Dynamics Games, Bayesian-Nash Equilibria, and Introduction to Industrial Organisation[1]

1 Private provision of public good

Consider the following public good game, where Player 1 plays rows and Player 2 plays columns:

	Contribute	Don't
Contribute	$1-c_1, 1-c_2$	$1-c_1, 1$
Don't	$1, 1-c_2$	$0, 0$

The benefits (1 to each player if at least one contributes) are commonly known, but the costs of contributing c_i is private information to player i. Any strategy for player i in this game specifies an action ('Contribute' or 'Don't') for each possible value of c_i.

1.1

Suppose it is common knowledge that $c_1 = \frac{1}{4}$, but Player 1 does not know c_2. Player 1 believes $c_2 = \frac{1}{4}$ with probability $\frac{1}{2}$ and $c_2 = 2$ with probability $\frac{1}{2}$.

1.1.1

If $c_2 = 2$, does Player 2 have a dominant strategy? If so, what is it? Does Player 2 have a dominant strategy if $c_2 = \frac{1}{4}$?

If $c_2 = 2$ then the payoff matrix for Player 2 (who plays columns) is

(P2)	Contribute	Don't
Contribute	-1	1
Don't	-1	0

Clearly, 'Don't' dominates 'Contribute for Player 2[2]. If $c_2 = \frac{1}{4}$, then Player 2's payoff matrix is

(P2)	Contribute	Don't
Contribute	3/4	1
Don't	3/4	0

Thus, 'Don't' no longer strictly dominates 'Contribute', and Player 2's choice now depends upon the actions of Player 1. If Player 2 believes that Player 1 may play 'Don't', then Player 2 should play 'Contribute'.

[1]Solution to this problem set are based on solutions by A/Prof Ines Moreno de Barreda.
[2]Actually, any player who faces costs of 2 will never play 'Contribute'.

1.1.2

Let z denote the probability that Player 2 contributes if the cost is $c_2 = \frac{1}{4}$. What is Player 1's expected payoff from 'Don't', given his beliefs, in terms of z? Does Player 1 have a dominant strategy?

If $c_2 = \frac{1}{4}$ then $z = \frac{1}{2}$. Why? z is the probability that Player 2 plays 'Contribute' conditional on the fact that $c_2 = \frac{1}{4}$, and if we look at Player 2's payoff matrix when $c_2 = \frac{1}{4}$, we see that $z = \frac{1}{2}$. Then, recall that Player 1 believes that Player's cost, c_2, is either $\frac{1}{4}$ with probability $\frac{1}{2}$, or 2 with probability $\frac{1}{2}$, and thus from Player 1's perspective Player 2 will play 'Contribute' with probability $\frac{z}{2}$. With z defined, we can now construct Player 1's payoff matrix:

(P1)	Contribute $(z/2)$	Don't $(1 - z/2)$	Expected
Contribute	$1 - 1/4$	$1 - 1/4$	$3/4$
Don't	1	0	$z/2$

Thus, Player 1 should play 'Contribute' if $\frac{3}{2} > \frac{z}{2}$. But since z is a probability it is by construction $0 \leq z \leq 1$. So, we can conclude that based on his beliefs, Player 1 will choose to play 'Contribute'.

1.1.3

What is the Bayesian-Nash equilibrium?

Since Player 1 will always choose 'Contribute', Player 2 will always choose 'Don't'. Thus, by the iterated elimination of strictly dominated strategies, we arrive at the unique Bayesian Nash equilibrium.

1.2

Suppose that Player i believes

$$\Pr(c_j = 1/4) = \Pr(c_j = 2),\ i = 1, 2,\ j = 1, 2,\ j \neq i.$$

What is the Bayesian-Nash equilibrium of the game?

Recall that this is game is symmetrical, and we already found that a Player i will not play 'Contribute' if $c_i = 2$. Further, they will always play 'Contribute' when $c_i = \frac{1}{4}$ because when $c_j = \frac{1}{4}$, 'Contribute' dominates 'Don't'. Thus, the Bayesian Nash equilibrium of the game is for both players to play 'Contribute' with $c_i = \frac{1}{4},\ i \in \{1, 2\}$.

1.3

Suppose that both players believe costs are drawn independently from a uniform distribution on the interval $[0, 2]$. What is the Bayesian-Nash equilibrium now? Hint: Show that an equilibrium strategy has the trigger form of contributing whenever $c_i \leq c_i^*$, and note the uniform distribution costs.

