PhD Macroeconomics: The RBC Model

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Introduction

- Modern economies undergo significant short-run fluctuations in aggregate output and employment.
- These fluctuations don't really follow a pattern that we can heuristically predict or forecast easily.
- We do, however, know that these fluctuations have some intriguing characteristics.
- Understanding the causes and characteristics of these aggregate fluctuations is a central goal of macroeconomics.
- Critically, by understanding these factors, we can build models which can replicate business cycle moments and to hopefully consider optimal policy responses to these fluctuations.

Environment and Assumptions

- Build a dynamic stochastic general equilibrium (DSGE).
- Our DSGE model will feature perfectly competitive markets without externalities, asymmetric information, missing markets, or other imperfections.
- The Ramsey model seems like a very good candidate to start with.
- We know that absent of any shocks, the Ramsey model with converge to a balanced growth path, and then grows smoothly.
- It then seems sensible to incorporate business cycle fluctuations and shocks into the Ramsey model.
- These shocks will thus change the actual productive capacity of the economy. Hence, the modified Ramsey model is known as the Real Business Cycle (RBC) model.

The Ramsey Social Planner's Problem

The social planner's (centralised) problem I

- There are a few ways to set up an RBC model: either from the perspective of a benevolent social planner, or by setting up competitive markets and finding market equilibria.
- The Ramsey social planner seeks to maximise social welfare subject to the economy's resource constraints; whereas in competitive markets agents optimise their utility or profit given their endowments.
- In the RBC model, both approaches yield the same outcome – an important point that we will later come back to.
- In macroeconomic models, solving the Ramsey planner's problem yields the social welfare maximising, Pareto-efficient solution. This is because in other models we will look at, markets are not fully competitive or efficient, so the competitive equilibrium will be unable to achieve a first-best outcome.

The social planner's (centralised) problem II

- The Ramsey planner solves the following problem:

$$\max_{\{C_t, N_t\}} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s U(C_{t+s}, N_{t+s}), \tag{1}$$

  where C_t is aggregate consumption, N_t is hours worked or aggregate labour supply, and β is the representative household's rate of time preference (their discount factor).
- Note that the household experiences disutility from supplying labour.
- Alternatively we could write:

$$U(C_t, N_t) = u(C_t) - v(N_t),$$

  where $u(\cdot)$ and $v(\cdot)$ are subutility functions.
- In words, the Ramsey planner wishes to maximise households' welfare by assigning the optimal amounts of consumption and labour supply each period.

The social planner's (centralised) problem III

- Furthermore, the Ramsey planner wishes to maximise (1) subject to the following economy-wide resource constraints:

$$Y_t = C_t + I_t, \tag{2}$$
$$Y_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha}, \tag{3}$$
$$K_t = I_t + (1-\delta)K_{t-1}, \tag{4}$$

and a process for the technology shock term A_t:

$$\ln A_t = (1-\rho)\ln \bar{A} + \rho \ln A_{t-1} + \varepsilon_t, \ \varepsilon_t \overset{\text{IID}}{\sim} \mathcal{N}(0, \sigma_a^2), \tag{5}$$

where Y_t is output, I_t is investment into new capital, A_t is total factor productivity term, K_t is productive capital, and δ is the depreciation rate.

Stochastic dynamic programming? I

- The question that arises now is: how do we go about maximising (1)?
- The main issue is that we have a stream of future consumption and labour decisions to make, constrained to the fact that we don't know what A_t will be in the future. Technically, the best way to solve this problem is using stochastic dynamic programming.
- But we don't have time for that.
- Instead, we will use a trick and simplification: we treat the Ramsey problem as a deterministic problem and then substitute $\mathbb{E}_t X_{t+i}$ for X_{t+i}.

Stochastic dynamic programming? II

- Suppose

$$G(x) = \sum_{j=1}^{N} p_j F(a_j, x),$$

which is maximised by setting

$$G'(x) = \sum_{j=1}^{N} p_j F'(a_j, x) = \mathbb{E}_t F'(x) = 0,$$

so, the FOCs for maximising $\mathbb{E}_t F(x)$ are just $\mathbb{E}_t F'(x) = 0$.

Stochastic dynamic programming? III

- Now, we can combine our constraints to simply get:

$$A_t K_{t-1}^{\alpha} N_t^{1-\alpha} = C_t + K_t - (1-\delta) K_{t-1}. \tag{6}$$

Then, we can set up the Ramsey planner's problem as a Lagrangian:

$$\begin{aligned}\mathcal{L} = {} & \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left[U(C_{t+s}, N_{t+s}) \right] \\ & + \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \lambda_{t+s} \left[A_{t+s} K_{t+s-1}^{\alpha} N_{t+s}^{1-\alpha} + (1-\delta) K_{t+s-1} - C_{t+s} - K_{t+s} \right].\end{aligned} \tag{7}$$

- But this is still a hideous equation to work with.
- So, what can we do? This is macroeconomics, so we will use another trick/simplification.

Stochastic dynamic programming? IV

- We want to take a snapshot of how the variables behave in the period in which we are optimising in, t. Most of our variables are denoted in period t with the subscript t, so they're fine.
- But we have K_{t-1} and A_{t-1} in the law of motion equations for capital and technology, respectively.
- So, what we can do is set up the Lagrange with the objective function based in period t, a single constraint dated in period t, and then we can add in a second constraint from period $t+1$.
- Then, the period t variables appear as:

$$
\begin{aligned}
\mathcal{L} = U(C_t, N_t) + \lambda_t (A_t K_{t-1}^{\alpha} N_t^{1-\alpha} + (1-\delta)K_{t-1} - C_t - K_t) \\
+ \beta \mathbb{E}_t \lambda_{t+1} \left(A_{t+1} K_t^{\alpha} N_{t+1}^{1-\alpha} + (1-\delta)K_t - C_{t+1} - K_{t+1} \right) .
\end{aligned}
$$

- After that, the period t variables don't ever appear again.

Stochastic dynamic programming? V

- So, the FOCs for the period t variables consist of differentiating this equation with respect to these variables and setting the derivatives equal to zero.
- Then, the period $t + i$ variables appear exactly as the period t variables do, except that they are in expectation form and they are multiplied by the discount rate β^i.
- But this means that the FOCs for the period $t + i$ variables will be identical to those for period t variables.
- So differentiating this equation gives us the equations for the optimal dynamics at all times!

Stochastic dynamic programming? VI

- Thus, we yield the following FOCs:

$$\mathcal{L}_{C_t} = U_C(C_t) - \lambda_t = 0, \tag{8}$$

$$\mathcal{L}_{K_t} = -\lambda_t + \beta \mathbb{E}_t \lambda_{t+1} \left(\alpha \frac{Y_{t+1}}{K_t} + 1 - \delta \right) = 0, \tag{9}$$

$$\mathcal{L}_{N_t} = U_N(N_t) + \lambda_t (1 - \alpha) \frac{Y_t}{N_t} = 0, \tag{10}$$

$$\mathcal{L}_{\lambda_t} = A_t K_{t-1}^{\alpha} N_t^{1-\alpha} + (1 - \delta) K_{t-1} - C_t - K_t = 0. \tag{11}$$

Easy!

