pi^{\epsilon-1})^{-1}.$$

So then we can divide the two:

$$\begin{aligned}\frac{X_1/P^{\epsilon+b}}{X_2/P^{\epsilon-1}} &= MC\frac{1-\theta\beta\Pi^{\epsilon-1}}{1-\theta\beta\Pi^{\epsilon+b}} \\ MC &= \mathcal{M}^{-1}\left(\frac{\Pi^*}{\Pi}\right)^{1+b}\frac{(1-\theta\beta\Pi^{\epsilon+b})}{(1-\theta\beta\Pi^{\epsilon-1})}.\end{aligned} \tag{11.36}$$

In words, the steady state real marginal cost is inverse to the price markup. If $\Pi = 1$, then $MC = \mathcal{M}^{-1} = \frac{\epsilon-1}{\epsilon}$. In other words, if steady state net inflation is zero, then the steady state markup will be what it would be if prices were fully flexible (also corresponding to $\theta = 0$). If $\Pi \neq 1$, then $MC < \frac{\epsilon-1}{\epsilon}$, which means that the steady state markup will be higher than it would be if net inflation were zero.

Figure 11.3: Steady State Marginal Cost and Markup



$\epsilon = 10, \phi = 0.75, \beta = 0.99, \alpha = 0$

With the steady steady state marginal cost in hand, we now look at the labour supply condition. We have that

$$MC = \frac{W}{P} Y^{\frac{\alpha}{1-\alpha}},$$

since $A = 1$. The lower is the marginal cost, the bigger is the wedge between the wage and the marginal product of labour (i.e., the more distorted the economy is). Then we have:

$$\begin{aligned}
\varphi_0 N^{\varphi} &= \frac{W}{C^{\sigma} P} \\
\Leftrightarrow \varphi_0 N^{\varphi} &= \frac{W}{Y^{\sigma} P},
\end{aligned}$$

and since $Y = \frac{AN^{1-\alpha}}{V^P}$, we have:

$$\begin{aligned}
\varphi_0 N^{\varphi} &= \frac{W}{P} \left(\frac{AN^{1-\alpha}}{V^P} \right)^{-\sigma} \\
&= \frac{W}{P} \left(V^P\right)^{\sigma} N^{-\sigma(1-\alpha)} \\
&= \frac{MC}{Y^{\frac{\alpha}{1-\alpha}}} \left(V^P\right)^{\sigma} N^{-\sigma(1-\alpha)} \\
&= MC (V^P)^{\sigma} N^{-\sigma(1-\alpha)} \left[\frac{N^{1-\alpha}}{V^P} \right]^{-\frac{\alpha}{1-\alpha}} \\
\varphi_0 N^{\sigma(1-\alpha)+\alpha+\varphi} &= MC (V^P)^{\frac{\sigma(1-\alpha)+\alpha}{1-\alpha}} \\
\therefore N &= \left[\frac{MC}{\varphi_0} (V^P)^{\frac{\sigma(1-\alpha)+\alpha}{1-\alpha}} \right]^{\frac{1}{\sigma(1-\alpha)+\alpha+\varphi}}. \qquad (11.37)
\end{aligned}$$

Finally, since we have Y, steady state M/P is easy:

$$\frac{M}{P} = \frac{\zeta_M R}{i} Y^{\sigma}. \tag{11.38}$$

11.7 The flexible price equilibrium

We now consider the hypothetical equilibrium case where all prices are flexible (i.e., when $\theta = 0$). But, even under flexible prices, we still have monopolistic competition. Since we have no endogenous state variables when prices are flexible, we can solve for the flex price equilibrium by hand. In this section superscript n denotes the hypothetical flex price allocation.

When $\theta = 0$, firms will always price their goods optimally to P_t^*. Then, from the price dispersion equation we have:

$$V_t^{P,n} = \left(\frac{\Pi^*}{\Pi}\right)^{-\epsilon} = 1, \tag{11.39}$$

and combining this result with the (11.23) we have

$$\frac{P_t^*}{P_t} = \mathcal{M} MC_t(j).$$

But when prices are flexible, the equilibrium is symmetric and we have: $P_t^* = P_t$, $MC_t(j) = \mathcal{M}^{-1}$, $Y_t(j) = Y_t$, and $N_t(j) = N_t$, $\forall j$. In words, if prices are flexible, all firms charge the same price, and price dispersion is at its lower bound of 1 and marginal costs are constant. Since marginal cost is the inverse of the price markup, we can say that under a flex price equilibrium, firms will set prices equal to a fixed markup over marginal cost.

Then, from the average marginal cost equation (11.19), we have

$$\frac{W_t^n}{P_t} = \mathcal{M}^{-1}(1-\alpha)A_t(N_t^n)^{-\alpha}.$$

Use this with the labour supply condition to get:

$$\varphi_0 \frac{(N_t^n)^{\varphi}}{(C_t^n)^{-\sigma}} = \frac{W_t^n}{P_t} = \mathcal{M}^{-1}(1-\alpha)A_t(N_t^n)^{-\alpha}.$$

The aggregate resource constraint will give us:

$$C_t^n = Y_t^n = A_t(N_t^n)^{1-\alpha}.$$

These equations will allow us to price the flexible price – or "natural" – level of output This implies that the flexible price output is:

$$Y_t^n = \left(\frac{1-\alpha}{\varphi_0 \mathcal{M}}\right)^{\frac{1-\alpha}{\sigma(1-\alpha)+\alpha+\varphi}} A_t^{\frac{1+\varphi}{\sigma(1-\alpha)+\alpha+\varphi}}. \tag{11.40}$$

Note that if $\sigma = 1$, then N_t^n is a constant and not a function of A_t. In other words, if prices are flexible and $\sigma = 1$ (meaning we have log utility), labour hours would not react to technology shocks A_t. What is driving this is that, if $\sigma = 1$, then preferences are consistent with King, Plosser, and Rebelo (1988) preferences,[12] in which the income and substitution effects of changes in A_t exactly offset. When there is capital in the model, this offset only occurs in the long-run, so that labour hours are constant in the long-run, but not in the short-run as capital adjusts to steady state. Without capital, the cancellation of income and substitution effects holds at all times.

Also note that in the flex price equilibrium, nominal shocks have no real effects. This makes sense as we no longer have any nominal rigidities or stickiness. We still have monopolistic competition, but all the monopolistically competitive firms (the intermediate firms) are able to optimise their prices each and every period without any cost.

11.8 Quantitative analysis

We can solve the model quantitatively in Dynare using a first order approximation about the steady state. Below is some MATLAB code or the Dynare mod file. I calibrate the model using the parameter values as in Galí (2015).[13] Also, full credit to Dynare extraordinaire Johannes Pfeifer (who is also a great macroeconomist!) for doing the original replication – I've plugged his fantastic work already, but check out his GitHub for a collection of replication files he has done.

```
// Baseline NK Model as in Gali (2015)
// Code is an extension of Johannes Pfeifer's original replication
// Written by David Murakami

%define whether to use interest rate or money growth rate rule
@#define money_growth_rule=1

var C               ${C}$              (long_name='Consumption')
    W_real          ${\frac{W}{P}}$    (long_name='Real Wage')
    Pi              ${\Pi}$            (long_name='Inflation')
    A               ${A}$              (long_name='TFP')
    N               ${N}$              (long_name='Labour Supply')
    R               ${R^n}$            (long_name='Nominal Interest Rate')
    realinterest    ${R^{r}}$          (long_name='Real Interest Rate')
    Y               ${Y}$              (long_name='Output')
    YN              ${Y^n}$            (long_name='Flex Price Output')
```

12. See King, Plosser, and Rebelo (1988, 2002).

13. With some small adjustments – but because I've been fairly consistent with Galí's notation, you can adjust as you see fit.

```
    X                ${x}$            (long_name='Output Gap')
    Q                ${Q}$            (long_name='Bond Price')
    Z                ${Z}$            (long_name='HH Preferences')
    S                ${V^P}$          (long_name='Price Dispersion')
    Pi_star          ${\Pi^*}$        (long_name='Optimal Reset Inflation')
    x_aux_1          ${X^r_1}$        (long_name='Aux. Var. 1')
    x_aux_2          ${X^r_2}$        (long_name='Aux. Var. 2')
    MC               ${MC}$           (long_name='Real Marginal Cost')
    M_real           ${M/P}$          (long_name='Real Money Supply')
    i_ann            ${i^{ann}}$      (long_name='Ann. Nominal Interest Rate')
    pi_ann           ${\pi^{ann}}$    (long_name='Ann. Inflation Rate')
    r_real_ann       ${r^{r,ann}}$    (long_name='Ann. Real Interest Rate')
    P                ${P}$            (long_name='Price Level')
    log_m_nominal    ${m}$            (long_name='Log Nominal Money Supply')
    log_y            ${y}$            (long_name='Log Output')
    log_W_real       ${w}$            (long_name='Log Real Wage')
    log_N            ${n}$            (long_name='Log Labour Supply')
    log_P            ${p}$            (long_name='Log Price Level')
    log_A            ${a}$            (long_name='Log TFP')
    log_Z            ${z}$            (long_name='Log HH Preferences')
    @#if money_growth_rule==0
        nu                   ${\nu}$              (long_name='AR(1) monetary policy shock process')
    @#else
        money_growth         ${\Delta m_q}$       (long_name='money growth')
        money_growth_ann     ${\Delta m^{ann}}$   (long_name='money growth annualized')
    @#endif
    ;

varexo eps_a         ${\varepsilon^a}$    (long_name='TFP Shock')
```

```
        eps_z          ${\varepsilon^z}$   (long_name='Preference
           Shock')
        @#if money_growth_rule==0
             eps_nu  ${\varepsilon_\nu}$ (long_name='monetary policy
                 shock')
        @#else
            eps_m     ${\varepsilon_m}$   (long_name='money supply
               shock innovation')
        @#endif
        ;

parameters alppha        ${\alpha}$       (long_name='capital share')
    betta                ${\beta}$        (long_name='discount factor
       ')
    rho_a                ${\rho_a}$       (long_name='autocorrelation
        technology shock')
    @#if money_growth_rule==0
        rho_nu           ${\rho_{\nu}}$   (long_name='autocorrelation
            monetary policy shock')
    @#else
        rho_m            ${\rho_{\zeta}}$ (long_name='
           autocorrelation monetary demand shock')
    @#endif
    rho_z                ${\rho_{z}}$     (long_name='autocorrelation
        monetary demand shock')
    siggma               ${\sigma}$       (long_name='inverse EIS')
    varphi               ${\varphi}$      (long_name='inverse Frisch
       elasticity')
    varphi0              ${\varphi_0}$    (long_name='labour supply
       disutility parameter')
    zetaM                ${\zeta_M}$      (long_name='money
       preference parameter')
    phi_pi               ${\phi_{\pi}}$   (long_name='inflation
       feedback Taylor Rule')
    phi_y                ${\phi_{y}}$     (long_name='output feedback
        Taylor Rule')
    eta                  ${\eta}$         (long_name='semi-elasticity
        of money demand')
    epsilon              ${\epsilon}$     (long_name='demand
```

```
        elasticity')
    theta               ${\theta}$      (long_name='Calvo parameter
        ')
    tau                 ${\tau}$        (long_name='labor subsidy')
    ;


// PARAMETERS, p. 67  and p. 113-115
siggma = 1;
varphi = 5;
varphi0 = 1;
zetaM = 1;
phi_pi = 1.5;
phi_y = 0.125;
theta = 3/4;
@#if money_growth_rule==0
    rho_nu = 0.5;
@#else
    rho_m = 0.5; %footnote 11, p. 115
@#endif
rho_z  = 0.5;
rho_a  = 0.9;
betta  = 0.99;
eta = 3.77; %footnote 11, p. 115
alppha = 1/3;
epsilon = 10;
tau = 0; //1/epsilon;


// EQUILIBRIUM CONDITIONS
model;
    [name='FOC Wages, eq. (2)']
    W_real = C^siggma*varphi0*N^varphi;

    [name='Euler equation eq. (3)']
    Q = betta*(C(+1)/C)^(-siggma)*(Z(+1)/Z)/Pi(+1);

    [name='Definition nominal interest rate), p. 22 top']
    R = 1/Q;
```

```

    [name='Aggregate output, above eq. (14)']
    Y = A*(N/S)^(1-alppha);

    [name='Definition Real interest rate']
    R = realinterest*Pi(+1);

    @#if money_growth_rule==0
        [name='Monetary Policy Rule, p. 26 bottom/eq. (22)']
        R = 1/betta*Pi^phi_pi*(Y/steady_state(Y))^phi_y*exp(nu);
    @#endif

    [name='Market Clearing, eq. (15)']
    C=Y;

    [name='Technology Shock, eq. (6)']
    log(A) = rho_a*log(A(-1)) + eps_a;

    [name='Preference Shock, p.54']
    log(Z) = rho_z*log(Z(-1)) - eps_z;

    @#if money_growth_rule==0
        [name='Monetary policy shock']
        nu = rho_nu*nu(-1) + eps_nu;
    @#endif

    @#if money_growth_rule==1
        [name='definition nominal money growth']
        money_growth = log(M_real/M_real(-1)*Pi);

        [name='exogenous process for money supply growth rate, eq.
           (35)']
        money_growth = rho_m*money_growth(-1)+eps_m;

        [name='definition annualized nominal money growth']
        money_growth_ann = 4*money_growth;
    @#endif

    [name='Marginal Cost']
```

```
    MC = W_real/((1-alppha)*Y/N*S);

    [name='LOM prices, eq. (7)']
    1 = theta*Pi^(epsilon-1)+(1-theta)*(Pi_star)^(1-epsilon);

    [name='LOM price dispersion']
    S = (1-theta)*Pi_star^(-epsilon/(1-alppha)) + theta*Pi^(epsilon
        /(1-alppha))*S(-1);

    [name='FOC price setting']
    Pi_star^(1 + epsilon*(alppha/(1-alppha))) = x_aux_1/x_aux_2*(1-
        tau)*epsilon/(epsilon-1);

    [name='Auxiliary price setting recursion 1']
    x_aux_1 = Z*C^(-siggma)*Y*MC + betta*theta*Pi(+1)^(epsilon +
        alppha*epsilon/(1-alppha))*x_aux_1(+1);

    [name='Auxiliary price setting recursion 2']
    x_aux_2 = Z*C^(-siggma)*Y + betta*theta*Pi(+1)^(epsilon-1)*
        x_aux_2(+1);

    [name='Definition log output']
    log_y = log(Y);

    [name='Definition log real wage']
    log_W_real=log(W_real);

    [name='Definition log hours']
    log_N=log(N);

    [name='Annualized inflation']
    pi_ann=4*log(Pi);

    [name='Annualized nominal interest rate']
    i_ann=4*log(R);

    [name='Annualized real interest rate']
    r_real_ann=4*log(realinterest);

