Consumer and Producer Theory

1 Utility maximisation

1.1

A consumer has utility function

$$u(x_1, x_2) = (1 + x_1)\sqrt{x_2}$$

defined for quantities of two products, 1 and 2. Product 1 can be consumed in continuous quantities (i.e. x_1 can be any non-negative real number), while product 2 is a discrete good (i.e., the possible levels of consumption of product 2 are $x_2 = 0$ or $x_2 = 1$). The consumer's consumption must satisfy her budget constraint $P_1 x_1 + P_2 x_2 \leq w$, where $P_i > 0$ is the unit price of product $i = 1, 2$ and $w > 0$ is her wealth. When does the consumer choose to buy product 2? Is product 1 a necessity or a luxury?

The consumer's problem can be written as:

$$\max \quad u(x_1, x_2) = (1 + x_2)\sqrt{x_2},$$

subject to:

$$P_1 x_1 + P_2 x_2 \leq w, \; x_1 \geq 0, \; x_2 \in \{0, 1\}.$$

Assuming that the consumer's budget constraint binds, then there are two regimes we need to assess: 1) When $x_2 = 0$, and 2) when $x_2 = 1$. When the consumer does NOT buy any x_2 then the amount of x_1 she purchases will be

$$x_1^* = \frac{w}{P_1},$$

and her utility will be

$$u(x_1, x_2) = (1 + x_2^*)\sqrt{x_1^*} = \sqrt{\frac{w}{P_1}}. \tag{1}$$

If she chooses to buy one unit of x_2 then we have

$$x_2^* = 1,$$
$$x_1^* = \frac{w - P_2}{P_1},$$

and so her utility will be

$$u(x_1, x_2) = (1 + 1)\sqrt{\frac{w - P_2}{P_1}}. \tag{2}$$

She will be indifferent between her different consumption bundles when we set the two utility levels equal to one another

$$\begin{aligned} \sqrt{\frac{w}{P_1}} &= 2\sqrt{\frac{w-P_2}{P_1}} \\ w^{\frac{1}{2}} &= 2(w-P_2)^{\frac{1}{2}} \\ w &= 4w - 4P_2 \\ \implies P_2 &= \frac{3w}{4}, \end{aligned}$$

implying that when $P_2 = \frac{3w}{4}$ she will be indifferent to either consuming or not consuming x_2. We can also clearly see that when she will definitely consume x_2 when $P_2 < \frac{3w}{4}$ since 2 is greater than 1.

Is x_1 a necessity or luxury? This is regime dependent. When $x_2^* = 0$, then x_1 is a normal good with unit income elasticity. When $x_2^* = 1$, then x_1 is a luxury good with income elasticity of greater than unity $(w/(w-P_2))$. Alternatively we can look at the demand function for x_1:

$$x_1(P_1, P_2, w) = \begin{cases} \frac{w}{P_1} & \text{if } P_2 > \frac{3w}{4}, \\ \frac{w-P_2}{P_1} & \text{if } P_2 < \frac{3w}{4}. \end{cases}$$

The budget share for x_1 of the consumer is unity in w for small values of w, then jumps discontinuously down when w reaches $\frac{4}{3}P_2$, and then increases again with w. Thus, this product is neither (globally) a necessity or a luxury.

1.2

A consumer has utility function

$$u(x_1, x_2) = (x_1 + 1)x_2$$

over goods $x_1, x_2 \geq 0$, and faces budget constrain $P_1x_1 + P_2x_2 \leq w$.

1.2.1

Show that the utility function is strictly quasi-concave.

Recall that

- A necessary condition for a function to be quasi-concave is that the even-numbered principle minors of the bordered Hessian be non-negative and the odd-numbered principle minors be non-positive; and
- A sufficient condition for a function to be quasi-concave is that the even-numbered principle minors of the bordered Hessian be strictly positive and the odd-numbered principle minors be strictly negative.

