Static Optimisation

1 Unconstrained optimisation

Find all the critical points (solutions to the first order conditions) of the function

$$f(x, y) = x^4 + x^2 - 6xy + 3y^2,$$

and classify each one (if possible) as a local maximum, local minimum, or saddle point.

The partial derivatives for the problem are as follows:

$$\frac{\partial f}{\partial x} = 4x^3 + 2x - 6y = 0. \quad (1)$$

$$\frac{\partial f}{\partial y} = -6x + 6y = 0. \quad (2)$$

From (2) we can infer that $x = y$. Substituting this back into (1) gives us the following: $4x^3 = 4x$. Which gives us critical points at (0,0), (1,1), and (-1,-1).

We need to check for local minimum, maximum, or saddle points. Therefore, we need to check second order sufficiency conditions. The Hessian matrix for this problem is:

$$\mathbf{H} = \begin{bmatrix} 12x^2 + 2 & -6 \\ -6 & 6 \end{bmatrix}.$$

At $(0, 0)$

$$\mathbf{H} = \begin{bmatrix} 2 & -6 \\ -6 & 6 \end{bmatrix}$$

which implies that $|\mathbf{H}_2| = -24$, so $\mathbf{H}$ is indefinite, and this implies that $(0, 0)$ is a saddle point.

At $(1, 1)$

$$\mathbf{H} = \begin{bmatrix} 14 & -6 \\ -6 & 6 \end{bmatrix}$$

which implies that $|\mathbf{H}_2| = 48$ and $|\mathbf{H}_1| = 14$, so $\mathbf{H}$ is positive definite, and this implies that $(1, 1)$ is a local minimum of $f(1, 1) = -1$.

At $(-1, -1)$

$$\mathbf{H} = \begin{bmatrix} 14 & -6 \\ -6 & 6 \end{bmatrix}$$

which implies that, like above, there is a local minimum of $f(-1, -1) = -1$.

2 Constrained optimisation

Use the Lagrangian method to find a maximum and minimum value of

$$x^2 + y^2,$$

subject to

$$x^2 + xy + y^2 = 3,$$

checking the constraint qualification and the second-order conditions.

The Lagrange for this problem is:

$$Z = x^2 + y^2 - \lambda(x^2 + xy + y^2 - 3),$$

where λ is the Lagrangian multiplier. Checking the gradient vector for the constraint function, we can see that it is of full rank, and therefore the constraint condition is fulfilled:

$$\nabla h(x, y)^\top = \begin{bmatrix} 2x + y & 2y + x \end{bmatrix},$$

where $h(x, y)$ is the constraint function. The first order conditions (FOCs) to this problem are:

$$\frac{\partial Z}{\partial x} = 2x - \lambda(2x + y) = 0, \tag{3}$$

$$\frac{\partial Z}{\partial y} = 2y - \lambda(2y + x) = 0. \tag{4}$$

From (3) we get an expression for λ as

$$\lambda = \frac{2x}{2x + y}.$$

Substituting this into (4) we attain a solution of $y^2 = x^2$, which implies that

$$x = \pm y.$$

If $x = y$, the constraint function becomes

$$3y^2 = 3,$$

which gives two solutions the FOCs:

$$\begin{aligned} x &= y = 1, \text{ and} \\ x &= y = -1, \\ \lambda &= \frac{2}{3}. \end{aligned}$$

If $x = -y$, the constraint function becomes

$$y^2 = 3,$$

which gives us two more solutions to the FOCs:

$$
\begin{aligned}
&x = \sqrt{3},\ y = -\sqrt{3},\ \text{and} \\
&x = -\sqrt{3},\ y = \sqrt{3}, \\
&\lambda = 2.
\end{aligned}
$$

The bordered Hessian matrix for this problem is:

$$
(\mathbf{H}) = \begin{bmatrix} 0 & 2x+y & 2y+x \\ 2x+y & 2-2\lambda & -\lambda \\ 2y+x & -\lambda & 2-2\lambda \end{bmatrix},
$$

and we need to check the signs of the determinants of the last $n-m$ principal minors. Here, $n-m = 2-1 = 1$, so for a local max we need $|(\mathbf{H})| > 0 = \text{sign}(-1)^n$; and for a local min we need $|(\mathbf{H})| < 0 = \text{sign}(-1)^m$.