The consumption Euler equation I

- Define the marginal value of an additional unit of capital next year as

$$R^k_{t+1} = \mathbb{E}_t \frac{\left(\alpha \frac{Y_{t+1}}{K_t} + 1 - \delta\right) Q_{t+1}}{Q_t}, \tag{12}$$

  where Q_t is the real price of capital. So Q_{t+1}/Q_t can be considered as the "capital gain".
- Recall that I mentioned that the RBC model is nested in a environment of perfectly competitive markets, with full information and complete asset markets. We're going to make use of that last point here – because of complete financial markets, we're going to declare the following "no-arbitrage condition":

$$R_t = R^k_t. \tag{13}$$

The consumption Euler equation II

- For reasons that we will explore later, this condition just says that the gross return on capital, R_t^k, is equal to the risk-free gross real interest rate, R_t. Our assumption of perfectly competitive markets also means that the price of capital is constant, $Q_t = Q_{t+1} = ... = Q_{t+s}, \forall s > 1$.
- The FOC for capital (9) can be written as:

$$\lambda_t = \beta \mathbb{E}_t \left[\lambda_{t+1} R_{t+1} \right],$$

and this can be combined with the FOC for consumption (8) to yield:

$$U_C(C_t) = \beta \mathbb{E}_t \left[U_C(C_{t+1}) R_{t+1} \right], \tag{14}$$

which is nothing but the consumption Euler equation – sometimes referred to as the Keynes-Ramsey condition.

The consumption Euler equation III

- As a quick refresher, we can interpret the Keynes-Ramsey condition as: decreasing consumption by Δ today at the cost of $U_C(C_t)\Delta$ in utility; invest to get $R_{t+1}\Delta$ tomorrow; that investment is worth $\beta\mathbb{E}_t\left[U_C(C_{t+1})R_{t+1}\Delta\right]$ in terms of utility today; and, along the optimal path, an agent must be indifferent between these options.
- If we assume CRRA utility and a simple linear technology for the disutility from supplying labour, we can write the utility function as:

$$U(C_t, N_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \eta N_t,$$

then the Keynes-Ramsey condition (14) becomes:

$$C_t^{-\sigma} = \beta\mathbb{E}_t C_{t+1}^{-\sigma} R_{t+1},$$

The consumption Euler equation IV

and the intratemporal Euler equation for labour and leisure, derived from (10), becomes:

$$-\eta + C_t^{-\sigma}(1-\alpha)\frac{Y_t}{N_t} = 0.$$

Equilibrium and log-linearisation I

- The RBC model can be defined by the following seven equations:

$$Y_t = C_t + I_t, \tag{15}$$

$$Y_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha}, \tag{16}$$

$$K_t = I_t + (1-\delta) K_{t-1}, \tag{17}$$

$$R_t = \alpha \frac{Y_t}{K_{t-1}} + 1 - \delta, \tag{18}$$

$$C_t^{-\sigma} = \beta \mathbb{E}_t \left[C_{t+1}^{-\sigma} R_{t+1} \right], \tag{19}$$

$$\frac{Y_t}{N_t} = \frac{\eta}{1-\alpha} C_t^{\sigma}, \tag{20}$$

$$\ln A_t = \rho \ln A_{t-1} + \varepsilon_t, \tag{21}$$

Equilibrium and log-linearisation II

so we have seven equations in seven unknown variables. Notice that a lot of the RBC model equations are non-linear – and we haven't discussed any strategies of solving systems of stochastic non-linear equations.

- So what can we do? Again, this is macroeconomics, so there's a trick: we linearise the model equations via log-linearisation, from which we can then solve the model.
- The idea is to use Taylor series approximations. In general, any non-linear function $F(x_t, y_t)$ can be approximated around any point $F(x_t^*, y_t^*)$ using the formula:

$$\begin{aligned} F(x_t, y_t) = {} & F(x_t^*, y_t^*) + F_x(x_t^*, y_t^*)(x_t - x_t^*) + F_y(x_t^*, y_t^*)(y_t - y_t^*) \\ & + F_{xx}(x_t^*, y_t^*)(x_t - x_t^*)^2 + F_{yy}(x_t^*, y_t^*)(y_t - y_t^*)^2 + F_{xy}(x_t^*, y_t^*)(x_t - x_t^*)(y_t - y_t^*) + \dots \end{aligned}$$

Equilibrium and log-linearisation III

- If the gap between (x_t, y_t) and (x_t^*, y_t^*) is small, then terms in second and higher powers and cross-terms will all be very small and can be ignored (i.e. a first-order Taylor series approximation will suffice), leaving something like:

$$F(x_t, y_t) \approx \alpha + \beta_1 x_t + \beta_2 y_t.$$

- Many DSGE solution methods use a particular version of this technique. They take logs and then linearise the logs of variables about a simple "steady-state" path in which all real variables are growing at the same rate.
- The steady-state path is relevant because the stochastic economy will, on average, tend to fluctuate around the values given by this path, making the approximation an accurate one.
- This will give us a set of linear equations in terms of deviations of the logs of these variables from their steady-state values.

Equilibrium and log-linearisation IV

- Remember that log-differences are approximately percentage deviations:

$$\log X - \log Y \approx \frac{X - Y}{Y},$$

  so this approach gives us a system that expresses variables in terms of their percentage deviations from the steady-state paths.
- In other words, it can be thought of as giving a system of variables that represents the business-cycle component of the model! Coefficients are elasticities and IRFs are easy to interpret.
- Also, believe it or not, log-linearisation is easy.

Equilibrium and log-linearisation V

- From here, it's important to note down some notation. Let "hatted" variables (e.g. $\hat{X}_t$) denote log-deviations of variables from their steady-state values, denoted by a "bar" (e.g. $\bar{X}$):

$$\hat{X}_t = \log X_t - \log \bar{X}.$$

- The key to the log-linearisation method is that every variable can be written as:

$$X_t = \bar{X}\frac{X_t}{\bar{X}} = \bar{X}e^{\hat{X}_t},$$

and the big trick is that a first-order Taylor approximation of $e^{\hat{X}_t}$ is given by:

$$e^{\hat{X}_t} \approx 1 + \hat{X}_t.$$

Equilibrium and log-linearisation VI

- So, we can write variables as:
$$X_t \approx \bar{X}(1+\hat{X}_t).$$

- The next trick is for variables multiplying each other such as:
$$X_t Y_t \approx \bar{X}\bar{Y}(1+\hat{X}_t)(1+\hat{Y}_t) \approx \bar{X}\bar{Y}(1+\hat{X}_t+\hat{Y}_t),$$
because you set terms like $\hat{X}_t\hat{Y}_t = 0$ since we're looking at small deviations from steady-state and multiplying these small deviations together gives a term close to zero.
- Anything else? Nope, that's it. It's also worth noting, however, that there are a few ways to do log-linearisation. The above gives a short-cut, broad picture approach to log-linearisation.

Equilibrium and log-linearisation VII

- "A Toolkit for Analyzing Nonlinear Dynamic Stochastic Models Easily" Uhlig (1998) gives a very rigorous treatment of log-linearisation.
- It's probably best that we go through a few examples (and that you practice) in order to nail down how log-linearisation works.

The Taylor expansion (standard) method I

- Consider a nonlinear model that can represented by a set of equations of the general form

$$F(\boldsymbol{X}_t) = \frac{G(\boldsymbol{X}_t)}{H(\boldsymbol{X}_t)}, \tag{22}$$

  where $\boldsymbol{X}_t$ is a vector of the variables of the model that can include forward-looking variables and lagged variables (jump and state variables, respectively), in addition to contemporaneous variables.
- The process of log-linearisation is to first take the logs of the functions $F(\cdot)$, $G(\cdot)$, and $H(\cdot)$, and then take a first-order Taylor series approximation.