The question is quite misleading. First, suppose we looked at this game from the perspective of Player 1, denoting the probability that Player 2 plays 'Contribute' be equal to μ:

(P1)	Contribute (μ)	Don't$(1-\mu)$	Expected
Contribute	$1-c_1$	$1-c_1$	$1-c_1$
Don't	1	0	μ

If we define ρ to be the probability which Player 1 plays 'Contribute', then from his payoffs we can see that he will play 'Contribute' if and only if

$$\begin{aligned} 1-c_1 &\geq \mu \\ \Leftrightarrow c_1 &\leq 1-\mu, \end{aligned}$$

and thus

$$\rho = \Pr(c_1 \leq 1-\mu) = \frac{1-\mu}{2}, \tag{1}$$

noting that the denominator is 2 due to the uniform distribution of the costs. We could do the same exercise from the perspective of Player 2, which would yield, by symmetry:

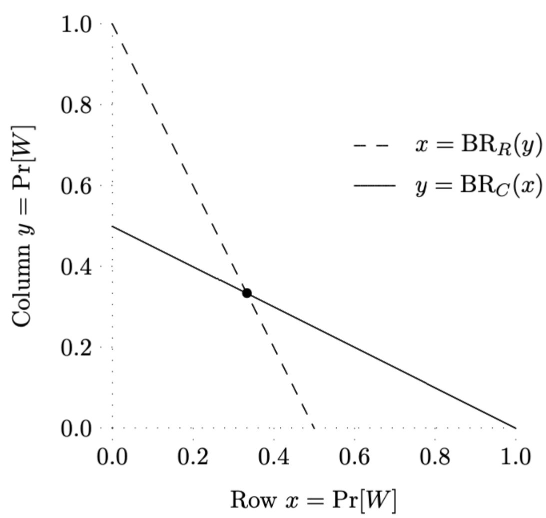
$$\mu = \Pr(c_2 \leq 1-\rho) = \frac{1-\rho}{2}. \tag{2}$$

Substituting (2) into (1) yields

$$\begin{aligned} \rho &= \frac{1-\frac{1-\rho}{2}}{2} \\ 2\rho &= 1-\frac{1}{2}+\frac{\rho}{2} \\ \frac{3\rho}{2} &= \frac{1}{2} \\ \rho &= \frac{1}{3}. \end{aligned}$$

The Bayesian reaction functions are depicted in 1, and the critical point is $c_i^* = 1-\rho = \frac{2}{3}$.

Figure 1: Figure for question 1



2 A Bayesian trading game

Suppose that a buyer has a valuation for a good v_b, uniformly distributed on $[0,1]$. The seller has a valuation v_s uniformly distributed on $[0,1]$. They each observe their own valuation, but not that of the other player. Simultaneously they each announce a price, p_b and p_s, respectively. If $p_b \geq p_s$ a sale takes place at a price halfway between their announcements, $p = (p_b + p_s)/2$, and the buyer receives the good, yielding payoffs of

$$u_b(p, v_b) = v_b - p$$

and

$$u_s(p, v_s) = p - v_s,$$

otherwise there is no sale and both players receive 0.[3]

2.1

Suppose that the seller uses a linear strategy of the form

$$p_s(v_s) = \alpha + \beta v_s.$$

Show that the buyer's expected payoff may be written as

$$\frac{p_b - \alpha}{\beta}\left[v_b - \frac{1}{2}\left(p_b + \frac{\alpha + p_b}{2}\right)\right].$$

The expected payoff of the buyer is

$$\begin{aligned}
\mathbb{E}[u_b] &= \Pr(p_b \geq p_s)\,(v_b - p) + \Pr(p_b \leq p_s) \cdot 0 \\
&= \Pr(p_b \geq p_s)\left(v_b - \frac{p_b + \mathbb{E}[p_s|p_b \geq p_s]}{2}\right).
\end{aligned}$$

The seller uses a linear strategy to determine his sale price, and since $v_s \sim U[0,1]$, it follows that

$$p_s(v_s) \sim U[\alpha, \alpha + \beta],$$

and the plot of the density is given in Figure 2. The probability that the buyer's bid price is greater than the sellers asking price, $p_b \geq p_s$ is

$$\Pr(p_b \geq p_s) = \frac{p_b - \alpha}{\beta},$$

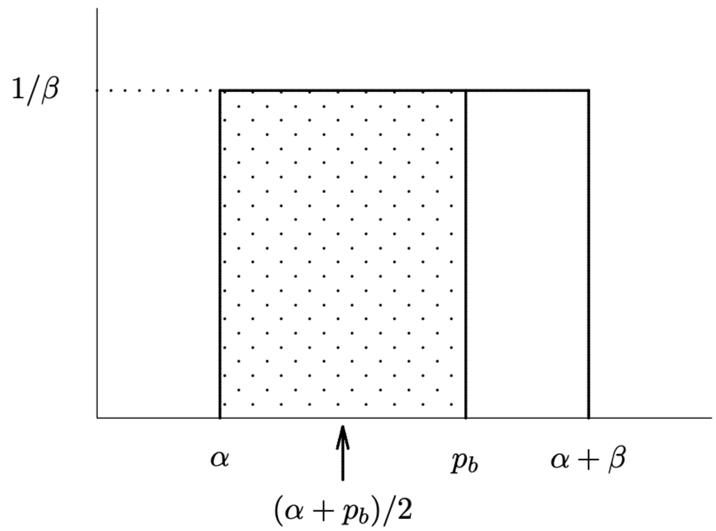
and so the expected utility of the buyer can be written as

