PhD Macroeconomics: Prerequisites for DSGE Models

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Introduction

- Before we tackle the RBC model, it's worth going over a few important mathematical concepts.
- Some of the concepts are essential to understand now, but some of the other concepts, such as solution methods for DSGE models can be revisited later.
- But it's good to be aware of them now, and keep them in mind as we move on in the course.

Vector Autoregressions

A brief recap I

- As we saw in the first lecture, AR models are useful tools for understanding the dynamics of individual variables such as output or consumption, but they ignore the interrelationships between variables.
- A vector autoregression (VAR) model captures the dynamics of n different variables allowing each variable to depend on lagged values of all variables.
- More specifically, with VAR models we can examine the impulse responses of all n variables to all n shocks.
- Consider the following simple VAR(1) model with two variables and one lag:

$$y_{1,t} = a_{11} y_{1,t-1} + a_{12} y_{2,t-1} + e_{1,t},$$
$$y_{2,t} = a_{21} y_{1,t-1} + a_{22} y_{2,t-1} + e_{2,t},$$

  where $e_{1,t}$ and $e_{2,t}$ are shocks to the system. What are these shocks?
- They could be shocks which macroeconomists are interested in.

A brief recap II

- VARs are a very common framework for modelling macroeconomic dynamics and the effects of shocks.
- But while VARs can describe how things work, they cannot explain why things work – hence why we need models based on economic theory (e.g., DSGE models!).
- These VAR models were introduced to the economics discipline by Sims (1980).
- Sims was telling macroeconomists to "get real" – move on from overly stylised models (e.g. IS-LM models).
- VARs were an alternative that allowed one to model macroeconomic data accurately, without having to impose lots of incredible restrictions.
- In the phrase used in an earlier paper by Sargent and Sims (who shared the Nobel prize) it was "macro modelling without pretending to have too much a priori theory".
- We will see that VARs are not theory free. But they do make the role of theoretical identifying assumptions far clearer than was the case for the types of models Sims was criticising.

Matrix representation of VARs and VMAs I

- Let's consider our simple VAR(1) model:

$$y_{1,t} = a_{11}y_{1,t-1} + a_{12}y_{2,t-1} + e_{1,t},\ e_{1,t} \overset{iid}{\sim} \mathcal{N}(0, \sigma_1^2),$$
$$y_{2,t} = a_{21}y_{1,t-1} + a_{22}y_{2,t-1} + e_{2,t},\ e_{2,t} \overset{iid}{\sim} \mathcal{N}(0, \sigma_2^2),$$

which we can express more compactly using matrices. Let

$$\boldsymbol{Y}_t = \begin{bmatrix} y_{1,t} \\ y_{2,t} \end{bmatrix},\ \boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix},\ \boldsymbol{e}_t = \begin{bmatrix} e_{1,t} \\ e_{2,t} \end{bmatrix},$$

and so we can write the simple VAR(1) model as:

$$\boldsymbol{Y}_t = \boldsymbol{A}\boldsymbol{Y}_{t-1} + \boldsymbol{e}_t. \tag{1}$$

Matrix representation of VARs and VMAs II

- VARs express variables as a function of what happened yesterday and today's shocks. But what happened yesterday depended on yesterday's shocks and on what happened the day before, and so on.
- So with a bit of recursion, and like we do with AR(1) models, we can express the VAR(1) model as a vector moving average (VMA) model:

$$\begin{aligned} \boldsymbol{Y}_t &= \boldsymbol{e}_t + \boldsymbol{A}\boldsymbol{Y}_{t-1} \\ &= \boldsymbol{e}_t + \boldsymbol{A}\left[\boldsymbol{e}_{t-1} + \boldsymbol{A}\boldsymbol{Y}_{t-2}\right] \\ &= \boldsymbol{e}_t + \boldsymbol{A}\boldsymbol{e}_{t-1} + \boldsymbol{A}^2\left[\boldsymbol{e}_{t-2} + \boldsymbol{A}\boldsymbol{Y}_{t-3}\right] \\ &\vdots \\ \boldsymbol{Y}_t &= \boldsymbol{e}_t + \boldsymbol{A}\boldsymbol{e}_{t-1} + \boldsymbol{A}^2\boldsymbol{e}_{t-2} + \boldsymbol{A}^3\boldsymbol{e}_{t-3} + \ldots + \boldsymbol{A}^t\boldsymbol{e}_0. \end{aligned}$$