We see that the determinants of the bordered Hessian are: -24, -24, 24, and 24 when evaluated at the candidate solution points. This confirms global maxima at (-1,-1,$\frac{2}{3}$) and (1,1,$\frac{2}{3}$), and global minima at ($\sqrt{3}$,$-\sqrt{3}$, 2) and ($-\sqrt{3}$,$\sqrt{3}$, 2), since the set of points satisfying the constraint is compact.

3 Karush-Kuhn-Tucker Theory

A consumer obtains utility $u = x^2y$ from consuming quantities x and y of goods X and Y. The price of good X is 2, the price of good Y is 3, and his total income is 9. He chooses x and y to maximise his utility subject to his budget constraint:

$$\max_{x,y} \; x^2y$$

subject to

$$\begin{aligned} 2x + 3y &\leq 9 \\ x &> 0 \\ y &> 0. \end{aligned}$$

3.1

Explain why his choice must satisfy $2x + 3y = 9$, and $x > 0$, $y > 0$.

$2x + 3y = 9$ represents the consumer's feasible set of consumption possibilities – any point above or below this equality is either unaffordable or suboptimal. The strictly positive orthant constraints are to a) prevent negative consumption of a good, and b) to define the existence of a utility function.

Treating the problem as an equality problem also simplifies the mathematics (no need for Karush-Kuhn-Tucker (KKT) Theory).

3.2

Write down the Lagrangian for this problem, and solve the FOCs.

The consumer wishes to maximise $u(x, y) = x^2y$ subject to her budget constraint. The consumer's problem can be represented by the following Lagrangian function:

$$Z = x^2y - \lambda(2x + 3y - 9),$$

with the following FOCs:

$$\frac{\partial Z}{\partial x} = 2xy - 2\lambda = 0, \tag{5}$$

$$\frac{\partial Z}{\partial y} = x^2 - 3y = 0. \tag{6}$$

Using the FOCs and the constraint function, we attain optimal values $x^* = 3$, $y^* = 1$, and $\lambda^* = 3$.

3.3

Show by checking the second order conditions that the solution you have found is a strict local maximum.

Evaluating the bordered Hessian matrix at the optimal point yields:

$$(\mathbf{H}) = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 2 & 6 \\ 3 & 6 & 0 \end{bmatrix},$$

and we need to check the signs of the determinants of the last $n - m$ principal minors. Here, $n - m = 2 - 1 = 1$, so for a local max, we need $|(\mathbf{H})| > 0 = \text{sign}(-1)^n$, and for a local min we need $|(\mathbf{H})| < 0 = \text{sign}(-1)^m$.

For this problem evaluating at the optimal gives $|(\mathbf{H})| = 54$, which is the same sign as $(-1)^n$ since $n = 2$. This point gives a strict local maximum.

3.4

Explain why this must also be the global maximum.

We have a single candidate for a maximum of the objective function on the set of points satisfying the constraints. Since this is a compact set, we know that the objective function achieves a global maximum on the set, so the one we have found must be it.

4 Convexity

Show that the function

$$f(x, y) = x^2 + 2y^2 + xy + 3x + 19y - 4$$

is strictly convex, by examining the determinant and trace of the Hessian and then using the eigenvalue test, or otherwise.

The gradient vector (of partial derivatives) for this problem is:

$$\nabla f(x, y) = \begin{bmatrix} 2x + y + 3 \\ 4y + x + 19 \end{bmatrix},$$

and the matrix of second order partial derivatives is:

$$\nabla^2 f(x, y) = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} = \mathbf{H},$$

which yields the Hessian matrix. We find that $|\mathbf{H}| = 7$, so the eigenvalues have the same sign; and $\text{tr}(\mathbf{H}) = 6$, so that common sign is positive. The determinant of the Hessian implies convexity, and its trace implies positive definiteness since each trace element is greater than 0. Alternatively, we can look at the leading principal minors which 2 and 7, implying positive definiteness.