The Taylor expansion (standard) method II

- Taking logs of (22) gives:
$$\ln F(\boldsymbol{X}_t) = \ln G(\boldsymbol{X}_t) - \ln H(\boldsymbol{X}_t),$$
and taking the first-order Taylor series expansion around the steady state, $\bar{\boldsymbol{X}}$, gives
$$\ln F(\bar{\boldsymbol{X}}) + \frac{F'(\bar{\boldsymbol{X}})}{F(\bar{\boldsymbol{X}})}(\boldsymbol{X}_t - \bar{\boldsymbol{X}}) \approx \ln G(\bar{\boldsymbol{X}}) + \frac{G'(\bar{\boldsymbol{X}})}{G(\bar{\boldsymbol{X}})}(\boldsymbol{X}_t - \bar{\boldsymbol{X}}) - \ln H(\bar{\boldsymbol{X}}) - \frac{H'(\bar{\boldsymbol{X}})}{H(\bar{\boldsymbol{X}})}(\boldsymbol{X}_t - \bar{\boldsymbol{X}}),$$
where the notation $X'(\bar{\boldsymbol{X}})$ is used to indicate the gradient at the steady state.

The Taylor expansion (standard) method III

- Notice that the model is now linear in $\boldsymbol{X}_t$, since $F'(\bar{\boldsymbol{X}})/F(\bar{\boldsymbol{X}})$, $G'(\bar{\boldsymbol{X}})/G(\bar{\boldsymbol{X}})$, $H'(\bar{\boldsymbol{X}})/H(\bar{\boldsymbol{X}})$, $\ln F(\bar{\boldsymbol{X}})$, $\ln G(\bar{\boldsymbol{X}})$, and $\ln H(\bar{\boldsymbol{X}})$ are constants. Since the following holds:

$$\ln F(\bar{\boldsymbol{X}}) = \ln G(\bar{\boldsymbol{X}}) - \ln H(\bar{\boldsymbol{X}}),$$

  we can eliminate the three log components and the previous expression simplifies to

$$\frac{F'(\bar{\boldsymbol{X}})}{F(\bar{\boldsymbol{X}})}(\boldsymbol{X}_t - \bar{\boldsymbol{X}}) \approx \frac{G'(\bar{\boldsymbol{X}})}{G(\bar{\boldsymbol{X}})}(\boldsymbol{X}_t - \bar{\boldsymbol{X}}) - \frac{H'(\bar{\boldsymbol{X}})}{H(\bar{\boldsymbol{X}})}(\boldsymbol{X}_t - \bar{\boldsymbol{X}}).$$

- The implicit assumption here is that if we stay close enough to the steady state, $\bar{\boldsymbol{X}}$, we can ignore the second-order or higher terms of the Taylor expansion – i.e., a first-order approximation is sufficient to capture the dynamics of the model.
- Let's work through some examples.

The Taylor expansion (standard) method IV

- **Example: The production function.** This method works particularly well when you have multiplicative terms. So let's start with our production technology:

$$Y_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha}, \tag{23}$$

and then take logs:

$$\log Y_t = \log A_t + \alpha \log K_{t-1} + (1-\alpha) \log N_t,$$

where we know that

$$\ln X_t = \ln \bar{X} + \frac{X_t - \bar{X}}{\bar{X}},$$

and so we have:

$$\ln \bar{Y} + \frac{Y_t - \bar{Y}}{\bar{Y}} \approx \ln \bar{A} + \frac{A_t - \bar{A}}{\bar{A}} + \alpha \left[\ln \bar{K} + \frac{K_{t-1} - \bar{K}}{\bar{K}} \right] + (1-\alpha) \left[\ln \bar{N} + \frac{N_t - \bar{N}}{\bar{N}} \right],$$

The Taylor expansion (standard) method V

and we know that in the steady-state we have $\ln \bar{Y} = \ln \bar{A} + \alpha \ln \bar{K} + (1-\alpha)\ln \bar{N}$, so

$$\frac{Y_t - \bar{Y}}{\bar{Y}} \approx \frac{A_t - \bar{A}}{\bar{A}} + \alpha \frac{K_{t-1} - \bar{K}}{\bar{K}} + (1-\alpha)\frac{N_t - \bar{N}}{\bar{N}}$$
$$\Leftrightarrow \hat{Y}_t = \hat{A}_t + \alpha \hat{K}_{t-1} + (1-\alpha)\hat{N}_t. \tag{24}$$

- **Example: The growth model.** Consider a simple growth model where the representative agent maximises

$$\sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} \right),$$

subject to the budget constraint,

$$C_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha} + (1-\delta)K_{t-1} - K_t. \tag{25}$$

The Taylor expansion (standard) method VI

The first-order conditions are:

$$C_t^{-\sigma} = \beta \mathbb{E}_t \left[\alpha A_{t+1} K_t^{\alpha-1} N_{t+1}^{1-\alpha} + (1-\delta) \right] C_{t+1}^{-\sigma}, \tag{26}$$

$$N_t^{\varphi} = C_t^{-\sigma} \left[(1-\alpha) A_t K_{t-1}^{\alpha} N_t^{-\alpha} \right]. \tag{27}$$

Taking logs of the budget constraint and first order conditions gives:

$$\begin{aligned} \ln C_t &= \ln \left[A_t K_{t-1}^{\alpha} N_t^{1-\alpha} + (1-\delta) K_{t-1} - K_t \right], \\ -\sigma \ln C_t &= \ln \beta + \ln \left[\alpha A_{t+1} K_t^{\alpha-1} N_{t+1}^{1-\alpha} + (1-\delta) \right] - \sigma \ln C_{t+1}, \\ \varphi \ln N_t &= -\sigma \ln C_t + \ln(1-\alpha) + \ln A_t + \alpha \ln K_{t-1} - \alpha \ln N_t. \end{aligned}$$

The Taylor expansion (standard) method VII

Then, take the first order Taylor expansions about the steady state (this is always an algebraic nightmare):

$$
\begin{aligned}
\ln \bar{C} + \frac{1}{\bar{C}}(C_t - \bar{C}) \approx \ln \left[\bar{A}\bar{K}^{\alpha}\bar{N}^{1-\alpha} + (1-\delta)\bar{K} - \bar{K}\right] + \frac{\bar{K}^{\alpha}\bar{N}^{1-\alpha}}{\bar{A}\bar{K}^{\alpha}\bar{N}^{1-\alpha} + (1-\delta)\bar{K} - \bar{K}}(A_t - \bar{A}) \\
+ \frac{\alpha\bar{A}\bar{K}^{\alpha-1}\bar{N}^{1-\alpha} + 1 - \delta}{\bar{A}\bar{K}^{\alpha}\bar{N}^{1-\alpha} + (1-\delta)\bar{K} - \bar{K}}(K_{t-1} - \bar{K}) + \frac{(1-\alpha)\bar{A}\bar{K}^{\alpha}\bar{N}^{-\alpha}}{\bar{A}\bar{K}^{\alpha}\bar{N}^{1-\alpha} + (1-\delta)\bar{K} - \bar{K}}(N_t - \bar{N}) \\
+ \frac{-1}{\bar{A}\bar{K}^{\alpha}\bar{N}^{1-\alpha} + (1-\delta)\bar{K} - \bar{K}}(K_t - \bar{K}),
\end{aligned}
$$

