PhD Macroeconomics: Introduction

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3 April 2024

About me

- David Hiroyuki Murakami – but just call me David.
  - Sadly, I am not related to Haruki Murakami.
- 2nd year PhD student here at University of Milan and University of Pavia, supervised by Guido Ascari and co-supervised by Andrea Ferrero and Francesco Zanetti.
- Research: Monetary policy, financial frictions, and international macroeconomics & finance.
  - But interested in all topics in macro.
  - (Weakness: Empirical or applied work.)
- I will try to upload TA material on my `GitHub Site`.
- Shoot me an email at `david.murakami@unimi.it`.

Plan for these classes

1. Review of economic growth vs business cycles
2. Solution methods for solving DSGE models
3. Rational Expectations and the RBC model
4. The New Keynesian model
5. Topics in macro-finance
6. Using `Dynare` to solve and compute your DSGE models
7. Problem sets and/or presentations

We will have to “learn on the fly” and adjust the schedule, depending on how challenging you find the material that Guido presents.

Today's plan

1. Getting to know you.
2. Review of economic growth verse business cycles
3. The consumption Euler equation and a [very] simple general equilibrium model

Self Introduction

Tips and advice

Some friendly advice on how to survive as a PhD econ student:

- **Socialise**! Cannot stress this enough.
- Despite time constraints, try and find the time to attend seminars.
  - Always available at UniMi/Pv, but also check out Bocconi, Bicocca, and Cattolica!
  - Avoid being unaware of forefront or topical issues, especially in your field.
  - Great way to meet other researchers.
- Form bonds and friendships with your colleagues – within and across institutions.
- Try to read top journal articles regularly – even just the introductions and conclusions.
  - Attend a reading group if you need a commitment device.
- Learn to read, write, and communicate well. Poorly written paper or presentation $=$ useless.
- Get comfortable with a programming language.

Tips and advice for macro

Some handy resources and advice for this macro course:

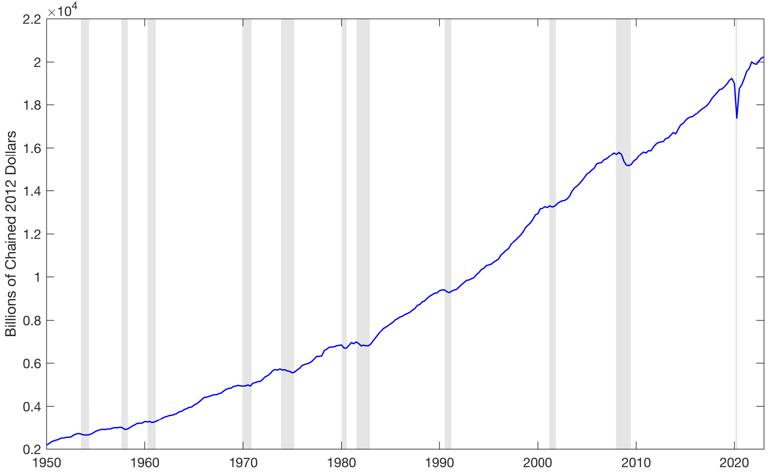
- Great textbooks: McCandless (2008), Walsh (2010), Galí (2015), Romer (2012), and Miao (2020)
- Not good [for this course]: Woodford (2003) and Ljungqvist and Sargent (2018) – controversial, I know. Don't get me wrong, LS is a great book but these are texts for maybe an advanced 1st or 2nd year grad student.
- Check out publicly available lecture notes – e.g. from distinguished professors such as Ben Moll, Thomas Drechsel, and the recent collaborative book from Azzimonti et al.
- Pay attention to CEPR, NBER, central banks, and top macro field journals such as AEJ: Macroeconomics, Journal of Monetary Economics, Journal of Economic Dynamics and Control, Journal of Money, Credit and Banking, and so on...
- Follow younger researchers on Twitter and join forums (e.g. Stack Exchange, GitHub, reddit, Dynare, etc) – lots of good advice (and bad) on there.

Let’s begin...

What is macroeconomics?

- At the heart of modern macroeconomic models is the belief that growth and "business cycles" should be explained by making explicit assumptions regarding the "deep" structural parameters of the economy, namely:
  - tastes and preferences of agents;
  - production technology; and
  - market structure.
- Today we will focus on how to represent agents in a simple economy, define the business cycle, and talk about stylised facts of economic growth.
- These observations will guide and discipline our choice of modelling, giving us a basis to assess a given model's performance.
- Finally, we will focus on the consumption Euler equation – a key equation which we will revisit time and time again.

Should we care about business cycles? I



US Real GDP. Source: FRED. Shaded regions denote NBER recession periods.

Should we care about business cycles? II

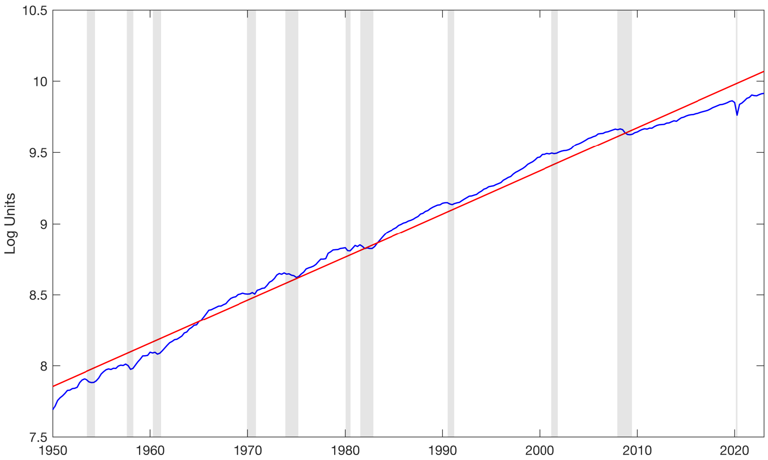
- Let's use the simplest tool, a log-linear trend, to try and break down the cyclical components of the real GDP time series to estimate the following regression

$$\ln Y_t = y_t = \alpha + gt + \epsilon_t. \tag{1}$$

- We can define the log difference in real GDP, Δy_t, as having two components: constant trend growth g and the change in the cyclical component $\Delta\epsilon_t$. We thus have:

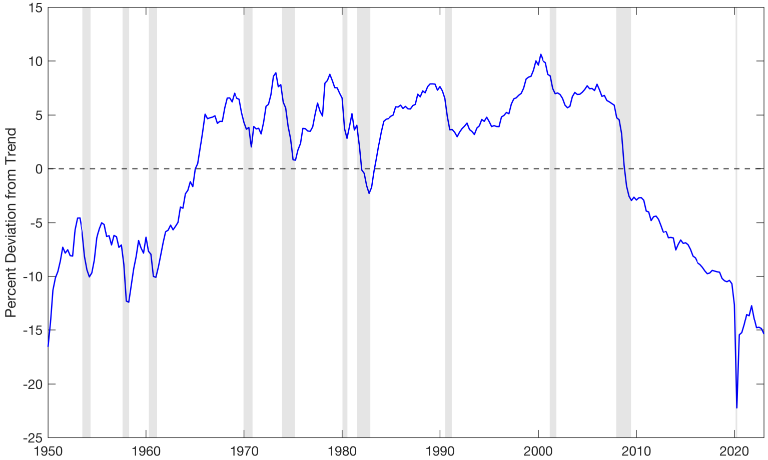
$$\begin{aligned}
\Delta y_t &= y_t - y_{t-1} \\
&= \alpha + gt + \epsilon_t - \alpha - g(t-1) - \epsilon_{t-1} \\
&= g + \epsilon_t - \epsilon_{t-1} \\
&= g - \Delta\epsilon_t.
\end{aligned}$$

Should we care about business cycles? III



SERIES AND TREND. Source: FRED. Shaded regions denote NBER recession periods.

Should we care about business cycles? IV



Deviations from Trend. Source: FRED. Shaded regions denote NBER recession periods.

Should we care about business cycles? V

- But drawing these straight lines to detrend a series can provide misleading results. For example, suppose that the correct model is

$$y_t = g + y_{t-1} + \epsilon_t, \tag{2}$$

  where growth has the constant component g and the random component ϵ_t.
- Then, the cycles here are just an accumulation of all the random shocks that have affected Δy_t over time.
- There is no tendency to revert to the trend, as the expected growth rate is always g no matter what happened in the past.
- In such a case, Δy_t is stationary: first differencing gets rid of the unit-root (non-stationary stochastic trend component) of the series.
- In this simple example, if we fit a model like (1) to a series like (2), there might appear to be mean-reverting cyclical component when there actually is not.

Should we care about business cycles? VI

- The simple takeaway is that detrending a time series – to understand the underlying trend, business cycle component, seasonality, and any other purely random fluctuations – is not as simple as fitting in a straight line.
- See the fantastic set of notes written by Dynare extraordinaire on this: Pfeifer (2013).

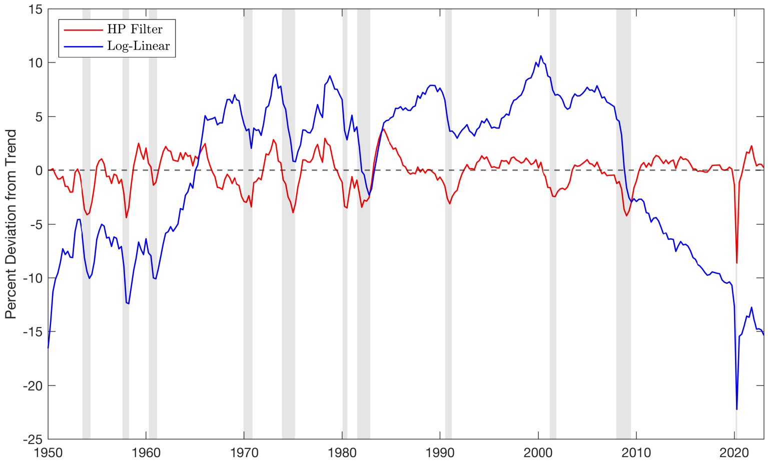
The HP Filter I

- So what can we do? Well, we can use a filter, such as the Hodrick-Prescott (HP) filter proposed by Hodrick and Prescott (1997) to try and break down a series into its various components.
- Hodrick and Prescott suggested choosing the time-varying trend Y_t^* so as to minimise the following

$$\min_{Y_t^*} \sum_{t=1}^{N} \left[(Y_t - Y_t^*)^2 + \lambda \left(\Delta Y_t^* - \Delta Y_{t-1}^* \right) \right]. \tag{3}$$