The bordered Hessian for u is:

$$(\mathbf{H}) = \begin{bmatrix} 0 & x_2 & 1+x_1 \\ x_2 & 0 & 1 \\ 1+x_1 & 1 & 0 \end{bmatrix}.$$

The principal minors are $|(\mathbf{H})_1| = -x_2^2 < 0$ and $|(\mathbf{H})_2| = -x_2(-(1+x_1)) > 0$ which verifies strict quasi-concavity.

1.2.2

If $\mathbf{P} = (P_1, P_2)^\top = (4, 1)$ and $w = 2$, what is the consumer's demand for x_1 and x_2?

Set up the consumer's problem as:

$$\max \quad u(x_1, x_2) = (1+x_1)x_2,$$

subject to:

$$P_1 x_1 + P_2 x_2 \leq w, \; x_1 \geq 0, \; x_2 \geq 0.$$

We setup the following Lagrangian:

$$Z = (1+x_1)x_2 + \lambda(w - P_1 x_1 - P_2 x_2),$$

and the KKT conditions:

$$\begin{aligned} Z_1 &= x_2 - \lambda P_1 \leq 0, \; x_1 \geq 0, \; x_1 Z_1 = 0, \\ Z_2 &= 1 + x_2 - \lambda P_2 \leq 0, \; x_2 \geq 0, \; x_2 Z_2 = 0, \\ Z_\lambda &= w - P_1 x_1 - P_2 x_2 \geq 0, \; \lambda \geq 0, \; \lambda Z_\lambda = 0. \end{aligned}$$

To verify slackness, we know that our parameters are all greater than 0. From Z_λ, this implies that x_1 and x_2 cannot both be zero. From Z_2, if $x_1 = 0$ then it implies that $\lambda = 1$ which implies that $x_2 > 0$ from Z_1. This means that Z_2 and Z_λ hold with strict equality. Thus, we have two regimes: $x_1 = 0, Z_1 \leq 0$ and $x_1 > 0, Z_1 = 0$.

If we have $x_1 > 0 \implies Z_1 = 0$, then from our KKT conditions we have:

$$\begin{aligned} x_2 &= 4\lambda \\ 1 + x_1 &= \lambda \\ 2 &= x_1 + 4x_2. \end{aligned}$$

But this regime cannot hold, as solving for x_1^* gives a value less than zero. Thus, we turn our attention to $x_1 = 0 \implies Z_1 \leq 0$. For our given parameters we attain the following optimal point:

$$x_1^* = 0, \; x_2^* = 2, \; \lambda = 1.$$

1.2.3

Derive the Walrasian (or Marshallian) demand functions $x_1(\mathbf{P}, w)$ and $x_2(\mathbf{P}, w)$ for general $(\mathbf{P}, w)$.

For the general case, we have the same KKT conditions as before. But assuming that prices and wages have no constraints, we assume that $x_1, x_2 > 0$ and that all constraints bind with equality. This gives the following optimal point:

$$\begin{aligned} x_1^* &= \frac{w - P_1}{2P_1}, \\ x_2^* &= \frac{P_1 + w}{2P_2}, \\ \lambda^* &= \frac{P_1^2 + P_1 w}{2P_2}, \end{aligned}$$

where $\mathbf{x}^*$ are our Marshallian demand functions.

1.2.4

What is the indirect utility function? Verify Roy's Identity.

The indirect utility function is the utility function evaluated with our Marshallian demand functions:

$$V(P, w) = u(\mathbf{x}^*(P, w)),$$

so

$$\begin{aligned} V(P, w) &= x_1^* x_2^* + x_2^* \\ &= \frac{w - P_1}{2P_1} \frac{P_1 + w}{2P_2} + \frac{P_1 + w}{2P_2}. \end{aligned}$$

Roy's Identity allows us the extract the Marshallian demand functions from the indirect utility function, and is defined as follows:

$$x_i^*(P, w) = -\frac{\partial V}{\partial P_i} / \frac{\partial V}{\partial w}.$$

For x_1 the ratio of partial derivatives using Roy's Identity gives us

$$x_1^* = -\frac{2(w + P_1)4P_1P_2 - (w + P_1)^2 4P_2}{16P_1^2 P_2^2} \times \frac{4P_1P_2}{2(w + P_1)},$$

and after rearranging we get:

$$\begin{aligned} x_1^* &= -\left[1 - \frac{w + P_1}{2P_1}\right] \\ &= \frac{w + P_1}{2P_1} - 1 = \frac{w - P_1}{2P_1}, \end{aligned}$$

which is the same result as above.