The Taylor expansion (standard) method VIII

$$
\begin{aligned}
-\sigma \ln \bar{C} - \sigma \frac{1}{\bar{C}}(C_t - \bar{C}) \approx \ln \beta + \frac{0}{\beta} + \ln\left(\alpha \bar{A}\bar{K}^{\alpha-1}\bar{N}^{1-\alpha} + 1 - \delta\right) \\
+ \frac{\alpha \bar{K}^{\alpha-1}\bar{N}^{1-\alpha}}{\alpha \bar{A}\bar{K}^{\alpha-1}\bar{N}^{1-\alpha} + 1 - \delta}(A_{t+1} - \bar{A}) + \frac{(\alpha - 1)\alpha \bar{A}\bar{K}^{\alpha-2}\bar{N}^{1-\alpha}}{\alpha \bar{A}\bar{K}^{\alpha-1}\bar{N}^{1-\alpha} + 1 - \delta}(K_t - \bar{K}) \\
+ \frac{(1 - \alpha)\alpha \bar{A}\bar{K}^{\alpha-1}\bar{N}^{-\alpha}}{\alpha \bar{A}\bar{K}^{\alpha-1}\bar{N}^{1-\alpha} + 1 - \delta}(N_{t+1} - \bar{N}) - \sigma \ln C - \sigma \frac{1}{\bar{C}}(C_{t+1} - \bar{C}),
\end{aligned}
$$

$$
\begin{aligned}
\varphi \ln \bar{N} + \varphi \frac{1}{\bar{N}}(N_t - \bar{N}) \approx -\sigma \ln \bar{C} - \sigma \frac{1}{\bar{C}}(C_t - \bar{C}) + \ln(1 - \alpha) + \frac{0}{1 - \alpha} + \ln \bar{A} + \frac{1}{\bar{A}}(A_t - \bar{A}) \\
+ \alpha \ln \bar{K} + \alpha \frac{1}{\bar{K}}(K_{t-1} - \bar{K}) - \alpha \ln \bar{N} - \alpha \frac{1}{\bar{N}}(N_t - \bar{N}).
\end{aligned}
$$

The Taylor expansion (standard) method IX

- Use steady state identities to get rid of the logs, and clean up:

$$\begin{aligned}\hat{C}_t =& \frac{1}{1-\delta\left(\bar{K}/\bar{N}\right)^{1-\alpha}}\hat{A}_t + \frac{\alpha\bar{K}^{\alpha-1}\bar{N}^{1-\alpha}+1-\delta}{\bar{K}^{\alpha-1}\bar{N}^{1-\alpha}-\delta}\hat{K}_{t-1} + \frac{1-\alpha}{1-\delta\left(\bar{K}/\bar{N}\right)^{1-\alpha}}\hat{N}_t \\ &- \frac{1}{\left(\bar{K}/\bar{N}\right)^{\alpha-1}-\delta}\hat{K}_t,\end{aligned} \tag{28}$$

$$\begin{aligned}-\sigma\hat{C}_t =& \frac{\alpha\bar{K}^{\alpha-1}\bar{N}^{1-\alpha}}{\alpha\bar{K}^{\alpha-1}\bar{N}^{1-\alpha}+1-\delta}\mathbb{E}_t\hat{A}_{t+1} + \frac{(\alpha-1)\alpha\bar{K}^{\alpha-1}\bar{N}^{1-\alpha}}{\alpha\bar{K}^{\alpha-1}\bar{N}^{1-\alpha}+1-\delta}\hat{K}_t \\ &+ \frac{(1-\alpha)\alpha\bar{K}^{\alpha-1}\bar{N}^{1-\alpha}}{\alpha\bar{K}^{\alpha-1}\bar{N}^{1-\alpha}+1-\delta}\mathbb{E}_t\hat{N}_{t+1} - \sigma\mathbb{E}_t\hat{C}_{t+1},\end{aligned} \tag{29}$$

$$\varphi\hat{N}_t = -\sigma\hat{C}_t + \hat{A}_t + \alpha\hat{K}_{t-1} - \alpha\hat{N}_t \tag{30}$$

The Uhlig method I

- As you can see, the standard Taylor expansion method is extremely cumbersome. Uhlig (1998) recommends using a simpler method for finding log-linear approximations of functions. His method does not require taking derivatives and gives the same results as the above method.
- We actually covered this briefly before. But just for clarification, consider an equation of a set of variables, X_t. Define $\hat{X}_t = \ln X_t - \ln \bar{X}$. One can write the original variable as

$$X_t = \bar{X} \exp \left\{ \hat{X}_t \right\},$$

since

$$\bar{X} \exp \hat{X}_t = \bar{X} \exp \left\{ \ln X_t - \ln \bar{X} \right\} = \bar{X} \exp \left\{ \ln \left(\frac{X_t}{\bar{X}} \right) \right\} = \bar{X} X_t / \bar{X} = X_t.$$

The Uhlig method II

Let's look at an example:

$$\frac{A_t B_t^{\alpha}}{C_t^{\delta}} = \frac{\bar{A} \exp \hat{A}_t \bar{B}^{\alpha} \exp \left\{ \alpha \hat{B}_t \right\}}{\hat{C}^{\delta} \exp \left\{ \delta \hat{C}_t \right\}},$$

and this becomes

$$\frac{\bar{A}\bar{B}^{\alpha}}{\bar{C}^{\delta}} \exp \left\{ \hat{A}_t + \alpha \hat{B}_t - \delta \hat{C}_t \right\}.$$

The Uhlig method III

- Now we take a Taylor expansion of the exponential term around the steady state – this time it's $\hat{X} = \ln \bar{X} - \ln \bar{X} = 0$!

$$\exp\left\{\hat{A}_t + \alpha\hat{B}_t - \delta\hat{C}_t\right\} \approx \exp\left\{\hat{A} + \alpha\hat{B} - \delta\hat{C}\right\} + \exp\left\{\hat{A} + \alpha\hat{B} - \delta\hat{C}\right\}(\hat{A}_t - \hat{A})$$
$$+ \alpha\exp\left\{\hat{A} + \alpha\hat{B} - \delta\hat{C}\right\}(\hat{B}_t - \hat{B}) - \delta\exp\left\{\hat{A} + \alpha\hat{B} - \delta\hat{C}\right\}(\hat{C}_t - \hat{C})$$
$$= 1 + \hat{A}_t + \alpha\hat{B}_t - \delta\hat{C}_t.$$

- Thus:

$$\frac{\bar{A}\bar{B}^{\alpha}}{\bar{C}^{\delta}}\exp\left\{\hat{A}_t + \alpha\hat{B}_t - \delta\hat{C}_t\right\} = \frac{\bar{A}\bar{B}^{\alpha}}{\bar{C}^{\delta}}\left(1 + \hat{A}_t + \alpha\hat{B}_t - \delta\hat{C}_t\right).$$

The Uhlig method IV

- **Example: The production function.** Suppose we have:

$$Y_t = A_t K_{t-1}^{\alpha} N_t^{1-\alpha}.$$

Substitute $X_t = \bar{X} \exp\left\{\hat{X}_t\right\}$ for each variable:

$$\bar{Y} \exp\left\{\hat{Y}_t\right\} = \bar{A}\bar{K}^{\alpha}\bar{N}^{1-\alpha} \exp\left\{\hat{A}_t + \alpha \hat{K}_{t-1} + (1-\alpha)\hat{N}_t\right\}$$