$$\begin{aligned}
\mathbb{E}[u_b] &= \frac{p_b - \alpha}{\beta}\left(v_b - \frac{p_b + \mathbb{E}[p_s|p_b \geq p_s]}{2}\right) \\
&= \frac{p_b - \alpha}{\beta}\left(v_b - \frac{1}{2}\left(p_b + \mathbb{E}[p_s|p_b \geq p_s]\right)\right) \\
&= \frac{p_b - \alpha}{\beta}\left(v_b - \frac{1}{2}\left(p_b + \frac{\alpha + p_b}{2}\right)\right), \qquad (3)
\end{aligned}$$

[3] It would be equivalent to assume $u_s = p$ in the case of a sale and $u_s = v_s$ when there is no sale. It's worthing thinking why this is the case.

where the expected sale price from the perspective of the buyer is $\mathbb{E}[p_s|p_b \geq p_s] = \frac{\alpha+p_b}{2}$.

Figure 2: Conditional expectation of p_s



2.2

Hence calculate the buyer's best reply to the seller's linear strategy, and show that it is also linear.

This requires us to maximise $\mathbb{E}[u_b]$ with respect to p_b:

$$\begin{aligned}
\frac{\partial \mathbb{E}[u_b]}{\partial p_b} &= \frac{\partial}{\partial p_b}\left\{\frac{p_b v_b - \alpha v_b}{\beta} - \frac{p_b - \alpha}{2\beta}\left(p_b + \frac{\alpha + p_b}{2}\right)\right\} = 0 \\
\implies 0 &= \frac{v_b}{\beta} - \left[\frac{1}{2\beta}\left(p_b + \frac{\alpha + p_b}{2}\right) + \frac{p_b - \alpha}{2\beta}\left(\frac{3}{2}\right)\right] \\
\frac{p_b}{2\beta} + \frac{\alpha + p_b}{4\beta} &= \frac{v_b}{\beta} - \frac{3p_b - 3\alpha}{4\beta} \\
\frac{p_b}{2} + \frac{p_b}{4} + \frac{3p_b}{4} &= v_b - \frac{3\alpha}{4} - \frac{\alpha}{4} \\
\therefore p_b &= \frac{2v_b}{3} + \frac{\alpha}{3}, \qquad (4)
\end{aligned}$$

which is clearly linear in v_b.

2.3

Now suppose the buyer uses a linear strategy of the form

$$p_b(v_b) = \gamma + \delta v_b.$$

By calculating the sellers expected payoff, find the best reply of the seller to this strategy, and show that is is also linear.

We essentially repeat the above exercise, this time from the seller's point of view. The expected payoff of the seller is

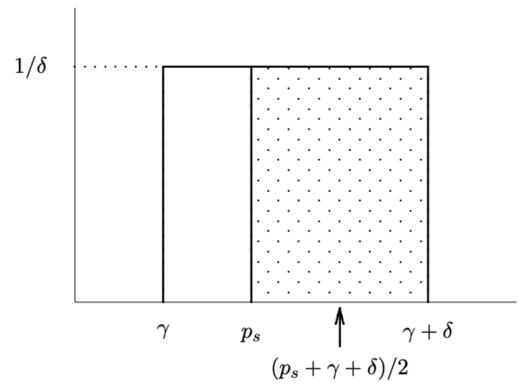
$$\begin{aligned}\mathbb{E}[u_s] &= \Pr(p_s \leq p_b)(p - v_s) + \Pr(p_s \geq p_b) \cdot 0 \\ &= \Pr(p_s \leq p_b)\left(\frac{\mathbb{E}[p_b|p_s \leq p_b] + p_s}{2} - v_s\right).\end{aligned}$$

Now, the buyer uses a linear strategy to determine his sale price, so we have $v_b \sim U[0, 1]$, and so

$$p_b(v_b) \sim U[\gamma, \gamma + \delta],$$

and the plot of the density is given in Figure 3.