Matrix representation of VARs and VMAs III

- This makes it clear how today's values for the series are the cumulation of all the shocks from the past. It is also useful for deriving predictions about the properties of VARs.

Impulse response functions I

- Suppose there is an initial shock identified as:

$$\boldsymbol{e}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

  and then all shock terms are zero afterwards, i.e., $\boldsymbol{e}_t = \mathbf{0}, \ \forall t > 0$.
- Using our VMA representation we see that the response in $\boldsymbol{Y}_t$ after n periods is

$$\boldsymbol{A}^n \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

- So the impulse response function (IRF) for VARs are directly analogous to the IRFs for AR(1) models that we looked at before.

Impulse response functions II

- VARs are often used for forecasting. Suppose we observe our vector of variables $\boldsymbol{Y}_t$. What is our forecast for $\boldsymbol{Y}_{t+1}$?
- Using forward iteration, we could write the following for the next period:

$$\boldsymbol{Y}_{t+1} = \boldsymbol{A}\boldsymbol{Y}_t + \boldsymbol{e}_{t+1}.$$

- But because $\mathbb{E}_t \boldsymbol{e}_{t+1} = 0$, an unbiased forecast at time t is $\boldsymbol{A}\boldsymbol{Y}_t$.
- In other words, $\mathbb{E}_t \boldsymbol{Y}_{t+1} = \boldsymbol{A}\boldsymbol{Y}_t$. The same reasoning tells us that $\boldsymbol{A}^2\boldsymbol{Y}_t$ is an unbiased forecast of $\boldsymbol{Y}_{t+2}$, and $\boldsymbol{A}^3\boldsymbol{Y}_t$ is an unbiased forecast of $\boldsymbol{Y}_{t+3}$, and so on.
- So once a VAR is estimated and organised in this form, it is very easy to construct forecasts.

Impulse response functions III

- What about more than one lagged term? It turns out the first-order matrix representation can represent VARs with longer lags. Consider the two-lag system:

$$y_{1,t} = a_{11}y_{1,t-1} + a_{12}y_{1,t-2} + a_{13}y_{2,t-1} + a_{14}y_{2,t-2} + e_{1,t}$$
$$y_{2,t} = a_{21}y_{1,t-1} + a_{22}y_{1,t-2} + a_{23}y_{2,t-1} + a_{24}y_{2,t-2} + e_{2,t},$$

  and define the vector

$$\boldsymbol{z}_t = \begin{bmatrix} y_{1,t} \\ y_{1,t-1} \\ y_{2,t} \\ y_{2,t-1} \end{bmatrix}.$$

Impulse response functions IV

- This system can be represented in matrix form as

$$\boldsymbol{Z}_t = \boldsymbol{A}\boldsymbol{Z}_{t-1} + \boldsymbol{e}_t, \tag{2}$$

where

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ 0 & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ \boldsymbol{e}_t = \begin{bmatrix} e_{1,t} \\ 0 \\ e_{2,t} \\ 0 \end{bmatrix}.$$

- The representation (2) is called the “companion form” matrix representation.

Interpreting shocks I

- The system we've been looking at is usually called a "reduced-form" VAR model. It is a purely econometric model, without any theoretical element, and fluctuations in the system are driven by the shocks $\boldsymbol{e}_t$. But how should we interpret these shocks?
- Suppose that $e_{1,t}$ is a shock that affects only $y_{1,t}$ on impact and $e_{2,t}$ is a shock that affects only $y_{2,t}$ on impact. For instance, one can use the IRFs generated from an inflation-output VAR to calculate the dynamic effects of "a shock to inflation" and "a shock to output".
- But we may imagine that the shocks are an "aggregate supply" shock and an "aggregate demand" shock and that both of these shocks have a direct effect on both inflation and output.
- How can we breakdown which "part" of $e_{1,t}$, say, affects only output?
- If we knew this or had an "identification strategy" to find this, then we could identify the component(s) of $\boldsymbol{e}_t$ that only affect inflation and that only affect output.

