Choice Under Uncertainty

1 Expected utility and insurance

An agent maximises her expected utility, and has a strictly increasing and concave utility function $u(\cdot)$ and initial wealth w. With probability α a bad event occurs which causes her to lose L. Thus without insurance her expected utility is

$$\alpha u(w - L) + (1 - \alpha)u(w).$$

1.1

Suppose a competitive market supplies insurance for this bad event at 'actuarially fair' rates. That is, if the agent asks for insurance which pays her C if the bad event occurs, she must pay an insurance premium equal to αC. The agent is free to choose her level of cover C. How much cover should the agent choose?

An individual has the following expected utility:

$$\mathbb{E}[u] = \alpha u(w - L) + (1 - \alpha)u(w), \tag{1}$$

and faces uncertainty regarding her wealth:

$$w' = \begin{cases} w - \alpha C - L + C & \text{w.p. } \alpha, \\ w - \alpha C & \text{w.p. } 1 - \alpha, \end{cases}$$

where w' is her wealth after the event occurs/does not occur. Thus, her expected wealth is the following

$$\begin{aligned} \mathbb{E}[w'] &= \alpha[w - \alpha C - L + C] + (1 - \alpha)[w - \alpha C] \\ &= \alpha w - \alpha^2 C - \alpha L + \alpha C + (1 - \alpha)w - (1 - \alpha)\alpha C \\ &= w - \alpha L. \end{aligned}$$

To find the optimal amount of insurance payout C, we attain the individual's marginal utility with respect to C and set it equal to 0. The intuition being that this FOC is the point at which the individual no longer attains extra utility from higher certainty insurance payouts (in the event of an accident) due to the cost of having to take out this insurance coverage, αC, and because $u(\cdot)$ is assumed to be strictly concave.

$$\begin{aligned} \frac{\partial \mathbb{E}[u]}{\partial C} = 0 &= (1 - \alpha)u'(w - \alpha C)(-\alpha) + \alpha u'(w - \alpha C - L + C)(1 - \alpha) \\ &= (\alpha^2 - \alpha)u'(w - \alpha C) + (\alpha - \alpha^2)u'(w - \alpha C - L + C), \end{aligned}$$

and rearranging to get

$$\begin{gathered} (\alpha^2 - \alpha)u'(w - \alpha C - L + C) = (\alpha^2 - \alpha)u'(w - \alpha C) \\ u'(w - \alpha C - L + C) = u'(w - \alpha C), \end{gathered}$$

and since $u(\cdot)$ is strictly concave, we can equalise the arguments:

$$\begin{aligned} w - \alpha C - L + C &= w - \alpha C \\ \therefore C &= L. \end{aligned}$$

Thus her expected wealth after taking out insurance coverage is $\mathbb{E}[w'] = w - \alpha C$. i.e. Her expected wealth is simply that of her initial wealth less the amount she chooses to spend on an optimal insurance policy.

1.2

A monopoly insurance company offers the agent full insurance (i.e., it will pay the agent L if the bad event occurs) in return for an insurance premium P. When $u(x) = -\exp(-rx)$ where $r > 0$, what is the largest value P can take which still makes the agent willing to buy the insurance? Find an approximate expression for P when L is small.

If the monopolist insurer offers full insurance, and assuming that the individual's utility function is exponential, her expected wealth and utility are as follows:

$$\begin{aligned} \mathbb{E}[w'] &= \alpha(w - P - L + L) + (1 - \alpha)(w - P) \\ &= \alpha w - \alpha P + w - P - \alpha w + \alpha P \\ &= w - P. \end{aligned}$$

$$\begin{aligned} \mathbb{E}[u(w')] &= -\alpha \exp(-r(w - P)) - (1 - \alpha) \exp(-r(w - P)) \\ &= -\exp(-r(w - P)). \end{aligned}$$