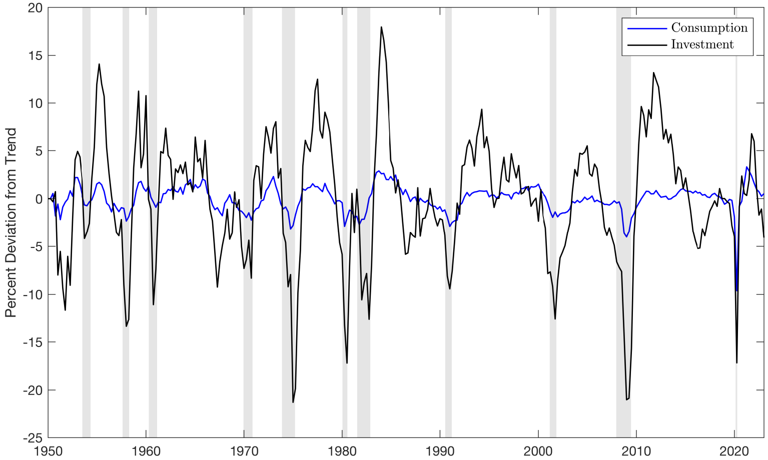
- This method tries to minimise the sum of squared deviations between output and its trend $(Y_t - Y_t^*)^2$, but it also contains a term that emphasises minimising the change in the trend growth rate, $\lambda(\Delta Y_t^* - \Delta Y_{t-1}^*)$.
- Here λ is a parameter that we have to set, and typically this is set to 1600 for quarterly data.

The HP Filter II



US REAL GDP FLUCTUATIONS. Source: FRED. Shaded regions denote NBER recession periods.

The HP Filter III

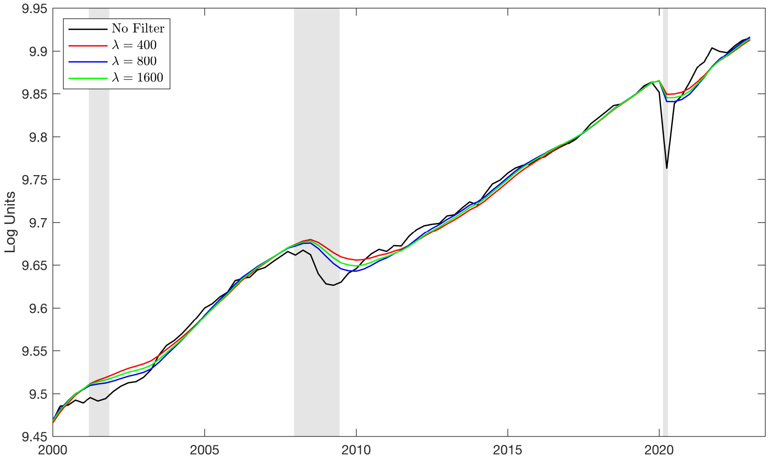


HP Filtered Consumption and Investment Fluctuations. Source: FRED. Shaded regions denote NBER recession periods.

The HP Filter IV

- As you can see, the HP-filter does seem to fit the quarterly data quite well, and that is probably one of the reasons why it has become an industry standard technique. However, there is also widespread concern about its use, mainly:
    1. Business cycle facts are not invariant to the detrending filter used.
    2. Other filters may be more optimal. A little bit of thought will reveal that if variables have different stochastic properties then a different detrending filter should be applied.
    3. The HP filter may produce spurious cycles.
- A well known result in the econometrics literature is by Nelson and Kang (1981), who show that if a linear time trend is fitted to a series which follows a random walk then the detrended data will display spurious cycles.

The HP Filter V



US GDP AND HP TRENDS. Source: FRED. Shaded regions denote NBER recession periods.

The HP Filter VI

- Even more strikingly, in the context of the Frisch-Slutsky paradigm, the HP filter can be dramatically misleading.
- Observed stylised facts about the business cycle reflect three factors:
  1. an impulse;
  2. a propagation mechanism;
  3. the data being detrended by the HP filter and the certain statistics reported.
- It can be shown that for a typical macroeconomic model 2. is unnecessary – merely assuming a process for the shock and applying the HP filter will be enough to generate business cycle patterns even if they are not there in the model.
- In other words, so called "stylised facts" are nothing more than artefacts. This is why some call the HP filter the "Hocus Pocus" filter – it can create business cycles from nothing.
- Hamilton (2018) provides a lengthy explanation of the HP filter's flaws, and provides an alternative filtering technique in his piece "Why You Should Never Use the Hodrick-Prescott Filter".

The Lucas calculation I

- Going back to the question: "should we care about business cycles?"
- How important are fluctuations away from trend growth compared to the importance of the actual growth rate g?
- Lucas considered a simple formulation to try and answer this question by looking at the "welfare cost" of business cycles.
- Suppose there are three economies: A, B, and C. Economy A grows at rate g but has business cycles, economy B grows at rate g too but does not have business cycles,

The Lucas calculation II

and lastly, economy C grows at rate $g' > g$ but has business cycle fluctuations. So to summarise,

$$c_t^A = \begin{cases} c_0(1+g)^t(1+f) & \text{w.p. } 0.5, \\ c_0(1+g)^t(1-f) & \text{w.p. } 0.5, \end{cases}$$

$$c_t^B = c_0(1+g)^t,$$

$$c_t^C = \begin{cases} c_0(1+g')^t(1+f) & \text{w.p. } 0.5, \\ c_0(1+g')^t(1-f) & \text{w.p. } 0.5. \end{cases}$$

The Lucas calculation III

- Clearly, economy B and C are better off than economy A, but the question is by how much? Suppose that the representative agent household in these economies has the following utility function:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}.$$

The Lucas calculation IV

- We can compute the welfare for economy A as follows:

$$
\begin{aligned}
\mathbb{W}^A(c_0, g, f) &= \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t^A\right) \\
&= \frac{1}{2} \sum_{t=0}^{\infty} \left(\beta^t \frac{1}{1-\sigma} \left[c_0 (1+g)^t (1+f) \right]^{1-\sigma} \right) \\
&\quad + \frac{1}{2} \sum_{t=0}^{\infty} \left(\beta^t \frac{1}{1-\sigma} \left[c_0 (1+g)^t (1-f) \right]^{1-\sigma} \right) \\
&= \frac{1}{2} \sum_{t=0}^{\infty} \beta^t \frac{1}{1-\sigma} \left(c_0 (1+g)^t \right)^{1-\sigma} \left[(1+f) + (1-f) \right]^{1-\sigma} \\
&= \frac{1}{2} \frac{c_0^{1-\sigma}}{1-\sigma} \left[\frac{(1+f)^{1-\sigma} + (1-f)^{1-\sigma}}{1-\beta(1+g)^{1-\sigma}} \right].
\end{aligned}
\tag{4}
$$