2 Expenditure minimisation

2.1

Suppose there are L products and a customer's expenditure function takes the Gorman Polar Form:

$$e(\mathbf{P}, u) = a(\mathbf{P}) + ub(P).$$

Show that the Engel curves are straight lines.

We need to obtain Walrasian/Marshallian demand functions to talk about Engel curves. Since, in general,

$$e(\mathbf{P}, V(\mathbf{P}, w)) = w,$$

it follows that

$$a(\mathbf{P}) + V(\mathbf{P}, w)b(\mathbf{P}) = w,$$

and hence

$$V(\mathbf{P}, w) = \frac{w - a(\mathbf{P})}{b(\mathbf{P})}.$$

Now, we can apply Roy's Identity

$$\begin{aligned}
x_i(\mathbf{P}, w) &= -\frac{\partial V}{\partial P_i} / \frac{\partial V}{\partial w}, \\
\frac{\partial V}{\partial P_i} &= \frac{\partial}{\partial P_i}\left[wb(\mathbf{P})^{-1} - \frac{a(\mathbf{P})}{b(\mathbf{P})} \right] \\
&= -wb(\mathbf{P})^{-2}b'(\mathbf{P}) - \frac{a'(\mathbf{P})b(\mathbf{P}) - a(\mathbf{P})b'(\mathbf{P})}{b(\mathbf{P})^2} \\
&= -\frac{wb'(\mathbf{P})}{b(\mathbf{P})^2} - \frac{a'(\mathbf{P})b(\mathbf{P}) - a(\mathbf{P})b'(\mathbf{P})}{b(\mathbf{P})^2} \\
&= \frac{a(\mathbf{P})b'(\mathbf{P}) - a'(\mathbf{P})b(\mathbf{P}) - wb'(\mathbf{P})}{b(\mathbf{P})^2},
\end{aligned}$$

$$\frac{\partial V}{\partial w} = \frac{1}{b(\mathbf{P})},$$

$$\begin{aligned}
\therefore -\frac{\partial V}{\partial P_i} / \frac{\partial V}{\partial w} &= -b(\mathbf{P})\left[\frac{a(\mathbf{P})b'(\mathbf{P}) - a'(\mathbf{P})b(\mathbf{P}) - wb'(\mathbf{P})}{b(\mathbf{P})^2} \right] \\
&= \frac{-b(\mathbf{P})a(\mathbf{P})b'(\mathbf{P}) + a'(\mathbf{P})b(\mathbf{P})^2 + b(\mathbf{P})wb'(\mathbf{P})}{b(\mathbf{P})^2} \\
&= a'(\mathbf{P}) + \frac{b(\mathbf{P})wb'(\mathbf{P}) - b(\mathbf{P})a(\mathbf{P})b'(\mathbf{P})}{b(\mathbf{P})^2} \\
&= a'(\mathbf{P}) + \frac{wb'(\mathbf{P}) - a(\mathbf{P})b'(\mathbf{P})}{b(\mathbf{P})} \\
\therefore x_i(\mathbf{P}, w) &= \frac{\partial a(\mathbf{P})}{\partial P_i} + \frac{\partial b(\mathbf{P})}{P_i}\frac{w - a(\mathbf{P})}{b(\mathbf{P})}.
\end{aligned}$$

We can see that the income effect does not depend on w. Strictly, Engel curves are depicted in product space. With two products, say, we need to show that x_2 is a linear function of x_1 as w varies, but this is obvious given that both products' demands are linear in w.