4.1

Find the pair (x, y) that minimises f.

The FOC, which is now necessary and sufficient for a global minimum, is

$$\nabla f(x, y) = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

this leads to $\arg\min f = (1, -5)$, with $f(1, -5) = -50$.

5 The firm's problem

A firm produces output $y > 0$ from inputs K and L. Its production possibilities are constrained by $K \geq 0$, $L \geq 0$, and $Y(K, L) \geq y$ where

$$Y(K, L) = (K+1)^{1-\alpha}(L+1)^{\alpha} - 1,\ 0 < \alpha < 1.$$

If the firm takes r and w, the strictly positive prices of K and L, as given, then what is the least-cost way of producing y, and what is this cost? Give bounds on the ratio r/w in terms of $\alpha$and y under which the solutions you have found are valid.

The firm's production problem is the following:

$$\begin{aligned} \min \quad & wL + rK, \\ \text{s.t.} \quad & (K+1)^{1-\alpha}(L+1)^{\alpha} - 1 \geq y, \\ & K \geq 0, L \geq 0. \end{aligned}$$

The constraint set is not compact, however we can compactify the set without losing generality by specifying upper bounds of K and L given by the wage-rental ratio. We only need compactness for existence – KKT conditions will find a global optimal if it does exist due to convexity/concavity. We now consider the constraint qualification:

$$\nabla \mathbf{g} = \begin{bmatrix} \frac{\partial g_1}{\partial K} & \frac{\partial g_1}{\partial L} \\ \frac{\partial g_2}{\partial K} & \frac{\partial g_2}{\partial L} \\ \frac{\partial g_3}{\partial K} & \frac{\partial g_3}{\partial L} \end{bmatrix} = \begin{bmatrix} (1-\alpha)\left[\frac{L+1}{K+1}\right]^{\alpha} & \alpha\left[\frac{L+1}{K+1}\right]^{\alpha-1} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The rank of the row vectors of $\nabla \mathbf{g}$ is of full rank, therefore the constraint qualification cannot fail for any combination of binding constraints, and so the KKT conditions are necessary for a global minimum. This gives a Lagrangian of the form:

$$Z = wL + rK - \lambda((K+1)^{1-\alpha}(L+1)^{\alpha} - 1 - y),$$

and the following KKT conditions:

$$Z_L = w - \alpha\lambda(K+1)^{1-\alpha}(L+1)^{\alpha-1} \geq 0,\ L \geq 0,\ LZ_L = 0, \tag{7}$$
$$Z_K = r - (1-\alpha)\lambda(K+1)^{-\alpha}(L+1)^{\alpha} \geq 0,\ K \geq 0,\ KZ_K = 0, \tag{8}$$
$$Z_\lambda = (K+1)^{1-\alpha}(L+1)^{\alpha} - 1 - y \leq 0,\ \lambda \leq 0,\ \lambda Z_\lambda = 0, \tag{9}$$

Since we know that $w, r, y > 0$, and so if we consider the regime where neither of the non-negativity constraints bind, we have

$$w = \alpha\lambda(K+1)^{1-\alpha}(L+1)^{\alpha-1}, \tag{10}$$
$$r = (1-\alpha)\lambda(K+1)^{-\alpha}(L+1)^{\alpha}, \tag{11}$$

and since both w and r are greater than zero, this implies that λ cannot be zero, and so Z_λ binds with equality to give

$$y = (K+1)^{1-\alpha}(L+1)^{\alpha} - 1. \tag{12}$$

From (12) we have

$$\begin{aligned} y &= (K+1)^{1-\alpha}(L+1)^{\alpha} - 1 \\ y + 1 &= (K+1)^{1-\alpha}(L+1)^{\alpha} \\ \implies (L+1)^{\alpha} &= \frac{y+1}{(K+1)^{1-\alpha}}, \end{aligned}$$

