PhD Macroeconomics: The Overlapping Generations Model

David Murakami

University of Milan and University of Pavia

5 April 2024

Introduction

- The endowment economy we examined previously had no firms and only financial capital.
- This isn't a crazy idea if labour is supplied inelastically and is the only input in production. However, physical capital is clearly important in the economy.
- We now allow agents in the economy to accumulate physical capital. To keep things simple, however, we will continue to assume that labour supply is exogenous.
- There are two important classes of models with capital accumulation: the overlapping generation (OLG) model and the representative agent model. We will start with the OLG model in this section.
- Peter Diamond produced a version of the OLG model introduced by Samuelson in which the savings rate is endogenous and can change with other parameters of the economy – addressing one of the biggest weaknesses of the Solow-Swan model.

Overlapping Generations

- Neoclassical growth models such as the OLG model typically have two entities (firms and households and three markets: goods, labour, and capital.
- While our analysis of the endowment economy focused mainly on the determination of interest rates, these models focus extensively on the determination of consumption and the capital stock.
- Agents in the OLG model live for two periods and must make decisions in the first period of their lives about their consumption in both periods of life.
- Sounds quite morbid, but it's a simplification we make for analytical tractability.
- People live for two periods in these economies because this is the smallest number of periods that permits a savings decision.

Basic Two-Period OLG Model

Setup and environment I

- Let there be an infinite sequence of time, $t = 0, 1, 2, ..., \infty$.
- The generation born in period t is referred to as generation t. There are $N(t)$ members of generation t, and people live for 2 periods.
- Generation t is young in t and old in $t+1$. Generation t does not exist in period $t+2$.
- A member h of generation t has utility

$$u_t^h(c_t^h(t), c_t^h(t+1)), \tag{1}$$

  where, to clarify the notation, $c_t^h(t+1)$ denotes the consumption of the aggregate consumption good by individual h of generation t in period $t+1$.
- Production takes place in competitive firms with homogenous of degree 1 (HOD1) production technology with constant returns to scale (CRS).

Setup and environment II

- Production in period t is given by

$$Y(t) = F(K(t), L(t)),$$

where $L(t)$ is the total labour used in production and $K(t)$ is the total capital.

- Individuals are endowed with lifetime endowment of labour given by

$$l_t^h = [l_t^h(t), l_t^h(t+1)].$$

- Total labour is given as

$$L(t) = \sum_{h=1}^{N(t)} l_t^h(t) + \sum_{h=1}^{N(t-1)} l_{t-1}^h(t).$$

Setup and environment III

- Aggregate labour of the young at time t is the first component of the RHS, and the aggregate labour of the old is the second component of the RHS.
- We also assume that $K(t)$ depreciates fully. This assumption removes the complication of a capital market between members of different generations.
- The economy has the following resource constraint:

$$Y(t) = F(K(t), L(t)) \geq \sum_{h=1}^{N(t)} c_t^h(t) + \sum_{h=1}^{N(t-1)} c_{t-1}^h(t) + K(t+1).$$

  The production of period t goes either to consumption of the young or the old or to capital for use in period $t+1$.
- We assume that the economic organisation of the economy is one of perfectly competitive markets where individuals are owners of their own labour.

Setup and environment IV

- Members of generation t earn income in period t by offering all their labour endowment to firms at market wage, w_t, and use income to fuel consumption in period t, to fund borrowing and lending to other members of generation t, and for accumulation of private capital.
- The budget constraint for individual h when they're young is

$$w_t l_t^h(t) = c_t^h(t) + a^h(t) + k^h(t+1), \tag{2}$$

  where $a^h(t)$ are net asset holdings of individual h.
- $a^h(t) < 0$ implies net borrowing from other members of generation t.
- Individuals cannot borrow or lend across generations so

$$\sum_{h=1}^{N(t)} a^h(t) = 0.$$

Setup and environment V

- In period $t + 1$ budget constraint for generation t individual is:

$$c_t^h(t+1) = w_{t+1} l_t^h(t+1) + R_t a^h(t) + R_{t+1} k^h(t+1), \tag{3}$$

  where R_t is the interest paid on loans between period t and $t + 1$.
- Factor prices are determined by their marginal products due to competitive equilibrium:

$$w_t = F_L(K(t), L(t)), \tag{4}$$
$$R_t = F_K(K(t), L(t)), \tag{5}$$

  where $F_i(\cdot)$ is the partial derivative of the production function with respect to its i-th component.
- Individuals are assumed to have perfect foresight, and no fraud is permitted.