2.1.1

A*ssume that a consumer's consumption set is* $X \subset \mathbb{R}^L$ *such that* $x_l \geq \gamma_l$ *for each* $l = 1, ..., L$, *where* $\gamma = (\gamma_1, ..., \gamma_L)$ *is a vector of parameters with* $\gamma_l \geq 0$. *Suppose that the consumer's utility function defined on* X *takes the Stone-Geary form*:

$$u(x) = \prod_{l=1}^{L} (x_l - \gamma_l)^{\alpha_l},$$

where each $\alpha_l > 0$ *and* $\sum_{l=1}^{L} \alpha_l = 1$. *Show that the consumer's expenditure function takes the form in the previous question. Interpret* $a(\mathbf{P})$ *as subsistence expenditure, and* $b(\mathbf{P})$ *as a price index which represents the marginal cost of living.*

Let $z_l = x_l - \gamma_l$, which allows us to write our problem as:

$$\max \quad \prod_{l=1}^{L} z_l^{\alpha_l},$$

which is a simple Cobb-Douglas maximisation problem for a consumer with $w - \boldsymbol{\gamma}\mathbf{P}$ income. An interior solution to this problem is:

$$z_l = \alpha_l \left(\frac{w - \boldsymbol{\gamma}\mathbf{P}}{P_l} \right),$$

and since $z_l = x_l - \gamma_l$:

$$x_l = z_l + \gamma_l = \alpha_l \left(\frac{w - \boldsymbol{\gamma}\mathbf{P}}{P_l} \right) + \gamma_l,$$

which we can rewrite as:

$$x_l = \alpha_l \left(\frac{w - \sum_{l=1}^{L} P_l \gamma_l}{P_l} \right) + \gamma_l,$$

and taking z_l and plugging it back into our utility function yields the indirect utility function:

$$\begin{aligned}
V(\mathbf{P}, w) &= \prod_{l=1}^{L} \left[\alpha_l \left(\frac{w - \sum_{l=1}^{L} P_l \gamma_l}{P_l} \right) \right]^{\alpha_l} \\
\prod_{l=1}^{L} P_l^{\alpha_l} V(\mathbf{P}, w) &= \prod_{l=1}^{L} \alpha_l^{\alpha_l} \left(w - \sum_{l=1}^{L} P_l \gamma_l \right) \\
\frac{\prod_{l=1}^{L} P_l^{\alpha_l}}{\prod_{l=1}^{L} \alpha_l^{\alpha_l}} V(\mathbf{P}, w) &= w - \sum_{l=1}^{L} P_l \gamma_l,
\end{aligned}$$

and since we always have the identity that

$$V(\mathbf{P}, e(\mathbf{P}, u)) = u,$$

$$\begin{aligned}\implies e(\mathbf{P}, u) &= \frac{\prod_{l=1}^{L} P_l^{\alpha_l}}{\prod_{l=1}^{L} \alpha_l^{\alpha_l}} u + \boldsymbol{\gamma}\mathbf{P} \\ &= \left[\prod_{l=1}^{L} \left(\frac{P_l}{\alpha_l}\right)^{\alpha_l}\right] u + \sum_{l=1}^{L} P_l \gamma_l.\end{aligned}$$

This takes the form in the first part of the question. The term $a(\mathbf{P}) = \sum_{l=1}^{L} P_l \gamma_l$ represents the minimum wealth needed to purchase the bundle of 'subsistence quantities' γ_l (utility is not defined unless the consumer has w at least equal to this level). The 'spare' wealth is $w - \sum_{l=1}^{L} P_l \gamma_l$, and this is used to generate utility. The cost of an extra unit of utility is

$$b(\mathbf{P}) = \prod_{l=1}^{L} \left(\frac{P_l}{\alpha_l}\right)^{\alpha_l},$$

which is clearly a price index: it equals a geometric weighted average of the individual prices, with more weight put on prices of more important products.

3 Constant elasticity of substitution

3.1

Consider the CES utility function

$$u(x_1, x_2) = x_1^\theta + x_2^\theta,$$

where $0 < \theta < 1$.