The Lucas calculation V

- Now, how do we compare this welfare to welfare in economies B and C?
- Rather, what fraction of their consumption every year would the households in economy A be prepared to give up in order to have the features of economies B or C?
- For economy B, this would mean we solve for some proportion λ^B in the following equation:

$$\mathbb{W}^A(c_0, g, f) = \mathbb{W}^B(\lambda^B c_0, g, f).$$

The Lucas calculation VI

- So, we have from (4):

$$\frac{1}{2}\frac{c_0^{1-\sigma}}{1-\sigma}\left[\frac{(1+f)^{1-\sigma}+(1-f)^{1-\sigma}}{1-\beta(1+g)^{1-\sigma}}\right]=\frac{\left(\lambda^B c_0\right)^{1-\sigma}}{1-\sigma}\frac{1}{1-\beta(1+g)^{1-\sigma}}$$

$$\implies \lambda^B=\left(\frac{1}{2}\left[(1+f)^{1-\sigma}+(1-f)^{1-\sigma}\right]\right)^{\frac{1}{1-\sigma}}. \qquad (5)$$

- If we parameterise $\beta=0.97$, $\sigma=2$, $g=0.015$, and $f=0.02$, we get $\lambda^B=0.9996$.
- What does this mean? Households in economy A would be willing to give up just 0.04% of initial consumption to eliminate fluctuations.

The Lucas calculation VII

- What about when we compare A to C? We get

$$\frac{c_0^{1-\sigma}}{1-\sigma}\frac{(1+f)^{1-\sigma}+(1-f)^{1-\sigma}}{1-\beta(1+g)^{1-\sigma}}=\frac{(\lambda^C c_0)^{1-\sigma}}{1-\sigma}\frac{(1+f)^{1-\sigma}+(1-f)^{1-\sigma}}{1-\beta(1+g')^{1-\sigma}}$$
$$\implies \lambda^C = 0.826,$$

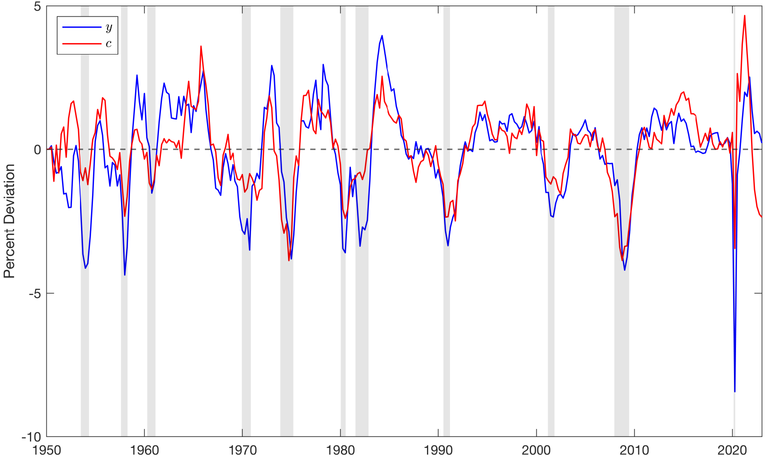
  when $g' = 0.025$.
- With this parameterisation $\lambda^C = 0.826$, which means that households in A would be willing to give up 17.4% of initial consumption to raise the rate of economic growth from 1.5% to 2.5% per year while also keeping fluctuations!
- So what does this simple exercise show? It seems to suggest that growth matters a lot more than business cycle fluctuations, which could probably be one reason why Lucas chose to focus on long-term growth rather than business cycle research.

The Lucas calculation VIII

- But there are some things that this hasn't addressed: distributional consequences of business cycles, other values of risk aversion, unemployment, and other social consequences of recessions (e.g. political instability and crime).
- So, we probably should care about business cycles.

Business Cycle Stylised Facts

Stylised facts of business cycles I



HP FILTERED CYCLES OF GDP AND CONSUMPTION. Source: FRED. Shaded regions denote NBER recession periods.

Stylised facts of business cycles II

Table CYCLICAL BEHAVIOUR OF THE US ECONOMY (1954Q1-1991Q2)

Variable	SD%	Cross-correlation of output with:								
		$t-4$	$t-3$	$t-2$	$t-1$	t	$t+1$	$t+2$	$t+3$	$t+4$
GNP	1.72	0.16	0.38	0.63	0.85	1.00	0.85	0.63	0.38	0.16
CND	0.86	0.40	0.55	0.68	0.78	0.77	0.64	0.47	0.27	0.06
CD	4.96	0.37	0.49	0.65	0.75	0.78	0.61	0.38	0.11	-0.13
H	1.59	0.09	0.30	0.53	0.74	0.86	0.82	0.69	0.52	0.32
$AveH$	0.63	0.16	0.34	0.48	0.63	0.62	0.52	0.37	0.23	0.09
L	1.14	0.04	0.23	0.46	0.69	0.85	0.86	0.76	0.59	0.40
GNP/L	0.90	0.14	0.20	0.30	0.33	0.41	0.19	0.00	-0.18	-0.25
$AveW$	0.55	0.25	0.21	0.14	0.09	0.03	-0.07	-0.09	-0.09	-0.09

Note: SD% denotes standard deviations, $t-j$ denotes the correlation between GNP at time t and the variable denoted by the first column at time $t-j$. CND stands for non-durable consumption, CD for durable consumption, H for total hours worked, $AveH$ is average hours worked per employee, L is employment, GNP/L is productivity, $AveW$ is average hourly wage based on national accounts. All unemployment data is based on household surveys. Source: "Frontiers of Business Cycle Research" (Cooley and Prescott, 1995).