Interpreting shocks II

- We could then interpret $\boldsymbol{e}_t$ as being the reduced-form shocks which are comprised of "structural shocks", ε_t.
- Suppose reduced-form and structural shocks are related by

$$e_{1,t} = c_{11}\varepsilon_{1,t} + c_{12}\varepsilon_{2,t},$$
$$e_{2,t} = c_{21}\varepsilon_{1,t} + c_{22}\varepsilon_{2,t},$$

  and in matrix form we can write this as

$$\boldsymbol{e}_t = \boldsymbol{C}\varepsilon_t.$$

Interpreting shocks III

- These two VMA representations describe the data equally well:

$$\begin{aligned} \boldsymbol{Y}_t &= \boldsymbol{e}_t + \boldsymbol{A}\boldsymbol{e}_{t-1} + \boldsymbol{A}^2\boldsymbol{e}_{t-2} + \boldsymbol{A}^3\boldsymbol{e}_{t-3} + \ldots + \boldsymbol{A}^t\boldsymbol{e}_0, \\ \Leftrightarrow \boldsymbol{Y}_t &= \boldsymbol{C}\varepsilon_t + \boldsymbol{A}\boldsymbol{C}\varepsilon_{t-1} + \boldsymbol{A}^2\boldsymbol{C}\varepsilon_{t-2} + \boldsymbol{A}^3\boldsymbol{C}\varepsilon_{t-3} + \ldots + \boldsymbol{A}^t\boldsymbol{C}\varepsilon_0. \end{aligned}$$

- We can interpret the model as one with reduced form shocks, $\boldsymbol{e}_t$, and IRFs given by $\boldsymbol{A}^n$; or as a model with structural shocks, ε_t, and IRFs are given by $\boldsymbol{A}^n\boldsymbol{C}$. We could do this for any $\boldsymbol{C}$ if we knew the structural shocks.
- Another way to see how reduced-form shocks can be different from structural shocks is if there are contemporaneous interactions between variables – which is likely in macroeconomics.

Interpreting shocks IV

- Consider the following model:

$$y_{1,t} = a_{12}y_{2,t} + b_{11}y_{1,t-1} + b_{12}y_{2,t-1} + \varepsilon_{1,t},$$
$$y_{2,t} = a_{21}y_{1,t} + b_{21}y_{1,t-1} + b_{22}y_{2,t-1} + \varepsilon_{2,t},$$

which can be written in matrix form as:

$$\boldsymbol{A}\boldsymbol{Y}_t = \boldsymbol{B}\boldsymbol{Y}_{t-1} + \varepsilon_t,$$

where

$$\boldsymbol{A} = \begin{bmatrix} 1 & -a_{12} \\ -a_{21} & 1 \end{bmatrix}, \ \boldsymbol{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Interpreting shocks V

- Now, if we estimate the "reduced-form" VAR model,

$$\boldsymbol{Y}_t = \boldsymbol{D}\boldsymbol{Y}_{t-1} + \boldsymbol{e}_t,$$

  then the reduced-form shocks and coefficients are:

$$\boldsymbol{D} = \boldsymbol{A}^{-1}\boldsymbol{B},$$
$$\boldsymbol{e}_t = \boldsymbol{A}^{-1}\varepsilon_t.$$