3.1.1

Show that u is a quasi-concave function in x_1 and x_2.

Much like the question 1.2, we need the bordered Hessian matrix of u to make inference on its quasi-concavity. The bordered Hessian for u is:

$$(\mathbf{H}) = \begin{bmatrix} 0 & \theta x_1^{\theta-1} & \theta x_2^{\theta-1} \\ \theta x_1^{\theta-1} & \theta(\theta-1)x_1^{\theta-2} & 0 \\ \theta x_2^{\theta-1} & 0 & \theta(\theta-1)x_2^{\theta-2} \end{bmatrix}.$$

The principal minors for $(\mathbf{H})$ are

$$\begin{aligned} |(\mathbf{H}_1)| &= -\theta^2 x_1^{2\theta-2} < 0, \\ |(\mathbf{H}_2)| &= -\theta x_1^{\theta-1}(\theta x_1^{\theta-1} - \theta(\theta-1)x_2^{\theta-2}) > 0, \end{aligned}$$

thus proving u is quasi-concave.

3.1.2

Derive the Walrasian (or Marshallian) demand functions and indirect utility function. Verify that these functions are homogeneous of degree zero in $(\mathbf{P}, w)$.

The consumer's problem is

$$\max \quad x_1^\theta + x_2^\theta$$

subject to

$$P_1 x_1 + P_2 x_2 = Y.$$

The Lagrangian is

$$Z = x_1^\theta + x_2^\theta + \lambda(Y - P_1 x_1 - P_2 x_2),$$

and the first order conditions are:

$$\begin{aligned} Z_1 &= \theta x_1^{\theta-1} - \lambda P_1 = 0 \\ Z_2 &= \theta x_2^{\theta-1} - \lambda P_2 = 0 \\ Z_\lambda &= Y - P_1 x_1 - P_2 x_2 = 0. \end{aligned}$$

Rearranging and solving for the optimal values gives us

$$x_1^* = \left(\frac{P_1}{\theta}\right)^{\frac{1}{\theta-1}} \frac{\theta Y}{P_1^{\frac{\theta}{\theta-1}} + P_2^{\frac{\theta}{\theta-1}}},$$
$$x_2^* = \left(\frac{P_2}{\theta}\right)^{\frac{1}{\theta-1}} \frac{\theta Y}{P_1^{\frac{\theta}{\theta-1}} + P_2^{\frac{\theta}{\theta-1}}},$$
$$\lambda^* = \left(\frac{\theta Y}{P_1^{\frac{\theta}{\theta-1}} + P_2^{\frac{\theta}{\theta-1}}}\right)^{\theta-1}.$$

The indirect utility function is thus

$$V(P,Y) = \left(\frac{P_1}{\theta}\right)^{\frac{\theta}{\theta-1}} \left(\frac{\theta Y}{P_1^{\frac{\theta}{\theta-1}} + P_2^{\frac{\theta}{\theta-1}}}\right)^{\theta} + \left(\frac{P_1}{\theta}\right)^{\frac{\theta}{\theta-1}} \left(\frac{\theta Y}{P_1^{\frac{\theta}{\theta-1}} + P_2^{\frac{\theta}{\theta-1}}}\right)^{\theta}.$$

Use x_1^* to show HOD0 where η^k denotes degree k of homogeneity:

$$\frac{(\eta P_1)^{\frac{1}{\theta-1}} \theta(\eta Y)}{(\eta\theta)^{\frac{1}{\theta-1}} (\eta P_1)^{\frac{1}{\theta-1}} + (\eta\theta)^{\frac{1}{\theta-1}} (\eta P_2)^{\frac{1}{\theta-1}}}$$
$$\implies \frac{\eta^{\frac{1+\theta-1}{\theta-1}}}{\eta^{\frac{\theta}{\theta-1}}} \frac{P_1{}^{\frac{1}{\theta-1}} \theta Y}{\theta^{\frac{1}{\theta-1}} P_1{}^{\frac{1}{\theta-1}} + \theta^{\frac{1}{\theta-1}} P_2{}^{\frac{1}{\theta-1}}}$$

where $\eta^0 \implies$ HOD0. By symmetry the same holds for x_2^*, and thus the indirect utility function is also HOD0.