Stylised facts of business cycles III

Table Business Cycle Statistics for the US Economy

Variable	SD	Relative SD	ρ	$\mathrm{corr}(\cdot, Y)$
Y	1.81	1.00	0.84	1.00
C	1.35	0.74	0.80	0.88
I	5.30	2.93	0.87	0.80
N	1.79	0.99	0.88	0.88
Y/N	1.02	0.56	0.74	0.55
w	0.68	0.38	0.66	0.12
r	0.30	0.16	0.60	-0.35
A	0.98	0.54	0.74	0.78

Note: All variables are in logarithms (with the exception of the real interest rate) and have been detrended with the HP filter. SD is standard deviation, ρ denotes a variable's first-order autocorrelation, and $\mathrm{corr}(\cdot, Y)$ is a variable's contemporaneous correlation with output. Data sources are described in Stock and Watson (1999), who also created the real rate series. Y is per capita output, C is per capita consumption, I is per capita investment, N is per capita hours, w is the real wage (compensation per hour), r is the real interest rate, and A is total factor productivity. Source: "Resuscitating Real Business Cycles" (King and Rebelo, 1999).

Stylised facts of business cycles IV

- There are six main stylised facts which emerge from Table 1:
  1. Consumption is smoother than output.
  2. Volatility in GNP is similar in magnitude to volatility in total hours.
  3. Volatility in employment is greater than volatility in average hours. Therefore most labour market adjustments operate on the extensive rather than intensive margin.
  4. Productivity is slightly pro-cyclical.
  5. Wages are less variable than productivity.
  6. There is no correlation between wages and output (nor with employment for that matter).
- Some facts that emerge from the King and Rebelo (1999) study are:
  1. Consumption of non-durables is less volatile than output.
  2. Consumer durables are more volatile than output.
  3. Investment is three times more volatile than output.
  4. Government expenditures are less volatile than output.
  5. Total hours worked are about the same volatility as output.
  6. Capital is much less volatile than output.

Stylised facts of business cycles V

7. Employment is as volatile as output, while hours per worker are much less volatile than output.
8. Labour productivity is less volatile than output
9. The real wage is much less volatile than output.

- Clearly, most macroeconomic series are pro-cyclical, exhibiting a positive contemporaneous correlation with output, and are very persistent with an autocorrelation order of roughly 0.8 to 0.9.
- There are three acyclical series: wages, government expenditures, and the capital stock.
- So, any model that we build will have to account and explain these facts, which we will soon find is quite a challenge.

Kaldor's Stylised Facts

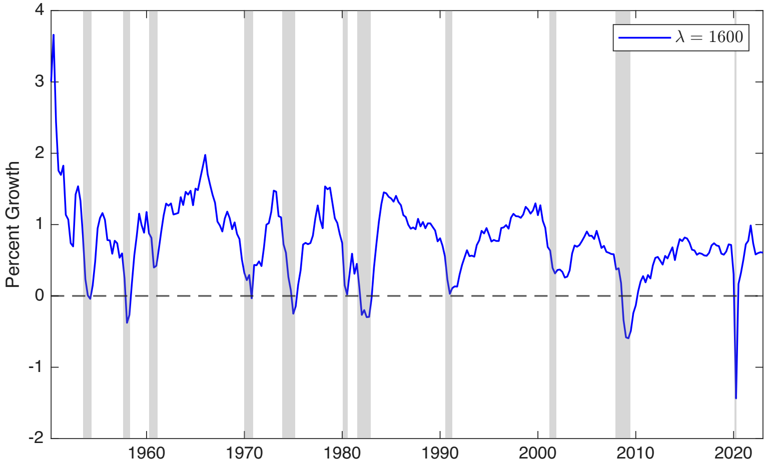
Stylised facts of economic growth I

- Statistical properties of long-term economic growth were first summarised by Kaldor (1957). These "remarkable historical constancies revealed by recent empirical investigations" quickly become known as the "Kaldor stylised facts."
- These stylised facts can be summarised as follows:
  1. Output per worker grows at a roughly constant rate that does not diminish over time. $\left(\frac{Y}{L}\right)\uparrow$
  2. Capital per worker grows over time. $\left(\frac{K}{L}\right)\uparrow$
  3. The capital/output ratio is roughly constant. $\overline{K/Y}$
  4. The rate of return to capital is constant. $\bar{r}^K$
  5. The share of capital and labour in net income are nearly constant. $\bar{\alpha}$
  6. Real wages grow over time. $w\uparrow$
  7. Constant ratios of consumption to GDP and investment to GDP. $\overline{C/Y}, \overline{I/Y}$