```

```
    [name='Real money demand, eq. (4), but we specify our own money
        demand']
    //M_real=Y/R^eta;
    M_real = zetaM*R/((R-1)*Z)*C^siggma;

    [name='definition nominal money stock']
    log_m_nominal=log(M_real*P);

    [name='Definition price level']
    Pi=P/P(-1);

    [name='Definition log price level']
    log_P=log(P);

    [name='Definition log TFP']
    log_A=log(A);

    [name='Definition log preference']
    log_Z=log(Z);

    [name='Flex price output']
    YN = ((1-alppha)/(varphi0*epsilon/(epsilon-1)))^((1-alppha)/(
        siggma*(1-alppha)+alppha+varphi))*A^((1+varphi)/(siggma*(1-
        alppha)+alppha+varphi));

    [name='Output gap']
    X = Y/YN;
end;

%----------------------------------------------------------------
%   Steady state values
%---------------------------------------------------------------

steady_state_model;
A=1;
Z=1;
S=1;
Pi_star=1;
P=1;
```

```
MC=(epsilon-1)/epsilon/(1-tau);
R=1/betta;
Pi=1;
Q=1/R;
realinterest=R;
N=((1-alppha)*MC)^(1/((1-siggma)*alppha+varphi+siggma));
C=A*N^(1-alppha);
W_real=C^siggma*N^varphi;
Y=C;
money_growth=0;
money_growth_ann=0;
nu=0;
x_aux_1=C^(-siggma)*Y*MC/(1-betta*theta*Pi^(epsilon/(1-alppha)));
x_aux_2=C^(-siggma)*Y/(1-betta*theta*Pi^(epsilon-1));
log_y = log(Y);
log_W_real=log(W_real);
log_N=log(N);
pi_ann=4*log(Pi);
i_ann=4*log(R);
r_real_ann=4*log(realinterest);
M_real = zetaM*R/((R-1)*Z)*C^siggma;
log_m_nominal=log(M_real*P);
log_P=log(P);
log_A=0;
log_Z=0;
YN = ((1-alppha)/(varphi0*epsilon/(epsilon-1)))^((1-alppha)/(siggma
    *(1-alppha)+alppha+varphi))*A^((1+varphi)/(siggma*(1-alppha)+
    alppha+varphi));
X = Y/YN;
end;

write_latex_dynamic_model;

resid;
steady;
check;

%----------------------------------------------------------------
%  define shock variances
```

```
%---------------------------------------------------------------

shocks;
    @#if money_growth_rule==0
        var eps_nu = 0.0025^2; //1 standard deviation shock of 25 basis points, i.e. 1 percentage point annualized
    @#else
        var eps_m = 0.0025^2; //1 standard deviation shock of 25 basis points, i.e. 1 percentage point annualized
    @#endif
end;

%----------------------------------------------------------------
% generate IRFs for monetary policy shock, replicates Figures 3.1, p. 69 (interest rate rule)
% 3.4, p. 76 (money supply rule)
%----------------------------------------------------------------
@#if money_growth_rule==0
    stoch_simul(order = 1,irf=20,graph_format=pdf, irf_plot_threshold=1e-6) pi_ann log_y log_N log_W_real log_P i_ann r_real_ann log_m_nominal nu;
@#else
    stoch_simul(order = 1,irf=20,graph_format=pdf, irf_plot_threshold=1e-6) pi_ann log_y log_N log_W_real log_P i_ann r_real_ann log_m_nominal money_growth_ann;
@#endif


%----------------------------------------------------------------
% generate IRFs for discount rate shock, replicates Figures 3.2, p. 70 (interest rate rule)
% 3.5, p. 78 (money supply rule)
%----------------------------------------------------------------
shocks;
    @#if money_growth_rule==0
        var eps_nu = 0;   //shut off monetary policy shock
    @#else
        var eps_m = 0;   //shut off monetary policy shock
    @#endif
```

```
var eps_z  = 0.005^2; //unit shock to technology
end;

stoch_simul(order = 1,irf=20,irf_plot_threshold=0,graph_format=pdf)
     pi_ann log_y log_N log_W_real log_P i_ann r_real_ann
    log_m_nominal log_Z;


%------------------------------------------------------------------
% generate IRFs, replicates Figures 3.3, p. 73 (interest rate rule)
% 3.6, p. 81 (money supply rule)
%------------------------------------------------------------------
shocks;
var eps_z = 0;   //shut off discount rate shock
var eps_a  = 0.01^2; //unit shock to technology
end;

stoch_simul(order = 1,irf=20,irf_plot_threshold=0,graph_format=pdf)
     pi_ann log_y log_N log_W_real log_P i_ann r_real_ann
    log_m_nominal log_A;
```

Figure 11.4 plots the dynamic responses to a 1% TFP shock. There are a couple if interesting things to point out here. Money is exogenous, and so $\Delta m_t = 0, \forall t$. The real interest rate increases quite substantially following the increase in TFP in order to keep the money supply fixed. This actually causes the TFP increase to be highly contractionary compared to the equilibrium under flexible prices.

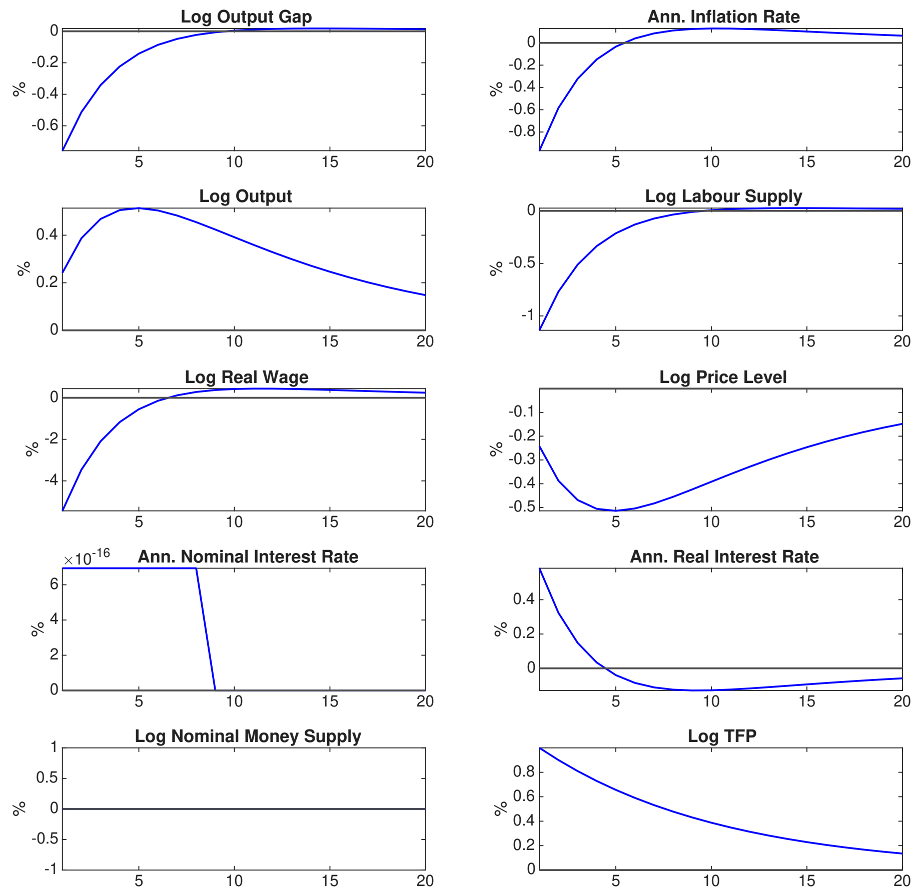
How can we intuit this? If $\theta = 0$, we see that output would respond significantly more to the productivity shock than in the baseline case where we used $\theta = 0.75$. Why? When prices are sticky, output becomes [partially] "demand-determined", and with exogenous money supply the way we have it here, price rigidity prevents demand from rising sufficiently when "supply" increases, so output rises by "too little" relative to what would happen with flexible prices. An easy way to see this is to look at the money demand relationship. In logs we have:

$$m_t - p_t = \ln \zeta_M + i_t - \ln i_t + \sigma y_t.$$

Since the nominal interest rate does not move, changes in output must be proportional to the movement in real money balances. Since m_t is fixed, real money balances must move through changes in p_t. That's why output and prices co-move – albeit inversely. But also, remember that prices are sticky so they cannot adjust quickly enough to their efficient level (the flexible level).

Next, consider a shock to the money supply, plotted in Figure 11.5. Here we have assumed

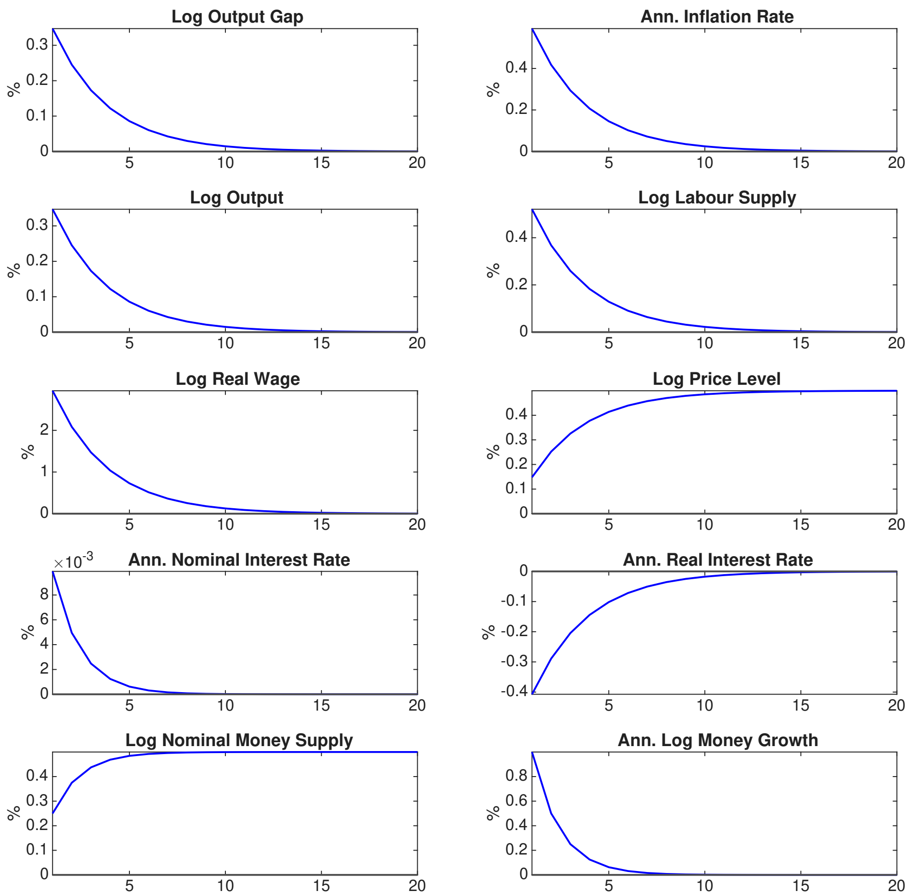
Figure 11.4: IRFs to Productivity Shock (Money Supply Rule)



that $\rho_m = 0.5$, and ε_t^m is 0.25 at the time of impact – corresponding to an annualised rate of 1% for the growth rate of money. Real marginal costs rise, which means that $w_t - p_t$ rises (since a_t is fixed): this is necessary to get workers to work more. The real interest rate falls, though the nominal interest rate hardly moves. Evidently, having sticky prices allows the nominal monetary shock to have real effects.

What is going on here? There are again a couple of ways to see this. Focusing on the money demand relationship, we again have the result that, for a fixed nominal interest rate, real balances and real GDP move together. When m_t increases, if prices were flexible p_t would increase by the same amount, so real balances wouldn't change, and hence y_t wouldn't change. But with sticky prices, p_t can't increase sufficiently, so real money balances increase, and therefore so too does output. Another way to see what is going on is by focusing on the real interest rate. If prices were flexible, the one time increase in m_t would be met by a one time permanent increase in p_t, so $\mathbb{E}_t p_{t+1} = p_t$, and therefore expected inflation would not react. With expected inflation fixed, and

Figure 11.5: IRFs to a Monetary Shock (Money Supply Rule)

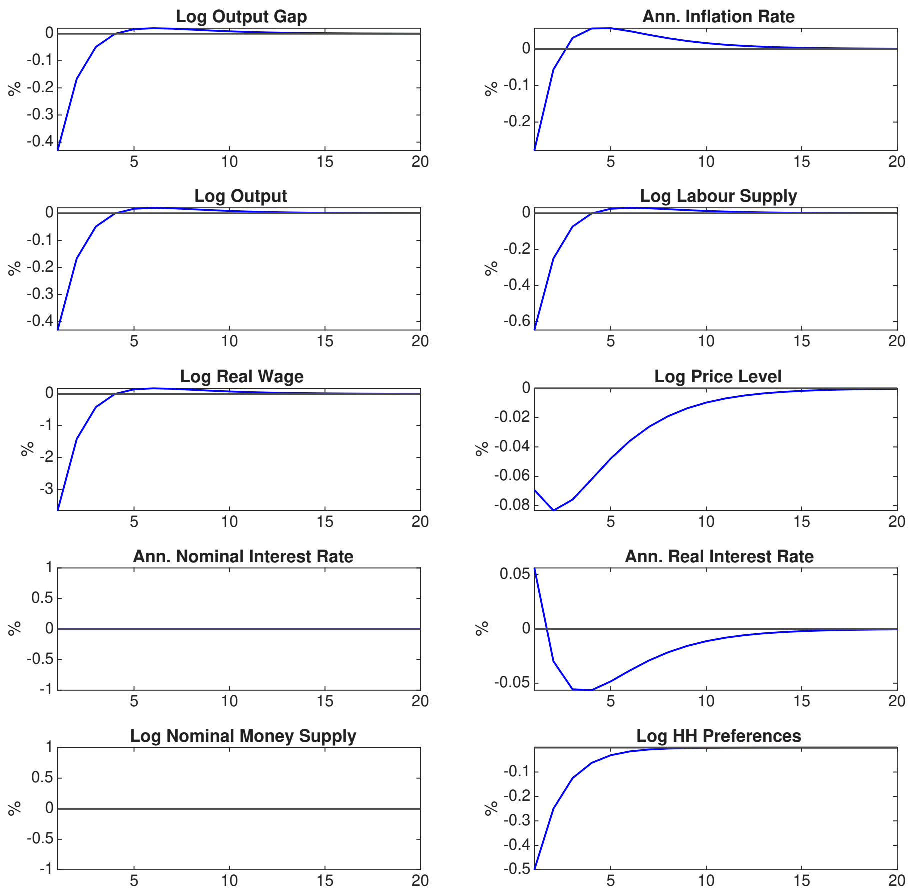


the nominal rate fixed, there would be no effect on the real interest rate. But with price stickiness, because not all firms can immediately adjust their prices, the aggregate price level adjusts slowly, and in particular $\mathbb{E}_t p_{t+1} > p_t$, so expected inflation rises. Higher expected inflation with a fixed nominal rate means a lower real interest rate, which stimulates expenditure and results in the output increase.

Finally, if you play with the code above and plot the flexible price output and output gap, you will notice something interesting. Since the flexible price level of output does not react to the monetary shock (since it's not a supply or "real" shock), the response of the gap is identical to the response of output.

Finally, for reference, I have included the dynamic responses of a preference shock in Figure 11.6, where households increase their patience. Galí (2015) discusses this in his textbook, with some comparisons to price level targeting. We won't get into that here, as I want to move on to log-linearising the NK model and getting rid of money with the Taylor rule. So, interested readers should consult the textbook.

Figure 11.6: IRFs to a Preference Shock (Money Supply Rule)



11.9 Log-linearising the canonical New Keynesian model

The vast majority of the macroeconomic literature presents the New Keynesian model in log-linear form. This will (on top of some other minor tweaks) allow us to write the model very compactly. The log-linearisation is a massive pain in the neck, but the final product is well worth it. For sanity, we will also assume that we log-linearise about a steady state with $\pi = 0$.[14] This also means that in the steady state, $P = P^* \implies MC = \mathcal{M}^{-1}$.[15]

14. Those interested in New Keynesian models approximated about non-zero trend inflation should read Ascari and Sbordone (2014) as a starting point.
15. This means that the steady state is inefficient – marginal cost is not equal to one, thus output is lower compared to its 'èfficient" level.