- Again, the following two decompositions both describe the data equally well:

$$\boldsymbol{Y}_t = \boldsymbol{e}_t + \boldsymbol{D}\boldsymbol{e}_{t-1} + \boldsymbol{D}^2\boldsymbol{e}_{t-2} + \boldsymbol{D}^3\boldsymbol{e}_{t-3} + \ldots,$$
$$\Leftrightarrow \boldsymbol{Y}_t = \boldsymbol{A}^{-1}\varepsilon_t + \boldsymbol{D}\boldsymbol{A}^{-1}\varepsilon_{t-1} + \boldsymbol{D}^2\boldsymbol{A}^{-1}\varepsilon_{t-2} + \ldots + \boldsymbol{D}^t\boldsymbol{A}^{-1}\varepsilon_0.$$

Interpreting shocks VI

- For the structural model, the impulse responses to the structural shocks from n periods are given by $\boldsymbol{D}^n\boldsymbol{A}^{-1}$. This is true for any matrix $\boldsymbol{A}$.
- So why should we care about this? There seems to be no problem with forecasting with reduced-form VARs: Once you know the reduced-form shocks and how they affected today's value of the variables, you can use the reduced-form coefficients to forecast, right?
- The problem comes when you start asking "what if" questions/counterfactuals.
  - For example, "what happens if there is a shock to the first variable in the VAR?"
- In practice, the error series in a reduced-form VAR are usually correlated with each other.
- So are you asking "What happens when there is a shock to the first variable only?" or, are you asking "What usually happens when there is a shock to the first variable given that this is usually associated with a corresponding shock to the second variable?"

Interpreting shocks VII

- Most interesting questions about the structure of the economy relate to the impact of different types of shocks that are uncorrelated with each other.
- A structural identification that explains how the reduced-form shocks are actually combinations of uncorrelated structural shocks is far more likely to give clear and interesting answers.

SVARs: A general formulation I

- In its general formulation, the structural VAR (SVAR) is:

$$\underset{n\times n}{\boldsymbol{A}}\underset{n\times 1}{\boldsymbol{Y}_t} = \underset{n\times n}{\boldsymbol{B}}\underset{n\times 1}{\boldsymbol{Y}_{t-1}} + \underset{n\times n}{\boldsymbol{C}}\underset{n\times 1}{\varepsilon_t}, \quad \varepsilon_t \overset{iid}{\sim} \mathcal{N}(\mathbf{0}, \Sigma). \tag{3}$$

- The model is fully described by the following parameters: n^2 parameters in $\boldsymbol{A}$, n^2 parameters in $\boldsymbol{B}$, n^2 parameters in $\boldsymbol{C}$, and $\frac{n^2+n}{2}$ parameters in Σ which describes the patterns of covariances of the underlying shock terms.
- Adding all these together, we see that the most general form of the SVAR is a model with $3n^2 + \frac{n^2+n}{n}$ parameters.

SVARs: A general formulation II

- But estimating the reduced-form VAR,

$$\boldsymbol{Y}_t = \boldsymbol{D}\boldsymbol{Y}_{t-1} + \boldsymbol{e}_t,$$

  only gives us information on $n^2 + \frac{n^2+n}{2}$ parameters: the coefficients in $\boldsymbol{D}$ and the estimated variance-covariance matrix of the reduced form errors.
- To obtain information about structural shocks, we thus need to impose $2n^2$ a priori theoretical restrictions on our SVAR.

The Cholesky decomposition I

- It's probably best to go through an example. Start with a reduced-form VAR with three variables and the errors, $e_{1,t}$, $e_{2,t}$, and $e_{3,t}$:

$$\boldsymbol{Y}_t = \boldsymbol{D}\boldsymbol{Y}_{t-1} + \boldsymbol{e}_t,$$
$$\Leftrightarrow \begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} e_{1,t} \\ e_{2,t} \\ e_{3,t} \end{bmatrix}, \tag{4}$$

where the joint distribution of $\boldsymbol{e}_t$ is:

$$\begin{bmatrix} e_{1,t} \\ e_{2,t} \\ e_{3,t} \end{bmatrix} \overset{iid}{\sim} \mathcal{N}(\mathbf{0}, \Sigma), \ \ \Sigma = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_y^2 & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_z^2 \end{bmatrix}.$$