Stylised facts of economic growth II

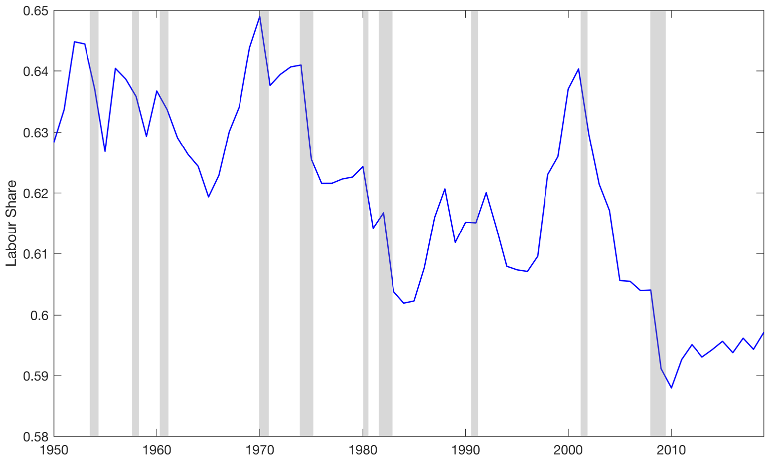
- The idea of Kaldor's stylised facts is not that these hold every period, rather that they hold when averaging data over long periods of time. This is exactly what the HP trend is designed to do, so if Kaldor is right we would expect to see fairly constant trend output per worker growth, and so on. Let's see how these stylised facts stack up:

Stylised facts of economic growth III



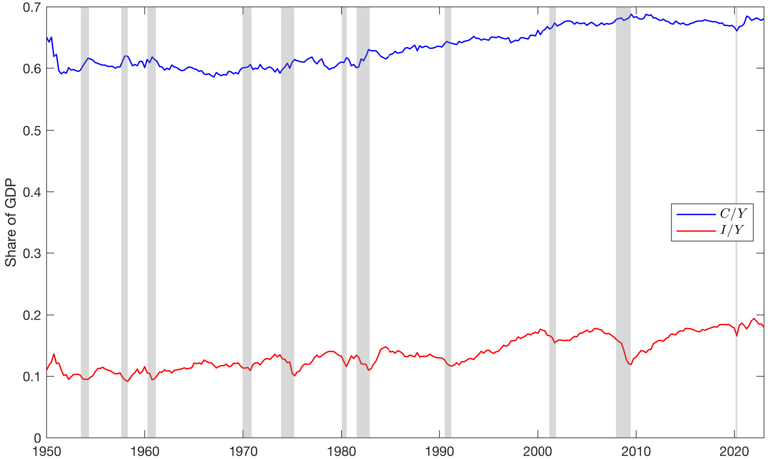
GDP Trend Growth. Source: FRED. Shaded regions denote NBER recession periods.

Stylised facts of economic growth IV



LABOUR SHARE OF OUTPUT. Source: FRED. Shaded regions denote NBER recession periods.

Stylised facts of economic growth V



Consumption and Investment Ratios. Source: FRED. Shaded regions denote NBER recession periods.

The Consumption Euler Equation and a Simple GE Model

Building a GE model I

- It's now time to begin building some models that can explain the stylised facts we've observed.
- We assume the existence of a utility function $u(c_t^i)$ where c_t^i is consumption of household i.
- Notice that utility depends only on current consumption – that is, preferences are intertemporally separable.
- Households have to make two decisions: (i) how much to spend, and (ii) how much to save.
- Households receive a gross interest rate, $R_t = 1 + r_t$, on any savings, and receive an endowment y_t^i each period.
- Both R_t and y_t^i are treated as beyond the household's control and are known with certainty into the infinite future.

Building a GE model II

- Assume that the household wishes to maximise the present value of the discounted stream of utility. That is

$$\max_{\{c^i_{t+s}, a^i_{t+s}\}} \sum_{s=0}^{\infty} \beta^s u(c^i_{t+s}), \tag{6}$$

subject to the following constraints:

$$c^i_{t+s} + a^i_{t+s} = y^i_{t+s} + R_{t+s-1} a^i_{t+s-1}, \tag{7}$$

$$\lim_{T\to\infty} \frac{a^i_T}{\prod_{s=t+1}^{T-1} R_s} = 0, \tag{8}$$

where a^i_t denotes the household's asset holdings and $\beta \in (0, 1)$ is the household's discount factor.

Building a GE model III

- Note that if we define $\tilde{R}_{t+1} = R_0 R_1 R_2 ... R_{t+1}$ for $t > 0$ then we can solve the period budget constraint (7) forward to get a present value budget constraint:

$$\sum_{t=0}^{\infty} \frac{c_t^i}{\tilde{R}_t} = a_0^i + \sum_{t=0}^{\infty} \frac{y_t^i}{\tilde{R}_t}. \tag{9}$$

- To see this, begin by writing (7) as

$$y_t^i + R_{t-1} a_{t-1}^i - c_t^i - a_t^i = 0,$$

Building a GE model IV

and then roll the budget constraint forward one period and then substitute the result for a_t^i back into the period t budget constraint:

$$
\begin{aligned}
0 &= y_{t+1}^i + R_t a_t^i - c_{t+1}^i - a_{t+1}^i \\
\implies a_t^i &= \frac{c_{t+1}^i + a_{t+1}^i - y_{t+1}^i}{R_t},
\end{aligned}
$$

put back into (7):

$$
y_t^i + R_{t-1} a_{t-1}^i - c_t^i - \left(\frac{c_{t-1}^i + a_{t-1}^i - y_{t+1}^i}{R_t} \right) = 0.
$$