This is approximated by a Taylor expansion (using the "Uhlig method"):

$$\bar{Y}(1+\hat{Y}_t) = \bar{A}\bar{K}^{\alpha}\bar{N}^{1-\alpha}\left(1 + \hat{A}_t + \alpha \hat{K}_{t-1} + (1-\alpha)\hat{N}_t\right),$$

Since in steady state we have $\bar{Y} \equiv \bar{A}\bar{K}^{\alpha}\bar{N}^{1-\alpha}$, we can then write:

$$\hat{Y}_t = \hat{A}_t + \alpha \hat{K}_{t-1} + (1-\alpha)\hat{N}_t.$$

The Uhlig method V

- **Example: The resource constraint.** Start with

$$Y_t = C_t + I_t,$$

now, we could take logs and then do some total derivatives to log-linearise, but it's far easier to use the methodology explained above (often referred to as the Uhlig method). Rewrite our equation as:

$$\bar{Y}e^{\hat{Y}_t} = \bar{C}e^{\hat{C}_t} + \bar{I}e^{\hat{I}_t}$$
$$\Leftrightarrow \bar{Y}(1+\hat{Y}_t) = \bar{C}(1+\hat{C}_t) + \bar{I}(1+\hat{I}_t),$$

and we know that in the steady-state $\bar{Y} \equiv \bar{C} + \bar{I}$, so terms cancel out, so

$$\bar{Y}\hat{Y}_t = \bar{C}\hat{C}_t + \bar{I}\hat{I}_t$$
$$\therefore \hat{Y}_t = \frac{\bar{C}}{\bar{Y}}\hat{C}_t + \frac{\bar{I}}{\bar{Y}}\hat{I}_t.$$

The Uhlig method VI

- So, in summary, his rules are:

$$\exp\left\{\hat{X}_t + a\hat{Y}_t\right\} \approx 1 + \hat{X}_t + a\hat{Y}_t, \tag{31}$$

$$\hat{X}_t\hat{Y}_t \approx 0, \tag{32}$$

$$a\mathbb{E}_t \exp\left\{\hat{X}_{t+1}\right\} \approx a + a\mathbb{E}_t\hat{X}_{t+1}, \tag{33}$$

$$\mathbb{E}_t X_{t+1} = \bar{X}\left(1 + \mathbb{E}_t\hat{X}_{t+1}\right). \tag{34}$$

The total derivative method I

- This method is a bit of a headache, but it does come in handy when we have to deal with messy expressions. It essentially uses the fact that the differential of a variable, say X_t, about its steady-state can be written as $\frac{1}{\bar{X}} dX_t$, where $dX_t = X_t - \bar{X}$.
- Again, it's better to demonstrate this, so let's take:

$$R_t = \alpha \frac{Y_t}{K_{t-1}} + 1 - \delta,$$

and don't bother taking logs (since we don't have to deal with any power terms); just take the total derivative:

$$dR_t = \alpha \frac{1}{\bar{K}} dY_t - \alpha \frac{\bar{Y}}{\bar{K}^2} dK_{t-1}$$
$$\Leftrightarrow R_t - \bar{R} = \alpha \frac{1}{\bar{K}} (Y_t - \bar{Y}) - \alpha \frac{\bar{Y}}{\bar{K}^2} (K_{t-1} - \bar{K}),$$

The total derivative method II

and then divide the LHS and RHS by $\bar{R}$, and then do some manipulation to the terms on the RHS:

$$\begin{aligned}\frac{R_t - \bar{R}}{\bar{R}} &= \frac{1}{\bar{R}}\left[\alpha\frac{1}{\bar{K}}(Y_t - \bar{Y})\frac{\bar{Y}}{\bar{Y}} - \alpha\frac{\bar{Y}}{\bar{K}^2}(K_{t-1} - \bar{K})\right] \\ \Leftrightarrow \hat{R}_t &= \frac{1}{\bar{R}}\left[\alpha\frac{\bar{Y}}{\bar{K}}\hat{Y}_t - \alpha\frac{\bar{Y}}{\bar{K}}\hat{K}_{t-1}\right],\end{aligned}$$

and then clean up a bit to get:

$$\hat{R}_t = \frac{\alpha}{\bar{R}}\frac{\bar{Y}}{\bar{K}}\left(\hat{Y}_t - \hat{K}_{t-1}\right). \tag{35}$$

The total derivative method III

- Now let's look at the Keynes-Ramsey condition since it has an exponent term:

$$C_t^{-\sigma} = \beta \mathbb{E}_t \left[C_{t+1}^{-\sigma} R_{t+1} \right] ,$$

and then take logs:

$$-\sigma \ln C_t = \ln \beta - \sigma \mathbb{E}_t \ln C_{t+1} + \mathbb{E}_t \ln R_{t+1},$$

then take total derivatives:

$$\begin{aligned}
\frac{-\sigma}{\bar{C}} dC_t &= \frac{-\sigma}{\bar{C}} \mathbb{E}_t dC_{t+1} + \frac{1}{\bar{R}} \mathbb{E}_t dR_{t+1} \\
\Leftrightarrow -\sigma \frac{C_t - \bar{C}}{\bar{C}} &= -\sigma \frac{\mathbb{E}_t C_{t+1} - \bar{C}}{\bar{C}} + \frac{\mathbb{E}_t R_{t+1} - \bar{R}}{\bar{R}} \\
-\sigma \hat{C}_t &= -\sigma \mathbb{E}_t \hat{C}_{t+1} + \mathbb{E}_t \hat{R}_{t+1} \\
\hat{C}_t &= \mathbb{E}_t \hat{C}_{t+1} - \frac{1}{\sigma} \mathbb{E}_t \hat{R}_{t+1}. \qquad (36)
\end{aligned}$$

Again, not too difficult since the terms were multiplicative.

Taylor approximation method: Single variable case I

- We won't use this method for the RBC model, but it can come in handy in future applications. Consider the following non-linear first-order difference equation:

$$X_t = f(X_{t-1}),$$

where f is any non-linear functional form you can think of (something not too crazy, though). A first-order Taylor expansion of the RHS about the steady-state gives:

$$X_t \approx f(\bar{X}) + f'(\bar{X})(X_{t-1} - \bar{X}),$$

and in the steady-state if we assume $\bar{X} = f(\bar{X})$,then our Taylor expansion becomes:

$$\begin{aligned} X_t &\approx \bar{X} + f'(\bar{X})(X_{t-1} - \bar{X}) \\ \Leftrightarrow X_t - \bar{X} &\approx f'(\bar{X})(X_{t-1} - \bar{X}) \end{aligned}$$

Taylor approximation method: Single variable case II

then divide this by $\bar{X}$:

$$\frac{X_t - \bar{X}}{\bar{X}} \approx f'(\bar{X})\frac{\bar{X}_{t-1} - \bar{X}}{\bar{X}},$$

and with a bit cleaning up we have:

$$\hat{X}_t = f'(\bar{X})\hat{X}_{t-1}. \tag{37}$$

- Consider the following example:

$$K_t = (1-\delta)K_{t-1} + AK_{t-1}^{\alpha},$$

and then apply the formula in (37) to get:

$$\hat{K}_t = \left[1 - \delta + \alpha A\bar{K}^{\alpha-1}\right]\hat{K}_{t-1}.$$