Figure 3: Conditional expectation of p_b



The probability that the seller's bid price is lower than the buyer's bid price, $p_s \leq p_b$ is

$$\Pr(p_s \leq p_b) = \frac{\gamma + \delta - p_s}{\delta}.$$

Next, we calculate the expected bid price from the perspective of the seller,

$$\mathbb{E}[p_b|p_s \leq p_b] = \frac{p_s + \gamma + \delta}{2},$$

and so the expected payoff for the seller is

$$\mathbb{E}[u_s] = \frac{\gamma + \delta - p_s}{\delta}\left(\frac{1}{2}\left[\frac{p_s + \gamma + \delta}{2} + p_s\right] - v_s\right). \tag{5}$$

Differentiating the expected payoff with respect to p_s and solving for the FOC gives the reaction function of the seller

$$\begin{aligned}\frac{1}{\delta}\left[\frac{1}{2}\left(p_s + \frac{\gamma + \delta + p_s}{2}\right) - v_s\right] &= \frac{3}{4}\left(\frac{\gamma + \delta - p_s}{\delta}\right) \\ \therefore p_s &= \frac{2v_s}{3} + \frac{1}{3}(\gamma + \delta), \end{aligned} \tag{6}$$

which is also linear in v_s.

2.4

Calculate values of α, β, γ, and δ, such that these strategies constitute a linear Bayesian-Nash equilibrium of the trading game.

The two reaction functions we calculated are

$$p_b = \frac{2v_b}{3} + \frac{\alpha}{3},$$
$$p_s = \frac{2v_s}{3} + \frac{1}{3}(\gamma + \delta),$$

and after matching coefficients against the linear strategies of the players given to us, we have

$$\alpha = \frac{1}{3}(\gamma + \delta), \ \beta = \frac{2}{3}, \ \gamma = \frac{1}{3}\alpha, \ \delta = \frac{2}{3}.$$

Solving these out allows us to write the Bayesian-Nash equilibrium as

$$p_b(v_b) = \frac{2}{3}v_b + \frac{1}{12},$$
$$p_s(v_s) = \frac{2}{3}v_s + \frac{1}{4}.$$

2.5

For what values of v_b and v_s is trade mutually advantageous? For what values of v_b and v_s does trade take place? Comment.

Trade is mutually advantageous whenever the value of the buyer exceeds the value of the seller. i.e. When $v_b \geq v_s$. However, trade only occurs when

$$p_b \geq p_s$$
$$\Leftrightarrow \frac{2}{3}v_b + \frac{1}{12} \geq \frac{2}{3}v_s + \frac{1}{4}$$
$$\Leftrightarrow v_b \geq v_s + \frac{1}{4}.$$

There are, therefore, a range of values of v_b and v_s for which trade would be mutually beneficial, but for which trade does not take place – clearly these are inefficient, as both parties would benefit from trade if it were to occur.

2.6

Now suppose the buyer and seller use the following strategies:

$$p_b = \begin{cases} \frac{1}{2} & \text{if } v_b \geq \frac{1}{2}, \\ 0 & \text{if } v_b < \frac{1}{2}, \end{cases}$$
$$p_s = \begin{cases} \frac{1}{2} & \text{if } v_s \leq \frac{1}{2}, \\ 1 & \text{if } v_s > \frac{1}{2}. \end{cases}$$

Argue that these strategies constitute a Bayesian-Nash equilibrium of the trading game. Without doing any further calculations, are there any other Bayesian-Nash equilibria of this game? Are any of these efficient?

Let's suppose that the buyer uses the strategy suggested. A sketch of the argument is as follows: If $v_s > \frac{1}{2}$ then announcing any $p_s < \frac{1}{2}$ is suboptimal. Doing so either results in a sale (when $p_b = \frac{1}{2} > p_s$) and trade takes place at a price halfway between $\frac{1}{2}$ and p_s, yielding a negative payoff for the seller; or it does not result in a sale (when $p_b = 0$). The seller is better off setting $p_s = 1$.

If $p_s \leq \frac{1}{2}$, a sale can be profitable for the seller, and it most profitable when p_s is chosen as high as possible whilst still allowing trade. The seller should choose $p_s = \frac{1}{2}$. Any higher, and no sale will take place; any lower an the seller's payoff could be improved by increasing p_s. An analogous argument shows that the buyer's strategy is optimal given the seller's. Thus this is indeed an equilibrium.