Building a GE model V

- Do this again for a^i_{t+1} to get

$$y^i_t + \frac{y^i_{t+1}}{R_t} + R_{t-1}a^i_{t-1} - c^i_t - \frac{c^i_{t+1}}{R_t} - \frac{1}{R_t}\left(\frac{c^i_{t+2} + a^i_{t+2} - y^i_{t+2}}{R_{t+1}}\right) = 0$$

$$\Leftrightarrow y^i_t + \frac{y^i_{t+1}}{R_t} + \frac{y^i_{t+2}}{R_tR_{t+1}} + R_{t-1}a^i_{t-1} - c^i_t - \frac{c^i_{t+1}}{R_t} - \frac{c^i_{t+2}}{R_tR_{t+1}} - \frac{1}{R_t}\frac{1}{R_{t+1}}a^i_{t+2} = 0,$$

and eventually we have

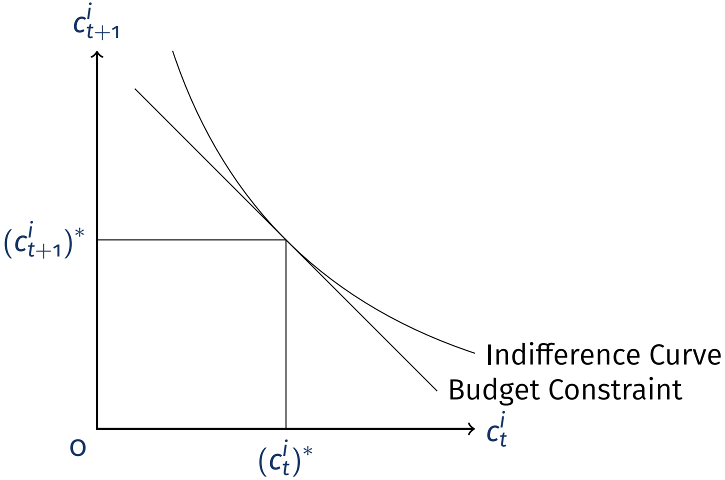
$$\sum_{s=0}^{\infty} \frac{y^i_{t+s}R_t}{\prod_{j=0}^{s} R_{t+j}} + R_{t-1}a^i_{t-1} - \sum_{s=0}^{\infty} \frac{c^i_{t+s}R_t}{\prod_{j=0}^{s} R_{t+j}} - \frac{a^i_{t+\infty}}{R_tR_{t+1}...R_{t+\infty-1}} = 0.$$

- Rearrange and assume that $t-1$ is period 0 to get (9).

How to solve the household's problem? I

- We could solve the problem graphically...
- Consider two consecutive periods, t and $t+1$, in the maximisation problem (6).
- From the utility function we can draw the indifference curves in $(c_t^i, c_{t+1}^i) \in \mathbb{R}^2$ space.

How to solve the household's problem? II



How to solve the household's problem? III

- The utility function is

$$u(c_t^i) + \beta u(c_{t+1}^i) + \sum_{s=2}^{\infty} \beta^s u(c_{t+s}^i),$$

and the slope of an indifference curve can be calculated by total differentiation of the utility function and is given by

$$0 = u_{c,t} dc_t^i + \beta u_{c,t+1} dc_{t+1}^i$$
$$\implies \frac{dc_{t+1}^i}{dc_t^i} = -\frac{1}{\beta} \frac{u_{c,t}}{u_{c,t+1}},$$

and this is what we call the marginal rate of substitution (MRS).

How to solve the household's problem? IV

- We then add the budget constraint with a slope given by iterating the budget constraint forward:

$$a_t^i = R_{t-1} a_{t-1}^i + y_t^i - c_t^i,$$
$$a_{t+1}^i = R_t a_t^i + y_{t+1}^i - c_{t+1}^i,$$

to then get

$$\frac{a_{t+1}^i - y_{t+1}^i + c_{t+1}^i}{R_t} = R_{t-1} a_{t-1}^i + y_t^i - c_t^i,$$

where it's clear that

$$-\frac{1}{R_t} dc_{t+1}^i = dc_t^i$$
$$\implies \frac{dc_{t+1}^i}{dc_t^i} = -R_t,$$

How to solve the household's problem? V

which is the marginal rate of transformation (MRT).

- Use basic microeconomic theory to justify that the solution to the household's problem is where MRS = MRT:

$$\begin{aligned} \frac{1}{\beta}\frac{u_{c,t}}{u_{c,t+1}} &= R_t \\ \Leftrightarrow u_{c,t} &= \beta R_t u_{c,t+1}, \end{aligned} \tag{10}$$

which is the consumption Euler equation.

Direct substitution/sledgehammer approach

- Next is the most brute-force method of solving the household's problem.
- Simply rearrange the budget constraint (7) to get a_t^i in terms of the other variables, and then substitute into the objective function (6):

$$\max_{\{a_{t+s}^i\}} \sum_{s=0}^{\infty} \beta^s u(R_{t+s-1}a_{t+s-1}^i + y_{t+s}^i - a_{t+s}^i).$$

- Differentiating the above summation term with respect to a_t, and setting the derivative equal to zero gives

$$-u_{c,t} + u_{c,t+1}\beta R_t = 0,$$

and after rearranging we get

$$u_{c,t} = \beta R_t u_{c,t+1},$$

which is simply (10).

Value function approach I

- This is the dynamic programming approach, which has a large range of uses in macroeconomics.
- Write the value function as

$$V(a_{t-1}^i) = \max_{a_t^i} \left[u(R_{t-1}a_{t-1}^i + y_t^i - a_t^i) + \beta V(a_t^i) \right], \quad (11)$$

  noting that a_{t-1}^i is the state variable.
- The first order condition (FOC) with respect to assets a_t^i is

$$\begin{aligned} 0 &= -u_{c,t} + \beta V'(a_t^i) \\ \implies u_{c,t} &= \beta V'(a_t^i). \end{aligned} \quad (12)$$

Value function approach II

- As is usual in dynamic programming, we do not know the form of the value function $V(a^i_{t-1})$, but we do know its first derivative $V'(a^i_{t-1})$.
- Differentiating the value function (11) yields

$$V'(a^i_{t-1}) = u_{c,t} R_{t-1},$$

and if we roll one period ahead (envelope condition)

$$V'(a^i_t) = u_{c,t+1} R_t,$$

then substitute into (12), we get

$$u_{c,t} = \beta R_t u_{c,t+1},$$

which is nothing but the consumption Euler equation.