Taylor approximation method: Multivariate case I

- The Taylor approximation has a vector version as well as a scalar version. Suppose have:

$$X_t = f(X_{t-1}, Y_t),$$

where f is a non-linear function. The vector (bivariate) version of a first-order Taylor expansion about the steady-state is:

$$X_t = f(\bar{X}, \bar{Y}) + f_X(\bar{X}, \bar{Y})(X_{t-1} - \bar{X}) + f_Y(\bar{X}, \bar{Y})(Y_t - \bar{Y}),$$

and again, set the steady-state condition $\bar{X} = f(\bar{X}, \bar{Y})$, and with a bit of rearranging we get:

$$X_t - \bar{X} = f_X(\bar{X}, \bar{Y})(X_{t-1} - \bar{X}) + f_Y(\bar{X}, \bar{Y})(Y_t - \bar{Y}),$$

and then divide through by $\bar{X}$:

$$\frac{X_t - \bar{X}}{\bar{X}} = f_X(\bar{X}, \bar{Y})\frac{(X_{t-1} - \bar{X})}{\bar{X}} + f_Y(\bar{X}, \bar{Y})\frac{(Y_t - \bar{Y})}{\bar{X}},$$

Taylor approximation method: Multivariate case II

use the steady-state trick on the second term on the RHS:

$$\frac{X_t - \bar{X}}{\bar{X}} = f_X(\bar{X}, \bar{Y}) \frac{(X_{t-1} - \bar{X})}{\bar{X}} + f_Y(\bar{X}, \bar{Y}) \frac{(Y_t - \bar{Y})}{\bar{X}} \frac{\bar{Y}}{\bar{Y}},$$

and then clean up

$$\hat{X}_t \approx f_X(\bar{X}, \bar{Y}) \hat{X}_{t-1} + f_Y(\bar{X}, \bar{Y}) \frac{\bar{Y}}{\bar{X}} \hat{Y}_t. \tag{38}$$

- Consider the following example

$$K_t = (1 - \delta) K_{t-1} + s Z_t K_{t-1}^{\alpha},$$

and so taking partial derivatives and following formula in (38) gives (you can try and verify it yourself):

$$\hat{K}_t = \left[(1 - \delta) + \alpha s \bar{Z} \bar{K}^{\alpha - 1} \right] \hat{K}_{t-1} + \left[s \bar{K}^{\alpha} \right] \frac{\bar{Z}}{\bar{Y}} \hat{Z}_t.$$

Log-linearised system and the steady state I

- The full log-linearised system is given by following seven equations:

$$\hat{Y}_t = \frac{\bar{C}}{\bar{Y}}\hat{C}_t + \frac{\bar{I}}{\bar{Y}}\hat{I}_t, \tag{39}$$

$$\hat{Y}_t = \hat{A}_t + \alpha\hat{K}_{t-1} + (1-\alpha)\hat{N}_t, \tag{40}$$

$$\hat{K}_t = \frac{\bar{I}}{\bar{K}}\hat{I}_t + (1-\delta)\hat{K}_{t-1}, \tag{41}$$

$$\hat{R}_t = \frac{\alpha}{\bar{R}}\frac{\bar{Y}}{\bar{K}}\left[\hat{Y}_t - \hat{K}_{t-1}\right], \tag{42}$$

$$\hat{C}_t = \mathbb{E}_t\hat{C}_{t+1} - \frac{1}{\sigma}\mathbb{E}_t\hat{R}_{t+1}, \tag{43}$$

$$\hat{N}_t = \hat{Y}_t - \sigma\hat{C}_t, \tag{44}$$

$$\hat{A}_t = \rho\hat{A}_{t-1} + \varepsilon_t. \tag{45}$$

Log-linearised system and the steady state II

- We are almost ready to take this basic RBC model to the computer (e.g., Dynare).
- We simply need to calibrate the model (macroeconomist speak for assigning values to our structural parameters), and to solve for steady-state values.
- In other words, we need to obtain numerical values for $\frac{\bar{C}}{\bar{Y}}, \frac{\bar{I}}{\bar{K}}, \frac{\alpha}{\bar{R}}\frac{\bar{Y}}{\bar{K}}$.
- We can do this by taking the original non-linearised RBC model and figuring out what things look like along a balanced-growth path.
- Start with the steady-state interest rate. This is linked to consumption behaviour via the consumption Euler equation:

$$1 = \beta \mathbb{E}_t \left(\frac{C_t}{C_{t+1}} \right)^{\sigma} R_{t+1}.$$

Log-linearised system and the steady state III

- Because we have no trend growth in technology in our model, the steady-state features consumption, investment, and output all taking on constant values with no uncertainty. Thus, in steady-state, we have $\bar{C}_t = \bar{C}_{t+1} = \bar{C}$, so

$$\bar{R} = \frac{1}{\beta}. \tag{46}$$

  In other words, in a no-growth economy, the rate of return on capital is determined by the rate of time preference.
- Next, take the equation for the rate of return on capital (in period t):

$$R_t = \alpha \frac{Y_t}{K_{t-1}} + 1 - \delta.$$

  In the steady-state we have:

$$\bar{R} = \frac{1}{\beta} = \alpha \frac{\bar{Y}}{\bar{K}} + 1 - \delta,$$

Log-linearised system and the steady state IV

thus, with a bit of rearranging, we get:

$$\frac{\bar{Y}}{\bar{K}} = \frac{\beta^{-1} + \delta - 1}{\alpha}. \tag{47}$$

So we have

$$\begin{aligned} \frac{\alpha}{\bar{R}} \frac{\bar{Y}}{\bar{K}} &= [\alpha\beta] \left[\frac{\beta^{-1} + \delta - 1}{\alpha}\right] \\ &= 1 - \beta(1 - \delta), \end{aligned} \tag{48}$$

which is one of the steady-state values we needed.

Log-linearised system and the steady state V

- Now, look at the law of motion of capital:

$$K_t = I_t + (1-\delta)K_{t-1},$$

and use the fact that in the steady-state we have $\bar{K}_t = \bar{K}_{t-1} = \bar{K}$, so:

$$\frac{\bar{I}}{\bar{K}} = \delta, \tag{49}$$

which is also what we were looking for.

Log-linearised system and the steady state VI

- Putting things together, we have:

$$\frac{\bar{I}}{\bar{Y}} = \frac{\frac{\bar{I}}{\bar{K}}}{\frac{\bar{Y}}{\bar{K}}} = \frac{\delta}{\frac{\beta^{-1}+\delta-1}{\alpha}} = \frac{\alpha\delta}{\beta^{-1}+\delta-1}, \tag{50}$$

and

$$\frac{\bar{C}}{\bar{Y}} = 1 - \frac{\bar{I}}{\bar{Y}} = 1 - \frac{\alpha\delta}{\beta^{-1}+\delta-1}. \tag{51}$$

Log-linearised system and the steady state VII

- So the final, log-linearised RBC model is:

$$\hat{Y}_t = \left[1 - \frac{\alpha\delta}{\beta^{-1} + \delta - 1}\right]\hat{C}_t + \left[\frac{\alpha\delta}{\beta^{-1} + \delta - 1}\right]\hat{I}_t, \tag{52}$$

$$\hat{Y}_t = \hat{A}_t + \alpha\hat{K}_{t-1} + (1-\alpha)\hat{N}_t, \tag{53}$$

$$\hat{K}_t = \delta\hat{I}_t + (1-\delta)\hat{K}_{t-1}, \tag{54}$$

$$\hat{R}_t = [1 - \beta(1-\delta)]\left[\hat{Y}_t - \hat{K}_{t-1}\right], \tag{55}$$

$$\hat{C}_t = \mathbb{E}_t\hat{C}_{t+1} - \frac{1}{\sigma}\mathbb{E}_t\hat{R}_{t+1}, \tag{56}$$

$$\hat{N}_t = \hat{Y}_t - \sigma\hat{C}_t, \tag{57}$$

$$\hat{A}_t = \rho\hat{A}_{t-1} + \varepsilon_t. \tag{58}$$

Log-linearised system and the steady state VIII

- We will explore the performance of this model via numerical simulation. But first, let's compare this Ramsey social planner equilibrium to the decentralised equilibrium.