Setup and environment VI

- We can combine the budget constraints of the young and old. From (2):

$$a^h(t) = w_t l_t^h(t) - c_t^h(t) - k^h(t+1),$$

and substitute this expression into (3) to get:

$$c_t^h(t+1) = w_{t+1} l_t^h(t+1) + R_t w_t l_t^h(t) - R_t c_t^h(t) - R_t k^h(t+1) + R_{t+1} k^h(t+1),$$

collecting terms, we can yield an expression for $c_t^h(t)$,

$$c_t^h(t) = \frac{w_{t+1} l_t^h(t+1) - c_t^h(t+1)}{R_t} + w_t l_t^h(t) - k^h(t+1)\left[1 - \frac{R_{t+1}}{R_t}\right].$$

Setup and environment VII

- Since we assume that there are no arbitrage opportunities, the return on capital should equal the return on loans amongst members of a particular cohort, $R_t = R_{t+1}$. Thus the budget constraint becomes:

$$c_t^h(t) + \frac{c_t^h(t+1)}{R_t} = w_t l_t^h(t) + \frac{w_{t+1} l_t^h(t+1)}{R_t}. \tag{6}$$

- In words: The present value of lifetime consumption must equal the present value of lifetime wage income.

Equilibrium I

- A competitive equilibrium consists of a sequence of prices

$$\{w_t, R_t\}_{t=0}^{\infty},$$

and quantities

$$\left\{\{c_t^h(t)\}_{h=1}^{N(t)}, \{c_{t-1}^h(t)\}_{h=1}^{N(t-1)}, K(t+1)\right\}_{t=0}^{\infty},$$

such that each member h of each generation $t > 0$ maximises utility (1) subject to their lifetime budget constraint given by (6), and so that the equilibrium conditions

$$\begin{aligned}
R_{t+1} &= R_t, \\
w_t &= F_L(K(t), L(t)), \\
R_t &= F_K(K(t), L(t)), \\
L(t) &= \sum_{h=1}^{N(t)} l_t^h(t) + \sum_{h=1}^{N(t-1)} l_{t-1}^h(t),
\end{aligned}$$

Equilibrium II

hold each period.

- Note that in the above definition we did not define the individual holdings of either lending or of capital.
- This is because they offer exactly the same return and there are an infinite number of distributions of lending and capital holdings among members of a generation that would meet the equilibrium conditions.
- Two example distributions for an economy where all members of a generation are identical are
  - i) person $h = 1$ borrows from everyone else and holds all the capital; and
  - ii) no one borrows and each person holds $K(t+1)/N(t)$ units of capital.
  - These two distributions would result in the same total capital stock and the same equilibrium as the above definition.

Getting laws of motion I

- Now, substitute the lifetime budget constraint (6) into the utility function, to set up household h of generation t's problem

$$\max_{c_t^h(t)} u\left(c_t^h(t), R_t w_t l_t^h(t) - w_{t+1} l_t^h(t+1) - R_t c_t^h(t)\right),$$

where, for individual h, the assumption of perfect foresight means that the values of all the other parameters are known. The FOC is:

$$\begin{aligned} & u_1\left(c_t^h(t), R_t w_t l_t^h(t) + w_{t+1} l_t^h(t+1) - R_t c_t^h(t)\right) \\ & \quad = R_t u_2\left(c_t^h(t), R_t w_t l_t^h(t) + w_{t+1} l_t^h(t+1) - R_t c_t^h(t)\right), \end{aligned} \tag{7}$$