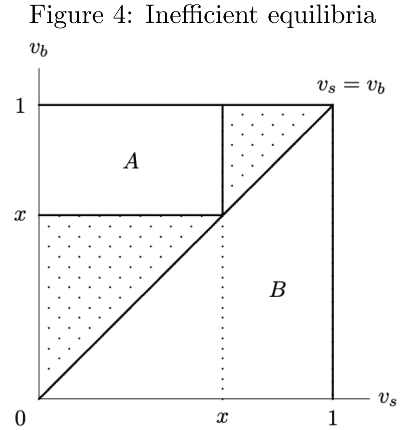
In fact, there is nothing special about the number $\frac{1}{2}$ in this context. In fact, any strategy pair of the form

$$p_b = \begin{cases} x & \text{if } v_b \geq x, \\ 0 & \text{if } v_b < x, \end{cases}$$

$$p_s = \begin{cases} x & \text{if } v_s \leq x, \\ 1 & \text{if } v_s > x, \end{cases}$$

where $x \in [0, 1]$, forms a Bayesian-Nash equilibrium of the game. Of course, none is efficient as there is always a range of values where $v_s \leq v_b$ (so that trade should take place) but $p_s > p_b$ so that trade does not in fact take place. The following figure illustrates.

Figure 4: Inefficient equilibria



For some value of x, the area A represents the values of v_s and v_b for which trade takes place (and should do so, as this area lies above the line $v_s = v_b$). Region B indicates values of v_s and v_b where trade would be suboptimal (and indeed, does not take place).

The inefficiency of this equilibrium is illustrated by the shaded areas, where trade does not take place even though it would be mutually beneficial.

There are many equilibria of this game (including those above). However, none are efficient – in fact, the linear one is the best of them. For a proof of this statement (and more) see Myerson & Satterthwaite (1983).

3 Two stage game

Consider the following simultaneous-move stage game:

	L	C	R
T	3,1	0,0	5,0
M	2,1	1,2	3,1
B	1,2	0,1	4,4

This game is played twice, with the outcome from the first stage observed before the second stage begins. There is no discounting. Can the payoff (4,4) be achieved in the first stage in a pure strategy sub-game perfect Nash equilibrium of the two stage game? If so, describe a strategy profile that does so and prove that it is a sub-game perfect Nash equilibrium. If not, prove why not.

The pure strategy Nash equilibria are underlined in the normal form representation of the game:

Table 1: **Normal Form Representation of the Two Stage Game**

	L	C	R
T	3,1	0,0	5,0
M	2,1	1,2	3,1
B	1,2	0,1	4,4

It follows that there are multiple equilibria in this final sub-game. Note, also, that we are only looking pure strategy sub-game perfect Nash equilibria. We may construct a sub-game perfect equilibrium for the whole game that involves conditional equilibrium selection. Consider the following strategy profile:

- In the first stage, the players agree to choose B and R, respectively.
- In the second stage, the action depends on the actions observed in the first stage:
  - If Row played B in the first period, then choose T, L.
  - If Row deviated, then choose M, C.

There is no opportunity to deviate in the second period, since players are adopting a Nash equilibrium in any sub-game that they play. Conciser the first period. Any deviation by the Column player has no effect on the equilibrium played in the second period. Furthermore, any deviation could only result in a payoff lower than 4 in the first period. Hence, Column is optimising in both stages. Turn to the Row player. If the Row player deviates in the first period, he gains at most 1 by deviating to play T. However, he then loses 2, since the second stage game equilibrium M, C is played in the second period. Hence Row is optimising in both stages, and this is a sub-game perfect equilibrium.

Note also we could also looked for mixed strategy sub-game perfect Nash equilibria (SPNE). The one shot game has several equilibria, so a SPNE does not need to involve playing a NE in each stage game. The NE are: (T, L), (M, C), and $(\sigma_1(T) = \frac{1}{2}$, $\sigma_1(M) = \frac{1}{2}$, $\sigma_2(L) = \frac{1}{2}$, $\sigma_2(C) = \frac{1}{2})$.

Proof: Suppose P1 plays T with w.p. p and M with w.p. $1-p$, and P2 plays L w.p. q and C w.p. $1-q$. P1 is indifferent between T and M when

$$\begin{aligned} 3q &= 2q + (1-q) \\ \implies q &= \frac{1}{2}. \end{aligned}$$

P2 is indifferent between L and C when

$$\begin{aligned} p + (1-p) &= 2(1-p) \\ \implies p &= \frac{1}{2}. \end{aligned}$$

- The expected payoffs for the mixed strategy are $u = (\frac{3}{2}, 1)$.
Proof: Summing across the probability weighted payoffs, given that $q = \frac{1}{2}$ and $p = \frac{1}{2}$ yields:

$$\begin{aligned} \frac{1}{2^2}3 + \frac{1}{2^2}2 + \frac{1}{2^2} &= \frac{3}{2}, \\ \frac{1}{2^2}1 + \frac{1}{2^2} + \frac{1}{2^2}2 &= 1. \end{aligned}$$

- The strategy (B, R) Pareto dominates all equilibria of the game. Can this be reached in the first period of the game as part of a SPNE? Yes.
Proof: Suppose that the players agree to (B, R) in period 1 and, conditional on (B, R) being observed in period 1, play (T, L) in period 2. If (B, R) is not observed, then players retaliate by playing their mixed strategies. We know that the final period of a finite stage game features no profitable deviations for the players, and thus must end in players playing a NE. As this is a two stage game, we can consider this strategy profile.