The Decentralised Problem

The decentralised equilibrium

- As I mentioned previously, there are alternatives in how to set up the RBC model, but these will give us the same outcome.
- We will now setup the RBC model without the Ramsey social planner.
- General equilibrium will be achieved via competitive markets as households and firms optimise over their endowments.
- Furthermore, I will make the assumption that firms own the capital stock in the economy, while the households own the firms – again, whether the firms own the capital stock or households own the capital stock, both will lead to the same outcome.

Households I

- Let there be a continuum of households indexed by $i \in [0, 1]$.
- Each household allocates its time between work and leisure, and it picks a stream of consumption $\{C_t^i\}_{t=0}^{\infty}$ to maximise its present discounted value of lifetime utility.
- In exchange for supplying labour, households earn a competitive wage, w_t, which they takes as given.
- In order to insure themselves against any idiosyncratic risk, each individual household can write and issue state-contingent securities, B_t^i, whereby the counterparty is another household $j \neq i$.
- Because there is a continuum of households, and because financial markets are complete, the households are fully insured against idiosyncratic risk.
    - In other words, these securities can be considered as Arrow-Debreu securities.
- The only risk they face is the risk arising from aggregate shocks, A_t.

Households II

- Additionally, in equilibrium, securities are in “zero net supply”:

$$\int_0^1 B_t^i di = B_t = 0.$$

- All households write and enter into debt contracts with one another; however it's because of this that overall – in aggregate – the sum of all debts must be zero in equilibrium.
- Again, like most things in macroeconomics, things will be clearer after a bit of derivation.
- In addition to taking wages as given, households also take the market clearing real interest rate as given.

Households III

- Additionally, since households own firms, they earns firms' profit in the form of dividend imputations, $\mathcal{D}_t^i$. So, the household problem can be written as:

$$\max_{\{C_t^i, N_t^i, B_t^i\}} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s U(C_{t+s}^i, N_{t+s}^i),$$

subject to

$$C_t^i + B_t^i \leq w_t N_t^i + \mathcal{D}_t^i + R_t B_{t-1}^i. \tag{59}$$

- Note the timing I have used in the budget constraint for the assets. Here I have assumed "end-of-period" timing.
- This basically means that households start the period by inheriting assets B_{t-1}, and in the current period t they must pick the asset amount B_t.

Households IV

- The assets they picked in period $t-1$ pay out a gross real return of R_t today, and assets picked today, B_t, are expected to pay out a return of R_{t+1} in the next period.
- Forming a Lagrangian by assuming that an individual's budget constraint (59) binds with equality (and using the trick in (7)) gives us:

$$\mathcal{L}^i = U(C_t^i, N_t^i) + \lambda_t^i \left(w_t N_t^i + \mathcal{D}_t^i + R_t B_{t-1}^i - C_t^i - B_t^i \right)$$
$$+ \beta \mathbb{E}_t \lambda_{t+1}^i \left(w_{t+1} N_{t+1}^i + \mathcal{D}_{t+1}^i + R_t B_t^i - C_{t+1}^i - B_{t+1}^i \right),$$

Households V

and the following FOCs:

$$\mathcal{L}_C^i = U_C(C_t^i) - \lambda_t^i = 0, \tag{60}$$

$$\mathcal{L}_N^i = U_N(N_t^i) + \lambda_t^i w_t = 0, \tag{61}$$

$$\mathcal{L}_B^i = -\lambda_t^i + \beta \mathbb{E}_t \lambda_{t+1}^i R_{t+1} = 0, \tag{62}$$

$$\mathcal{L}_\lambda^i = w_t N_t^i + \mathcal{D}_t^i + R_t B_{t-1}^i - C_t^i - B_t^i = 0. \tag{63}$$

- These seem very familiar. They're essentially identical to the problem which the Ramsey planner solved.
- Let's park them here for a bit – we'll return to them later.

Firms and production I

- There is a representative firm.
  - You can either think of a continuum of perfectly competitive firms that employ workers and capital to produce, or you can just aggregate them all together to make one representative firm since they all behave identically. It doesn't matter in the RBC model.
- The firm wants to maximise the present discounted value of real net profits. It discounts future cash flows by a household stochastic discount factor (SDF). The way we'll define the SDF puts cash flows (measured in goods) in terms of current consumption since these firms are ultimately owned by households and the households care about consumption. Define the SDF as:

$$M^i_{t,t+s} = \beta^s \mathbb{E}_t \frac{U_C(C^i_{t+s})}{U_C(C^i_t)}, \; s > t, \tag{64}$$

where t is the current period.

Firms and production II

- Why do the firms use this formulation for the stochastic discount factor? Because this is how consumers value future dividend flows. One unit of dividends returned to the household at time $t+s$ generates $U_C(C_{t+s})$ additional units of utility, which must be discounted back to the present period, by β^s. Dividing by $U_C(C_t^i)$ gives the current consumption equivalent value of the future utils.
- But, look at (60), (62), and (64). If we combine these equations we can write:

$$\mathbb{E}_t \frac{1}{R_{t+1}} = \beta \mathbb{E}_t \frac{U_C(C_{t+1}^i)}{U_C(C_t^i)} = M_{t,t+1}^i, \quad \forall i.$$

- We just did something – well, a few things – pretty great.
  * First, we have written an expression for the household's FOC wrt assets B_t using the definition of the household's SDF.

Firms and production III

  - Next, we've managed to define the gross real interest rate as being the inverse of the SDF. Recall back to your undergrad lectures and you should realise that for zero-coupon bonds there is an inverse relationship between yields and prices. For these one-period assets, you can basically think of the SDF as the price and R_t as the yield.
  - Finally, we have – admittedly in an ad-hoc and hand waiving way – managed to drop the index i. This means that we can write the household problem using a representative household setup!

- The firm produces output, Y_t, with a CRS production function,

$$Y_t = A_t F(K_{t-1}, N_t),$$

with the usual assumptions that we make. It hires labour, purchases new capital goods, and issues one-period debt promises, D_t. The firm also pays R_t^k on debt issued in the previous period, and the interest paid on debt is equal to the interest paid on assets due to a no-arbitrage condition.