Getting laws of motion II

where $u_i(\cdot,\cdot)$ is the partial derivative of the utility function with respect to its i-th element. Using the budget constraint when young (2), we can find a savings function for individual h of generation t, $s_t^h(\cdot)$, where

$$s_t^h(w_t, w_{t+1}, R_t) = a^h(t) + k^h(t+1).$$

- Summing the savings of all members of generation t, we define an aggregate savings function $S(\cdot)$, as equal to

$$S_t(\cdot) = \sum_{h=1}^{N(t)} s_t^h(\cdot) = \sum_{h=1}^{N(t)} a^h(t) + \sum_{h=1}^{N(t)} k^h(t+1).$$

Getting laws of motion III

- Given that, in equilibrium,

$$\sum_{h=1}^{N(t)} a^h(t) = 0,$$

and

$$K(t+1) = \sum_{j=1}^{N(t)} k^h(t+1),$$

the aggregate savings equation can be written as

$$S_t(w_t, w_{t+1}, R_t) = K(t+1).$$

Getting laws of motion IV

- Substituting R_{t+1} for R_t, and using the equilibrium conditions for factor prices ((4) and (5)) in periods t and $t+1$, we can write aggregate savings as:

$$S_t\left(\underbrace{F_L(K(t),L(t))}_{w_t},\underbrace{F_L(K(t+1),L(t+1))}_{w_{t+1}},\underbrace{F_K(K(t+1),L(t+1))}_{R_t}\right)=K(t+1).$$

- The above expression gives $K(t+1)$ as an implicit functions of the labour supplies in each periods, $L_t(t), L_{t-1}(t), L_t(t+1)$, the parameters of the utility functions and the production function, and $K(t)$.

Getting laws of motion V

- Since, as the model is constructed, all of these except $K(t)$ are constants through time, one can find the capital stock in $t+1$ as a function of the capital stock in time t:

$$K(t+1) = G(K(t)). \tag{8}$$

- This is a first-order difference equation/law of motion that describes the growth path of the economy.

An example OLG economy I

- Suppose that the agents in our model possessed log-utility:

$$u(c_t) = \ln c_t,$$

  and the production technology is of Cobb-Douglas form:

$$F(K(t), L(t)) = K(t)^{\alpha} L(t)^{1-\alpha}, \ \alpha \in (0, 1).$$

An example OLG economy II

- Our problem would be

$$\max_{c_t^h(t)} \ln c_t^h(t) + \beta \ln c_t^h(t+1),$$

and using our lifetime budget constraint (6) we can write this as

$$\max_{c_t^h(t)} \ln c_t^h(t) + \beta \ln \left(R_t w_t l_t^h(t) - w_{t+1} l_t^h(t+1) - R_t c_t^h(t) \right),$$

and with the following FOC:

$$0 = \frac{1}{c_t^h(t)} - \frac{\beta R_t}{\underbrace{R_t w_t l_t^h(t) - w_{t+1} l_t^h(t+1) - R_t c_t^h(t)}_{c_t^h(t+1)}}$$

$$\implies 1 = \beta \frac{R_t c_t^h(t)}{c_t^h(t+1)}.$$

An example OLG economy III

- The above equation is nothing but the consumption Euler equation. Now, substitute the optimal consumption given by the Euler equation back into the budget constraint:

$$
\begin{aligned}
c_t^h(t) + \frac{c_t^h(t+1)}{R_t} &= w_t l_t^h(t) + \frac{w_{t+1} l_t^h(t+1)}{R_t} \\
\implies \quad c_t^h(t) + \frac{1}{R_t}\left[\beta R_t c_t^h(t)\right] &= w_t l_t^h(t) + \frac{w_{t+1} l_t^h(t+1)}{R_t} \\
c_t^h(t)(1+\beta) &= w_t l_t^h(t),
\end{aligned}
$$

where we also assume that the agent does not work when they're old, so we have

$$
c_t^h(t) = \frac{w_t l_t^h(t)}{1+\beta}. \tag{9}
$$

An example OLG economy IV

- Now that have consumption per period for an individual h of generation t in period t, we want to pin down aggregate savings, which help us get the law of motion of capital in this model. But first, we need our factor prices:

$$\frac{\partial Y(t)}{\partial K(t)} = R_t = \alpha \left[\frac{K(t)}{L(t)}\right]^{\alpha-1} = \alpha k(t)^{\alpha-1},$$
$$\frac{\partial Y(t)}{\partial L(t)} = w_t = (1-\alpha) \left[\frac{K(t)}{L(t)}\right]^{\alpha} = (1-\alpha) k(t)^{\alpha},$$

  and from our household FOC, we have

$$c_t^h(t) = \frac{w_t l_t^h(t)}{1+\beta} = \frac{l_t^h(t)}{1+\beta}(1-\alpha)k(t)^{\alpha},$$

An example OLG economy V

and aggregating across the cohort yields

$$
\begin{aligned}
C_t(t) &= \frac{1}{1+\beta}(1-\alpha)K(t)^{\alpha}L(t)^{1-\alpha} \\
&= \left(\frac{1-\alpha}{1+\beta}\right)Y(t).
\end{aligned}
$$

- So savings is given by

$$
\begin{aligned}
S(t) &= Y(t) - C_t(t) \\
&= Y(t) - \left(\frac{1-\alpha}{1+\beta}\right)Y(t) \\
&= \left(\frac{\alpha+\beta}{1+\beta}\right)Y(t),
\end{aligned}
$$

An example OLG economy VI

and since $S(t) = K(t+1)$,

$$K(t+1) = \left(\frac{\alpha+\beta}{1+\beta}\right) Y(t).$$

- If we assume that labour is supplied inelastically by the young, then the law of motion of capital can be written as

$$K(t+1) = \left(\frac{\alpha+\beta}{1+\beta}\right) K(t)^{\alpha}. \tag{10}$$

An example OLG economy VII

- The steady-state capital stock, $\bar{K}$, is given by

$$\begin{aligned} \bar{K} &= \left(\frac{\alpha+\beta}{1+\beta}\right)\bar{K}^{\alpha} \\ \bar{K}^{1-\alpha} &= \left(\frac{\alpha+\beta}{1+\beta}\right) \\ \therefore \bar{K} &= \left(\frac{\alpha+\beta}{1+\beta}\right)^{\frac{1}{1-\alpha}} \quad . \end{aligned} \tag{11}$$

- In other words, $\bar{K}$ satisfies the condition $\Delta K(t+1) = 0$:

$$\Delta K(t+1) = 0 = K(t+1) - K(t),$$

and this satisfies these conditions for two values of $\bar{K}$: $\bar{K} = 0$ and the value for $\bar{K}$ in (11).

An example OLG economy VIII

- Actually, also, we could log-linearise the law of motion of capital (10):

$$\ln K(t+1) = \ln\left(\frac{\alpha+\beta}{1+\beta}\right) + \alpha \ln K(t)$$
$$\ln \bar{K} + \frac{1}{\bar{K}}(K(t+1) - \bar{K}) \approx \ln\left(\frac{\alpha+\beta}{1+\beta}\right) + \alpha \ln \bar{K} + \frac{\alpha}{\bar{K}}(K(t) - \bar{K}),$$

  and we know from (11) that in the steady-state $\ln \bar{K} = \ln\left(\frac{\alpha+\beta}{1+\beta}\right) + \alpha \ln \bar{K}$, so we have:

$$\frac{1}{\bar{K}}(K(t+1) - \bar{K}) = \frac{\alpha}{\bar{K}}(K(t) - \bar{K})$$
$$\therefore \hat{K}(t+1) = \alpha \hat{K}(t).$$

Convergent dynamics I

- The behaviour of this model out of a steady-state is similar to that of the Solow-Swan model.
- If the initial capital stock is between the two steady-states, $0 < K(0) < \left(\frac{\alpha+\beta}{1+\beta}\right)^{\frac{1}{1-\alpha}}$, the capital stock will grow, converging on the positive steady-state.
- This can be seen by simply looking for the range of initial capital stocks for which $K(t+1) > K(t)$, or where

$$\Delta K(t+1) = \left(\frac{\alpha+\beta}{1+\beta}\right) K(t)^{\alpha} - K(t) > 0.$$

- This condition holds for positive $K(t)$ when $K(t) < \left(\frac{\alpha+\beta}{1+\beta}\right)^{\frac{1}{1-\alpha}}$.