In the first period, the most profitable deviation for the players are that P1 play T while P2 remains committed to R. This leads to a deviation payoff for the two periods of $5 + \frac{3}{2}$ which is less than the agreed payoffs of $4 + 3$. Thus, for P1, there is no incentive to deviate from the agreed strategy. By an analogous argument, P2 also has no incentive to deviate. Thus, this satisfies the One-Deviation Principle.

- What if we have the following strategy profile: play (B, R) in the first period then (M, C) in the second period if (B, R) is observed in the first period. Otherwise, play mixed strategies.
- P1 would have an incentive to deviate since the deviation payoff is $5 + \frac{3}{2}$ which is less than the agreed strategy payoff of $4 + 1$. But P2 knows that P1 has incentive to deviate with this agreement, and P1's agreed upon payoffs in the previous agreement are strictly higher. This is clearly a violation of the One-Deviation Principle.
- Thus, the strategy profile where (B, R) is played in the first period is a SPNE.

4 Repeated game

Consider an infinitely repeated game where the stage game is

	L	R
U	9,9	1,10
D	10,1	7,7

Players discount the future using the common discount factor δ.

4.1

What outcomes in the stage-game are consistent with Nash equilibrium play?

For the Row player, when Column plays L his best response is U. When Column plays R his best response is U. For the Column player, when Row plays U his best response his D, and when Column plays D his best response is R. Hence, the unique pure strategy Nash equilibrium is D, R.

4.2

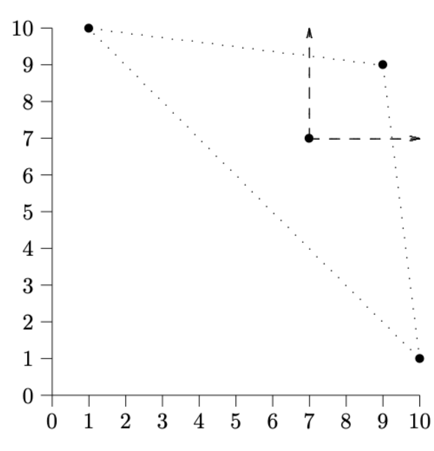
Let v_R and v_C be the repeated game payoffs to Row and Column, respectively. Draw the set of feasible payoffs from the repeated game, explaining any normalisation you use.

Present discounted value of a payoff will be

$$v_i = (1-\delta)\sum_{t=0}^{\infty} \delta^t v_{i,t},\ i \in \{R, C\},$$

where the factor $(1-\delta)$ is introduced as a normalisation, so that utility remains bounded even when $\delta \to 1$. Now, to consider the feasible payoffs, we take the convex hull of payoffs from the payoff profiles in the stage game. This is shown in the dotted line below:

Figure 5: Feasible payoffs



4.3

Are all the payoffs in the feasible set obtainable from mixed-strategy combinations in the stage game? That is, for every point in the feasible set, can you find a p such that $0 \leq q \leq 1$ *and a q such that* $0 \leq p \leq 1$ *that will give those expected payoffs from a single play?*

No, not all the payoffs shown in Figure 5 are achievable via mixed strategies. One can obtain them from randomisations *between strategy profiles* rather than combinations of independent randomisations between strategies themselves. Some payoff profiles, therefore, may require a public randomisation device. Given a high enough δ, all feasible payoffs may be mimicked by a long run combination of pure strategies. In fact, for combinations of p and q we have:

$$\text{Row} = 9pq + 1p(1-q) + 10(1-p)q + 7(1-p)(1-q),$$
$$\text{Column} = 9pq + 10p(1-q) + 1(1-p)q + 7(1-p)(1-q).$$

We can illustrate these payoffs in a diagram. For each $p \in [0, 1]$, we can consider the payoffs available to both players as a function of q. These payoffs are then linear in q. Thus, for a fixed p, the feasible payoffs form a line. The endpoints of this line correspond to $q = 0$. We can therefore calculate these endpoints as a function of p. These are assemlbed in the following table:

	Row	Column
Start ($q = 0$)	$p + 7(1-p)$	$10p + 7(1-p)$
End ($q = 1$)	$9p + 10(1-p)$	$9p + (1-p)$

For each p, draw such a line. By varying p from 0 to 1, sketch out the feasible set of payoffs from independent randomisations in the stage game: see Figure 6. In order to make the diagram look more balanced, I have also completed the reverse operation. Allowing q to range from 0 to 1, in step size of 0.1, I gave illustrated lines showing feasible payoffs for values of p.