Expected utility without any insurance is thus:

$$\mathbb{E}[u(w)] = -\alpha \exp(-r(w - L)) - (1 - \alpha) \exp(-rw).$$

Thus, the risk premium, P, is the amount the individual is willing to pay to eliminate all risk:

$$\begin{aligned} \mathbb{E}[u(w')] &= \mathbb{E}[u(w)] \\ \exp(-r(w - P)) &= \alpha \exp(-r(w - L)) + (1 - \alpha) \exp(-rw) \\ \exp(-rw + rP) &= \alpha \exp(-rw + rL) + (1 - \alpha) \exp(-rw) \\ \exp(-rw) \exp(rP) &= \alpha \exp(-rw) \exp(rL) + (1 - \alpha) \exp(-rw) \\ \exp(rP) &= \alpha \exp(rL) + (1 - \alpha) \qquad (2) \end{aligned}$$

let the term on the RHS of the above equation be some constant, A, then take logs:

$$\begin{aligned} rP &= \ln A \\ \therefore P &= \frac{\ln A}{r}. \end{aligned}$$

We can see that as the loss, L, increases and discount factor increases, the risk premium increases.

Focusing on the second part of the question, we know that when utility is exponential (like we have here), absolute risk aversion is constant (we have constant absolute risk aversion (CARA) preferences). For example, with a risky outcome, say z, and initial wealth x_0, the risk premium P satisfies

$$\begin{aligned} u(x_0 - P) &= \mathbb{E}[u(x_0 + z)] \\ \Leftrightarrow \exp\left(-r(x_0 - P)\right) &= \mathbb{E}\left[\exp\left(-r(x_0 + z)\right)\right] \\ \implies \exp(rP) &= \mathbb{E}[\exp(-rz)], \end{aligned}$$

so with CARA preferences there are no 'income effects' in the agent's demand for insurance, and so the agent's willingness to pay for insurance does not depend on initial wealth. The agent is willing to pay the premium P which equates the two expressions given in (2). By a first order Taylor approximation of $P(L)$ around $L = 0$, we have $P \approx \alpha L$. This arises from when $L \approx 0 \implies P \approx 0$, and so $\exp(rL) \approx 1 + rL$ and $\exp(rP) \approx 1 + rP$, giving our result. Thus, the agent is willing to pay an actuarially fair premium to insure against a small risk.

Definition: Recall that a first order Taylor approximation of $f(x)$ around a is given by

$$f(x) \approx f(a) + f'(a)(x - a),$$

for our example we have an approximation of $\exp(rL)$ around $L = 0$

$$\begin{aligned} &\exp(rL) \approx \exp(0) + r\exp(0)(L - 0) \\ \implies &\exp(rL) \approx 1 + rL, \end{aligned}$$

and we have

$$\begin{aligned} &\exp(rP) \approx \exp(0) + r\exp(0)(P - 0) \\ \implies &\exp(rP) \approx 1 + rP, \end{aligned}$$

and from (2) we have

$$\begin{aligned} 1 + rP &= \alpha(1 + rL) + 1 - \alpha \\ rP &= \alpha rL \\ \therefore P &\approx \alpha L, \end{aligned}$$

where we use an approximation since our result is based on the first order Taylor approximation.

2 Absolute and relative risk aversion

An agent has initial wealth $w > 0$ and is considering how much of a single risky asset to buy. This asset has a random rate of return R and, if the agent spends a on the asset, her final wealth is $w - a + a(1 + R) = w + aR$. The agent has an increasing and strictly concave utility function $u(\cdot)$ and chooses $a \geq 0$ to maximise her expected utility

$$\mathbb{E}[u(w + aR)]. \tag{3}$$

2.1

Show that the agent chooses $a > 0$ if and only if $\mathbb{E}[R] > 0$. (Thus even a risk-averse agent wishes to buy some of this risky asset, so long as the expected return is positive.)

First get marginal utility by differentiating (3) wrt a:

$$\frac{d\mathbb{E}[u(w + aR)]}{da} = \mathbb{E}[u'(w + aR)R] = \mathbb{E}[R]\mathbb{E}[u'(w + aR)].$$

Now suppose if $\mathbb{E}[R] > 0$, then the agent will choose at least some a such that $a > 0$. Next, we have to prove that $\mathbb{E}[R] > 0$. Starting by taking the second derivative of (3):

$$\frac{d^2\mathbb{E}[u(w + aR)}{da^2} = \mathbb{E}[R^2]\mathbb{E}[u''(w + aR)],$$

and we know that since $u(\cdot)$ is strictly concave, $u''(\cdot)$ must be $< 0 \implies \mathbb{E}[R^2] > 0$. Next, we know from our FOC that an optimal choice of a, say $a^* > 0$, will give

$$\mathbb{E}[R]\mathbb{E}[u'(w + a^*R)] = 0,$$

and we also know that from strict concavity that $\mathbb{E}[R]\mathbb{E}[u'(w)] > 0$, so it must be the case that $\mathbb{E}[R] > 0$.