Firms and production IV

- The firm's problem can be written as:

$$\mathbb{V}_t = \max_{\{N_t, I_t, D_t, K_t\}} \mathbb{E}_t \sum_{s=0}^{\infty} M_{t,t+s} \left[A_{t+s} F(K_{t-1+s}, N_{t+s}) - w_{t+s} N_{t+s} - I_{t+s} + D_{t+s} - R_{t+s} D_{t-1+s} \right],$$

subject to

$$K_t = I_t + (1-\delta) K_{t-1}.$$

Firms and production V

- Rearranging the law of motion for capital, and substituting for I_t in the objective function gives us:

$$
\begin{aligned}
\mathbb{V}_t &= \max_{\{N_t, D_t, K_t\}} \mathbb{E}_t \sum_{s=0}^{\infty} M_{t,t+s} \begin{pmatrix} A_{t+s}F(K_{t-1+s}, N_{t+s}) - K_{t+s} + (1-\delta)K_{t-1+i} \\ -w_{t+s}N_{t+s} + D_{t+s} - R^k_{t+s}D_{t-1+s} \end{pmatrix} \\
&= \max_{\{N_t, D_t, K_t\}} \left\{ \begin{array}{c} A_tF(K_{t-1}, N_t) - K_t + (1-\delta)K_{t-1} - w_tN_t + D_t - R^k_t D_{t-1} \\ \mathbb{E}_t M_{t,t+1} \left(A_{t+1}F(K_t, N_{t+1}) - K_{t+1} + (1-\delta)K_t - w_{t+1}N_{t+1} + D_{t+1} - R^k_{t+1}D_t \right) \end{array} \right\}
\end{aligned}
$$

  which basically says that the firm's revenue each period is equal to output, and that its costs each period are the wage bill, investment in new physical capital, and servicing costs on its debt.

Firms and production VI

- The FOCs from the firm problem are:

$$\begin{aligned}
\frac{\partial \mathbb{V}_t}{\partial N_t} &= A_t F_N(K_{t-1}, N_t) - w_t = 0 \\
\implies \quad w_t &= A_t F_N(K_{t-1}, N_t), && (65) \\
\frac{\partial \mathbb{V}_t}{\partial D_t} &= 1 - \mathbb{E}_t M_{t,t+1} R^k_{t+1} = 0 \\
\implies \quad U_C(C_t) &= \beta \mathbb{E}_t U_C(C_{t+1}) R^k_{t+1}, && (66) \\
\frac{\partial \mathbb{V}_t}{\partial K_t} &= -1 + \mathbb{E}_t M_{t,t+1} A_{t+1} F_K(K_t, N_{t+1}) + (1-\delta) = 0 \\
\implies \quad U_C(C_t) &= \beta \mathbb{E}_t U_C(C_{t+1}) A_{t+1} F_K(K_t, N_{t+1}) + (1-\delta). && (67)
\end{aligned}$$

Firms and production VII

- Let's interpret these FOCs a bit. (65) is pretty intuitive: The wage rate w_t is equal to the marginal productivity of labour.
- However, look at (66) and (67) – they're essentially the same, and must therefore hold in equilibrium as long as the household is optimising. In fact they're directly analogous to the consumption Euler equation or Keynes-Ramsey condition in (14)!
- This means that the amount of debt the firm issues is indeterminate, since the condition will hold for any choice of D_t. This is essentially the Modigliani-Miller theorem (Modigliani and Miller, 1958): it doesn't matter how the firm finances its purchases of new capital – debt or equity – and hence the debt/equity mix is indeterminate.

Technology process

- In order to close the model, we need to specify a stochastic process for the exogenous variable(s).
- The only exogenous variable in this model is A_t – which is the same as before when we solved for the Ramsey social planner equilibrium:

$$\ln A_t = \rho \ln A_{t-1} + \varepsilon_t. \tag{68}$$

Competitive equilibrium I

- A competitive equilibrium is a set of prices $\{R_t, w_t\}$ and allocations $\{C_t, N_t, K_t, D_t, B_t\}$ taking K_{t-1}, D_{t-1}, B_{t-1}, A_{t-1} and the stochastic process for A_t as given; the optimality conditions (60)-(67); the labour and bond market clearing conditions ($N_t^d = N_t^s$ and $B_t = D_t,\ \forall t$); and both budget constraints holding with equality.
- Consolidating the household and firm budget constraints gives:

$$\begin{aligned} C_t + B_t &= w_t N_t + R_{t-1} B_{t-1} + A_t F(K_{t-1}, N_t) - w_t N_t - I_t + D_t - R_{t-1} D_{t-1} \\ \implies A_t F(K_{t-1}, N_t) &= C_t + I_t, \end{aligned}$$

  in other words, bond market-clearing plus both budget constraints holding just gives the standard accounting identity that output must be consumed or invested.

Competitive equilibrium II

- If you combine the household's FOC for labour supply (61) with the firm's FOC, you get:

$$-U_N(N_t) = U_C(C_t)A_tF_N(K_{t-1}, N_t). \tag{69}$$

- The FOC for bonds/debt (67) along with the FOC for the firm's choice of its capital stock (66) imply that:

$$\beta\mathbb{E}_tU_C(C_{t+1})\left[A_{t+1}F_K(K_t, N_{t+1}) + (1-\delta)\right] = \beta\mathbb{E}_tR^k_{t+1}U_C(C_{t+1}),$$

which can be rewritten as:

$$R^k_{t+1} = A_{t+1}F_K(K_t, N_{t+1}) + 1 - \delta, \tag{70}$$

which is what we proposed in (12).

Competitive equilibrium III

- Putting all the equations together, the equilibrium conditions for the decentralised RBC model are:

$$U_C(C_t) = \beta \mathbb{E}_t U_C(C_{t+1}) \left[A_{t+1} F_K(K_t, N_{t+1}) + 1 - \delta\right], \tag{71}$$

$$U_N(N_t) = U_C(C_t) A_t F_N(K_{t-1}, N_t), \tag{72}$$

$$K_t = A_t F(K_{t-1}, N_t) - C_t + (1-\delta) K_{t-1}, \tag{73}$$

$$\ln A_t = \rho \ln A_{t-1} + \varepsilon_t, \tag{74}$$

$$Y_t = A_t F(K_{t-1}, N_t), \tag{75}$$

$$Y_t = C_t + I_t, \tag{76}$$

$$U_C(C_t) = \beta \mathbb{E}_t U_C(C_{t+1}) R_{t+1}, \tag{77}$$

$$w_t = A_t F_N(K_{t-1}, N_t), \tag{78}$$

$$R_t = A_t F_K(K_{t-1}, N_t). \tag{79}$$

Competitive equilibrium IV

- But these are nothing the same as the equilibrium conditions for when we solved for the Ramsey social planner. Why is this the case?

First and Second Welfare Theorems of Economics

Welfare Theorems I

- Recall the fundamental welfare theorems of economics from Mas-Colell, Whinston, and Green (1995).
- **The First Fundamental Welfare Theorem**: If the economy is described by complete markets, no externalities or non-convexities then every equilibrium of the competitive market is socially optimal.
- **The Second Fundamental Welfare Theorem**: If household preferences and firm production sets are convex, there is a complete set of markets with publicly known prices, and every agent acts as a price taker, then any Pareto optimal outcome can be achieved as a competitive equilibrium if appropriate lump-sum transfers of wealth are arranged.
- The result that the competitive equilibrium of a representative agent economy and that of a perfectly competitive one, that is otherwise identical, is not surprising.

Conclusion

- Next class we will work through the problem set, and use `Dynare` to compute an RBC model.
- If I give you a functional form for utility and production, as well as calibrated parameters, you can simulate the RBC model on your computer!
- For further details on the RBC model, once again a very good reference is McCandless (2008).

References I

Mas-Colell, Andreu, Michael D. Whinston, and Jerry R. Green. 1995. *Microeconomic Theory.* Oxford University Press.

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