Nash Equilibria and Dominated Strategies[1]

1 Dominance solvable

Each of n people announces an integer between 1 and 100. A prize of $1 is split equally between all the people whose number is closes to $\frac{2}{3}$ of the average number.

1.1

Show that the game has a unique mixed strategy Nash equilibrium, in which each player's strategy is pure.

Let k^* denote the largest number to which any player's strategy assigns positive probability in a mixed strategy equilibrium and assume that player i's strategy does so. We now argue as follows.

- In order for player i's strategy to be optimal his payoff from the pure strategy k^* must be equal to her equilibrium payoff.
- In any equilibrium player i's expected payoff is positive, since for any strategies of the other players she has a pure strategy that for some realisation of the other players' strategies is at least as close to $\frac{2}{3}$ of the average number as any other player's number.
- Some other player should be playing k^* with positive probability, since by the previous two points player i's payoff (from playing k^*) is positive and hence no other player's number can be closer to $\frac{2}{3}$ of the average number than k^*. (Note that all the other numbers cannot be less than $\frac{2}{3}$ of the average number.)
- If $k^* > 1$. player i can increase his playoff by playing $k^* - 1$ rather than k^* in her mixed strategy. By making this change she becomes the outright winner in the cases in which she was tying with at least on other player. Note that for $k^* > 1$ to be a winning strategy there should be a positive probability of no one choosing $k^* - 1$.

The remaining is that $k^* = 1$; every player uses the pure strategy in which she announces the number 1.

1.2

Show that in fact the equilibrium strategy is the only rationalisable strategy.

Since the game is symmetric, the set of rationalisable actions is the same, say Z, for all players. Let k^* be the largest number in Z. By the argument above the action k^* is a best response to a belief whose support is a subset of Z only if $k^* = 1$.

[1]Solution to this problem set are based on solutions by A/Prof Ines Moreno de Barreda.

2 Mixed equilibrium

Suppose three identical firms must decide simultaneously and irreversibly whether to enter a new market which can accommodate only two of them. If all three firms enter, all get a payoff of 0; otherwise, entrants get 9 and firms that stay out get 8. Identify the unique (completely) mixed-strategy equilibrium and describe the resulting probability distribution of the total ex-post number of entrants. (There are also three pure strategy equilibria, in which exactly two firms enter; but these equilibria are arguably unattainable in a one-shot game in the absence of prior agreement or precedent. The mixed-strategy equilibrium is symmetric, hence attainable.)

Suppose that each firm $F_i, i \in \{1, 2, 3\}$ plays a mixed strategy between enter, E, and stay, S, with probability e_i and $1 - e_i$, respectively. The normal form game can be represented as in Table 1. Note that S, E, E, E, S, E, and E, E, S are the unattainable pure strategy NE.

Table 1: **Normal form representation for firm decision and payoffs**

F_1 \ F_2	E	S
E	0,0,0	9,8,9
S	8,9,9	8,8,9

$E(F_3)$

F_1 \ F_2	E	S
E	9,9,8	9,8,8
S	8,9,8	8,8,8

$S(F_3)$

To find the mixed strategy NE, we find the probability e_i for firm i at which they are indifferent in their expected payoff to playing E or S. For F_1 this is

$$\begin{aligned} &0e_2e_3 + 9(1-e_2)e_3 + 9e_2(1-e_3) + 9(1-e_2)(1-e_3) \\ &= 8e_2e_3 + 8(1-e_2)e_3 + 8e_2(1-e_3) + 8(1-e_2)(1-e_3), \end{aligned}$$

and by symmetry we can write the same conditions for F_2 and F_3:

$$\begin{aligned} &0e_1e_3 + 9(1-e_1)e_3 + 9e_1(1-e_3) + 9(1-e_1)(1-e_3) \\ &= 8e_1e_3 + 8(1-e_1)e_3 + 8e_1(1-e_3) + 8(1-e_1)(1-e_3), \end{aligned}$$

$$\begin{aligned} &0e_1e_2 + 9(1-e_1)e_2 + 9e_1(1-e_2) + 9(1-e_1)(1-e_2) \\ &= 8e_1e_2 + 8(1-e_1)e_2 + 8e_1(1-e_2) + 8(1-e_1)(1-e_2). \end{aligned}$$