3.1.3

Show that the elasticity of substitution between goods 1 and 2 is constant and equal to $\frac{1}{1-\theta}$. *Note that the elasticity of substitution between goods 1 and 2 is defined to be*

$$\epsilon(\mathbf{P}, w) = -\frac{\partial \left[\frac{x_1(\mathbf{P},w)}{x_2(\mathbf{P},w)}\right]}{\partial \left[\frac{P_1}{P_2}\right]} \frac{\frac{P_1}{P_2}}{\frac{x_1(\mathbf{P},w)}{x_2(\mathbf{P},w)}}.$$

We wish to prove that $\epsilon(\mathbf{P}, w) = \frac{1}{1-\theta}$. Define

$$A = \theta P_1^{\theta} + \theta P_2^{\theta},$$

and

$$x_1^* = \frac{\theta Y P_1^{\frac{1}{\theta-1}}}{A^{\frac{1}{\theta-1}}},$$
$$x_2^* = \frac{\theta Y P_2^{\frac{1}{\theta-1}}}{A^{\frac{1}{\theta-1}}}.$$

Start with

$$\frac{x_1^*}{x_2^*} = \frac{\theta Y P_1^{\frac{1}{\theta-1}}}{A^{\frac{1}{\theta-1}}} \frac{A^{\frac{1}{\theta-1}}}{\theta Y P_2^{\frac{1}{\theta-1}}} = \left(\frac{P_1}{P_2}\right)^{\frac{1}{\theta-1}},$$

and for the second term of ϵ:

$$\frac{P_1}{P_2} / \left(\frac{P_1}{P_2}\right)^{\frac{1}{\theta-1}} = \left(\frac{P_1}{P_2}\right)^{1-\frac{1}{\theta-1}} = \left(\frac{P_1}{P_2}\right)^{\frac{\theta-2}{\theta-1}}.$$

Then, differentiating the expression for the ratio of the Marshallian demand curves with respect to P_1/P_2 yields:

$$\frac{1}{\theta-1}\left(\frac{P_1}{P_2}\right)^{\frac{2-\theta}{\theta-1}},$$

which gives our result for the elasticity, ϵ:

$$\epsilon = -\left[\frac{1}{\theta-1}\left(\frac{P_1}{P_2}\right)^{\frac{2-\theta}{\theta-1}}\left(\frac{P_1}{P_2}\right)^{\frac{\theta-2}{\theta-1}}\right] = \frac{1}{1-\theta}.$$

4 Cost minimisation

This question is about a profit maximising firm. However, it could also apply to a utility maximising consumer whose level of utility is given (i.e. expenditure minimisation). Let $c(\mathbf{W}, q)$ be a firm's minimum cost of producing q units of a single output when input prices are $\mathbf{W} = (w_1, ..., w_L)^\top$, and let $\mathbf{z}(\mathbf{W}, q) = (z_1(\mathbf{W}, q), ..., z_L(\mathbf{W}, q))$ be the choice of inputs which minimise its cost of producing this output (so $\mathbf{z}(\mathbf{W}, q)$ is the conditional factor demand function). Define $s_{ij} = \frac{\partial z_i(\mathbf{W},q)}{\partial w_j}$ for $i, j = 1, ..., L$. The $L \times L$ matrix whose (i, j)'th element is s_{ij} is denoted S.

4.1

Why is S negative semi-definite and symmetric?