The Cholesky decomposition II

- Now, we want to express the shocks $\boldsymbol{e}_t$ as a function of structural shocks. Apply the following restriction:

$$
\begin{aligned}
e_{1,t} &= c_{11}\varepsilon_{1,t} \\
e_{2,t} &= c_{21}\varepsilon_{1,t} + c_{22}\varepsilon_{2,t} \\
e_{3,t} &= c_{31}\varepsilon_{1,t} + c_{32}\varepsilon_{2t} + c_{33}\varepsilon_{3,t},
\end{aligned}
$$

or, in matrix form:

$$
\begin{bmatrix} e_{1,t} \\ e_{2,t} \\ e_{3,t} \end{bmatrix} = \begin{bmatrix} c_{11} & 0 & 0 \\ c_{21} & c_{22} & 0 \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{bmatrix}, \tag{5}
$$

$$
\Leftrightarrow \boldsymbol{e}_t = \boldsymbol{C}\boldsymbol{\varepsilon}_t,
$$

The Cholesky decomposition III

where the joint distribution of ε_t is:

$$\begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{bmatrix} \overset{iid}{\sim} \mathcal{N}\left(\mathbf{0}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right).$$

- How do we get $\boldsymbol{C}$? Estimation is one option, but the easier way is to use the Choleski decomposition for the variance-covariance matrix $\boldsymbol{\Sigma}$:

$$\begin{aligned} \boldsymbol{\Sigma} &= \boldsymbol{C}\boldsymbol{C}^\top, \\ \implies \boldsymbol{C}^{-1}\boldsymbol{\Sigma}(\boldsymbol{C}^\top)^{-1} &= \boldsymbol{I}_n. \end{aligned}$$

The Cholesky decomposition IV

- To see how this works, first note that the transpose of equation (5) is:

$$\begin{aligned} \boldsymbol{e}_t^\top &= (\boldsymbol{C}\varepsilon_t)^\top \\ &= \varepsilon_t^\top \boldsymbol{C}^\top, \end{aligned}$$

so post multiply (5) with $\boldsymbol{e}_t^\top$ to get:

$$\boldsymbol{e}_t\boldsymbol{e}_t^\top = \boldsymbol{C}\varepsilon_t\varepsilon_t^\top\boldsymbol{C}^\top, \tag{6}$$

or equivalently:

$$\begin{bmatrix} e_{1,t} \\ e_{2,t} \\ e_{3,t} \end{bmatrix} \begin{bmatrix} e_{1,t} & e_{2,t} & e_{3,t} \end{bmatrix} = \begin{bmatrix} c_{11} & 0 & 0 \\ c_{21} & c_{22} & 0 \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t} & \varepsilon_{2,t} & \varepsilon_{3,t} \end{bmatrix} \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ 0 & c_{22} & c_{32} \\ 0 & 0 & c_{33} \end{bmatrix},$$

The Cholesky decomposition V

and if we take expectations of (6), we get:

$$\begin{aligned} \mathbb{E}_t \left[\boldsymbol{e}_t \boldsymbol{e}_t^\top \right] &= \mathbb{E}_t \left[\boldsymbol{C} \varepsilon_t \varepsilon_t^\top \boldsymbol{C}^\top \right] \\ \Leftrightarrow \Sigma &= \boldsymbol{C} \boldsymbol{I}_n \boldsymbol{C}^\top = \boldsymbol{C}\boldsymbol{C}^\top . \end{aligned} \tag{7}$$

- Identification done! We have shown that we can get $\boldsymbol{C}$ by a Choleski decomposition for the variance-covariance matrix.