2.2

Write $a(w)$ for the optimal choice of a when the agent's initial wealth is w. Assuming $\mathbb{E}[R] > 0$, show that the agent chooses to buy more of the asset when her initial wealth w locally increases whenever:

$$\mathbb{E}[Ru''(w + a(w)R)] \geq 0.$$

We know from the previous question that the FOC for $a(w)$ to be optimal is

$$\mathbb{E}[Ru'(w + a(w)R)] = 0,$$

and if we differentiate this wrt w we get

$$\frac{d}{dw}\mathbb{E}[Ru'(w + a(w)R)] = \mathbb{E}[R]\mathbb{E}[u''(w + a(w)R)(1 + a'(w)R)].$$

Setting this derivative equal to 0 and solving for $a'(w)$ gives

$$\begin{aligned} 0 &= \mathbb{E}[R]\mathbb{E}[u''(w + a(w)R)(1 + a'(w)R)] \\ \implies -\mathbb{E}[R]\mathbb{E}[a'(w)R]\mathbb{E}[u''(w + a(w)R)] &= \mathbb{E}[R]\mathbb{E}[u''(w + a(w)R)] \\ a'(w) &= -\frac{\mathbb{E}[R]\mathbb{E}[u''(w + a(w)R)]}{\mathbb{E}[R^2]\mathbb{E}[u''(w + a(w)R)]}. \end{aligned} \tag{4}$$

What is this equation telling us? That the marginal asset allocation as a function of wealth, $a'(w)$, is equal to 0, which must be a condition of optimality for the agent. At this point, the agent will no longer purchase additional risky assets, nor will she sell any. Finally, we know that the denominator of (4) is negative and combining this with what've been told in the question, that means that the sign of $a'(w)$ must be the same as the numerator.

2.3

If the agent exhibits decreasing absolute risk aversion (DARA), show that (regardless of whether R is positive or negative)

$$Ru''(w + aR) \geq \frac{u''(w)}{u'(w)} Ru'(w + aR).$$

Deduce that the agent buys more of the asset when she is wealthier (so the asset is a normal good).

Recall that the basic definition of DARA preferences is that the agent becomes less risk averse as her wealth increases (her risk aversion coefficient is decreasing in w). Suppose first that $R > 0$, then DARA preferences imply that

$$\begin{aligned} \frac{u''(w + aR)}{u'(w + aR)} &\geq \frac{u''(w)}{u'(w)} \text{ or,} \\ -\frac{u''(w + aR)}{u'(w + aR)} &\leq -\frac{u''(w)}{u'(w)}, \end{aligned}$$

where we know that the coefficient of relative risk aversion, ρ, say is defined as

$$\rho(\cdot) = -\frac{u''(\cdot)}{u'(\cdot)}.$$

What are those equations saying? $u'(w)$ is the agent's marginal utility of wealth, and $u''(w)$ denotes the curvature or concavity of the agent's utility from wealth. Then the ratio $\frac{u''(w)}{u'(w)}$ must be smaller than the ratio $\frac{u''(w+aR)}{u'(w+aR)}$ if the agent has DARA preferences. This is because the agent becomes increasingly favourable to purchasing the risky asset as a means of increasing her expected utility as her wealth increases. Put algebraically, $\rho(w + aR) \leq \rho(w)$.