The Lagrangian approach I

- This should also be very familiar from undergraduate macroeconomics.
- Begin by setting up the Lagrangian:

$$\mathcal{L}^i = \sum_{s=0}^{\infty} \beta^s u(c^i_{t+s}) + \sum_{s=0}^{\infty} \lambda^i_{t+s}\beta^s \left(R_{t+s-1}a_{t+s-1} + y^i_{t+s} - c^i_{t+s} - a^i_{t+s} \right).$$

- This is the present value formulation of the Lagrangian as the Lagrangian multiplier, λ^i_{t+s}, is discounted by β^s back to its present value.
- It is equally valid to work with the current value Lagrangian and write the second term without discounting, i.e. $\tilde{\lambda}^i_{t+s} = \lambda^i_{t+s}\beta^s$.
  - They are mathematically equivalent but sometimes it is more convenient to work with one than the other.

The Lagrangian approach II

- The FOCs with respect to c_t^i, c_{t+1}^i, and a_t^i are

$$
\begin{aligned}
u_{c,t} &= \lambda_t^i, \\
u_{c,t+1} &= \lambda_{t+1}^i, \\
\lambda_{t+1}^i \beta R_t - \lambda_t^i &= 0.
\end{aligned}
$$

- Do some substitution and rearranging and then we get

$$u_{c,t} = \beta R_t u_{c,t+1},$$

which is the consumption Euler equation.

Implications of the Euler equation I

- We will soon assign particular functional forms to the utility function, but from (10) we can already notice some of the major implications of the neoclassical model for consumption.
- What determines the growth in the marginal utility of consumption is the interest rate, R_t.
- In our model we have assumed that the consumer can only invest in one asset, a_t^i. However, Equation (10) holds for any asset the consumer invests in so we should think of R_t more widely as the return on any asset.
- To see this more clearly, assume that $R_t = \bar{R}$ and that $\beta\bar{R} = 1$. This then implies that $\mathbb{E}_t \left[u_{c,t+1} / u_{c,t} \right] = 1$ so that agents do not expect their marginal utility to change between time periods.
- As a consequence, they are not expecting their consumption to change either.
- The only thing which determines consumption growth is the rate of return/interest rate and not income.

Implications of the Euler equation II

- Intuition: If consumers know that savings this period are going to earn a high rate of return, there is an incentive for them to smooth their consumption by saving more now.
- For a given end of period consumption level, the lower the level of initial consumption the faster is the growth rate of consumption.

Econometric evidence on the consumption Euler equation I

- The first paper to examine the consumption Euler equation was Hall (1978).
  - He focused on utility functions which were well approximated by quadratic functions and assumed a constant interest rate which satisfies $\beta R = 1$
- The result of this model is that consumption changes should be unpredictable. This paper sparked one of the largest literature fields in applied econometrics.
- Hall found that consumption growth was not predicted by income growth, but could be forecast by stock market prices.
- He interpreted this as a mild victory for the model.
- Subsequent work has been less kind to the model and has found that consumption growth does display a small but significant dependence on past income growth.
- However, what if we let interest rates to be time varying and assume a standard form for preferences: $u(c^i) = (c^i)^{1-\sigma}/(1-\sigma)$.

Econometric evidence on the consumption Euler equation II

- With some other assumptions, the Euler equation for consumption is then:

$$\Delta \ln c^i_{t+1} = \alpha + \frac{1}{\sigma} \ln R_t + \varepsilon^i_t. \tag{13}$$

- Take a closer look at what (13) is saying though:
  - First, it's saying that consumption growth is positively correlated with changes in interest rates;
  - and secondly, if we calibrate σ to something in the range of 2 to 5, then σ^{-1} is small – somewhere in the range of 0.2 to 0.5.
  - The problem is these implications are strongly rejected by the data (Ascari, Magnusson, and Mavroeidis, 2021).
- Interest rates tend to be counter-cyclical – they tend to be high in recessions and low in growth periods – and consumption growth is strongly procyclical.
- More obviously: consumption growth should be affected by other factors besides interest rates, suggesting that α and ε^i_t in (13) are misspecified.

Taking the model to GE I

- Our simple model is nice but too flawed:
  - “Everything should be made as simple as possible, but not simpler.”
- Before we finish, let’s take it to general equilibrium though.
- Assume log utility: $u(c_t^i) = \log c_t^i$.
- consumption Euler equation is therefore

$$c_{t+1}^i = \beta R_t c_t^i.$$

- In general equilibrium all markets clear (Walras’ Law).
- Since there is no aggregate savings device, market clearing requires that individual net claims must sum to zero and $\sum_i a_t^i = 0, \ \forall t$.
- In this case, all the endowment is consumed each period and $\sum_i y_t^i = \sum_i c_t^i, \ \forall t$.

Taking the model to GE II

- When we aggregate the individual consumption Euler equations with logarithmic utility, we find that $\sum_i c^i_{t+1} = \beta R_t \sum_i c^i_t$, and hence:

$$\sum_i y^i_{t+1} = \beta R_t \sum_i y_t.$$

- Defining $\bar{y}_t$ and $\bar{y}_{t+1}$ as the average endowments in periods t and $t+1$, we see that the rate of interest is determined by the ratio of endowments in the two periods

$$\beta R_t = \frac{\bar{y}_{t+1}}{\bar{y}_t} = \frac{\bar{c}_{t+1}}{\bar{c}_t}.$$

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