The Cholesky decomposition VI

- So, from (4), if we substitute in equation (5), we have

$$\begin{bmatrix} x_t \\ y_t \\ z_t \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{t-1} \\ y_{t-1} \\ z_{t-1} \end{bmatrix} + \begin{bmatrix} c_{11} & 0 & 0 \\ c_{21} & c_{22} & 0 \\ c_{31} & c_{32} & c_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{1,t} \\ \varepsilon_{2,t} \\ \varepsilon_{3,t} \end{bmatrix}$$

$$\Leftrightarrow \boldsymbol{Y}_t = \boldsymbol{D}\boldsymbol{Y}_{t-1} + \boldsymbol{C}\varepsilon_t,$$

  which can be transformed into:

$$\boldsymbol{C}^{-1}\boldsymbol{Y}_t = \boldsymbol{C}^{-1}\boldsymbol{D}\boldsymbol{Y}_{t-1} + \varepsilon_t, \tag{8}$$

  which is nothing but a SVAR with what macro-econometricians like to call a "short-run restriction."
- Note now that, by construction, the ε_t shocks constructed in this way are uncorrelated with each other.

The Cholesky decomposition VII

- This method posits a sort of “causal chain” of shocks.
  - The first shock affects all of the variables at time t.
  - The second only affects two of them at time t.
  - The last shock only affects the last variable at time t.
- There is a serious drawback to this however: The causal ordering is not unique.
- Any one of the VAR variables can be listed first, and any one can be listed lasted. This means there are $n! = 1 \times 2 \times 3 \times \ldots \times n$ possible recursive orderings.
- We need to think very carefully about our own prior thinking about causation!

Solving Models with Rational Expectations

Introducing expectations I

- A key sense in which DSGE models differ from VARs is that while VARs just have backward-looking dynamics, DSGE models have both backward-looking and forward-looking dynamics.
- The backward-looking dynamics stem, for instance, from identities linking today's capital stock with last period's capital stock and this period's investment. For example:

$$K_t = (1 - \delta)K_{t-1} + I_t.$$

- The forward-looking dynamics stem from optimising behaviour: What agents expect to happen tomorrow is very important for what they decide to do today – think about our consumption Euler equation.
- Modelling this idea requires an assumption about how people formulate their expectations.

Introducing expectations II

- This approach was criticised in the 1970s by economists such as Robert Lucas and Thomas Sargent.
- Lucas and Sargent instead promoted the use of an approach which they called “Rational Expectations”.
- In economics, rational expectations usually means two things:
  1. Agents use publicly available information in an efficient manner.
  2. That agents understand the structure of the model economy and base their expectations of variables on this knowledge.
- Rational Expectations is a strong assumption. No one truly understands the structure of an economy – not even macroeconomists.
- But one reason for using Rational Expectations as a baseline assumption is that once one has specified a particular model of the economy, any other assumption about expectations means that people are making systematic errors, which seems inconsistent with rationality.

Introducing expectations III

- In other words, we think it's entirely reasonable to presume that agents are optimising to get what's best for them.
- We can easily disagree on what "the best" is for them, but I think we can agree that they will *try to* act optimally.

First-order stochastic difference equations I

- A lot of models in economics take the form:

$$y_t = x_t + a\mathbb{E}_t y_{t+1}, \tag{9}$$

  which just says that y today is determined by x and by tomorrow's expected value of y given the information we have today.

- But what determines this expected value? Rational Expectations implies a very specific answer.
- Under Rational Expectations, the agents in the economy understand the equation and formulate their expectation in a way that is consistent with it:

$$\mathbb{E}_t y_{t+1} = \mathbb{E}_t x_{t+1} + a\mathbb{E}_t \mathbb{E}_{t+1} y_{t+2},$$

First-order stochastic difference equations II

where we can simplify the second expression on the RHS by the law of iterated expectations (LIE):

$$\mathbb{E}_t y_{t+1} = \mathbb{E}_t x_{t+1} + a\mathbb{E}_t y_{t+2}.$$

- Substituting our expression for $\mathbb{E}_t y_{t+1}$ into our expression for y_t yields:

$$y_t = x_t + a\mathbb{E}_t x_{t+1} + a^2\mathbb{E}_t y_{t+2},$$

and if we kept repeating this by substituting for $\mathbb{E}_t y_{t+2}$, then $\mathbb{E}_t y_{t+3}$, and so on, we would get:

$$\begin{aligned} y_t &= x_t + a\mathbb{E}_t x_{t+1} + a^2\mathbb{E}_t x_{t+2} + \ldots + a^{N-1}\mathbb{E}_t x_{t+N-1} + a^N\mathbb{E}_t y_{t+N}, \\ \Leftrightarrow y_t &= \sum_{j=0}^{N-1} a^j\mathbb{E}_t x_{t+j} + a^N\mathbb{E}_t y_{t+N}, \end{aligned}$$