Start with Euler equation for consumption:

$$\begin{aligned} C_t^{-\sigma} &= \beta \mathbb{E}_t \frac{C_{t+1}^{-\sigma} R_t P_t}{P_{t+1}} \frac{Z_{t+1}}{Z_t} \\ \Leftrightarrow Y_t^{-\sigma} &= \beta \mathbb{E}_t \frac{Y_{t+1}^{-\sigma} R_t P_t}{P_{t+1}} \frac{Z_{t+1}}{Z_t} \\ \Leftrightarrow Y_t^{-\sigma} &= \beta \mathbb{E}_t \frac{Y_{t+1}^{-\sigma} R_t}{\Pi_{t+1}} \frac{Z_{t+1}}{Z_t}, \end{aligned}$$

and then take logs:

$$\begin{aligned} -\sigma \ln Y_t &= \ln \beta - \sigma \mathbb{E}_t \ln Y_{t+1} + \ln R_t - \mathbb{E}_t \ln \Pi_{t+1} + \ln Z_{t+1} - \ln Z_t \\ \Leftrightarrow -\sigma y_t &= \ln \beta - \sigma \mathbb{E}_t y_{t+1} + i_t - \mathbb{E}_t \pi_{t+1} + (\rho_z - 1) z_t, \end{aligned} \tag{11.41}$$

then use our log-linear rules (I use a Taylor series expansion about the steady state):

$$\begin{aligned} -\sigma y - \frac{\sigma}{Y}(Y_t - Y) = \ln \beta - \sigma y - \frac{\sigma}{Y}(\mathbb{E}_t Y_{t+1} - Y) + i + (i_t - i) \\ - \pi - (\mathbb{E}_t \pi_{t+1} - \pi) + (\rho_z - 1) z + (\rho_z - 1)(z_t - z), \end{aligned}$$

where the $-\sigma y$ terms cancel out:

$$-\frac{\sigma}{Y}(Y_t - Y) = \ln \beta - \frac{\sigma}{Y}(\mathbb{E}_t Y_{t+1} - Y) + i + (i_t - i) - \pi - (\mathbb{E}_t \pi_{t+1} - \pi) + (\rho_z - 1) z + (\rho_z - 1)(z_t - z),$$

and then we know that in the steady state, $\pi = 0$, which would imply that by the Fisher equation $i = \rho$. Since we're taking logs $\ln R_t = i_t$, and we know $1 + \rho = \frac{1}{\beta}$, hence $\ln(1 + \rho) = -\ln \beta$, and so we can write $-\ln \beta = i$. We also have that $z = 0$ in the steady state. Thus, we have

$$\begin{aligned} -\frac{\sigma}{Y}(Y_t - Y) &= -\frac{\sigma}{Y}(\mathbb{E}_t Y_{t+1} - Y) + (i_t - i) + (\mathbb{E}_t \pi_{t+1} - \pi) + (\rho_z - 1) z_t \\ \Leftrightarrow -\sigma \hat{y}_t &= -\sigma \mathbb{E}_t \hat{y}_{t+1} + \hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1} + (\rho_z - 1) \hat{z}_t, \end{aligned}$$

where $\hat{y}_t = y_t - y \approx \frac{Y_t - Y}{Y}$ denotes percent (log) deviations from steady state for Y_t, and variables already in rate form are expressed as absolute deviations (e.g. $\hat{\pi}_t = \pi_t - \pi$ and $\hat{i}_t = i_t - i$). We can rewrite the above equation as:

$$\hat{y}_t = \mathbb{E}_t \hat{y}_{t+1} - \sigma^{-1}\left(\hat{i}_t - \mathbb{E}_t \hat{\pi}_{t+1}\right) + \sigma^{-1}(1 - \rho_z) \hat{z}_t, \tag{11.42}$$

which is called the "New Keynesian IS Curve" or "Dynamic IS Equation" (DISE). The naming is a bit odd: in old Keynesian models, IS stood for "Investment = Saving", but here we don't have any investment. But if you stare at (11.42) long enough you can see that it states an inverse relationship between consumption today and the real interest rate.

Alternatively – or equivalently, rather – we could take (11.41) and write the DISE in log-levels as Galí (2015) does:

$$y_t = \mathbb{E}_t y_{t+1} - \frac{1}{\sigma}(i_t - \mathbb{E}_t \pi_{t+1} - \rho) + \frac{1}{\sigma}(1-\rho_z) z_t. \tag{11.43}$$

You can see that (11.42) and (11.43) are equivalent; the former is just approximated about the steady state given by the Fisher equation, $i = \rho$.

Now, let's look at the labour supply equation:[16]

$$\varphi_0 N_t^{\varphi} = C_t^{-\sigma} \frac{W_t}{P_t},$$
$$\implies \ln \varphi_0 + \varphi n_t = w_t - p_t - \sigma c_t,$$

but we know that:

$$mc_t = w_t - p_t - \ln(1-\alpha) + \frac{\alpha}{1-\alpha} y_t - \frac{1}{1-\alpha} a_t$$
$$\Leftrightarrow w_t - p_t = mc_t - \frac{\alpha}{1-\alpha} y_t - \frac{1}{1-\alpha} a_t + \ln(1-\alpha)$$

and since $c_t = y_t$, we can write the labour supply condition as by substituting for $w_t - p_t$:

$$\varphi n_t = mc_t - \frac{[\alpha + \sigma(1-\alpha)]}{1-\alpha} y_t + \frac{1}{1-\alpha} a_t + \ln(1-\alpha) - \ln \varphi_0.$$

Note that if we subtract φn from both the LHS and RHS, we could write everything in terms of log-deviations from steady state:

$$\varphi \hat{n}_t = \hat{mc}_t - \frac{[\alpha + \sigma(1-\alpha)]}{1-\alpha} \hat{y}_t + \frac{1}{1-\alpha} \hat{a}_t, \tag{11.44}$$

where "hatted" variables – as before when we log-linearised the RBC model – denote the percent or log deviation from steady state of a variable. Why do we sometimes look at log-linearised equations in terms of log-levels as opposed to log-deviations? Well, for some variables and equations, like interest rates and inflation, it's helpful to look at log-levels rather than log-deviations. For others, such as the labour supply condition above or production function, log-deviations are a bit "cleaner" as we get rid of constants and parameters. I'll try and be explicit when switching between log-levels and log-deviations, but just be aware of the subtle differences between the two.

Next, we take logs of the production function:

$$y_t = a_t + (1-\alpha) n_t - v_t^P.$$

What is v_t^P? This is going to be messy... First, start by taking logs of the price dispersion equation:

$$V_t^P = \left[(1-\theta)(\Pi_t^*)^{-\frac{\epsilon}{1-\alpha}} \Pi_t^{\frac{\epsilon}{1-\alpha}} + \theta \Pi_t^{\frac{\epsilon}{1-\alpha}} (V_{t-1}^P)^{\frac{1}{1-\alpha}} \right]^{1-\alpha}$$

16. You can see why some people just assume that $\varphi_0 = 1$, so $\ln \varphi_0 = 0$.

$$\ln V_t^P \equiv v_t^P = (1-\alpha)\ln\left[(1-\theta)(\Pi_t^*)^{-\frac{\epsilon}{1-\alpha}}\Pi_t^{\frac{\epsilon}{1-\alpha}} + \theta\Pi_t^{\frac{\epsilon}{1-\alpha}}(V_{t-1}^P)^{\frac{1}{1-\alpha}}\right]$$

Now, totally differentiate the above to get:[17]

$$v_t^P - v^P = \frac{(1-\alpha)}{V^P}\left\{\begin{array}{c} -\frac{\epsilon}{1-\alpha}(1-\theta)(\Pi^*)^{-\frac{\epsilon}{1-\alpha}-1}\Pi^{\frac{\epsilon}{1-\alpha}}(\pi_t^* - \pi^*) + \frac{\epsilon}{1-\alpha}(1-\theta)(\Pi^*)^{-\frac{\epsilon}{1-\alpha}}\Pi^{\frac{\epsilon}{1-\alpha}-1}(\pi_t - \pi) \\ +\frac{\epsilon}{1-\alpha}\theta\Pi^{\frac{\epsilon}{1-\alpha}-1}V^P(\pi_t - \pi) + \frac{1}{1-\alpha}\theta\Pi^{\frac{\epsilon}{1-\alpha}}(V^P)^{\frac{1}{1-\alpha}-1}(V_{t-1}^P - V^P)\end{array}\right\},$$

simplify things by using our facts about $V^P = 1$ and $\pi = \pi^* = 0$:

$$\begin{aligned} v_t^P &= \frac{(1-\alpha)}{V^P}\left[-\frac{\epsilon}{1-\alpha}(1-\theta)\pi_t^* + \frac{\epsilon}{1-\alpha}(1-\theta)\pi_t + \frac{\epsilon}{1-\alpha}\theta V^P\pi_t + \frac{\theta}{1-\alpha}(V_{t-1}^P - V^P)\right], \\ \implies v_t^P &= -\epsilon(1-\theta)\pi_t^* + \epsilon(1-\theta)\pi_t + \epsilon\theta\pi_t + \theta v_{t-1}^P, \end{aligned}$$

and this can be written as:

$$v_t^P = -\epsilon(1-\theta)\pi_t^* + \epsilon\pi_t + \theta v_{t-1}^P. \tag{11.45}$$

Next, log-linearise the equation for the evolution of inflation:

$$\begin{aligned} \Pi_t^{1-\epsilon} &= (1-\theta)(\Pi_t^*)^{1-\epsilon} + \theta \\ \implies (1-\epsilon)\ln\Pi_t &= \ln\left[(1-\theta)(\Pi_t^*)^{1-\epsilon} + \theta\right] \\ \therefore (1-\epsilon)\pi_t &= \ln\left[(1-\theta)(\Pi_t^*)^{1-\epsilon} + \theta\right], \end{aligned}$$

and then totally differentiate:

$$(1-\epsilon)(\pi_t - \pi) = \Pi^{\epsilon-1}\left[(1-\epsilon)(1-\theta)(\Pi^*)^{-\epsilon}(\pi_t^* - \pi^*)\right],$$

where $\Pi^{\epsilon-1}$ shows up because the term inside the large square parentheses is equal to $\Pi^{1-\epsilon}$ in the steady state, and when we take the derivative of the log this term gets inverted at the steady state. We can use what we know about the zero inflation steady state to write:

$$\begin{aligned} (1-\epsilon)\pi_t &= (1-\epsilon)(1-\theta)\pi_t^* \\ \Leftrightarrow \pi_t &= (1-\theta)\pi_t^*. \end{aligned} \tag{11.46}$$

In words, actual inflation is just proportional to reset price inflation, where the constant is equal to the fraction of firms that are updating their prices. Now use this by substituting it into the expression for price dispersion (11.45) to get:

$$\begin{aligned} v_t^P &= -\epsilon(1-\theta)\pi_t^* + \epsilon(1-\theta)\pi_t + \epsilon\theta\pi_t + \theta v_{t-1}^P \\ \therefore v_t^P &= \theta v_{t-1}^P. \end{aligned} \tag{11.47}$$

17. Remember:

$$\frac{X_t - X}{X} \approx \ln X_t - \ln X = x_t - x = \hat{x}_t.$$

This is a fairly important equation to note. If we are approximating about the zero inflation steady state where $V^P = 1$ or $v^P = 0$, then we're starting from a position in which $v_{t-1}^P = 0$, which means that $v_t^P = 0, \forall t$. In other words, about a zero inflation steady state, price dispersion is a second order phenomenon, and we can just ignore it up to a first order approximation about a zero inflation steady state.[18]

Alternatively, another way to show that price dispersion is the following. Let $\tilde{p}_t(j) \equiv p_t(j) - p_t$, and notice that from (11.16):

$$\begin{aligned}\left[\frac{P_t(j)}{P_t}\right]^{1-\epsilon} &\approx \exp\left[(1-\epsilon)\tilde{p}_t(j)\right] \\ &\approx 1 + (1-\epsilon)\tilde{p}_t(j) + \frac{(1-\epsilon)^2}{2}\tilde{p}_t(j)^2, \end{aligned} \tag{11.48}$$

where we used the fact that

$$1 = \int_0^1 \left[\frac{P_t(j)}{P_t}\right]^{1-\epsilon} dj. \tag{11.49}$$

Substituting (11.48) into (11.49) gives:

$$\begin{aligned} 1 &= \int_0^1 \left[1 + (1-\epsilon)\tilde{p}_t(j) + \frac{(1-\epsilon)^2}{2}\tilde{p}_t(j)^2\right] dj \\ 0 &= \int_0^1 \left[(1-\epsilon)\tilde{p}_t(j) + \frac{(1-\epsilon)^2}{2}\tilde{p}_t(j)^2\right] dj \\ \int_0^1 \tilde{p}_t(j) dj &= \int_0^1 \frac{\epsilon - 1}{2}\tilde{p}_t(j)^2 dj \\ \Leftrightarrow \mathbb{E}_j \tilde{p}_t(j) &= \frac{\epsilon-1}{2}\mathbb{E}_j\tilde{p}_t(j)^2. \end{aligned} \tag{11.50}$$

Then, take a second-order approximation of:

$$\begin{aligned}\left[\frac{P_t(j)}{P_t}\right]^{-\frac{\epsilon}{1-\alpha}} &= \exp\left[-\frac{\epsilon}{1-\alpha}\tilde{p}_t(j)\right] \\ &= 1 - \frac{\epsilon}{1-\alpha}\tilde{p}_t(j) + \frac{1}{2}\left(\frac{\epsilon}{1-\alpha}\right)^2 \tilde{p}_t(j)^2. \end{aligned} \tag{11.51}$$

Combine what we did with (11.50) and (11.51) to write:

$$\begin{aligned}\int_0^1 \left[\frac{P_t(j)}{P_t}\right]^{-\frac{\epsilon}{1-\alpha}} dj &= \int_0^1 \left[1 - \frac{\epsilon}{1-\alpha}\tilde{p}_t(j) + \frac{1}{2}\left(\frac{\epsilon}{1-\alpha}\right)^2 \tilde{p}_t(j)^2\right] dj \\ 1 &= 1 - \frac{\epsilon}{1-\alpha}\int_0^1 \tilde{p}_t(j) dj + \int_0^1 \frac{1}{2}\left(\frac{\epsilon}{1-\alpha}\right)^2 \tilde{p}_t(j)^2 dj \end{aligned}$$

18. This is why we can equate Calvo pricing to Rotemberg pricing up to a first order. Remember that there is no price dispersion in the Rotemberg model as all firms price identically.

$$
\begin{aligned}
&= 1 - \frac{\epsilon}{1-\alpha}\int_0^1 \frac{\epsilon-1}{2}\tilde{p}_t(j)^2 dj + \int_0^1 \frac{1}{2}\left(\frac{\epsilon}{1-\alpha}\right)^2 \tilde{p}_t(j)^2 dj \\
&= 1 + \left[\frac{\epsilon(1-\epsilon)}{2(1-\alpha)} + \frac{1}{2}\left(\frac{\epsilon}{1-\alpha}\right)^2\right]\int_0^1 \tilde{p}_t(j)^2 dj \\
&= 1 + \frac{\epsilon}{2(1-\alpha)}\left(1-\epsilon+\frac{\epsilon}{1-\alpha}\right)\int_0^1 \tilde{p}_t(j)^2 dj \\
&= 1 + \frac{\epsilon}{2(1-\alpha)}\left(\frac{1-\alpha-\epsilon(1-\alpha)+\epsilon}{1-\alpha}\right)\int_0^1 \tilde{p}_t(j)^2 dj \\
&= 1 + \frac{1}{2}\left(\frac{\epsilon}{1-\alpha}\right)\frac{1}{\Theta}\text{var}_j\left\{p_t(j)\right\}, \qquad (11.52)
\end{aligned}
$$

which is what Galí (2015) has in the Appendix of Chapter 3, and where $\Theta = \frac{1-\alpha}{1-\alpha+\alpha\epsilon} \in (0,1]$. To justify the final variance term, let's go back to (11.49) and write it as:

$$
1 = \int_0^1 \exp\left\{(1-\epsilon)\left[p_t(j)-p_t\right]\right\} dj.
$$

Then, approximate this about $p_t(j) = p_t$:

$$
\begin{aligned}
1 &\approx \int_0^1 \left\{1 + (1-\epsilon)\left[p_t(j)-p_t\right] + \frac{(1-\epsilon)^2}{2}\left[p_t(j)-p_t\right]^2\right\} dj \\
&= 1 - (1-\epsilon)p_t + (1-\epsilon)\int_0^1 p_t(j)dj + \frac{(1-\epsilon)^2}{2}\int_0^1 \left[p_t(j)-p_t\right]^2 dj \\
p_t &= \int_0^1 p_t(j)dj + \frac{1-\epsilon}{2}\int_0^1 \left[p_t(j)-p_t\right]^2 dj \\
\Leftrightarrow p_t &= \mathbb{E}_j p_t(j) + \frac{1-\epsilon}{2}\mathbb{E}_j \tilde{p}_t(j)^2.
\end{aligned}
$$

Hence, price dispersion is a second-order phenomenon since $p_t = \mathbb{E}_j p_t(j)$ at up to first-order. Therefore:

$$
\begin{aligned}
\int_0^1 \tilde{p}_t(j)^2 dj = \int_0^1 \left[p_t(j)-p_t\right]^2 dj &\approx \int_0^1 \left[p_t(j)-\mathbb{E}_j p_t(j)\right]^2 dj \\
&\equiv \text{var}_j p_t(j). \qquad (11.53)
\end{aligned}
$$

Finally, this means we can write:

$$
\begin{aligned}
V_t^P &= \left\{\int_0^1 \left[\frac{P_t(j)}{P_t}\right]^{-\frac{\epsilon}{1-\alpha}} dj\right\}^{1-\alpha} \\
\implies \ln V_t^P \equiv v_t^p &= (1-\alpha)\ln \int_0^1 \left[\frac{P_t(j)}{P_t}\right]^{-\frac{\epsilon}{1-\alpha}} dj \\
&\approx (1-\alpha)\ln\left[1 + \frac{1}{2}\left(\frac{\epsilon}{1-\alpha}\right)\frac{1}{\Theta}\text{var}_j\left\{p_t(j)\right\}\right]
\end{aligned}
$$

$$\approx \frac{\epsilon}{2\Theta} \text{var}_j \left\{ p_t(j) \right\}.$$

We've probably offended countless mathematicians with our abuse of notation, but oh well.