which we then substitute into (11) to get

$$\begin{aligned} r &= (1-\alpha)\lambda(K+1)^{-\alpha}(L+1)^{\alpha}, \\ r &= (1-\alpha)\lambda(K+1)^{-\alpha}\left[\frac{y+1}{(K+1)^{1-\alpha}}\right] \\ r &= (1-\alpha)\lambda(K+1)^{-\alpha}(y+1)(K+1)^{\alpha-1} \\ r &= (1-\alpha)\lambda(K+1)^{-1}(y+1) \\ \implies K^* + 1 &= \frac{(1-\alpha)\lambda(y+1)}{r}. \end{aligned}$$

Looking at (12) again we have

$$\begin{aligned} y &= (K+1)^{1-\alpha}(L+1)^{\alpha} - 1 \\ (K+1)^{1-\alpha} &= \frac{(y+1)}{(L+1)^{\alpha}}, \end{aligned}$$

which we can substitute into (10):

$$\begin{aligned} w &= \alpha\lambda(K+1)^{1-\alpha}(L+1)^{\alpha-1}, \\ w &= \alpha\lambda\left[\frac{(y+1)}{(L+1)^{\alpha}}\right](L+1)^{\alpha-1} \\ w &= \alpha\lambda(y+1)(L+1)^{\alpha-1-\alpha} \\ \implies L^* + 1 &= \frac{\alpha\lambda(y+1)}{w}. \end{aligned}$$

Then we can substitute $K^* + 1$ and $L^* + 1$ into (12):

$$\begin{aligned} y &= (K^*+1)^{1-\alpha}(L^*+1)^{\alpha} - 1 \\ y + 1 &= \left[\frac{(1-\alpha)\lambda(y+1)}{r}\right]^{1-\alpha}\left[\frac{\alpha\lambda(y+1)}{w}\right]^{\alpha} \\ &= \frac{(1-\alpha)^{1-\alpha}\lambda^{1-\alpha}(y+1)^{1-\alpha}}{r^{1-\alpha}}\frac{\alpha^{\alpha}\lambda^{\alpha}(y+1)^{\alpha}}{w^{\alpha}} \\ &= (y+1)\lambda\left[\frac{1-\alpha}{r}\right]^{1-\alpha}\left[\frac{\alpha}{w}\right]^{\alpha} \\ \implies \lambda^* &= \left[\frac{r}{1-\alpha}\right]^{1-\alpha}\left[\frac{w}{\alpha}\right]^{\alpha}, \end{aligned} \tag{13}$$

which can sub into K^* and L^* to get :

$$\begin{aligned} K^* &= \frac{(1-\alpha)(y+1)}{r}\left[\frac{r}{1-\alpha}\right]^{1-\alpha}\left[\frac{w}{\alpha}\right]^{\alpha} - 1 \\ &= (y+1)(1-\alpha)^{1-1+\alpha} r^{1-\alpha-1}\left[\frac{w}{\alpha}\right]^{\alpha} - 1 \\ \therefore K^* &= (y+1)\left[\frac{1-\alpha}{r}\right]^{\alpha}\left[\frac{w}{\alpha}\right]^{\alpha} - 1, \end{aligned}$$

and

$$\begin{aligned} L^* &= \frac{\alpha(y+1)}{w}\left[\frac{r}{1-\alpha}\right]^{1-\alpha}\left[\frac{w}{\alpha}\right]^{\alpha} - 1. \\ &= (y+1)\alpha^{1-\alpha} w^{\alpha-1}\left[\frac{r}{1-\alpha}\right]^{1-\alpha} - 1 \\ \therefore L^* &= (y+1)\left[\frac{w}{\alpha}\right]^{\alpha-1}\left[\frac{r}{1-\alpha}\right]^{1-\alpha} - 1. \end{aligned}$$