Convergent dynamics II

- In addition, the rate of growth of the capital stock declines as it grows. Define the gross rate of growth of capital as $\Delta K(t) = K(t+1)/K(t)$. This can be written as

$$\Delta K(t) = \left(\frac{\alpha+\beta}{1+\beta}\right)\frac{K(t)^{\alpha}}{K(t)} = \left(\frac{\alpha+\beta}{1+\beta}\right)K(t)^{\alpha-1}.$$

- Taking the derivative of the growth rate with respect to the capital stock yields:

$$\frac{d\Delta K(t)}{dK(t)} = (\alpha-1)\left(\frac{\alpha+\beta}{1+\beta}\right)K(t)^{\alpha-2} < 0.$$

- As in the Solow-Swan model, the larger the initial capital stock, the slower the growth rate of capital.

Convergent dynamics III

- In addition, since output is defined by a Cobb-Douglas technology, the gross growth rate of output, $\Delta Y(t) = Y(t+1)/Y(t)$ is equal to

$$\Delta Y(t) = \frac{K(t+1)^{\alpha} L(t+1)^{1-\alpha}}{K(t)^{\alpha} L(t)^{1-\alpha}} = \frac{K(t+1)^{\alpha}}{K(t)^{\alpha}} = \Delta K(t)^{\alpha},$$

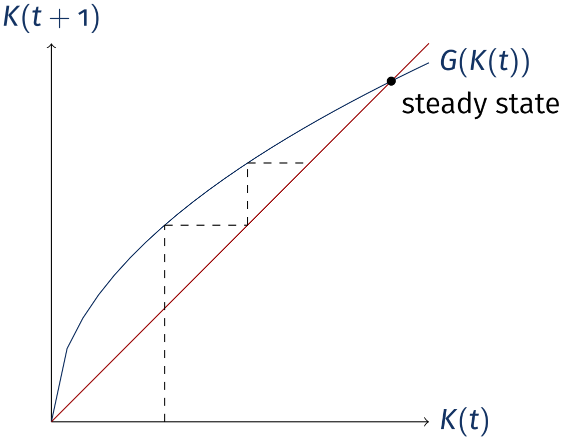
where the second equality is given because we have inelastic labour supply equal to unity.

- The derivative of the gross growth rate of output with respect to the capital stock is:

$$\frac{d\Delta Y(t)}{dK(t)} = \frac{d\left[\left(\frac{\alpha+\beta}{1+\beta}\right) K(t)^{\alpha-1}\right]^{\alpha}}{dK(t)} = \alpha \left[\left(\frac{\alpha+\beta}{1+\beta}\right) K(t)^{\alpha-1}\right]^{\alpha-1} (\alpha-1) \left(\frac{\alpha+\beta}{1+\beta}\right) K(t)^{\alpha-2} < 0,$$

so output growth slows as the capital stock increases.

Convergent dynamics IV



Convergent dynamics V

- As the figure shows, we have monotonic convergent dynamics, although there is a possibility of dynamic inefficiency as it may be possible to generate Pareto improvements by transferring resources from each young generation to the current old generation.
- ...which is what a government can do, right?