To prove normality, use the inequality given to us in the question and take expectations to get:

$$\begin{aligned}\mathbb{E}[Ru''(w+aR)] &\geq \mathbb{E}\left[\frac{u''(w)}{u'(w)}Ru'(w+aR)\right] \\ \mathbb{E}[Ru''(w+aR)] &\geq \frac{u''(w)}{u'(w)}\underbrace{\mathbb{E}\left[Ru'(w+aR)\right]}_{=0} \\ \therefore \mathbb{E}[Ru''(w+aR)] &\geq 0,\end{aligned}$$

where $\mathbb{E}[Ru''(w+aR)] = 0$ comes from the FOC of $a(w)$. Recall from (4) we had

$$a'(w) = -\frac{\mathbb{E}[R]\mathbb{E}[u''(w+a(w)R)]}{\mathbb{E}[R^2]\mathbb{E}[u''(w+a(w)R)]},$$

and we deduced that $a'(w)$ must have the same sign as the numerator since the denominator is negative. Well, we now know that the numerator is ≥ 0, and so must be $a'(w) > 0$. What does this mean? That a is at least weakly increasing in wealth, which proves that the asset is a normal good.

2.4

Suppose the agent exhibits decreasing relative risk aversion, so that

$$-\frac{wu''(w)}{u'(w)}$$

is decreasing in w. Show that the agent then devotes a larger fraction of her wealth to the asset when she is wealthier. i.e. $\frac{a(w)}{w}$ rises in w so that the asset is a luxury good.

Recall, again, that the FOC for $a(w)$ is

$$\mathbb{E}[Ru'(w+a(w)R)] = 0,$$

and let us define $b(w) = \frac{a(w)}{w}$ and rewrite the FOC as

$$\mathbb{E}\left[Ru'\left(w+wb(w)R\right)\right] = 0.$$

Differentiate this with respect to w, and setting equal to 0 to get

$$\begin{aligned}0 &= \mathbb{E}\left[R\right] \times \mathbb{E}\left[u''\left(w(1+b(w)R)\right)\right] \times \mathbb{E}\left[1+b(w)R+wb'(w)R\right] \\ &= \mathbb{E}\left[R \times u''\left(w+b(w)R\right) \times \left(1+b(w)R+wb'(w)R\right)\right],\end{aligned}$$

and rearrange to get

$$\begin{aligned}\mathbb{E}\left[Ru''\left(w+b(w)R\right)\right]\mathbb{E}\left[wb'(w)R\right] &= -\mathbb{E}\left[Ru''\left(w+b(w)R\right)\right]\left(1+b(w)\mathbb{E}[R]\right) \\ \therefore b'(w) &= -\frac{\mathbb{E}\left[Ru''\left(w+b(w)R\right)\right]\left(1+b(w)\mathbb{E}[R]\right)}{w\mathbb{E}\left[R^2u''\left(w+b(w)R\right)\right]}. \qquad (5)\end{aligned}$$

Again, we know that the denominator is negative, and so $b'(w)$ must have the same sign as the numerator. Since the agent has decreasing relative risk aversion, if $R > 0$ we have

$$-w\frac{u''(w)}{u(w)} \geq -[w+wb(w)R]\frac{u''(w+wb(w)R)}{u'(w+wb(w)R)}, \qquad (6)$$

where we define the coefficient of relative risk aversion as

$$\sigma(w) = -w\frac{u''(w)}{u'(w)},$$

so

$$\sigma(w) \geq \sigma(w + wb(w)R).$$

As before, we can rearrange (6) and use FOCs to prove that $b'(w) \geq 0$

$$\begin{aligned}
-w\frac{u''(w)}{u(w)}u'(w + wb(w)R) &\geq -[w + wb(w)R]u''(w + wb(w)R) \\
w\frac{u''(w)}{u(w)}u'(w + wb(w)R) &\leq [w + wb(w)R]u''(w + wb(w)R) \\
w\mathbb{E}[R]\mathbb{E}\left[\frac{u''(w)}{u(w)}u'(w + wb(w)R)\right] &\leq \mathbb{E}[R]\mathbb{E}\left[(w + wb(w)R)u''(w + wb(w)R)\right] \\
\mathbb{E}[R]\mathbb{E}\left[\frac{u''(w)}{u(w)}u'(w + wb(w)R)\right] &\leq \mathbb{E}[R]\mathbb{E}\left[(1 + b(w)R)u''(w + wb(w)R)\right],
\end{aligned}$$

where the LHS is equal to 0 since $u'(w + wb(w)R) = 0$ and the term on the RHS is the numerator in (5), and so $b'(w) \geq 0$.