Expanding out the expressions gives us three equations for three unknowns. Solving the system of equations leads to our mixed strategy NE conditions:

$$e_1^* = e_2^* = e_3^* = \frac{1}{3}.$$

The resulting probability distribution for when 0, 1, 2, or 3 firms enter is a binomial distribution:

$$\Pr(E = 0) = \binom{3}{0} e_i^0 (1 - e_i)^3 = \frac{8}{27},$$
$$\Pr(E = 1) = \binom{3}{1} e_i^1 (1 - e_i)^2 = \frac{12}{27},$$
$$\Pr(E = 2) = \binom{3}{2} e_i^2 (1 - e_i)^1 = \frac{6}{27},$$
$$\Pr(E = 3) = \binom{3}{3} e_i^3 (1 - e_i)^0 = \frac{1}{27}.$$

3 Switching costs

In many markets, consumers have switching costs. Consider the following simple model of such a market: Two firms A and B simultaneously and non-cooperatively set prices in a single period for a commodity that they can each produce at zero cost. There are $n + s$ customers, where $n > 0$ and $s > 0$ all with reservation price R. Because of switching costs, $s/2$ customers can only by from A and $s/2$ can only buy from B. The n new customers buy from the cheapest firm, if at all.

3.1

Show that there are no pure strategy equilibria.

Let p denote the price that a firm sets, and that a firm will never set $p_i \leq 0$, $i = \{A, B\}$. A firm would (at beast) receive zero profits. But of course, by setting $p_i = R$, it would at least attract $\frac{s}{2}$ 'locked in' customers, and make a profit of at least $\frac{p_i s}{2} = \frac{Rs}{2}$. Thus $p_i = 0$ is strictly dominated because of the existence of customers who cannot switch away from either A or B. Any pure strategy equilibrium must, therefore, involve $p_A > 0$ and $p_B > 0$. Similarly, setting $p > R$ is a dominated strategy by the same logic. Thus we have

$$0 < p_i \leq R,$$

where any price less than 0 or more than R is strictly dominated. Consider the remaining cases:

- Suppose that we have a pure strategy Nash equilibrium with

  $$0 < p_A = p_B \leq R.$$

  This cannot be an equilibrium. One firm must be attracting less than n of the floating customers. It could cut its price by some finite small amount ϵ and attract all n of the floaters, increasing its profits.

- Suppose that we have a pure strategy Nash equilibrium with

  $$0 < p_B < p_A \leq R.$$

  Firm B attracts all the floaters as well as its locked in customers. It could raise its price without losing demand to increase profits – this cannot be an equilibrium. Similarly for

  $$0 < p_A < p_B \leq R.$$

Thus, there is no pure strategy Nash equilibrium. Indeed, the firms are competing Bertrand style for the n floating customers. But pricing at marginal cost (in this case zero) cannot be an equilibrium, since a firm can always chase its locked in customers.

3.2

Find the mixed-strategy Nash equilibrium in which each firm chooses price p according to a distribution $F(p)$. Hint: First check that F is continuous (with no atoms) and strictly *increasing over an interval (no gaps in the support of the distribution). Now, for a given p, calculate expected profit. What can you say about all profits in the support of the mix? What value does p take when $F(p) = 1$? Over what interval of prices do the firms mix? You should now be able to find $F(p)$.*

If both firms choose a common price p from the same cumulative distribution function $F(p)$, then we are looking for a symmetric equilibrium. Recall that a firm can guarantee a profit of $\frac{Rs}{2}$ by setting $p = R$. At price $p \neq R$, the best that a firm can do is to attract the whole of the floating n customers as well as the locked in $\frac{s}{2}$ customers. This means that a necessary condition for it to be willing to charge p is that

$$p\left(n + \frac{s}{2}\right) \geq \frac{Rs}{2}$$
$$\Leftrightarrow p \geq \frac{Rs}{2n+s}.$$

In other words, any p less than this is a dominated strategy. So, this means that

$$F(p) = 0 : p < \frac{Rs}{2n+s},$$

and we must have

$$F(R) = 1,$$

since $p > R$ is strictly dominated.