Fiscal policy and non-Ricardian equivalence I

- With the possibility of efficiency gains by reallocating resources from the old to the young, we now amend the basic OLG model slightly to see the effects of government spending. To make things simple, I will use a change of notation here.
- Let fiscal policy in the two period OLG model be given by $\{G_t, T_t^0, T_t^1, B_t\}$ where G_t is government spending which directly benefits the young (such as schooling), T_t^0 is taxes on the young, T_t^1 is taxes on the old, B_t is government debt.
- Interest on government debt is given by $R_t^b = R_t$, so the government budget constraint is:

$$B_{t+1} = G_t - T_t^0 - T_t^1 + R_t^b B_t.$$

Fiscal policy and non-Ricardian equivalence II

- The household problem is now:

$$\max_{C_t^0, C_{t+1}^1} \; u(C_t^0 + G_t) + \beta u(C_{t+1}^1),$$

subject to

$$C_t^0 + B_{t+1}^1 + K_{t+1}^1 = w_t - T_t^0, \tag{12}$$

$$C_{t+1}^1 = (B_{t+1}^1 + K_{t+1}^1)R_{t+1} - T_t^1, \tag{13}$$

where B_{t+1}^1 is saving by the household in the form of government debt.

Fiscal policy and non-Ricardian equivalence III

- Assuming a well-behaved utility function and $T_t^1 = 0$ so there are no taxes on the old, the FOC with respect to K_{t+1}^1 yields the consumption Euler equation:

$$C_{t+1}^1 = \beta \underbrace{R_{t+1}}_{F_K(K_{t+1}^1)} (C_t^0 + G_t), \tag{14}$$

  and with the household budget constraints and market clearing factor prices, we can find the law of motion of capital. From (13) and (14), we have

$$(B_{t+1}^1 + K_{t+1}^1)R_{t+1} = \beta R_{t+1}(C_t^0 + G_t),$$

Fiscal policy and non-Ricardian equivalence IV

and then substitute in the value for C_t^o from (12),

$$\begin{aligned}
(B_{t+1}^1 + K_{t+1}^1)R_{t+1} &= \beta R_{t+1}(w_t - T_t^o - B_{t+1}^1 - K_{t+1}^1 + G_t) \\
(B_{t+1}^1 + K_{t+1}^1)R_{t+1} + \beta R_{t+1}K_{t+1}^1 &= \beta R_{t+1}(w_t - T_t^o - B_{t+1}^1 + G_t) \\
B_{t+1}^1 + K_{t+1}^1 + \beta K_{t+1}^1 &= \beta(w_t - T_t^o - B_{t+1}^1 + G_t) \\
K_{t+1}^1 &= \frac{\beta}{1+\beta}(w_t - T_t^o - B_{t+1}^1 + G_t) - \frac{B_{t+1}^1}{1+\beta},
\end{aligned}$$

which gives:

$$K_{t+1}^1 = \frac{\beta}{1+\beta}\left(F_L(K_t^1) - T_t^o + G_t\right) - B_{t+1}^1.$$

- Ricardian equivalence states that it does not matter whether a given sequence of government spending is funded through taxes or debt.

Fiscal policy and non-Ricardian equivalence V

- To see where this fails in OLG models, we fix the sequence of government spending to $G_t = \bar{G}$ at time t, and $G_{t+i} = 0, \ \forall i > 0$.
- Now consider two financing schemes:
  * The first funds the one-off government spending by a tax on the young so that $T_t^o = G_t$.
  * The second places no taxes in period t and instead borrows $B_{t+1} = G_t$ and taxes the young $R_{t+1}^b G_t$ in period $t+1$ to repay the debt.
- If Ricardian equivalence holds then these two financing schemes will have equivalent aggregate effects.
- In the first case, government policy does nothing:

$$K_{t+1}^1 = \frac{\beta}{1+\beta}\left(F_L(K_t^1) - T_t^o + G_t\right) = \frac{\beta}{1+\beta}F_L(K_t^1),$$

so if the economy is in steady-state at time t it will stay there.