We now move onto log-linearising the aggregate output identity. As we just showed above, in the vicinity of the steady state $v_t^P = 0$, and so we can write:

$$y_t = a_t + (1-\alpha) n_t. \tag{11.54}$$

Then, substitute $n_t = (y_t - a_t)/(1-\alpha)$ into the labour supply condition (11.44) to get

$$\begin{aligned}
\varphi \left(\frac{y_t - a_t}{1-\alpha} \right) &= mc_t - \frac{[\alpha + \sigma(1-\alpha)]}{1-\alpha} y_t + \frac{1}{1-\alpha} a_t + \ln(1-\alpha) - \ln \varphi_0 \\
\implies mc_t &= \varphi \left(\frac{y_t - a_t}{1-\alpha} \right) + \frac{\alpha + \sigma(1-\alpha)}{1-\alpha} y_t - \frac{1}{1-\alpha} a_t - \ln(1-\alpha) + \ln \varphi_0 \\
mc_t &= \frac{\sigma(1-\alpha) + \alpha + \varphi}{1-\alpha} y_t - \frac{(1+\varphi)}{1-\alpha} a_t - \ln(1-\alpha) + \ln \varphi_0, \qquad (11.55)
\end{aligned}$$

where just for completeness I used the log-level version of (11.44). Now, we know that

$$Y_t^n = \left(\frac{1-\alpha}{\varphi_0 \mathcal{M}} \right)^{\frac{1-\alpha}{\sigma(1-\alpha)+\alpha+\varphi}} A_t^{\frac{1+\varphi}{\sigma(1-\alpha)+\alpha+\varphi}},$$

and log-linearising this we have

$$y_t^n = \left(\frac{1-\alpha}{\sigma(1-\sigma)+\alpha+\varphi} \right) [\ln(1-\alpha) - \ln \varphi_0 - \ln \mathcal{M}] + \left(\frac{1+\varphi}{\sigma(1-\alpha)+\alpha+\varphi} \right) a_t. \tag{11.56}$$

But then notice that we can use this expression to write:

$$\begin{aligned}
\left(\frac{1-\alpha}{\sigma(1-\alpha)+\alpha+\varphi} \right) \ln \mathcal{M} &= -y_t^n + \left(\frac{1-\alpha}{\sigma(1-\alpha)+\alpha\varphi} \right) [\ln(1-\alpha) - \ln \varphi_0] + \left(\frac{1+\varphi}{\sigma(1-\alpha)+\alpha+\varphi} \right) a_t \\
\mu_t \equiv \ln \mathcal{M} &= -\left(\frac{\sigma(1-\alpha)+\alpha+\varphi}{1-\alpha} \right) y_t^n + \frac{1+\varphi}{1-\alpha} a_t + \ln(1-\alpha) - \ln \varphi_0 \qquad (11.57) \\
\Leftrightarrow \mu_t &= p_t - \psi_t,
\end{aligned}$$

where μ_t is the [log] average price markup and $\psi_t = \ln \Psi_t$ is the log nominal marginal cost.[19] When prices are flexible ($\theta = 0$), the average markup is constant and equal to the "desired markup" μ, so (11.57) evaluated at the flexible price equilibrium is:

$$\mu = -\left(\frac{\sigma(1-\alpha)+\alpha+\varphi}{1-\alpha} \right) y_t^n + \left(\frac{1+\varphi}{1-\alpha} \right) a_t + \ln(1-\alpha) - \ln \varphi_0. \tag{11.58}$$

19. I'm trying to map what we have done here to Galí (2015). So, just to be clear:

$$mc_t = \psi_t - p_t = -\mu_t.$$

Then we can use this expression to write y_t^n compactly as:

$$y_t^n = \psi_{ya} a_t + \psi_y, \tag{11.59}$$

where

$$\psi_{ya} \equiv \frac{1+\varphi}{\sigma(1-\alpha)+\alpha+\varphi}, \quad \psi_y \equiv -\frac{(1-\alpha)\left[\mu + \ln\varphi_0 - \ln(1-\alpha)\right]}{\sigma(1-\alpha)+\alpha+\varphi}.$$

Alright, hopefully you're still following – I've done all this derivation because I want to be explicit about the different ways of writing these expressions, and also so that we can follow the notation of Galí (2015). Having said that, I will deviate slightly from what Galí does and instead of using μ_t I will use the expression for the real marginal cost (11.55). Substitute for a_t using the expression for the flexible price level of output ((11.56) or (11.59)) to write:[20]

$$\begin{aligned}
mc_t &= \frac{\sigma(1-\alpha)+\alpha+\varphi}{1-\alpha} y_t - \left(\frac{1+\varphi}{1-\alpha}\right)\left(\frac{y_t^n - \psi_y}{\psi_{ya}}\right) - \ln(1-\alpha) + \ln\varphi_0 \\
&= \frac{\sigma(1-\alpha)+\alpha+\varphi}{1-\alpha}(y_t - y_t^n) + \mu, \\
\implies \widehat{mc}_t &= \frac{\sigma(1-\alpha)+\alpha+\varphi}{1-\alpha}\tilde{y}_t. \qquad (11.60)
\end{aligned}$$

In words, deviations of real marginal cost are proportional to the output gap, $\tilde{y}_t \equiv y_t - y_t^n$. Recall that real marginal cost is the inverse of the price markup. So if the gap is zero, then markups are equal to the desired fixed steady state markup of $\mathcal{M} = \frac{\epsilon}{\epsilon-1}$. If the output gap is positive, then real marginal cost is above its steady state, so markups are lower than desired (equivalently, the economy is less distorted). The converse is true when the gap is negative. So, equivalently, as in Galí (2015):

$$\hat{\mu}_t = -\left(\frac{\sigma(1-\alpha)+\alpha+\varphi}{1-\alpha}\right)\tilde{y}_t. \tag{11.61}$$

We now need to log-linearise the auxiliary variables:

$$\begin{aligned}
\frac{X_{1,t}}{P_t^{\epsilon+b}} &= C_t^{-\sigma} MC_t Y_t + \theta\beta\mathbb{E}_t \Pi_{t+1}^{\epsilon+b} \frac{X_{1,t+1}}{P_{t+1}^{\epsilon+b}}, \\
\frac{X_{2,t}}{P_t^{\epsilon-1}} &= C_t^{-\sigma} Y_t + \theta\beta\mathbb{E}_t \Pi_{t+1}^{\epsilon-1} \frac{X_{2,t+1}}{P_{t+1}^{\epsilon+b}}.
\end{aligned}$$

This is going to be messy... Let's start with $X_{1,t}$. I'm actually going to just make *another* pair of auxiliary variables because it'll make things a lot easier. Let's define

$$\tilde{X}_{1,t} \equiv \frac{X_{1,t}}{P_t^{\epsilon+b}} = C_t^{-\sigma} MC_t Y_t + \theta\beta\mathbb{E}_t \Pi_{t+1}^{\epsilon+b} \tilde{X}_{1,t+1}, \tag{11.62}$$

$$\tilde{X}_{2,t} \equiv \frac{X_{2,t}}{P_t^{\epsilon-1}} = C_t^{-\sigma} Y_t + \theta\beta\mathbb{E}_t \Pi_{t+1}^{\epsilon-1} \tilde{X}_{2,t+1}. \tag{11.63}$$

20. We could simply assume $\alpha = 0$, and this would look a lot nicer.

Then, because we have $Y_t = C_t$, we can write the following:

$$\tilde{x}_{1,t} = \ln\left[Y_t^{1-\sigma} MC_t + \theta\beta\mathbb{E}_t\Pi_{t+1}^{\epsilon+b}\tilde{X}_{1,t+1}\right]$$

and totally differentiating we have:

$$\begin{aligned}
\tilde{x}_{1,t} - \tilde{x}_1 &\approx \frac{1}{\tilde{X}_1}\begin{Bmatrix} Y^{1-\sigma}(MC_t - MC) + (1-\sigma)MCY^{-\sigma}(Y_t - Y) \\ +(\epsilon+b)\theta\beta\Pi^{\epsilon+b-1}\tilde{X}_1(\mathbb{E}_t\pi_{t+1} - \pi) + \theta\beta\mathbb{E}_t\Pi^{\epsilon+b}(\mathbb{E}_t\tilde{X}_{1,t+1} - \tilde{X}_1) \end{Bmatrix} \\
\hat{\tilde{x}}_{1,t} &= \frac{Y^{1-\sigma}}{\tilde{X}_1}(MC_t - MC) + \frac{(1-\sigma)Y^{-\sigma}MC}{\tilde{X}_1}(Y_t - Y) \\
&\quad + \frac{(\epsilon+b)\theta\beta\Pi^{\epsilon+b-1}\tilde{X}_1}{\tilde{X}_1}(\mathbb{E}_t\pi_{t+1} - \pi) + \frac{\theta\beta\Pi^{\epsilon+b}}{\tilde{X}_1}(\mathbb{E}_t\tilde{X}_{1,t+1} - X_1) \\
&= \frac{Y^{1-\sigma}MC}{\tilde{X}_1}\widehat{mc}_t + \frac{(1-\sigma)Y^{1-\sigma}MC}{\tilde{X}_1}\hat{y}_t + (\epsilon+b)\theta\beta\mathbb{E}_t\pi_{t+1} + \theta\beta\mathbb{E}_t\hat{\tilde{x}}_{1,t+1},
\end{aligned}$$

and in the steady state we know that $\tilde{X}_1 = Y^{1-\sigma}MC/(1-\theta\beta)$, which yields

$$\hat{\tilde{x}}_{1,t} = (1-\theta\beta)\widehat{mc}_t + (1-\sigma)(1-\theta\beta)\hat{y}_t + (\epsilon+b)\theta\beta\mathbb{E}_t\pi_{t+1} + \theta\beta\hat{\tilde{x}}_{1,t+1}. \tag{11.64}$$

Now we can deal with $x_{2,t}$. Start by taking logs:

$$x_{2,t} = \ln\left[Y_t^{1-\sigma} + \theta\beta\mathbb{E}_t\Pi_{t+1}^{\epsilon-1}\tilde{X}_{2,t+1}\right],$$

and then totally differentiate:

$$\begin{aligned}
\tilde{x}_{2,t} - \tilde{x}_2 &\approx \frac{1}{\tilde{X}_2}\begin{Bmatrix} (1-\sigma)Y^{-\sigma}(Y_t - Y) + (\epsilon-2)\theta\beta\Pi^{\epsilon-2}\tilde{X}_2(\mathbb{E}_t\pi_{t+1} - \pi) \\ +\theta\beta\Pi^{\epsilon-1}(\mathbb{E}_t\tilde{X}_{2,t+1} - \tilde{X}_2) \end{Bmatrix} \\
\hat{\tilde{x}}_{2,t} &= \frac{(1-\sigma)Y^{-\sigma}(Y_t - Y)}{\tilde{X}_2} + (\epsilon-1)\theta\beta\Pi^{\epsilon-2}(\mathbb{E}_t\pi_{t+1} - \pi) + \frac{\theta\beta\Pi^{\epsilon-1}(\mathbb{E}_t\tilde{X}_{2,t+1} - \tilde{X}_2)}{\tilde{X}_2} \\
&= \frac{(1-\sigma)Y^{1-\sigma}}{\tilde{X}_2}\hat{y}_t + (\epsilon-1)\theta\beta\pi_t + \theta\beta\mathbb{E}_t\hat{\tilde{x}}_{2,t+1},
\end{aligned}$$

and we know $\tilde{x}_2 = \frac{Y^{1-\sigma}}{1-\theta\beta}$, so we have:

$$\hat{\tilde{x}}_{2,t} = (1-\sigma)(1-\theta\beta)\hat{y}_t + (\epsilon-1)\theta\beta\mathbb{E}_t\pi_{t+1} + \theta\beta\mathbb{E}_t\hat{\tilde{x}}_{2,t+1}. \tag{11.65}$$

Subtracting $\hat{\tilde{x}}_{2,t}$ from $\hat{\tilde{x}}_{1,t}$ yields:

$$\begin{aligned}
\hat{\tilde{x}}_{1,t} - \hat{\tilde{x}}_{2,t} &= (1-\theta\beta)\widehat{mc}_t + (1-\sigma)(1-\theta\beta)\hat{y}_t + (\epsilon+b)\theta\beta\mathbb{E}_t\pi_{t+1} + \theta\beta\mathbb{E}_t\hat{\tilde{x}}_{1,t+1} \\
&\quad - (1-\sigma)(1-\theta\beta)\hat{y}_t - (\epsilon-1)\theta\beta\mathbb{E}_t\pi_{t+1} - \theta\beta\mathbb{E}_t\hat{\tilde{x}}_{2,t+1} \\
&= (1-\theta\beta)\widehat{mc}_t + \theta\beta(1+b)\mathbb{E}_t\pi_{t+1} + \theta\beta(\mathbb{E}_t\hat{\tilde{x}}_{1,t+1} - \mathbb{E}_t\hat{\tilde{x}}_{2,t+1}).
\end{aligned}$$

Next we log-linearise the reset price expression in order to find $\hat{\tilde{x}}_{1,t} - \hat{\tilde{x}}_{2,t}$:

$$\begin{aligned}
(\Pi_t^*)^{1+b} &= \mathcal{M}\frac{X_{1,t}/P_t^{\epsilon+b}}{X_{2,t}/P_t^{\epsilon-1}}\Pi_t^{1+b}, \\
(\Pi_t^*)^{1+b} &= \mathcal{M}\frac{\tilde{X}_{1,t}}{\tilde{X}_{2,t}}\Pi_t^{1+b}, \\
\implies (1+b)\pi_t^* &= \mu + \tilde{x}_{1,t} - \tilde{x}_{2,t} + (1+b)\pi_t \\
\pi_t^* &= \frac{\mu + \tilde{x}_{1,t} - \tilde{x}_{2,t}}{1+b} + \pi_t \\
\Leftrightarrow \tilde{x}_{1,t} - \tilde{x}_{2,t} &= (1+b)(\pi_t^* - \pi_t) - \mu.
\end{aligned}$$

But from (11.46) we have:

$$\pi_t^* = \frac{1}{1-\theta}\pi_t,$$

and so

$$\tilde{x}_{1,t} - \tilde{x}_{2,t} = \frac{(1+b)\theta}{1-\theta}\pi_t - \mu.$$

Putting things together, we can then write:

$$\hat{\tilde{x}}_{1,t} - \hat{\tilde{x}}_{2,t} = \frac{(1+b)\theta}{1-\theta}\pi_t = (1-\theta\beta)\widehat{mc}_t + \theta\beta(1+b)\mathbb{E}_t\pi_{t+1} + \theta\beta\frac{(1+b)\theta}{1-\theta}\mathbb{E}_t\pi_{t+1},$$

and with a bit of cleaning up:

$$\begin{aligned}
\pi_t &= \frac{(1-\theta)(1-\theta\beta)}{(1+b)\theta}\widehat{mc}_t + \beta\mathbb{E}_t\pi_{t+1} \\
\Leftrightarrow \pi_t &= \lambda\widehat{mc}_t + \beta\mathbb{E}_t\pi_{t+1} \\
\Leftrightarrow \pi_t &= \beta\mathbb{E}_t\pi_{t+1} - \lambda\hat{\mu}_t,
\end{aligned} \tag{11.66}$$

where

$$\lambda \equiv \frac{(1-\theta)(1-\theta\beta)}{\theta}\Theta$$

Expression (11.66) is called the "New Keynesian Phillips Curve" (NKPC). It is "new" because it is forward-looking unlike classic Phillips Curves, but it's a Phillips Curve in the sense that it captures a relationship between inflation and some real measure. We can re-write the NKPC in terms of the output gap by using (11.60):

$$\begin{aligned}
\pi_t &= \lambda\frac{(\sigma(1-\alpha)+\alpha+\varphi)}{1-\alpha}\tilde{y}_t + \beta\mathbb{E}_t\pi_{t+1} \\
\Leftrightarrow \pi_t &= \kappa\tilde{y}_t + \beta\mathbb{E}_t\pi_{t+1},
\end{aligned} \tag{11.67}$$

where $\kappa = \lambda(\sigma(1-\alpha)+\alpha+\varphi)/(1-\alpha)$ and is often referred to as the slope of the NKPC. Using the terminal condition that inflation will return to steady state eventually, using (11.66) and the law of iterated expectations we can solve the NKPC forward to get:

$$\begin{aligned}\pi_t &= \lambda\widehat{mc}_t + \beta\mathbb{E}_t\left[\lambda\widehat{mc}_{t+1} + \beta\mathbb{E}_{t+1}\left[\widehat{mc}_{t+2} + \beta\mathbb{E}_{t+2}\left[\lambda\widehat{mc}_{t+3}...\right]\right]\right] \\ &= \lambda\sum_{j=0}^{\infty}\beta^j\widehat{mc}_{t+j}.\end{aligned}$$

In words, current inflation is proportional to the present discounted value of expected real marginal cost. Real marginal cost is the inverse of the price markup. In the model without price rigidity, firms desire constant markups – the "desired markup". If expected future marginal cost is high, then firms will have to lower markups. Firms given the option of updating prices today will try to increase prices today (since they may be stuck with that price in the future) to hit their desired price markup (and vice-versa if they wish to cut prices), putting upward pressure on current inflation (or deflation). Thus, the slope of the NKPC is decreasing in θ: when θ is large, the coefficient on marginal cost (or the gap) is small, suggesting that real movements put little upward pressure on inflation.