First-order stochastic difference equations III

where usually we assume that

$$\lim_{N\to\infty} a^N \mathbb{E}_t y_{t+N} = 0.$$

- So, the solution is:

$$y_t = \sum_{k=0}^{\infty} a^k \mathbb{E}_t x_{t+k}. \tag{10}$$

- This solution underlies the logic of a very large amount of modern macroeconomics.

Example: Lucas tree model I

- Consider an asset that can be purchased today for price P_t and which yields a dividend D_t.
- Suppose there is a close alternative to this asset that will yield a guaranteed rate of return of r.
- Then, a risk neutral investor will only invest in the asset if it yields the same rate of return, i.e., if

$$\frac{D_t + \mathbb{E}_t P_{t+1}}{P_t} = 1 + r. \tag{11}$$

Example: Lucas tree model II

- We can rearrange this to get:

$$P_t = \frac{D_t}{1+r} + \frac{\mathbb{E}_t P_{t+1}}{1+r},$$

  and then iterating forward we get:

$$P_t = \sum_{j=0}^{\infty} \left(\frac{1}{1+r} \right)^{j+1} \mathbb{E}_t D_{t+j}. \tag{12}$$

- This equation, which states that asset prices should equal a discounted present-value sum of expected future dividends is usually known as the dividend-discount model.

Forward and backward solutions I

- The model
$$y_t = x_t + a\mathbb{E}_t y_{t+1} \tag{13}$$
can also be written as:
$$y_t = x_t + a y_{t+1} + a\varepsilon_{t+1},$$
where ε_{t+1} is a forecast error that cannot be predicted at date t.
- Moving the time subscripts back one period and rearranging this yields:
$$y_t = a^{-1} y_{t-1} - a^{-1} x_{t-1} - \varepsilon_t.$$

Forward and backward solutions II

- This backward-looking equation which can also be solved via recursive substitution to give:

$$y_t = -\sum_{j=0}^{\infty} a^{-j}\varepsilon_{t-j} - \sum_{j=1}^{\infty} a^{-j}x_{t-j}. \tag{14}$$

- The forward and backward solutions are both correct solutions to the first-order stochastic difference equation (as are all linear combinations of them).
- Which solution we choose to work with depends on the value of the parameter a.
  - If $|a| > 1$, then the weights on future values of x_t in the forward solution (10) will explode.
  - In this case, it is most likely that the forward solution will not converge to a finite sum.
  - But this may not make sense; may think that practical applications should focus on the backwards solutions.
- However, the equation holds for any set of shocks ε_t such that $\mathbb{E}_{t-1}\varepsilon_t = 0$.

Forward and backward solutions III

- So the solution is indeterminate: We can't actually predict what will happen with y_t even if we knew the full path for x_t.
- But if $|a| < 1$, then the weights in the backwards solution are explosive and the forward solution is the one to focus on. Also, this solution is determinate.
- Knowing the path of x_t will tell you the path of y_t. In most cases, it is assumed that $|a| < 1$, and we can assume that

$$\lim_{n \to \infty} a^n \mathbb{E}_t y_{t+n} = 0,$$

amounts to a statement that y_t can't grow too fast.