Fiscal policy and non-Ricardian equivalence VI

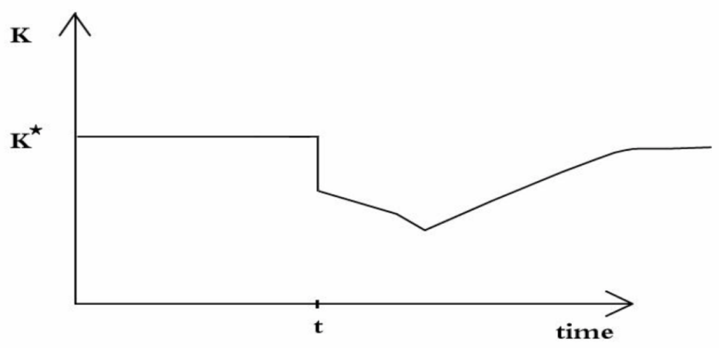
- In contrast, with the second policy we have that:

$$
\begin{aligned}
K_{t+1}^{1} &= \frac{\beta}{1+\beta} F_L(K_t^1) - \frac{1}{1+\beta} G_t \\
K_{t+2}^{1} &= \frac{\beta}{1+\beta} \left(F_L(K_t^1) - T_{t+1}^{o} \right) \\
K_{t+i}^{1} &= \frac{\beta}{1+\beta} F_L(K_{t+i-1}^1), \ \forall i > 3.
\end{aligned}
$$

- So financing matters and Ricardian equivalence fails to hold in the OLG model.
- If the economy starts in steady-state then it will deviate from steady-state for several periods:

Fiscal policy and non-Ricardian equivalence VII

Figure Path of Capital Stock After a Bond Financed Fiscal Expansion



Adding Technological Change I

- The model so far has abstracted from technological change. In general, we can think of technological change as entering the production function $Y_t = F(K_t, L_t, \theta_t)$, where θ_t is a technology term that is given exogenously.
- Fortunately, there are cases where this allows for a simple characterisation of a balanced growth path that satisfies the Kaldor facts.
- Example production functions which account for technological changed are:
  * The Hicks-neutral production function $Y_t = \theta_t F(K_t, L_t)$;
  * The capital-augmenting production function $Y_t = F(\theta_t K_t, L_t)$; and
  * The labour-augmenting production function $Y_t(K_t, \theta_t L_t)$.
- The latter of these technologies allows a characterisation consistent with the Kaldor stylised facts!

Adding Technological Change II

- Consider the labour augmenting production function. Also, let's assume that households in that model possess a constant-relative risk aversion (CRRA) utility function of the form

$$U(c) = \frac{c^{1-\sigma}}{1-\sigma},$$

and let's assume that the law of motion for technological change is:

$$\theta_{t+1} = (1+g)\theta_t,$$

where g is the growth rate of the economy.

Adding Technological Change III

- Now, in a model with exogenous labour, we have the following equilibrium condition (from the consumption Euler equation):

$$
\begin{aligned}
u'(C_t) &= \beta R_{t+1} u'(C_{t+1}) \\
u'(w_t - K^1_{t+1}) &= \beta R_{t+1} u'(K^1_{t+1} R_{t+1}) \\
u'(\theta_t F_L(K^1_t, \theta_t L_t) - K^1_{t+1}) &= \beta F_K(K^1_{t+1}, \theta_{t+1} L_{t+1}) u'(K_{t+1} F_K(K^1_{t+1}, \theta_{t+1} L_{t+1})) \\
u'\left(\theta_t F_L\left(\frac{K^1_t}{\theta_t}, 1\right) - K^1_{t+1}\right) &= \beta F_K\left(\frac{K^1_{t+1}}{\theta_{t+1}}, 1\right) u'\left(K^1_{t+1} F_K\left(\frac{K^1_{t+1}}{\theta_{t+1}}, 1\right)\right), \qquad (15)
\end{aligned}
$$

since labour is exogenous at $L_t = 1,\ \forall t$.