$c(\mathbf{W}, q)$ is concave in $\mathbf{W}$ and by Shephard's Lemma

$$\frac{\partial c(\mathbf{W}, q)}{\partial W_i} = z_i(\mathbf{W}, q).$$

Since the matrix of second order derivatives is symmetric, and here is equal to the matrix of derivatives of $\mathbf{z}$, it follows that the matrix of derivatives of $\mathbf{z}$ is both symmetric and negative semidefinite (NSD), implying that all the diagonal elements are negative. Recall that for a matrix A:

- If $|A_i| \geq 0, 1 \leq i \leq n$, then A is positive semi-definite;
- If $|A_i| \leq 0$ for i is odd and $|A_i| \geq 0$ for i is even, then A is negative semi-definite.

Since S is a matrix of first order partial derivatives of the firm's conditional factor demand functions (by Shephard's lemma) we know that the diagonal elements of S essentially capture the substitution effect of input i and its factor price – and we know by Slutsky's equation that the substitution effect is negative. Finally, a NSD matrix implies that the underlying cost function is convex, and that we have a global minimum. Combining these facts completes our requirement for NSD.

As for symmetry, the off-diagonal elements essentially give us our cross-price effects of input i and j for $i \neq j$. Using Shephard's lemma and Young's theorem, we know that these cross-price effects must be symmetrical.

Let $\bar{x}_i$ be the conditional factor demand for input i, c be the firm's cost function, and P_i be the price of input i. By Shephard's lemma we have:

$$\frac{\partial \bar{x}_i}{\partial P_j} = \frac{\partial}{\partial P_j} \frac{\partial c}{\partial P_i}$$

and by Young's theorem:

$$\begin{aligned} &= \frac{\partial^2 c}{\partial P_j \partial P_i} = \frac{\partial^2 c}{\partial P_i \partial P_j} \\ &= \frac{\partial \bar{x}_j}{\partial P_i}. \end{aligned}$$

4.2

Show that for each $i = 1, ..., L$ we have

$$\sum_{j=1}^{L} w_j \frac{\partial z_i(\mathbf{W}, q)}{\partial W_j} = 0$$

Deduce that the determinant of S is zero.

We can use duality theory to justify this, and look at a simple two input case. If

$$w_1 \frac{\partial z_1}{\partial w_1} + w_2 \frac{\partial z_1}{\partial w_2} = 0$$

then

$$w_1 \frac{\partial z_1}{\partial w_1} = -w_2 \frac{\partial z_1}{\partial w_2}.$$

We can set the ratio of input prices equal to the ratio of partial derivatives of conditional demands wrt inputs:

$$-\frac{w_1}{w_2} = \frac{\partial z_1}{\partial w_1} / \frac{\partial z_1}{\partial w_2},$$

which is analogous to the standard consumer utility maximisation/firm profit maximisation problem, thus verifying the general case for the input shares summing to zero.

4.3

The matrix below shows S for a profit maximising firm with three inputs at the input prices $\mathbf{W} = (1, 2, 6)^\top$:

$$S = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} = \begin{bmatrix} -10 & ? & ? \\ ? & -4 & ? \\ 3 & ? & ? \end{bmatrix}$$

Using Young's Theorem we can start to fill in the matrix, giving us:

$$\begin{bmatrix} -10 & -4 & 3 \\ -4 & -4 & 2 \\ 3 & 2 & -7/6 \end{bmatrix}.$$

To start, we know by Young's Theorem that $s_{31} = s_{13}$. Next we can get s_{12} using the properties from the previous question

$$\begin{aligned} 1 \times -10 + 2 \times s_{12} + 6 \times 3 = 0 \\ \implies s_{12} = -4, \\ \implies s_{21} = -4, \end{aligned}$$

and repeat this process for the other elements. Then check if S is NSD.

$$\begin{aligned} (-1)|S_1| &= 10 \\ (-1)^2|S_2| &= 24 \\ |S_3| &= 0, \end{aligned}$$

therefore S is NSD. We know that if Hicksian demand functions have a matrix of derivatives which is symmetric and NSD, we can find a utility function/production set which generates these functions. Thus, S possesses all the required properties.