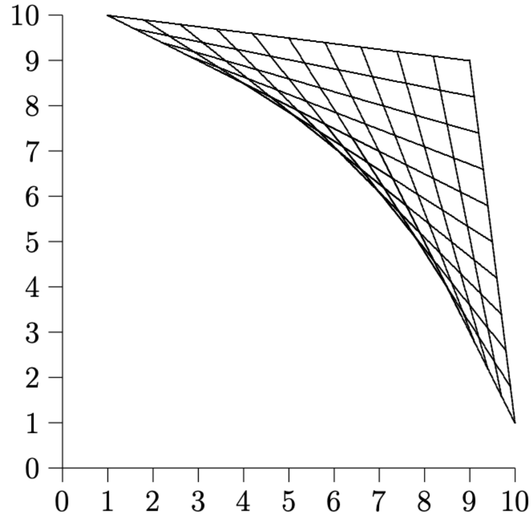
To be explicit: not all payoffs in the feasible set (the convex hull of Figure 5) can be achieved as outcomes of pairs of mixed trategies i nthe stage game. Only those in Figure 6 can. In a one-shot game, a public randomisation device (which would allow mixed strategies to be correlated) would allow us to achieve the payoffs that are in the convex hull but not in Figure 6. Of course, once the game is infinitely repeated, all payoffs in the convex hull may be achieved. In this sense, repitition plays the role of the correlation device.

4.4

What are the players' minmax values? Show the individually rational feasible set.

Minmax values are straightforward. No matter what, a column player will respond with a stage best response of R. By playing D, Row player can hold her down to a payoff of 7. Thus, (7,7) are the minmax payoffs for the two players. Payoffs above the minmax values are individually rational, and are depicted in Figure 5.

Figure 6: Independent randomisation



4.5

Find a Nash equilibrium in which the players obtain the (9,9) payoff each period forever. What restrictions on δ are necessary?

The players will choose U and L and get payoffs of (9,9) if these strategies have always happened before. If either player deviates, then both players choose D and R forever. Is this an equilibrium? We only need to check for Nash. On the equilibrium path, a player obtains:

$$9\sum_{t=0}^{\infty}\delta^t = \frac{9}{1-\delta}.$$

A deviating player obtains:

$$10 + 7\sum_{t=0}^{\infty}\delta^t = \frac{(1-\delta)10+\delta 7}{1-\delta}.$$

Hence, this is an equilibrium if:

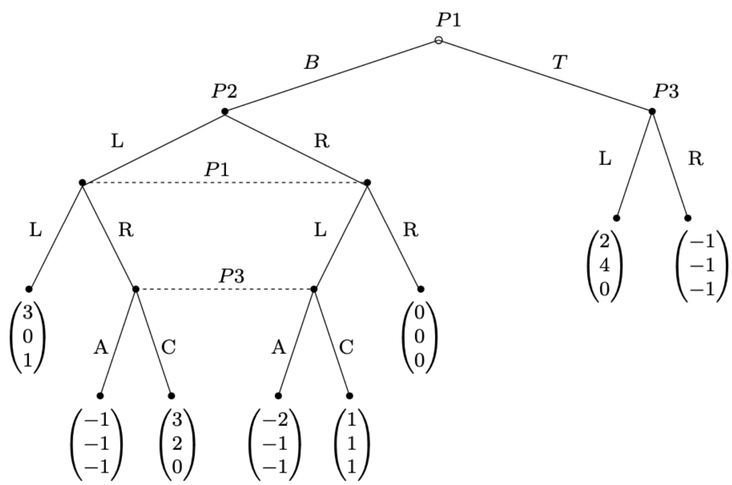
$$\begin{aligned}\frac{9}{1-\delta} &\geq \frac{(1-\delta)10+\delta 7}{1-\delta} \\ 9 &\geq 10-\delta 3 \\ \implies \delta &\geq \frac{1}{3}.\end{aligned}$$

It is also worth noting that this equilibrium is also subgame perfect: off the equilibrium path, players adopt a stage NE which is also a SPNE.

5 Sequential equilibrium

Consider the following extensive form game:

Figure 7: Sequential game



Find the set of pure strategy Nash, subgame perfect, and sequential equilibria, and their payoffs. Is there any mixed sequential equilibrium of this game? If so, which ones?