Adding Technological Change IV

- To see why we can write this, consider the case where Y_t is a Cobb-Douglas technology:

$$
\begin{aligned}
F(K_t, \theta_t L_t) &= Y_t = K_t^{\alpha}(\theta_t L_t)^{1-\alpha}, \\
\implies F_L(K_t, \theta_t L_t) &= (1-\alpha) K_t^{\alpha} \theta_t^{1-\alpha} L_t^{-\alpha} \\
&= (1-\alpha) \left(\frac{K_t}{L_t}\right)^{\alpha} \theta_t^{1-\alpha} \\
\implies F_L(K_t, \theta_t) &= (1-\alpha) K_t^{\alpha} \theta_t^{1-\alpha} \\
&= (1-\alpha) \frac{K_t^{\alpha}}{\theta_t^{\alpha-1}},
\end{aligned}
$$

Adding Technological Change V

and divide through by θ_t to get:

$$
\begin{aligned}
(1-\alpha)\frac{K_t^{\alpha}}{\theta_t^{\alpha-1}}\theta_t^{-1} &= (1-\alpha)\left(\frac{K_t}{\theta_t}\right)^{\alpha} \\
&= F_L\left(\frac{K_t}{\theta_t}, 1\right).
\end{aligned}
$$

- We could also easily show this for marginal product of capital.
- It's also worth noting that the model will have a balanced growth path property if it can be written as a dynamic equation in K_t/θ_t.
- When the production function has constant returns to scale then derivatives are HOD0, which means that a doubling of all inputs leads to a doubling of output, and doubling of factor prices and resources has no effect in input demand.

Adding Technological Change VI

- Now, assuming we have CRRA utility, (15) yields:

$$\left(\theta_t F_L\left(\frac{K_t^1}{\theta_t},1\right) - K_{t+1}^1\right)^{-\sigma} = \beta F_K\left(\frac{K_{t+1}^1}{\theta_{t+1}},1\right)\left[K_{t+1}^1 F_K\left(\frac{K_{t+1}^1}{\theta_{t+1}},1\right)\right]^{-\sigma},$$

divide through by $\theta_{t+1}^{-\sigma}$ from our law of motion of technological growth:

$$\begin{aligned}
&\left(\frac{\theta_t}{\theta_{t+1}} F_L\left(\frac{K_t^1}{\theta_t},1\right) - \frac{K_{t+1}^1}{\theta_{t+1}}\right)^{-\sigma} = \beta F_K\left(\frac{K_{t+1}^1}{\theta_{t+1}},1\right)\left[\frac{K_{t+1}^1}{\theta_{t+1}} F_K\left(\frac{K_{t+1}^1}{\theta_{t+1}},1\right)\right]^{-\sigma},\\
\Leftrightarrow &\left(\frac{1}{1+g} F_L\left(\frac{K_t^1}{\theta_t},1\right) - \frac{K_{t+1}^1}{\theta_{t+1}}\right)^{-\sigma} = \beta F_K\left(\frac{K_{t+1}^1}{\theta_{t+1}},1\right)\left[\frac{K_{t+1}^1}{\theta_{t+1}} F_K\left(\frac{K_{t+1}^1}{\theta_{t+1}},1\right)\right]^{-\sigma},
\end{aligned}$$

which is now a first order difference equation in K_t/θ_t with only a minor change compared to what we saw in the baseline OLG model without technological change.

Adding Technological Change VII

- If we assumed log utility, where $\sigma = 1$ and a Cobb-Douglas production technology, then the law of motion for capital relative to technology becomes:

$$\frac{K^1_{t+1}}{\theta_{t+1}} = \frac{\beta(1-\alpha)}{(1+\beta)(1+g)} \left(\frac{K^1_t}{\theta_t}\right)^\alpha.$$

- Not much has changed, however. $g > 0$ changes the speed of convergence but nothing else.
- The model with exogenous technological change has the same qualitative properties as the model without it.

Comments and key readings

- An OLG model allows us to make the savings decision endogenous to the model in a relatively simple way. Since agents live only two periods, their optimisation problem involves only those two periods.
- In the version shown here, we did not make the labour supply decision endogenous, but this can be done relatively easily, since it adds only two more variables to the decision problem of each agent: the labour to supply when young and that when old.
- Those interest in learning more about the OLG model should see Acemoglu (2009), Blanchard and Fischer (1989), Ljungqvist and Sargent (2018), McCandless (2008), and Romer (2012). For a more rigorous treatment of OLG models see McCandless and Wallace (1992) and Sargent (1987).

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