Now, we need to check that the CDF is continuous and that $\nexists$ an atom in F. Let us do this proof by contradiction. Suppose there is an atom in F at the point $\hat{p}$. This means that

$$F(\hat{p}) > \lim_{p \to \hat{p}} F(p).$$

In other words, firm A (suppose) chooses $\hat{p}$ with strictly positive probability – a discontinuity. Of course, it must be that $\hat{p} > 0$, by the previous argument showing that small p values are strictly dominated. Can this be an equilibrium? No. If $\hat{p}$ is chosen with positive probability by firm A, then firm B will never choose it. Why? Well, by charging $p = \hat{p} - \epsilon$, B shaves its price by a finitely small amount and gets the entire amount of n floating customers.This event occurs with strictly positive probability, leading to a jump upwards in expected profits. Thus there is no atom in the distribution for firm B, completing our proof by contradiction.

We now know that if there is a symmetric mixed equilibrium, it involves a nice continuous distribution – firms mix continuously over an interval. Can portions of this interval be chosen with zero probability? The answer is no. Suppose that there is an interval $[p_L, p_H]$ such that $F(p_H) = F(p_L) > 0$ and $p_H > p_L$. Will a firm ever charge $p = p_L - \epsilon$? The answer is no. By increasing to p_H, it loses demand with arbitrarily small probability, yet gains $p_H - p_L$ on $\frac{s}{2}$ customers. But this, of course, means that the interval over which the 'gap' occurs has no lower bound – and therefore does not exist.

We are dealing with a strictly increasing and continuous distribution function $F(p)$ over an interval. Abusing notation, call this interval $[p_L, p_H]$. A firm must be indifferent between prices in this interval. Suppose that it charges $p = p_H$. It receives demand only from its locked in $\frac{s}{2}$ customers. Hence

$$\pi(p_H) = p_H \frac{s}{2},$$

and suppose that it charges $p = p_L$, winning all the floating n customers, and gains profits of

$$\pi(p_L) = p_L \left(n + \frac{s}{2}\right).$$

For indifference we require

$$\begin{aligned} \pi(p_H) &= \pi(p_L) \\ \Leftrightarrow p_H \frac{s}{2} &= p_L \left(n + \frac{s}{2}\right) \\ p_L &= \frac{p_H s}{2n + s}. \end{aligned}$$

Of course, we also need to ensure that prices outside the mix are not optimal. We already that a firm can charge $p = R$ and obtain profits of $\frac{Rs}{2}$. Thus we need

$$\begin{aligned} \pi(p_H) &\geq \frac{Rs}{2} \\ \Leftrightarrow p_H \frac{s}{2} &\geq \frac{Rs}{2} \\ p_H &\geq R, \end{aligned}$$

but we know that any $p > R$ is dominated, so

$$\implies p_H = R.$$

Which also implies that for our indifference condition

$$p_L = \frac{Rs}{2n + s}.$$

What about prices on the interior of this interval? These yield profits of

$$\begin{aligned} \pi(p) = p\left[n(1 - F(p)) + \frac{s}{2}\right] &= \pi(R) = \frac{Rs}{2} \\ \implies F(p) = 1 - \frac{s(R - p)}{2np}. \end{aligned}$$

This is the unique symmetric mixed Nash equilibrium to this game. Players are indifferent between all prices in the mix, which occurs on the interval:

$$\frac{Rs}{2n + s} \leq p \leq R.$$

Prices below the lower bound are strictly dominated, and prices about the upper bound yield no demand and are also dominated.