The values for K^*, L^*, and λ^* for regime 1 satisfy the FOCs, non-negativity of multipliers and KKT conditions. The production constraint is met, but the non-negativity constraints on K and L are met iff:

$$(y+1)^{-\frac{1}{1-\alpha}}\left[\frac{1-\alpha}{\alpha}\right] \leq \frac{r}{w} \leq (y+1)^{\frac{1}{\alpha}}\left[\frac{1-\alpha}{\alpha}\right].$$

Why? From K^* when $K = 0$ we have

$$\begin{aligned} 1 &= (y+1)\left[\frac{1-\alpha}{r}\right]^{\alpha}\left[\frac{w}{\alpha}\right]^{\alpha} \\ \frac{r^\alpha}{w^\alpha} &= (y+1)\left[\frac{1-\alpha}{r}\right]^{\alpha} \\ \implies \frac{r}{w} &= (y+1)^{\frac{1}{\alpha}}\left[\frac{1-\alpha}{\alpha}\right], \end{aligned}$$

and from L^* we have

$$\begin{aligned} 1 &= (y+1)\left[\frac{w}{\alpha}\right]^{\alpha-1}\left[\frac{r}{1-\alpha}\right]^{1-\alpha} \\ &= (y+1)\frac{w^{\alpha-1}}{\alpha^{\alpha-1}}\frac{r^{1-\alpha}}{(1-\alpha)^{1-\alpha}} \\ &= (y+1)\frac{w^{\alpha-1}}{r^{\alpha-1}}\frac{\alpha^{1-\alpha}}{(1-\alpha)^{1-\alpha}} \\ \left[\frac{r}{w}\right]^{\alpha-1} &= (y+1)\left[\frac{1-\alpha}{\alpha}\right]^{\alpha-1} \\ \implies \frac{r}{w} &= (y+1)^{-\frac{1}{1-\alpha}}\left[\frac{1-\alpha}{\alpha}\right], \end{aligned}$$

which gives us our upper and lower bounds, respectively.

For the second regime, we assume that the non-negativity constraint on only K binds ($K = 0 \implies Z_K > 0$). But, we assume that $L > 0$. For convenience, it's worth looking at the KKT conditions again:

$$\begin{aligned}
Z_L &= w - \alpha\lambda(K+1)^{1-\alpha}(L+1)^{\alpha-1} \geq 0,\ L \geq 0,\ LZ_L = 0, \\
Z_K &= r - (1-\alpha)\lambda(K+1)^{-\alpha}(L+1)^{\alpha} \geq 0,\ K \geq 0,\ KZ_K = 0, \\
Z_\lambda &= (K+1)^{1-\alpha}(L+1)^{\alpha} - 1 - y \leq 0,\ \lambda \leq 0,\ \lambda Z_\lambda = 0.
\end{aligned}$$

Since $L > 0$, this implies that $\lambda \neq 0$, and so Z_λ binds with equality. Thus, we have

$$\begin{aligned}
w &= \alpha\lambda(L+1)^{\alpha-1}, \\
y &= (L+1)^{\alpha} - 1,
\end{aligned}$$

and we can solve for λ and L:

$$(y+1)^{\frac{1}{\alpha}} = L^* + 1,$$

which after substituting into w gives

$$\begin{aligned}
w &= \alpha\lambda(y+1)^{\frac{\alpha-1}{\alpha}} \\
\implies \lambda^* &= \frac{w}{\alpha(y+1)^{\frac{\alpha-1}{\alpha}}}.
\end{aligned}$$

The constraints imposed in this regime occur iff:

$$(y+1)^{\frac{1}{\alpha}}\left[\frac{1-\alpha}{\alpha}\right] \leq \frac{r}{w}.$$