5 Separability and quasi-linear utility

Consider the separable, quasi-linear utility function

$$u = v(x_1, x_2) + \gamma x_3$$

where

$$v(x_1, x_2) = \alpha \log x_1 + \beta \log x_2$$

5.1

Derive the demand for the three goods.

Our problem is the following:

$$\max \quad u(\mathbf{x}) = \alpha \log x_1 + \beta \log x_2 + \gamma x_3$$

subject to

$$P_1 x_1 + P_2 x_2 + P_3 x_3 = Y.$$

Our Lagrangian is:

$$Z = \alpha \log x_1 + \beta \log x_2 + \gamma x_3 + \lambda(Y - P_1 x_1 - P_2 x_2 - P_3 x_3),$$

with the following first order conditions:

$$\begin{aligned} Z_1 &= \frac{\alpha}{x_1} - \lambda P_1 = 0, \\ Z_2 &= \frac{\beta}{x_2} - \lambda P_2 = 0, \\ Z_3 &= \gamma - \lambda P_3 = 0, \\ Z_\lambda &= Y - P_1 x_1 - P_2 x_2 - P_3 x_3 = 0. \end{aligned}$$

With some rearranging, we get the following:

$$x_1^* = \frac{\alpha P_3}{\gamma P_1}, x_2^* = \frac{\beta P_3}{\gamma P_2}, x_3^* = \frac{Y}{P_3} - \frac{\alpha - \beta}{\gamma}, \lambda^* = \frac{\gamma}{P_3}.$$

5.2

Show that the income effects for goods 1 and 2 are zero.

The income effect is given by the Slutsky equation.

$$\frac{\partial x_i^*}{\partial P_i} = \frac{\partial \bar{x}_i}{\partial P_i} - x_i^* \frac{\partial x_i^*}{\partial Y}.$$

So, for x_1 and x_2:

$$\begin{aligned} x_1^* \frac{\partial x_1^*}{\partial Y} &= \frac{\alpha P_3}{\gamma P_1} \times 0 = 0, \\ x_2^* \frac{\partial x_2^*}{\partial Y} &= \frac{\beta P_3}{\gamma P_2} \times 0 = 0. \end{aligned}$$

5.3

Derive an expression for the expenditure on the separable group as a function of prices of all of the goods.

Spending allocated to x_1 and x_2 is

$$\begin{aligned} &P_1 \frac{\alpha P_3}{\gamma P_1} + P_2 \frac{\beta P_3}{\gamma P_2} \\ &= P_3 \left(\frac{\alpha + \beta}{\gamma} \right). \end{aligned}$$

5.4

Solve the sub-utility maximisation problem for goods 1 and 2 subject to the budget constraint derived in the previous question and show that the demands for goods 1 and good 2 are identical to those derived derived in the first part of this question.

Our Lagrangian is now

$$Z = \alpha \log x_1 + \beta \log x_2 + \lambda (P_3 \left(\frac{\alpha + \beta}{\gamma} \right) - P_1 x_1 - P_2 x_2),$$

with the following first order conditions:

$$\begin{aligned} Z_1 &: \frac{\alpha}{x_1 P_1} = \lambda, \\ Z_2 &: \frac{\beta}{x_2 P_2} = \lambda, \\ Z_\lambda &: P_3 \left(\frac{\alpha + \beta}{\gamma} \right) = +P_1 x_1 + P_2 x_2. \end{aligned}$$

From the above first order conditions, we get

$$x_1 = \frac{\alpha P_2 x_2}{\beta P_1},$$

which we substitute into our Z_λ condition to get an expression for x_2^* :

$$\begin{aligned} x_2^* &= \frac{P_3(\alpha + \beta)}{\gamma P_2 \left(\frac{\alpha}{\beta} + 1 \right)} \\ &= \frac{P_3(\alpha + \beta)\beta}{\gamma P_2 (\alpha + \beta)} \\ &= \frac{\beta P_3}{\gamma P_2} = x_2^*. \end{aligned}$$

By symmetry, we also get the same result for x_1^*:

$$x_1^* = \frac{\alpha P_3}{\gamma P_1}.$$