Forward and backward solutions IV

- What if it doesn't hold? Then the solution can have other elements. Let

$$y_t^* = \sum_{j=0}^{\infty} a^j \mathbb{E}_t x_{t+j},$$

and let $y_t = y_t^* + b_t$ be any other solution. The solution must satisfy

$$y_t^* + b_t = x_t + a\mathbb{E}_t y_{t+1}^* + a\mathbb{E}_t b_{t+1}.$$

- By construction, one can show that $y_t^* = x_t + a\mathbb{E}_t y_{t+1}^*$. Now, the above equation means that the additional component satisfies

$$b_t = a\mathbb{E}_t b_{t+1},$$

and because $|a| < 1$, this means that b is always expected to get bigger in absolute value, going to infinity in expectation.

Forward and backward solutions V

- This is a bubble. Note that the term bubble is usually associated with irrational behaviour by investors. But in this simple model, the agents have rational expectations. This is a rational bubble.
- There may be restrictions in the real economy that stop b from growing forever. But constant growth is not the only way to satisfy $b_t = a\mathbb{E}_t b_{t+1}$. The following process also works:

$$b_{t+1} = \begin{cases} (aq)^{-1} b_t + e_{t+1}, & \text{w.p.} \quad q, \\ e_{t+1}, & \text{w.p.} \quad 1 - q, \end{cases}$$

  where $\mathbb{E}_t e_{t+1} = 0$.
- This is a bubble that everyone knows is going to crash eventually. And even then, a new bubble can get going.
- Imposing $\lim_{n\to\infty} a^n \mathbb{E}_t y_{t+n} = 0$ rules out bubbles of this (or any other) form.

The DSGE Recipe

From structural to reduced-form relationships I

- The forward solution to (13),

$$y_t = \sum_{j=0}^{\infty} a^j \mathbb{E}_t x_{t+j},$$

provides useful insights into how the variable y_t is determined.

- However, without some assumptions about how x_t evolves over time, it cannot be used to give precise predictions about the dynamics of y_t (and ideally, we want to be able to simulate the behaviour of y_t).
- One reason why there is a strong linkage between DSGE modelling and VARs is because we assume that the exogenous "driving variables" such as x_t are generated by backward-looking time series models like in VARs.

From structural to reduced-form relationships II

- Consider for instance the case where the process driving x_t is AR(1),

$$x_t = \rho x_{t-1} + \varepsilon_t, \quad |\rho| < 1.$$

- In this case, we have

$$\mathbb{E}_t x_{t+j} = \rho^j x_t.$$

From structural to reduced-form relationships III

- Now the model's solution can be written as

$$y_t = \left[\sum_{j=0}^{\infty} (a\rho)^j \right] x_t,$$

and because $|a\rho| < 1$, the infinite sum converges to

$$\sum_{j=0}^{\infty} (a\rho)^j = \frac{1}{1 - a\rho}.$$

From structural to reduced-form relationships IV

- Which should look familiar if you did undergrad macro – it's how we derived the Keynesian multiplier formula. So, in this case, the model solution is

$$y_t = \frac{1}{1 - a\rho} x_t.$$

- Macroeconomists call this a reduced-form solution for the model. Together with the equation describing the evolution for x_t, it can be easily simulated on a computer (e.g., `Dynare` will do this for you automatically).

The DSGE recipe I

- While the previous example is obviously very simple, it illustrates the general principle for getting predictions from DSGE models:
  1. Obtain structural equations involving expectations of future driving variables (in this case, the $\mathbb{E}_t x_{t+j}$ terms).
  2. Make assumptions about the time series process for the driving variables (in this case, x_t).
  3. Solve for a reduced-form solution that can be simulated on the computer along with the driving variables.
- Finally, note that the reduced-form of this model also has a VAR-like representation, which can be shown as follows

$$\begin{aligned} y_t &= \frac{1}{1-a\rho}(\rho x_{t-1} + \varepsilon_t) \\ &= \rho y_{t-1} + \frac{1}{1-a\rho}\varepsilon_t. \end{aligned}$$

The DSGE recipe II

- So both the x_t and y_t series have purely backward-looking representations.
- Even this simple model helps to explain how theoretical models tend to predict that the data can be described well using a VAR.

References I

Sims, Christopher A. 1980. "Macroeconomics and Reality." *Econometrica* 48 (1): 1–48.