Finally, for the third regime, we assume that the non-negativity constraint on only L binds ($L = 0 \implies Z_L > 0$). But, we assume that $K > 0$, which implies that $\lambda \neq 0$, and so Z_λ binds with equality. Thus, we have

$$\begin{aligned}
r &= (1-\alpha)\lambda(K+1)^{-\alpha}, \\
y &= (K+1)^{1-\alpha} - 1,
\end{aligned}$$

and we can solve for λ and K:

$$(y+1)^{\frac{1}{1-\alpha}} = K^* + 1,$$

and substituting this into r gives

$$\begin{aligned}
r &= (1-\alpha)\lambda(y+1)^{-\frac{\alpha}{1-\alpha}} \\
\implies \lambda^* &= \frac{r(y+1)^{\frac{\alpha}{1-\alpha}}}{1-\alpha}.
\end{aligned}$$

The constraints imposed in this regime occur iff:

$$\frac{r}{w} \leq (y+1)^{-\frac{1}{1-\alpha}} \left[\frac{1-\alpha}{\alpha}\right].$$

For completion, we can summarise the three regimes as

$$\underbrace{\frac{r}{w} \leq}_{\text{regime 3}} (y+1)^{-\frac{1}{1-\alpha}} \left[\frac{1-\alpha}{\alpha}\right] \underbrace{\leq \frac{r}{w} \leq}_{\text{regime 1}} (y+1)^{\frac{1}{\alpha}} \left[\frac{1-\alpha}{\alpha}\right] \underbrace{\leq \frac{r}{w}}_{\text{regime 2}} .$$

6 The consumer's problem (again)

A household has preferences represented by the utility function $u(x_1, x_2)$ where

$$u(x_1, x_2) = x_1^\beta + x_2^\beta,\ 0 < \beta < 1,$$

it is constrained by $x_1 \geq 0$, $x_2 \geq 0$, and also by

$$P^s x_1 + x_2 \leq P^s m_1 + m_2,$$
$$P^b x_1 + x_2 \leq P^b m_1 + m_2,$$

where m_1, m_2 are fixed endowments, both strictly positive. The price of good 2 is normalised to 1, and we also have $0 < P^s < P^b$.
Draw the budget set. What is the interpretation of P^s and P^b? What choices should the household make if it wants to maximise utility? Give bounds on P^s and/or P^b in terms of β, m_1, and m_2 under which the solutions you have found are valid.

P^b and P^s can be interpreted as the relative price to borrow and save x_1, respectively, against x_2. For example, observing the budget set, the household is able to trade x_2 for x_1 at the price of P^s between points a and b. From points b to c, the household is only able to exchange x_1 for x_2 at the higher price of P^b.

To solve this optimisation problem, we can think of the household's problem as:

$$\begin{aligned} \min \quad & u(x_1, x_2) = x_1^\beta + x_2^\beta, \\ \text{s.t.} \quad & x_1 \geq 0, x_2 \geq 0, \\ & P^s x_1 + x_2 \leq P^s m_1 + m_2, \\ & P^b x_1 + x_2 \leq P^b m_1 + m_2. \end{aligned}$$

But if we look at the marginal rate of substitution (MRS), we get an expression which looks like:

$$MRS = \frac{MU_1}{MU_2} = \frac{\beta x_1^{\beta-1}}{\beta x_2^{\beta-1}}.$$

We can see that the MRS is undefined for values of x_1 and x_2 if either of them tend to zero. As such, we can ignore the positive constraints for our household's problem, reducing the number of constraints to two. The gradient vector looks like

$$\nabla \mathbf{g} = \begin{bmatrix} P^s & 1 \\ P^b & 1 \end{bmatrix},$$

and is of full rank, implying that the KKT conditions are necessary for a global max (note, we have a concave objective function subject to a set of convex constraint functions). The Lagrangian for our problem is:

$$Z = x_1^\beta + x_2^\beta - \lambda_1(P^s x_1 + x_2 - P^s m_1 - m_2) - \lambda_2(P^b x_1 + x_2 - P^b m_1 - m_2),$$

with the following KKT conditions:

$$Z_1 = \beta x_1^{\beta-1} - \lambda_1 P^s - \lambda_2 P^b \geq 0,\ x_1 \geq 0,\ x_1 Z_1 = 0, \tag{14}$$
$$Z_2 = \beta x_2^{\beta-1} - \lambda_1 - \lambda_2 \geq 0,\ x_2 \geq 0,\ x_2 Z_2 = 0, \tag{15}$$
$$Z_{\lambda_1} = P^s x_1 + x_2 - P^s m_1 - m_2 \leq 0,\ \lambda_1 \geq 0,\ \lambda_1 Z_{\lambda_1} = 0, \tag{16}$$
$$Z_{\lambda_2} = P^b x_1 + x_2 - P^b m_1 - m_2 \leq 0,\ \lambda_2 \geq 0,\ \lambda_2 Z_{\lambda_2} = 0. \tag{17}$$

Since we know that $x_1, x_2 > 0$, we know that (14) and (15) bind with equality. Thus we have three regimes: a) When both λ_1 and $\lambda_2 > 0$, b) when only $\lambda_1 > 0$ and $\lambda_2 = 0$, and c) when only $\lambda_2 > 0$ and $\lambda_1 = 0$.

When both $Z_{\lambda_1} = Z_{\lambda_2} = 0$, our KKT conditions become:

$$\beta x_1^{\beta-1} = \lambda_1 P^s + \lambda_2 P^b \tag{18}$$

$$\beta x_2^{\beta-1} = \lambda_1 + \lambda_2, \tag{19}$$

$$P^s x_1 + x_2 = P^s m_1 + m_2, \tag{20}$$

$$P^b x_1 + x_2 = P^b m_1 + m_2. \tag{21}$$

From (20) and (21):

$$\begin{aligned} P^s x_1 - P^s m_1 &= P^b x_1 - P^b m_1 \\ \implies x_1(P^s - P^b) &= P^s m_1 - P^b m_1 \\ x_1^* &= m_1, \end{aligned}$$

and substituting this value for x_1^* into either (20) or (21) yields

$$\begin{aligned} P^s m_1 + x_2 &= P^s m_1 + m_2 \\ \implies x_2^* &= m_2. \end{aligned}$$

We then substitute these values into (18) and (19) to solve for two equations in two unknowns:

$$\begin{aligned} \beta m_1^{\beta-1} &= \lambda_1 P^s + \lambda_2 P^b, \\ \beta m_2^{\beta-1} &= \lambda_1 + \lambda_2, \end{aligned}$$

so we have

$$\begin{aligned} \beta m_1^{\beta-1} &= (\beta m_2^{\beta-1} - \lambda_2) P^s + \lambda_2 P^b \\ &= \beta m_2^{\beta-1} P^s + \lambda_2 (P^b - P^s) \\ \implies \lambda_2^* &= \frac{\beta(m_1^{\beta-1} - m_2^{\beta-1} P^s)}{P^b - P^s}, \end{aligned}$$

and, by implication,

$$\begin{aligned} \lambda_1^* &= \beta m_2^{\beta-1} - \frac{\beta(m_1^{\beta-1} - m_2^{\beta-1} P^s)}{P^b - P^s} \\ &= \frac{\beta m_2^{\beta-1}(P^b - P^s) - \beta(m_1^{\beta-1} - m_2^{\beta-1} P^s)}{P^b - P^s} \\ &= \frac{\beta m_2^{\beta-1} P^b - \beta m_2^{\beta-1} P^s - \beta m_1^{\beta-1} + \beta m_2^{\beta-1} P^s}{P^b - P^s} \\ \lambda_1^* &= \frac{\beta(m_2^{\beta-1} P^b - m_2^{\beta-1} P^s - m_1^{\beta-1} + m_2^{\beta-1} P^s)}{P^b - P^s}. \end{aligned}$$