The rest of the equations are fairly straightforward. The expressions for a_t, z_t, and money growth are already log-linear so we have:

$$a_t = \rho_a a_{t-1} + \varepsilon_t^a, \tag{11.68}$$

$$z_t = \rho_z z_{t-1} + \varepsilon_t^z, \tag{11.69}$$

$$\Delta m_t = \Delta(m_t - p_t) + \pi_t \tag{11.70}$$

$$\Delta(m_t - p_t) + \pi_t = \rho_m\Delta(m_{t-1} - p_{t-1}) + \rho_m\pi_{t-1} + \varepsilon_t^m. \tag{11.71}$$

Then, log-linearise the money demand function (using what we know about y_t):

$$m_t - p_t = \ln\zeta_M + i_t - \ln(i_t) - z_t + \sigma y_t,$$

Which is pretty straightforward: Demand for real money balances is decreasing in the real interest rate and increasing in y_t (think of the LM curve).

Finally, there's one more thing we need to do: write the DISE (11.43) in terms of the output gap:

$$\begin{aligned}y_t - y_t^n - \mathbb{E}_t y_{t+1}^n &= \mathbb{E}_t(y_{t+1} - y_{t+1}^n) - y_t^n - \frac{1}{\sigma}(i_t - \mathbb{E}_t\pi_{t+1} - \rho) + \frac{1}{\sigma}(1-\rho_z)z_t \\ \tilde{y}_t - (\psi_{ya}\mathbb{E}_t a_{t+1} + \psi_y) &= \mathbb{E}_t\tilde{y}_{t+1} - (\psi_{ya}a_t + \psi_y) - \frac{1}{\sigma}(i_t - \mathbb{E}_t\pi_{t+1} - \rho) + \frac{1}{\sigma}(1-\rho_z)z_t \\ \tilde{y}_t &= \mathbb{E}_t\tilde{y}_{t+1} + (1-\rho_a)\psi_{ya}a_t - \frac{1}{\sigma}(i_t - \mathbb{E}_t\pi_{t+1} - \rho) + \frac{1}{\sigma}(1-\rho_z)z_t.\end{aligned}$$

Do a bit of cleaning up:

$$\tilde{y} = \mathbb{E}_t\tilde{y}_{t+1} - \frac{1}{\sigma}(i_t - \mathbb{E}_t\pi_{t+1} - r_t^n), \tag{11.72}$$

where r_t^n is the natural rate of interest – the real interest rate that would prevail in the flexible price equilibrium also known as the "Wicksellian natural rate of interest":[21]

$$r_t^n = \rho - \sigma(1-\rho_a)\psi_{ya}a_t + (1-\rho_z)z_t. \tag{11.73}$$

The complete log-linearised system of equations is:

$$\begin{aligned}
\tilde{y}_t &= \mathbb{E}_t\tilde{y}_{t+1} - \frac{1}{\sigma}(i_t - \mathbb{E}_t\pi_{t+1} - r_t^n), \\
\pi_t &= \beta\mathbb{E}_t\pi_{t+1} + \kappa\tilde{y}_t, \\
r_t^n &= \rho - \sigma(1-\rho_a)\psi_{ya}a_t + (1-\rho_z)z_t, \\
\Delta m_t &= \Delta(m_t - p_t) + \pi_t, \\
\Delta(m_t - p_t) + \pi_t &= \rho\Delta(m_{t-1} - p_{t-1}) + \rho_m\pi_{t-1} + \varepsilon_t^m, \\
m_t - p_t &= \ln\zeta_M + i_t - \ln(i_t) - z_t + \sigma y_t, \\
a_t &= \rho_a a_{t-1} + \varepsilon_t^a, \\
z_t &= \rho_z z_{t-1} + \varepsilon_t^z.
\end{aligned}$$

This is eight equations in eight variables. In true Keynesian fashion, "aggregate demand" is given by the DISE, "aggregate supply" is given by the NKPC, and we have equations for productivity shocks, preference shocks, money supply, money demand, and the real interest rate. But we can go simpler...

11.10 The canonical New Keynesian model with a Taylor Rule

So far, our model has contained an exogenous rule for money growth. But this doesn't seem to match how monetary policy is conducted. We want monetary policy to focus on changing the interest rate in response to endogenous changes in inflation and output.[22] A popular interest rate rule is the Taylor Rule:[23]

$$i_t = \rho + \phi_\pi\pi_t + \phi_y\widehat{y}_t + v_t, \tag{11.74}$$

where $\widehat{y} = y_t - y$; and ϕ_π, ϕ_y, and ρ_i are coefficients, with $\phi_\pi > 1$.[24] v_t is an exogenous monetary policy shock that evolves according to the AR(1) process:

$$v_t = \rho_v v_{t-1} + \varepsilon_t^v.$$

21. Wicksell's most influential contribution was his theory of interest, originally published in German as *Geldzins und Gíterpreise*, in 1898. The English translation *Interest and Prices* became available in 1936; a literal translation of the original title would read *Money Interest and Commodity Prices*. Wicksell invented the key term natural rate of interest and defined it at that interest rate which is compatible with a stable price level.

22. We shall later see that exogenous rules for the interest rate may lead to indeterminacy.

23. Taylor's original rule was

$$i_t^F = 4 + 1.5(\pi_t - 2) + 0.5(y_t - y_t^*),$$

where i_t^F is the Federal Funds Rate, π_t is annual inflation, and y_t^* was trend (log) GDP.

24. This is crucial for determinacy. More on this soon.

Notice that we could rewrite (11.74) in terms of the output gap:

$$i_t = \rho + \phi_\pi \pi_t + \phi_y \tilde{y}_t + \phi_y \widehat{y}_t^n + v_t,$$

where $\widehat{y}_t^n \equiv y_t^n - y$. Notice that money does not enter the interest rate rule. We can replace the money growth process with the interest rate rule and assume that the central bank provides sufficient money at all times to meet money demand at the interest rate. Given the preferences households have of money – that they are additively separable – we could actually ignore money altogether and assume a cashless economy. Thus, our log-linearised New Keynesian model with a Taylor rule can be compactly written as:

$$\text{DISE: } \tilde{y}_t = \mathbb{E}_t \tilde{y}_{t+1} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t \pi_{t+1} - r_t^n\right), \tag{11.75}$$

$$\text{NKPC: } \pi_t = \beta \mathbb{E}_t \pi_{t+1} + \kappa \tilde{y}_t, \tag{11.76}$$

$$\text{Taylor Rule: } i_t = \rho + \phi_\pi \pi_t + \phi_y \widehat{y}_t + v_t, \tag{11.77}$$

$$\text{Natural Rate: } r_t^n = \rho - \sigma(1-\rho_a)\psi_{ya} a_t + (1-\rho_z) z_t. \tag{11.78}$$

This is referred to in the macroeconomic literature as the "canonical New Keynesian model", often written down with just the first three equations, with the law of motion for r_t^n in the background (since it is driven by exogenous shocks).

Just for reference, and to appreciate the compactness of the log-linearised model, the full set of [non-linear] equilibrium conditions for the cashless economy are:

$$\begin{aligned}
C_t^{-\sigma} &= \beta \mathbb{E}_t C_{t+1}^{-\sigma} \Pi_{t+1} R_t \frac{Z_{t+1}}{Z_t}, \\
\varphi_0 N_t^{\varphi} &= C_t^{-\sigma} \frac{W_t}{P_t}, \\
\frac{M_t}{P_t} &= \zeta_M \frac{R_t}{i_t Z_t} C_t^{\sigma}, \\
MC_t &= \frac{W_t/P_t}{(1-\alpha) A_t (Y_t/A_t)^{\frac{-\alpha}{1-\alpha}}}, \\
C_t &= Y_t, \\
Y_t &= \frac{A_t N_t^{1-\alpha}}{V_t^P}, \\
(V_t^P)^{\frac{1}{1-\alpha}} &= (1-\theta)(\Pi_t^*)^{-\frac{\epsilon}{1-\alpha}} \Pi_t^{\frac{\epsilon}{1-\alpha}} + \theta \Pi_t^{\frac{\epsilon}{1-\alpha}} (V_{t-1}^P)^{\frac{1}{1-\alpha}}, \\
\Pi_t^{1-\epsilon} &= (1-\theta)(\Pi_t^*)^{1-\epsilon} + \theta, \\
(\Pi_t^*)^{1+b} &= \mathcal{M} \frac{\tilde{X}_{1,t}}{\tilde{X}_{2,t}} \Pi_t^{1+b}, \\
\tilde{X}_{1,t} &= C_t^{-\sigma} MC_t Y_t + \theta \beta \mathbb{E}_t \Pi_{t+1}^{\epsilon+b} \tilde{X}_{1,t+1}, \\
\tilde{X}_{2,t} &= C_t^{-\sigma} Y_t + \theta \beta \mathbb{E}_t \Pi_{t+1}^{\epsilon-1} \tilde{X}_{2,t+1},
\end{aligned}$$

$$
\begin{aligned}
z_t &= \rho_z z_{t-1} + \varepsilon_t^z, \\
a_t &= \rho_a a_{t-1} + \varepsilon_t^a \\
i_t &= \rho + \phi_\pi \pi_t + \phi_y \tilde{y}_t + \phi_y \widehat{y}_t^n + v_t.
\end{aligned}
$$

We got rid of the law of motion for money, and thus we can get rid of $\Delta(m_t - p_t)$. So that's one less variable and one less equation. We could even get rid of m_t entirely from the model, but keeping it for our simulations will prove to be useful – we want to see how real money balances are moving in the background.

One caveat worth pointing out: we may be tempted to also think that the steady state output gap is zero, i.e., $Y = Y^n$. This will only be the case if $\pi = 0$, otherwise $Y < Y^n$. From before we know that:

$$
Y^n = \left(\frac{1-\alpha}{\varphi_0 \mathcal{M}} \right)^{\frac{1-\alpha}{\sigma(1-\alpha)+\alpha+\varphi}},
$$

and for the sticky price economy we have:

$$
N = \left[\frac{MC}{\varphi_0} (V^P)^{\frac{\sigma(1-\alpha)+\alpha}{1-\alpha}} \right]^{\frac{1}{\sigma(1-\alpha)+\alpha+\varphi}}.
$$

We know that

$$
Y = \frac{\left[\frac{MC}{\varphi_0} (V^P)^{\frac{\sigma(1-\alpha)+\alpha}{1-\alpha}} \right]^{\frac{1-\alpha}{\sigma(1-\alpha)+\alpha+\varphi}}}{V^P}
$$

which implies that steady state output is

$$
Y = \left(\frac{MC}{\varphi_0} \right)^{\frac{1-\alpha}{\sigma(1-\alpha)+\alpha+\varphi}} (V^P)^{\frac{-\varphi}{\sigma(1-\alpha)+\alpha+\varphi}}.
$$

From the expressions for $\tilde{X}_{1,t}$ and $\tilde{X}_{2,t}$ at the steady state we know:

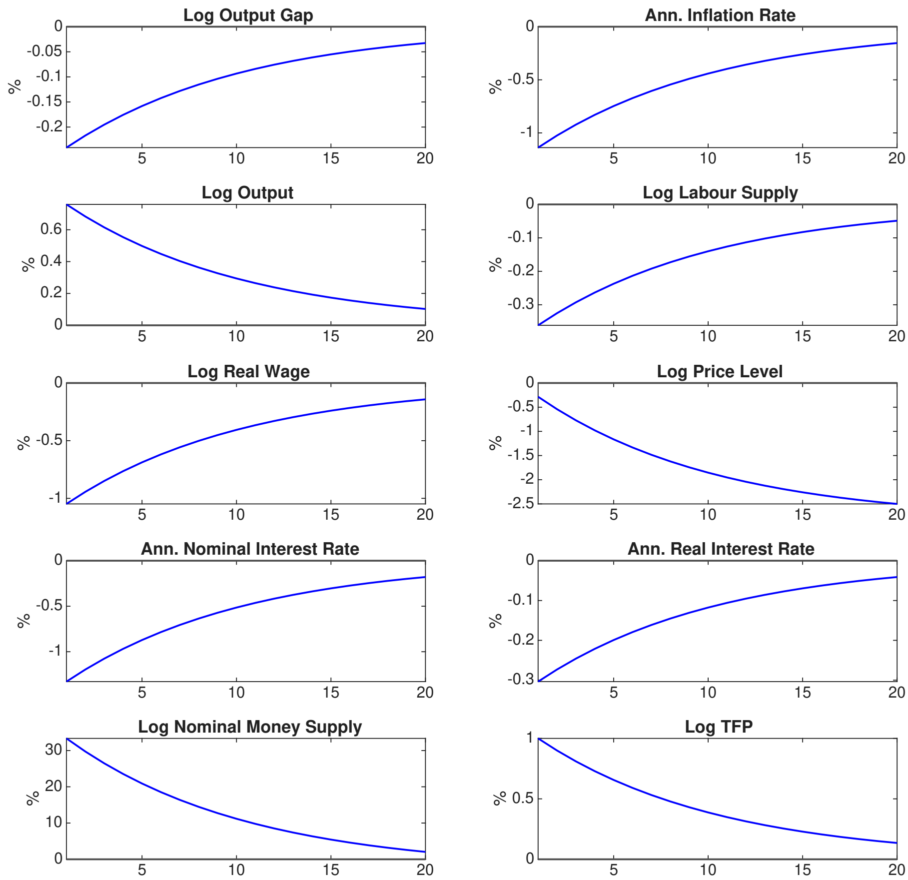
$$
MC = \mathcal{M}^{-1} \left(\frac{\Pi^*}{\Pi} \right)^{1+b} \frac{(1 - \theta\beta\Pi^{\epsilon+b})}{(1 - \theta\beta\Pi^{\epsilon-1})}
$$

If $\Pi = 1$, then $MC = \mathcal{M}^{-1}$ and $V^P = 1$, so this reduces to the same expression as Y^n, so we'll have $Y = Y^n$. But if $\Pi > 1$, you can show that $MC < \mathcal{M}^{-1}$, and we know that $v^P > 1$. Since the exponent on MC is positive, and the exponent on V^P negative, this means that $\Pi > 1$ will mean that $Y < Y^n$, which means that the steady state output gap will be negative, $\tilde{y} = y - y^n < 0$.

If you actually simulate the model with the Taylor rule against the model with the money supply rule, you will find some pretty stark differences. Consider the case of a productivity shock as shown in Figure 11.7.

Under the Taylor rule, output increases significantly more than under the money supply rule; there is a smaller drop in hours worked on impact; smaller increase in the real interest rate; lower inflation; the response of the price level seems to be more or less permanent, whereas under a

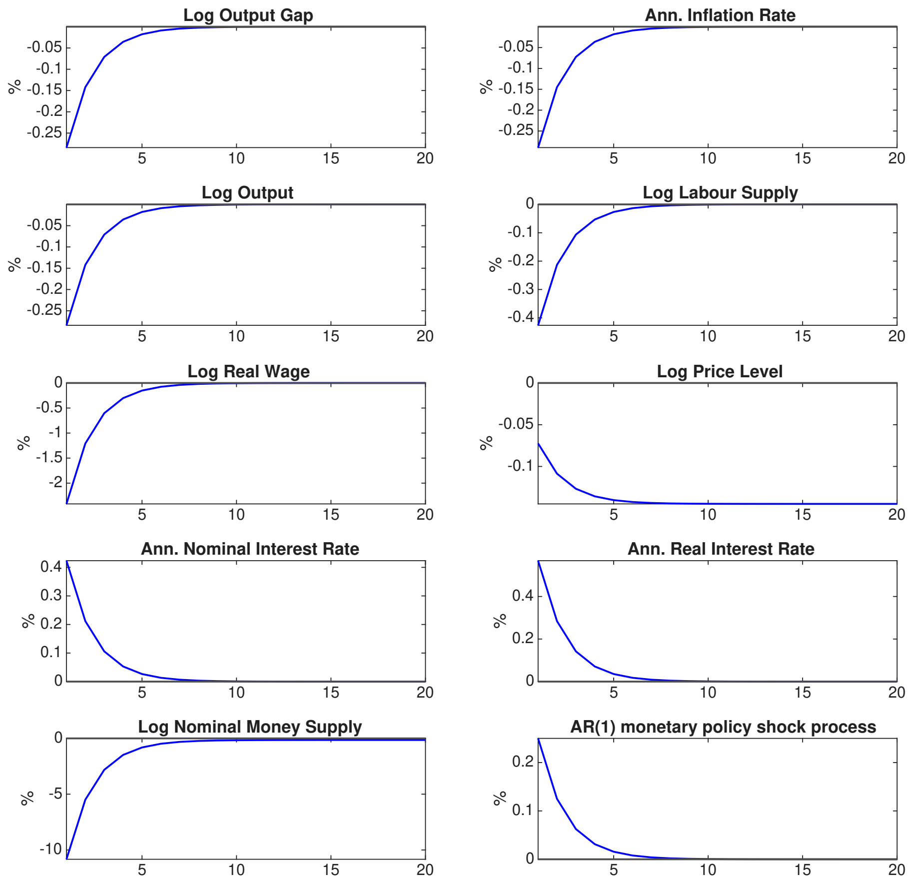
Figure 11.7: IRFs of Productivity Shock (Taylor Rule)



money supply rule it was mean reverting; and, nominal money supply increases significantly. In other words, under the Taylor rule, money supply is basically endogenous. An increase in output and transactional demand sees the central bank substantially increase nominal money balances, which also leads to an increase in real money balances. We don't have to rely on just the price level falling to get increases in real balances.

Next, consider a positive shock to the Taylor rule (a surprise monetary policy contraction), which raises the nominal interest rate. This coincides with a decline in the money supply, an increase in the real interest rate, and a decline in economic activity. IRFs are shown in Figure 11.8. The channels at play for why this nominal shock has real effects are the same as above when we thought about the nominal shock in terms of the money supply. There are two ways to think about this. First, the decrease in the money supply is matched by a less than proportional decrease in the price level because of price stickiness; this means that real balances decline, which via the basic logic previously necessitates a decline in output. It also has an effect of raising the real interest rate. The nominal rate rises, and because of price stickiness expected inflation does not rise enough, the real rate rises which leads to a reduction in demand.

Figure 11.8: IRFs of Monetary Policy Shock (Taylor Rule)

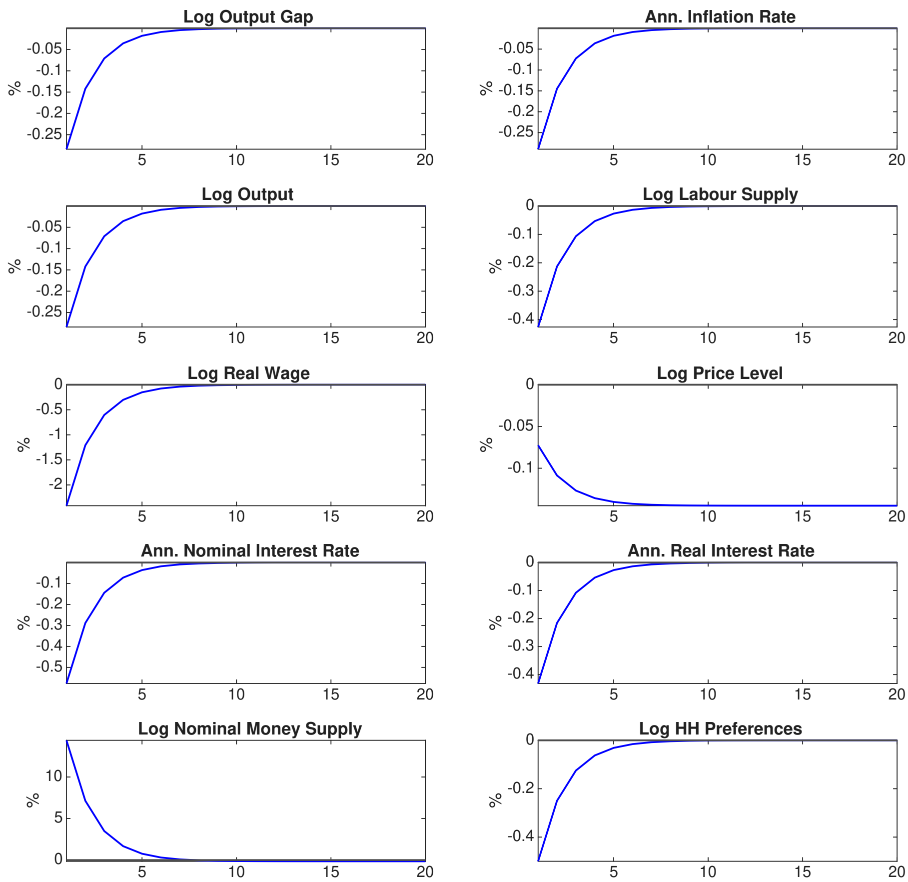


Finally, Figure 11.9 plots the dynamic responses to a household preference shock – again, we can interpret this as households increasing their patience. As before, $\varepsilon_t^z = -0.5$ percentage points and $\rho_z = 0.5$, so that $\partial r_t^n / \partial \varepsilon_t^z = (1 - \rho_z)\varepsilon_t^z = -0.25$, implying an increase on impact of one percentage point in the annualised natural rate of interest.

The decline z_t leads to a contraction in output, labour supply, inflation and the real wage. In fact, and given the normalisation of the size of the shock, the response of those variables is identical to that describing the effects of a monetary policy tightening, as in Figure 11.8.

But preference shocks and monetary policy shocks differ in the following ways: i) shifts in the discount factor have an effect on the natural rate of interest r_t^n, whereas monetary policy shocks don't; and ii) monetary policy shocks lead to changes in the nominal interest rate, for any given levels of inflation and output, whereas discount factor shocks don't.

Figure 11.9: IRFs of Preference Shock (Taylor Rule)



11.11 The method of undetermined coefficients and the Rational Expectations solution

The three equation New Keynesian (NK) model has two jump variables/control variables $(\pi_t, \tilde{y}_t)$ and an exogenous state variable, r_t^n.[25] We could solve for the policy functions mapping the states into the jump variables – Dynare will do this very easily for us. Or, since we don't have capital and investment in this model, we can use the method of undetermined coefficients.

The method of undetermined coefficients involves us guessing a policy function function (linear in this case since the system is log-linear), imposing that, and the solving a system of equations for the policy rule coefficients. In a small scale model without capital, this is pretty easy to do, and gives us nice analytical solutions.

25. If the Taylor rule contained a lagged interest rate, then it would be an endogenous state variable.

11.11.1 A simple example: Shocks with no persistence

Consider the three equation NK model with the exogenous process for r_t^n:

$$\begin{aligned}
\tilde{y}_t &= \mathbb{E}_t \tilde{y}_{t+1} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t \pi_{t+1} - r_t^n\right), \\
\pi_t &= \beta \mathbb{E}_t \pi_{t+1} + \kappa \tilde{y}_t, \\
i_t &= \rho + \phi_\pi \pi_t + \phi_y \tilde{y}_t + \phi_y \widehat{y}_t^n + v_t, \\
r_t^n &= \rho - \sigma(1-\rho_a)\psi_{ya} a_t + (1-\rho_z) z_t.
\end{aligned}$$

To demonstrate getting the policy functions and solving the model with the method of undetermined coefficients, let's shut down all shocks except for the TFP shock a_t, and furthermore let's assume that the shock is transitory so $\rho_a = 0$. We will also express our model in log deviations from steady (to eliminate some constants). Finally, we will use our definition of natural output (11.59) to replace y_t^n in the Taylor rule. So, we have:

$$\begin{aligned}
\tilde{y}_t &= \mathbb{E}_t \tilde{y}_{t+1} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t \pi_{t+1} - \hat{r}_t^n\right), \\
\pi_t &= \beta \mathbb{E}_t \pi_{t+1} + \kappa \tilde{y}_t, \\
\hat{i}_t &= \phi_\pi \pi_t + \phi_y \tilde{y}_t + \phi_y \psi_{ya} \varepsilon_t^a, \\
\hat{r}_t^n &= -\sigma \psi_{ya} \varepsilon_t^a.
\end{aligned}$$

Since i_t and r_t^n are not a state variables, we can substitute them out of the DISE:

$$\begin{aligned}
\tilde{y}_t &= \mathbb{E}_t \tilde{y}_{t+1} - \sigma^{-1}\left[\phi_\pi \pi_t + \phi_y \tilde{y}_t + \psi_{ya}(\phi_y + \sigma)\varepsilon_t^a\right], \\
\pi_t &= \beta \mathbb{E}_t \pi_{t+1} + \kappa \tilde{y}_t.
\end{aligned}$$

Now, we want to look at how our jump variables react to our shocks. First, guess that the policy functions look like:

$$\begin{aligned}
\tilde{y}_t &= A_y \varepsilon_t^a, \\
\pi_t &= A_\pi \varepsilon_t^a,
\end{aligned}$$

and then plug them into the DISE and the NKPC:

$$A_y \varepsilon_t^a = A_y \mathbb{E}_t \varepsilon_{t+1}^a - \sigma^{-1}\left[\phi_\pi A_\pi + \phi_y A_y + \psi_{ya}(\phi_y + \sigma)\right]\varepsilon_t^a, \tag{11.79}$$

$$A_\pi \varepsilon_t^a = \beta A_\pi \mathbb{E}_t \varepsilon_{t+1}^a + \kappa A_y \varepsilon_t^a. \tag{11.80}$$

Then, since the shocks are assumed to be IID with mean zero, we can simplify this down to:

$$A_y = -\sigma^{-1}\left[\phi_\pi A_\pi + \phi_y A_y + \psi_{ya}(\phi_y + \sigma)\right],$$

$$A_\pi = \kappa A_y.$$

Then we plug A_π back into the guessed DISE:

$$\begin{aligned}
A_y &= -\sigma^{-1}\left[\phi_\pi \kappa A_y + \phi_y A_y + \psi_{ya}(\phi_y + \sigma)\right] \\
A_y &= -\frac{\psi_{ya}(\phi_y + \sigma)}{\sigma + \phi_\pi \kappa + \phi_y}
\end{aligned}$$

So, our policy functions are

$$\tilde{y}_t = -\frac{\psi_{ya}(\phi_y + \sigma)}{\sigma + \phi_\pi \kappa + \phi_y}\varepsilon_t^a, \tag{11.81}$$

$$\pi_t = -\frac{\psi_{ya}(\phi_y + \sigma)\kappa}{\sigma + \phi_\pi \kappa + \phi_y}\varepsilon_t^a. \tag{11.82}$$

11.11.2 A Rational Expectations solution: Multiple shocks with persistence

Again, suppose we have the DISE, NKPC, Taylor rule, and a law of motion for the natural interest rate:

$$\begin{aligned}
\tilde{y}_t &= \mathbb{E}_t \tilde{y}_{t+1} - \frac{1}{\sigma}\left(i_t - \mathbb{E}_t \pi_{t+1} - r_t^n\right), \\
\pi_t &= \beta \mathbb{E}_t \pi_{t+1} + \kappa \tilde{y}_t, \\
i_t &= \rho + \phi_\pi \pi_t + \phi_y \tilde{y}_t + \phi_y \widehat{y}_t^n + v_t, \\
r_t^n &= \rho - \sigma(1-\rho_a)\psi_{ya} a_t + (1-\rho_z) z_t.
\end{aligned}$$

Combine these together to get:

$$\begin{aligned}
\tilde{y}_t &= \mathbb{E}_t \tilde{y}_{t+1} - \frac{1}{\sigma}\left[\underbrace{\rho + \phi_\pi \pi_t + \phi_y \tilde{y}_t + \phi_y \widehat{y}_t^n + v_t}_{i_t} - \mathbb{E}_t \pi_{t+1} \underbrace{-\rho + \sigma(1-\rho_a)\psi_{ya} a_t - (1-\rho_z) z_t}_{-r_t^n}\right], \\
\pi_t &= \beta \mathbb{E}_t \pi_{t+1} + \kappa \tilde{y}_t.
\end{aligned}$$

Do some rearranging and cleaning up to isolate the period t and $t+1$ jump variables, and define an auxiliary variable $u_t \equiv \widehat{r}_t^n - \phi_y \widehat{y}_t^n - v_t$:[26]

$$\begin{aligned}
(\sigma + \phi_y)\tilde{y}_t + \phi_\pi \pi_t &= \sigma \mathbb{E}_t \tilde{y}_{t+1} + \mathbb{E}_t \pi_{t+1} \underbrace{-\psi_{ya}(\phi_y + \sigma(1-\rho_a)) a_t + (1-\rho_z) z_t - \phi_y \widehat{y}_t^n - v_t}_{u_t}, \\
-\kappa \tilde{y}_t + \pi_t &= \beta \mathbb{E}_t \pi_{t+1}.
\end{aligned}$$

26. Note that $y = \psi_y$.

This can be written as:

$$\underbrace{\begin{bmatrix} \sigma + \phi_y & \phi_\pi \\ -\kappa & 1 \end{bmatrix}}_{\boldsymbol{A}_0} \begin{bmatrix} \tilde{y}_t \\ \pi_t \end{bmatrix} = \underbrace{\begin{bmatrix} \sigma & 1 \\ 0 & \beta \end{bmatrix}}_{\boldsymbol{A}_1} \begin{bmatrix} \mathbb{E}_t \tilde{y}_{t+1} \\ \mathbb{E}_t \pi_{t+1} \end{bmatrix} + \tilde{\boldsymbol{B}} u_t.$$

We need to then invert $\boldsymbol{A}_0$ in order to write the system as:

$$\begin{bmatrix} \tilde{y}_t \\ \pi_t \end{bmatrix} = \boldsymbol{A}_T \begin{bmatrix} \mathbb{E}_t \tilde{y}_{t+1} \\ \mathbb{E}_t \pi_{t+1} \end{bmatrix} + \boldsymbol{B}_T u_t. \tag{11.83}$$

We can do this matrix inversion very easily in MATLAB:

```
% Define symbolic variables
syms sigma phi_y phi_pi kappa rho beta;

% Define matrices A0, A1, and B symbolically
A0 = [sigma + phi_y, phi_pi; -kappa, 1];
A1 = [sigma, 1; 0, beta];
B = [1; 0];

% Compute the determinant of A0
det_A0 = det(A0);

% Compute the inverse of A0
A0_inv = inv(A0);

% Compute A0^{-1} A1
A0_inv_A1 = simplify(A0_inv * A1);

% Compute A0^{-1} B
A0_inv_B = simplify(A0_inv * B);

% Display the results
disp('A0^{-1} A1:');
disp(A0_inv_A1);
disp('A0^{-1} B:');
disp(A0_inv_B);
```

So, we get:

$$\boldsymbol{A}_T \equiv \Omega \begin{bmatrix} \sigma & 1 - \beta\phi_\pi \\ \sigma\kappa & \kappa + \beta(\sigma + \phi_y) \end{bmatrix}, \quad \boldsymbol{B}_T \equiv \Omega \begin{bmatrix} 1 \\ \kappa \end{bmatrix},$$

where $\Omega \equiv (\sigma + \phi_y + \kappa\phi_\pi)^{-1}$.