This point can be diagrammatically represented by the intersection point of the two budget constraints (m_1, m_2) – and this makes intuitive sense. If a consumer is given an initial endowment of m_1 and m_2, taking prices and preferences as given, then they can no better by exchanging x_1 for x_2. This occurs iff

$$P_s \leq \left[\frac{m_2}{m_1}\right]^{1-\beta} \leq P_b.$$

For the second regime ($\lambda_1 \neq 0 \implies Z_{\lambda_1} = 0$) we have the following from the KKT conditions:

$$\begin{aligned} \beta x_1^{\beta-1} &= \lambda_2 P^b, \\ \beta x_2^{\beta-1} &= \lambda_2, \\ P^s x_1 + x_2 &= P^s m_1 + m_2. \end{aligned}$$

So solving for x_1^* and x_2^* gives:

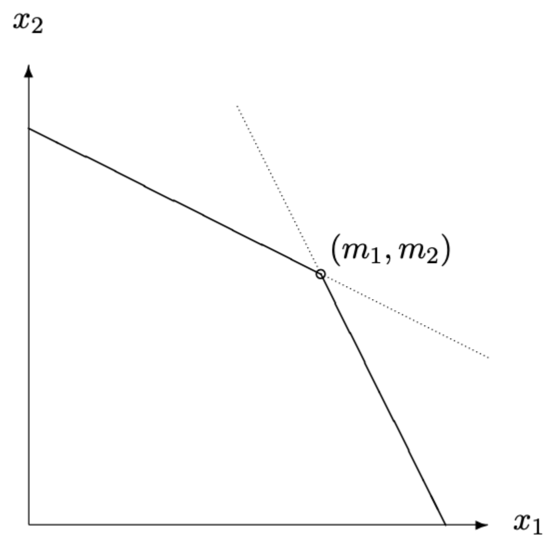
$$\begin{aligned} x_1^{\beta-1} &= x_2^{\beta-1} P^b \\ x_1 &= x_2 (P^b)^{\frac{1}{\beta-1}}, \\ P^s x_2 (P^b)^{\frac{1}{\beta-1}} + x_2 &= P^s m_1 + m_2 \\ x_2 (P^s (P^b)^{\frac{1}{\beta-1}} + 1) &= P^s m_1 + m_2 \\ \therefore x_2^* &= \frac{P^s m_1 + m_2}{1 + P^s (P^b)^{\frac{-1}{1-\beta}}}, \end{aligned}$$

and

$$\begin{aligned} x_1^{\beta-1} &= \beta x_2^{\beta-1} P^b \\ x_2^{\beta-1} &= \frac{x_1^{\beta-1}}{\beta P^b} \\ x_2 &= \frac{x_1}{(\beta P^b)^{\frac{1}{\beta-1}}}, \\ P^s x_1 + \frac{x_1}{(\beta P^b)^{\frac{1}{\beta-1}}} &= P^s m_1 + m_2 \\ x_1 \left(P^s + (\beta P^b)^{\frac{1}{1-\beta}}\right) &= P^s m_1 + m_2 \\ \implies x_1^* &= \frac{P^s m_1 + m_2}{P^s + (\beta P^b)^{\frac{1}{1-\beta}}}. \end{aligned}$$

The equality and strict inequality from the KKT conditions together with the fact that $P^b > P^s$ imply that $x_1 > m_1$ and $x_2 < m_2$, and so the solution is below and to the right of (m_1, m_2). Further, $m_2 > x_2 \implies P^b < \left(\frac{m_2}{m_1}\right)^{1-\beta}$.

Figure 1: Figure for Question 6



Now, for the third regime ($\lambda_2 \neq 0 \implies Z_{\lambda_2} = 0$), by symmetry we have:

$$x_1^* = \frac{P^b m_1 + m_2}{P^b + (\beta P^b)^{\frac{1}{1-\beta}}},$$
$$x_2^* = \frac{P^b m_1 + m_2}{1 + P^b (P^s)^{\frac{-1}{1-\beta}}},$$

when $P^s > \left(\frac{m_2}{m_1}\right)^{1-\beta}$.