Now we can use the method of undetermined coefficients to solve for the model, and see how $\tilde{y}_t$ and π_t respond to shocks. Assume that u_t follows a stationary AR(1) process with an autoregressive coefficient $\rho_u \in [0, 1)$.

Again, propose a guess for the policy function:

$$\tilde{y}_t = A_y u_t,$$
$$\pi_t = A_\pi u_t.$$

Plugging these guesses into (11.83), and use the fact that $\mathbb{E}_t u_{t+1} = \rho_u u_t$ to get:

$$A_y = (1 - \beta\rho_u)\Lambda_u, \tag{11.84}$$
$$A_\pi = \kappa\Lambda_u, \tag{11.85}$$

where

$$\Lambda_u \equiv \frac{1}{(1 - \beta\rho_u)\left[\sigma(1 - \rho_u) + \phi_y\right] + \kappa(\phi_\pi - \rho_u)},$$

and where $\Lambda_v > 0$ if the Taylor Principle[27] holds.

11.12 The New Keynesian model with Rotemberg pricing

Under Rotemberg pricing, all intermediate goods firms are able to adjust their pricing, but with a quadratic adjustment cost. In equilibrium they all behave identically, since the adjustment costs are identical across all firms, which makes aggregation work out nicely. Whether a NKPC is derived via Calvo or Rotemberg pricing makes little difference up to a first order approximation about a zero inflation steady state.

Intermediate firms still face the same demands from final goods firms under Rotemberg pricing and they produce output according to:

$$Y_t(j) = A_t N_t(j)^{1-\alpha}.$$

An intermediate firm j's cost minimisation problem is:

$$\min_{N_t(j)} \; W_t N_t(j),$$

subject to :

$$A_t N_t(j)^{1-\alpha} \geq Y_t(j) = \left[\frac{P_t(j)}{P_t}\right]^{-\epsilon} Y_t.$$

27. We will discuss this soon.

So the Lagrangian for the firm problem is:

$$\mathcal{L} = W_t N_t(j) - \Psi_t(j) \left\{ A_t N_t(j)^{1-\alpha} - \left[\frac{P_t(j)}{P_t} \right]^{-\epsilon} Y_t \right\},$$

which yields the following FOC:

$$\begin{aligned} \mathcal{L}_{N_t(j)} :\ W_t &= (1-\alpha)\Psi_t(j) A_t N_t(j)^{-\alpha} \\ \implies \Psi_t(j) &= \frac{W_t}{(1-\alpha)A_t N_t(j)^{-\alpha}}, \end{aligned} \tag{11.86}$$

which is exactly what we had in (11.18)! The real marginal cost is also the same as before:

$$MC_t(j) = \frac{\Psi_t(j)}{P_t} = \frac{W_t/P_t}{(1-\alpha)A_t N_t(j)^{-\alpha}}$$

I mentioned before that all firms behave the same under Rotemberg pricing, and so we could simply drop the j index. But let's keep it for now and formulate the profit maximisation problem for firm j.

The nominal flow profit for producer j is given by:

$$D_t(j) = P_t(j) Y_t(j) - W_t N_t(j) - \frac{\psi}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 P_t Y_t, \tag{11.87}$$

where ψ is the Rotemberg cost of price adjustment parameter, and is measured in units of the final good. Next, write the profit function in real terms:

$$\frac{D_t(j)}{P_t} = \frac{P_t(j)}{P_t} Y_t(j) - \frac{W_t}{P_t} N_t(j) - \frac{\psi}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 Y_t,$$

and since $W_t/P_t = (1-\alpha) MC_t A_t N_t(j)^{-\alpha}$:

$$\frac{D_t(j)}{P_t} = \frac{P_t(j)}{P_t} Y_t(j) - (1-\alpha) MC_t \underbrace{A_t N_t(j)^{-\alpha} N_t(j)}_{Y_t(j)} - \frac{\psi}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 Y_t,$$

and then sub in for $Y_t(j)$:

$$\begin{aligned} \frac{D_t(j)}{P_t} &= \frac{P_t(j)}{P_t} \left[\frac{P_t(j)}{P_t} \right]^{-\epsilon} Y_t - (1-\alpha) MC_t \left[\frac{P_t(j)}{P_t} \right]^{-\epsilon} Y_t - \frac{\psi}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 Y_t \\ &= \left[\frac{P_t(j)}{P_t} \right]^{1-\epsilon} Y_t - (1-\alpha) MC_t \left[\frac{P_t(j)}{P_t} \right]^{-\epsilon} Y_t - \frac{\psi}{2} \left(\frac{P_t(j)}{P_{t-1}(j)} - 1 \right)^2 Y_t. \end{aligned}$$

So firms choose $P_t(j)$ to maximise the expected present discounted value of flow profit each period, where discounting is done by the household's stochastic discount factor:

$$\frac{\partial \frac{D_t(j)}{P_t}}{\partial P_t(j)} = (1-\epsilon)\left[\frac{P_t(j)}{P_t}\right]^{-\epsilon}\frac{Y_t}{P_t} + \epsilon(1-\alpha)MC_t\left[\frac{P_t(j)}{P_t}\right]^{\epsilon-1}\frac{Y_t}{P_t} - \psi\left(\frac{P_t(j)}{P_{t-1}(j)} - 1\right)\frac{Y_t}{P_{t-1}(j)} + \psi\mathbb{E}_t M_{t,t+1}\left(\frac{P_{t+1}(j)}{P_t(j)} - 1\right)\frac{P_{t+1}(j)}{P_t(j)}\frac{Y_{t+1}}{P_t(j)} = 0, \tag{11.88}$$

and rearrange:

$$(\epsilon-1)\left[\frac{P_t(j)}{P_t}\right]^{-\epsilon}\frac{Y_t}{P_t} = \epsilon(1-\alpha)MC_t\left[\frac{P_t(j)}{P_t}\right]^{\epsilon-1}\frac{Y_t}{P_t} - \psi\left(\frac{P_t(j)}{P_{t-1}(j)} - 1\right)\frac{Y_t}{P_{t-1}(j)} + \psi\mathbb{E}_t M_{t,t+1}\left(\frac{P_{t+1}(j)}{P_t(j)} - 1\right)\left[\frac{P_{t+1}(j)}{P_t(j)}\right]\frac{Y_{t+1}}{P_t(j)},$$

then divide both the LHS and RHS by Y_t, multiply both the LHS and RHS by P_t, and note that gross inflation $\Pi_t = \frac{P_t}{P_{t-1}}$, and since all firms behave identically, $P_t(j) = P_t$:

$$\begin{aligned}\epsilon - 1 &= \epsilon(1-\alpha)MC_t - \psi\left(\frac{P_t}{P_{t-1}} - 1\right)\frac{P_t}{P_{t-1}} + \psi\mathbb{E}_t M_{t,t+1}\left(\frac{P_{t+1}}{P_t} - 1\right)\frac{P_{t+1}}{P_t}\frac{Y_{t+1}}{P_t}\frac{P_t}{Y_t} \\ &= \epsilon(1-\alpha)MC_t - \psi(\Pi_t - 1)\Pi_t + \psi\mathbb{E}_t M_{t,t+1}(\Pi_{t+1} - 1)\Pi_{t+1}\frac{Y_{t+1}}{Y_t}.\end{aligned}$$

Phew! Now, we need to log-linearise!

$$\ln(\epsilon - 1) = \ln\left\{\epsilon(1-\alpha)MC_t - \psi(\Pi_t - 1)\Pi_t + \psi\mathbb{E}_t M_{t,t+1}(\Pi_{t+1} - 1)\Pi_{t+1}\frac{Y_{t+1}}{Y_t}\right\},$$

then totally differentiate[28]

$$0 = \frac{1}{\epsilon - 1}\left\{\epsilon(1-\alpha)dMC_t - \psi d\Pi_t + \psi\beta\mathbb{E}_t d\Pi_{t+1}\right\},$$

and since we know that in the steady state $\pi = 0$ and $d\Pi_t = \pi_t$:

$$0 = \frac{\epsilon(1-\alpha)}{\epsilon - 1}dMC_t - \frac{\psi}{\epsilon - 1}\pi_t + \frac{\beta\psi}{\epsilon - 1}\mathbb{E}_t\pi_{t+1}.$$

Now, we need to use a little trick. We know that $\frac{\epsilon}{\epsilon-1} = \mathcal{M}$ is nothing but MC^{-1}, so the first term on the RHS is:

$$\frac{(1-\alpha)dMC_t}{MC} = (1-\alpha)\frac{MC_t - MC}{MC} = (1-\alpha)\widehat{mc}_t,$$

so we have:

$$0 = (1-\alpha)\widehat{mc}_t - \frac{\psi}{\epsilon - 1}\pi_t + \frac{\beta\psi}{\epsilon - 1}\mathbb{E}_t\pi_{t+1},$$

28. I'm skipping some steps here and using the fact that $\Pi - 1 = 0$ and $M = \beta$.

$$\implies \frac{\psi}{\epsilon-1}\pi_t = (1-\alpha)\widehat{mc}_t + \frac{\beta\psi}{\epsilon-1}\mathbb{E}_t\pi_{t+1}$$
$$\therefore \pi_t = \frac{(\epsilon-1)(1-\alpha)}{\psi}\widehat{mc}_t + \beta\mathbb{E}_t\pi_{t+1}, \quad (11.89)$$

which is nothing but the NKPC, and it will match the Calvo pricing-based NKPC if:

$$\psi = \frac{(\epsilon-1)(1-\alpha)\theta}{(1-\theta)(1-\theta\beta)\Theta}.$$

Finally, under Rotemberg pricing, the aggregate resource constraint comes out to:

$$Y_t = C_t + \frac{\psi}{2}(\Pi_t - 1)^2 Y_t, \quad (11.90)$$

and log-linearising this yields:

$$\ln Y_t = \ln\left[C_t + \frac{\psi}{2}(\Pi_t-1)^2 Y_t\right]$$
$$dY_t = dC_t + \frac{\psi}{2}2(\Pi-1)Y d\pi_t + \frac{\psi}{2}(\Pi-1)^2 dY_t$$
$$\frac{dY_t}{Y} = \frac{dC_t}{Y} + \psi(\Pi-1)d\pi_t + \frac{\frac{\psi}{2}(\Pi-1)^2 dY_t}{Y},$$
$$\hat{y}_t = \frac{dC_t}{Y},$$

but $Y = C$, so:

$$\therefore \hat{y}_t = \hat{c}_t. \quad (11.91)$$

11.13 Empirical evidence on the NKPC

11.13.1 The wrong sign!

The NKPC is perhaps the central relationship in the modern approach to monetary policy analysis (as the Euler equation was known long before the NKPC became popular in the 1990s). Despite this success, there are some well known problems with it as an empirical model of inflation. A practical problem is how to measure the output gap $\tilde{y}_t$. A reasonable approach would be to assume that, on average, output tends to return to its natural rate, so the natural rate can be proxied by a simple trend (as measured, for instance, by the HP filter). So instead of $\tilde{y}_t = y_t - y_t^n$, we could use $\tilde{y}_t = y_t - y_t^{tr}$, and estimate the NKPC with data:

$$\hat{\pi}_t = \beta\mathbb{E}_t\hat{\pi}_{t+1} + \kappa\tilde{y}_t, \quad (11.92)$$

where y_t^{tr} is estimated trend output. Since we can't observe $\mathbb{E}_t\hat{\pi}_{t+1}$ we can substitute the realised $\hat{\pi}_{t+1}$ and use an estimation technique such as instrumental variables to deal with the fact this is a noisy estimator of what we really want (i.e. we're facing a classical measurement error problem).

The problem is that when we estimate (11.92), the sign of κ usually comes out negative. This is shocking to some but actually not so surprising once you work through it. We already know the "accelerationist" fact that $\Delta\hat{\pi}_t$ is negatively correlated with unemployment rate. This means that it is positively correlated with the output gap. Because $\beta \approx 1$, we can proxy $\hat{\pi}_t - \beta\mathbb{E}_t\hat{\pi}_{t+1}$ with $\hat{\pi}_t - \hat{\pi}_{t+1} = -\Delta\hat{\pi}_{t+1}$. Looked at this way, it's not too surprising that the estimated output gap coefficient is negative – another reminder that, despite their apparent similarity, the new and older Phillips Curves are very different.

There are two possible responses to this failure: Either the model is wrong the output gap measure is wrong. In a famous paper, Galí and Gertler (1999) argued the latter. They suggested that deterministic trends do a bad job in capturing movements in the natural rate of output and suggested an alternative approach.

Remember that the "correct" variable driving inflation is the ratio of marginal cost to the price level. Galí and Gertler argue for proxying marginal cost with unit labour costs W_tL_t/Y_t so that the proxy for real marginal cost is the labour share of income:

$$S_t = \frac{W_tL_t}{P_tY_t}.$$

Galí and Gertler showed that estimating:

$$\hat{\pi}_t = \beta\mathbb{E}_t\hat{\pi}_{t+1} + \gamma\hat{S}_t,$$

finds a positive γ. This is a very popular, widely-cited result – seen as putting the NKPC back on sound empirical footing.

However, Rudd and Whelan (2007) were not quite convinced of this result. They show that updating Galí and Gertler's estimates, the estimated labour share coefficient is no longer statistically significant. Also, real marginal cost should be procylical, rising when output is above potential (due to overtime compensation, production bottlenecks, and so on). Labour's share, however, has generally moved in countercyclical fashion (see Figure 11.10) – it has generally spiked upwards in recessions. Maybe output is actually above potential during recessions (negative technology shocks) but this seems unlikely. Furthermore, in many countries, there has been a downward trend in the labour share. This is now evident in the US data for the period after the Galí and Gertler (1999) study (see Figure 11.11). Naive detrending methods may have problems, but they seem to give a better proxy for output's deviation from potential than the labour.

Figure 11.10: Labour Share in the US

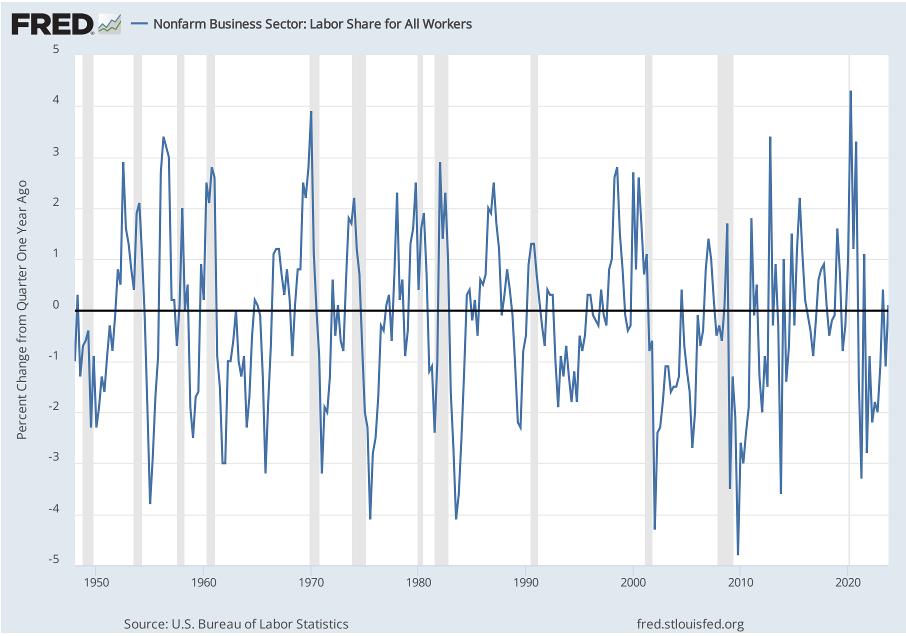
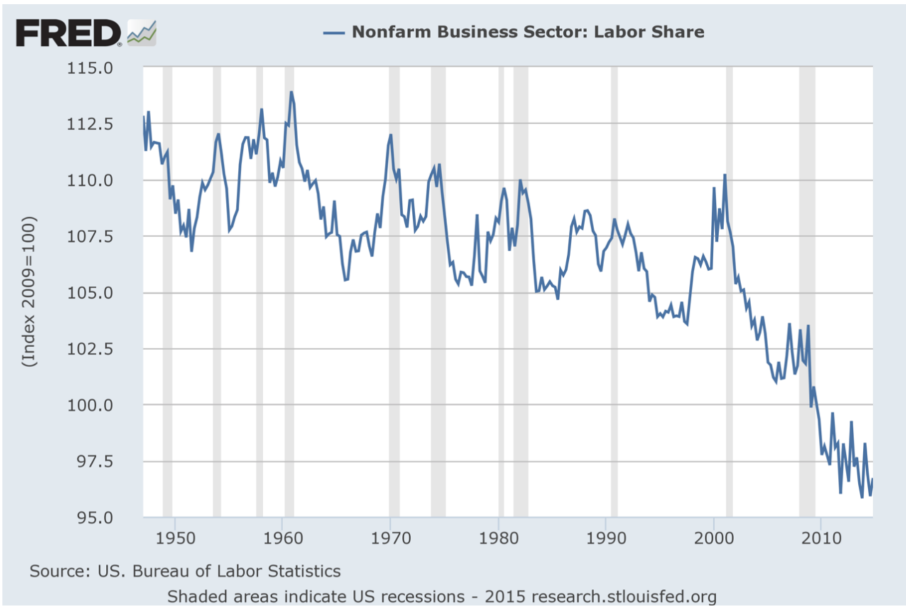


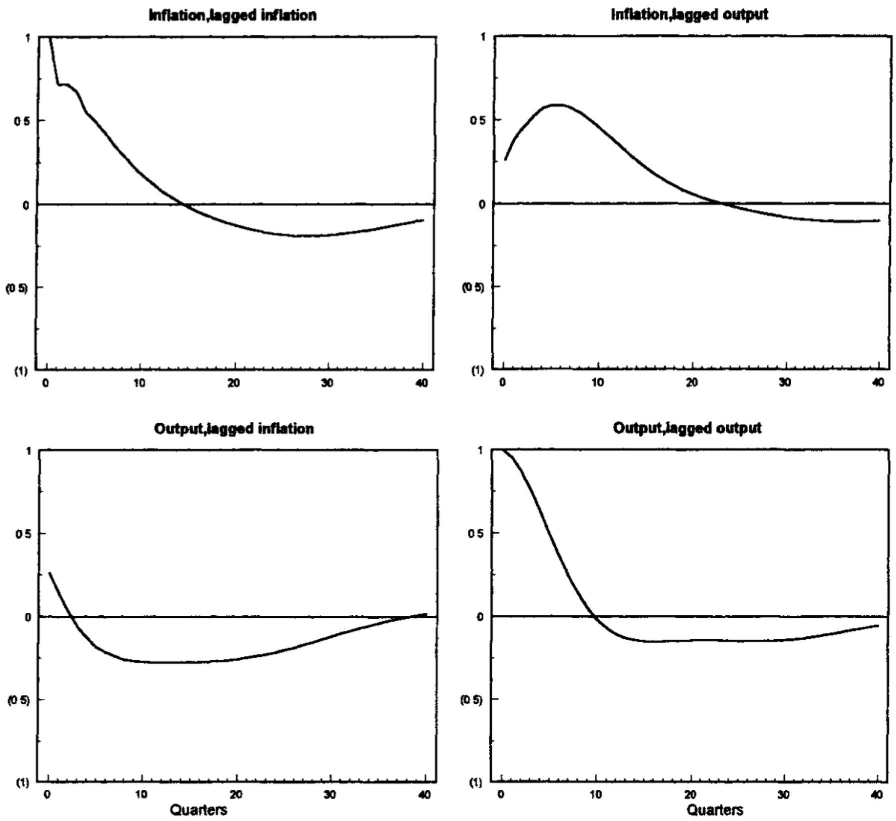
Figure 11.11: Declining Labour Share in the US



11.13.2 The inflation persistence problem

Fuhrer and Moore (1995) explored inflation persistence with a statistical VAR (unconstrained VAR in output and inflation), where they found that inflation was very inertial – its autocorrelation remains positive for about 4 years:

Figure 11.12: Autocorrelation Function, Vector Autoregression



Source: Fuhrer and Moore (1995).

Using recursion, the NKPC can be rewritten as:

$$\pi_t = \kappa \sum_{i=0}^{\infty} \beta^i \mathbb{E}_t \tilde{y}_{t+i}$$

So, according the NKPC, inflation is purely a forward-looking jump variable. There is absolutely no inflation inertia whatsoever. The canonical New Keynesian model is a model of sluggish price adjustment – not sluggish inflation adjustment. In fact, of the Phillips Curves that we have seen, the only one which implied inflation inertia was the accelerationist Phillips Curve.

This issue may be better illustrated with an example. Let's assume that $\tilde{y}_t$ follows an AR(1) process:

$$\tilde{y}_t = \rho \tilde{y}_{t-1} + e_t,$$

which would imply a solution for inflation of:

$$\pi_t = A\tilde{y}_t,$$

where A is some unknown constant. Period $t+1$ inflation would of course then be given by:

$$\mathbb{E}_t \pi_{t+1} = \mathbb{E}_t A\tilde{y}_{t+1} = \rho A\tilde{y}_t,$$

subbing this into the NKPC would give:

$$\begin{aligned} \pi_t &= \beta \mathbb{E}_t \pi_{t+1} + \kappa \tilde{y}_t \\ &= \beta\rho A\tilde{y}_t + \kappa\tilde{y}_t \\ &= (\beta\rho A + \kappa)\tilde{y}_t, \\ \implies A &= \frac{\kappa}{1-\beta\rho}. \end{aligned}$$

So we can use A and π_t to rewrite our AR(1) process as:

$$\pi_t = \rho\pi_{t-1} + Ae_t,$$

where we can see that inflation dynamics depends only on the serial correlation of $\tilde{y}_t$. There are no other endogenous mechanisms in the model to generate inflation dynamics.

One more point to consider is from Estrella and Fuhrer (2002). The NKPC implies:

$$\beta\mathbb{E}_t\pi_{t+1} - \hat{\pi}_t = -\kappa\tilde{y}_t,$$

and since $\hat{\pi}_{t+1} - \mathbb{E}_t\hat{\pi}_{t+1} = \varepsilon_{t+1}$ (the forecast error), we have:

$$\beta\pi_{t+1} - \pi_t = -\kappa\tilde{y}_t + \beta(\underbrace{\pi_{t+1} - \mathbb{E}_t\pi_{t+1}}_{\varepsilon_{t+1}}),$$

and since $\beta \approx 1$ in quarterly data, we have that:

$$\pi_{t+1} - \pi_t \approx -\kappa\tilde{y}_t + \varepsilon_{t+1}.$$

An increase in the output gap should lead to a fall in future inflation. In other words, an increase in unemployment should be associated with an increase in future/expected inflation (see the discussion above about the "wrong sign" of the NKPC).

Galí and Gertler (1999) also sought to address this in their paper using an econometric approach. Their main interest is in testing between the accelerationist and New Keynesian views. They begin by positing a "Hybrid Phillips Curve" (HPC) with backward looking and forward looking elements:

$$\pi_t = \gamma_b \pi_{t-1} + \gamma_f \mathbb{E}_t \pi_{t+1} + \kappa \widehat{mc}_t + e_t. \tag{11.93}$$

When output is above normal, marginal costs are high, which increases desired relative prices. In the model, for example, desired relative prices rise when output rises because the real wage increases. To reiterate what we said in the previous section, Galí and Gertler therefore try a more direct approach to estimating marginal costs. Real marginal cost equals the real wage divided by the marginal product of labour. If the production function is Cobb-Douglas, so that $Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$, the marginal product of labour is $(1-\alpha)Y_t/L_t$. Thus, real marginal cost is $w_t L_t/[(1-\alpha)Y_t]$, where w_t is the real wage. That is, marginal cost is proportional to the share of income going to labour.[29] Galí and Gertler therefore focus on the equation:

$$\hat{\pi}_t = \gamma_b \hat{\pi}_{t-1} + \gamma_f \hat{\pi}_{t+1} + \lambda \hat{S}_t + e_t,$$

where, as before, S_t is labour's share. Typical estimates for Galí and Gertler's methodology using quarterly US data for the period 1960-1997 are:

$$\hat{\pi}_t = \underset{(0.020)}{0.378} \hat{\pi}_{t-1} + \underset{(0.016)}{0.591} \mathbb{E}_t \hat{\pi}_{t+1} + \underset{(0.004)}{0.015} \hat{S}_t + e_t,$$

where the numbers in parentheses are standard errors. Thus their results appear to provide strong support for the importance of forward looking expectations. In a series of papers, however, Rudd and Whelan show that in fact the data provides little evidence for the NKPC and HPCs (see especially Rudd and Whelan (2005, 2006). They make two key points. The first, as previously mentioned, Rudd and Whelan dispute the inclusion of labour share to capture the rise in firms' marginal costs when output rises, finding that labour's share is low in booms and high in recessions. In Galí and Gertler's framework, this would mean that booms are times when the economy's flexible price level of output has risen even more than actual output, and when marginal costs are therefore unusually low. A much more plausible possibility, however, is that there are forces other than those considered by Galí and Gertler moving labour's share over the business cycle, and that labour's share is therefore a poor proxy for marginal costs.

Since labour's share is countercyclical, the finding of a large coefficient on expected future inflation and a positive coefficient on the share means that inflation tends to be above future inflation in recessions and below future inflation in booms. That is, inflation tends to fall in recessions and rise in booms, consistent with the accelerationist Phillips Curve and not with the NKPC.

Rudd and Whelan's second concern has to do with the information content of current inflation.

29. See also Sbordone (2002).

Replacing $\tilde{y}_t$ with a generic cost variable, $\widehat{mc}_t$, and then iterating the NKPC forward implies:

$$\begin{aligned}\pi_t &= \kappa \widehat{mc}_t + \beta \mathbb{E}_t \pi_{t+1} \\ &= \kappa \widehat{mc}_t + \beta \left[\kappa \mathbb{E}_t \widehat{mc}_{t+1} + \beta \mathbb{E}_t \pi_{t+2} \right] \\ &\vdots \\ &= \kappa \sum_{i=0}^{\infty} \beta^i \mathbb{E}_t \widehat{mc}_{t+i}. \end{aligned} \tag{11.94}$$

Thus the model implies that inflation should be a function of the expectation of future marginal costs, and thus that it should help predict marginal costs. Rudd and Whelan (2005) show, however, that the evidence for this hypothesis is minimal. When marginal costs are proxied by an estimate $\tilde{y}_t$, inflation's predictive power is small – and goes in the wrong direction as we previously discussed.

The bottom line of this analysis is twofold. First, the evidence we have on the correct form of the Phillips Curve is limited. The debate between Galí and Gertler and Rudd and Whelan, along with further analysis of the econometrics of the NKPC,[30] does not lead to clear conclusions on the basis of formal econometric studies. Second, although the evidence is not definitive, it points in the direction of inflation inertia and provides little support for the NKPC. So, despite its popularity in applications, the theoretical foundations for the various HPCs are weak. These models are just as open to the Lucas Critique as traditional ones. Nevertheless, we will explore two extremely popular approaches to the HPC.

11.14 Comments and key readings

There are plenty of references to be found throughout this chapter, and also in the texts Woodford (2003), Galí (2015), Walsh (2010), and Romer (2012). We will defer the discussion of medium-scale DSGE, such as Smets and Wouters (2003, 2007), Christiano, Eichenbaum, and Evans (2005), and Christiano, Trabandt, and Walentin (2011) to later chapters once we have gone over wage Phillips curves.

The "canonical three-equation" New Keynesian model we covered in this section is based on a log-linearisation about the zero inflation steady state. This has kept much of analysis very simple. But those interested in trend inflation should refer to Ascari (2004) and Ascari and Ropele (2009). There are also alternatives to the Calvo and Rotemberg sticky price models we worked with here: see, for example, Chari, Kehoe, and McGrattan (2000) and models of sticky information (Mankiw and Reis, 2002).

As Galí points out, the empirical performance of the NKPC has always been controversial. Many studies have criticised the NKPC for its poor performance. See, for example, Mavroeidis, Plagborg-Møller, and Stock (2014) which provides an overview of the empirical literature on the NKPC, with a focus on weak identification. But in recent years, a number of contributions have

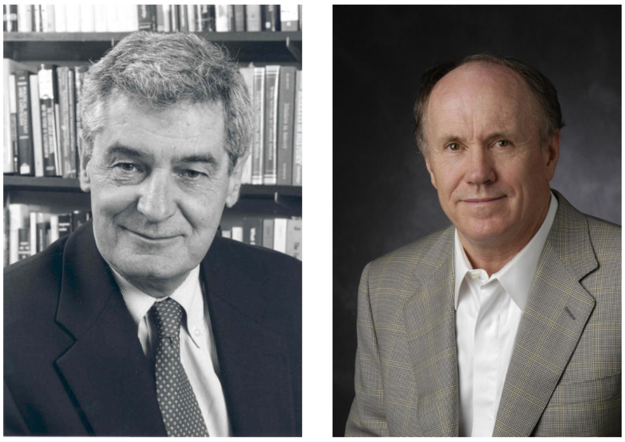
30. For example Mavroeidis (2005) and King and Plosser (2005).

found a close relationship between the empirical relevance of the NKPC when it is adjusted to account for labour markets (Benigno and Eggertsson, 2023, 2024; Siena and Zago, 2024). Perhaps for this reason, many simplified representations of the New Keynesian model model sticky wages instead of sticky goods prices.

But despite the many flaws of the NKPC and the New Keynesian DSGE framework more broadly, they have become workhorses of policy design, forecasting, and analysis in academia and especially in central banks. These DSGE models – in particular, the medium-scale New Keynesian models such as the Smets-Wouters model – have a lot of strengths: They fit the data well and allow us to answer many "what if" questions regarding policy and economic shocks.

However, they also have a list of weaknesses:

- A large number of ad-hoc economic mechanisms designed mainly to fit persistence properties of the data rather than because economists have a strong belief in these particular stories;
- A large amount of unexplained shocks which are often highly persistent;
- A minimal treatment of banking and financial markets (still true despite current ongoing work);
- Very limited modelling of policy tools or details of national accounts;
- Plenty of evidence that pure Rational Expectations assumption is flawed; and
- Claims that they are based on stable structural parameters and thus immune to the Lucas Critique are silly, and would most likely upset these two:



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Abstracting from New Keynesian models, DSGE models in general have come under heavy criticism – rather, the herd mentality and group think surrounding the DSGE models in modern macroeconomics has been heavily criticised. An example would be the paper "The Trouble with

Macroeconomics" by Paul Romer[31] – once you've read a few DSGE papers (or sat in a conference where these DSGE models are presented), I would highly recommend reading Romer's paper.

Finally, I would like to share a quote from Lawrence J. Christiano, Martin S. Eichenbaum, and Mathias Trabandt in their essay "On DSGE Models" (2018) – which was a response to critics like Paul Romer:

> The enterprise of dynamic stochastic general equilibrium modelling is an organic process that involves the constant interaction of data and theory. Pre-crisis DSGE models had shortcomings that were highlighted by the financial crisis and its aftermath. Substantial progress has occurred since then. We have emphasised the incorporation of financial frictions and heterogeneity into DSGE models. However, we should also mention that other exciting work is being done in this area, like research on deviations from conventional rational expectations. These deviations include k-level thinking, robust control, social learning, adaptive learning, and relaxing the assumption of common knowledge.
>
> Frankly, we do not know which of these competing approaches will play a prominent role in the next generation of mainstream DSGE models. Will the future generation of DSGE models predict the time and nature of the next crisis? Frankly, we doubt it. As far as we know, there is no sure, time-tested way of foreseeing the future. The proximate cause for the financial crisis was a failure across the economics profession, policymakers, regulators, and financial market professionals to recognise and to react appropriately to the growing size and leverage of the shadow-banking sector. DSGE models are evolving in response to that failure as well as to the treasure trove of micro data available to economists. We don't know where that process will lead. But we do know that DSGE models will remain central to how macroeconomists think about aggregate phenomena and policy. There is simply no credible alternative to policy analysis in a world of competing economic forces operating on different parts of the economy.

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31. https://paulromer.net/the-trouble-with-macro/WP-Trouble.pdf. See also: https://paulromer.net/trouble-with-macroeconomics-update/

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Part III

Optimal Policy

Part IV

Macro-Finance

Part V

Extensions and Closing Remarks