The extensive form game can be represented in the following normal form:

Table 2: Normal form representation of the sequential game

P1	P2 = L, P3: LA	LC	RA	RC	P2 = R, P3: LA	LC	RA	RC
BL	3,0,1	3,0,1	3,0,1	3,0,1	-2,-1,-1	1,1,1	-2,-1,-1	1,1,1
BR	-1,-1,-1	3,2,0	-1,-1,-1	3,2,0	0,0,0	0,0,0	0,0,0	0,0,0
TL	2,4,0	2,4,0	-1,-1,-1	-1,-1,-1	2,4,0	2,4,0	-1,-1,-1	-1,-1,-1
TR	2,4,0	2,4,0	-1,-1,-1	-1,-1,-1	2,4,0	2,4,0	-1,-1,-1	-1,-1,-1

There are 10 pure strategy NE for the entire game are: (BL, L, LA), (BL, L, RA), (BR, L, LC), (BR, L, RC), (BL, R, RC), (BR, R, RA), (TL, R, LA), (TL, R, LC), (TR, R, LA), and (TR, R, LC).

The set of Nash equilibrium payoffs are: $(3, 0, 1)$, $(3, 2, 0)$, $(1, 1, 1)$, $(0, 0, 0)$, and $(2, 4, 0)$. Which of these can we rule out as non-credible threats? To start, recall the definition of a subgame: the game following a singleton information set, where all successor information sets are accessible only from that singleton. We have one singleton information set, and two subgames:

- The subgame starting with $P2$, with four subgame perfect Nash equilibria (L, L, A), (R, L, C), (L, R, C), and (R, R, A); and
- The game game starting with $P3$ which has a unique Nash equilibrium, where $P3$ plays L.

Recall the definition of subgame perfect equilibrium: A strategy profile in an extensive form game is a subgame perfect Nash equilibrium if it induces a Nash equilibrium in any subgame. The four subgame perfect Nash equilibria are highlighted in the table below:

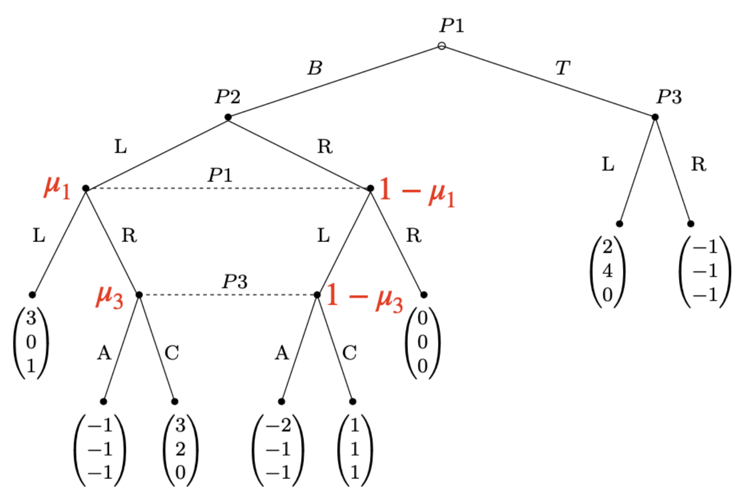
Table 3: Subgame perfect Nash equilibria

P2 →	L				R			
P1 \ P3	LA	LC	RA	RC	LA	LC	RA	RC
BL	3,0,1	3,0,1	3,0,1	3,0,1	-2,-1,-1	1,1,1	-2,-1,-1	1,1,1
BR	-1,-1,-1	3,2,0	-1,-1,-1	3,2,0	0,0,0	0,0,0	0,0,0	0,0,0
TL	2,4,0	2,4,0	-1,-1,-1	-1,-1,-1	2,4,0	2,4,0	-1,-1,-1	-1,-1,-1
TR	2,4,0	2,4,0	-1,-1,-1	-1,-1,-1	2,4,0	2,4,0	-1,-1,-1	-1,-1,-1

Why? First, $P3$ will never play R when she can play L, so that eliminates strategy profiles RA and RC. Then, if $P1$ plays TL and $P2$ plays R, $P3$ will play LC. Analogously, (TR, R, LA) is chosen as opposed to (TR, R, LC).

To find the sequential equilibria, assign μ_1 and μ_3 as the beliefs for the non-singleton information sets of $P1$ and $P3$, respectively:

Figure 8: Sequential game with beliefs



Start with $P3$, we can see that no matter what value of $\mu_3 \in [0, 1]$, she will always play C as it is strictly dominant. Moving up the tree, $P1$ knows that $P3$ will always play C, and so if $P1$ believes, for sure, that she is on the left information node (i.e. $\mu_1 = 1$), she is completely indifferent between playing L or R. However, if $\mu_1 < 1$, then she will play L in order to get a payoff of 1 (as opposed to 0), knowing that $P3$ will always C. Now, depending on μ_1, there are two different sequential equilibria:

- Suppose that $\mu_1 < 1$. $P2$ knows that $P1$ will play L for sure when $\mu_1 < 1$. Thus, it is optimal for $P2$ to play R, since this will result in a payoff of 1(as opposed to 0) based on the strategies and beliefs of $P1$ and $P3$. $P1$ knows that $P2$ will always play R when $\mu_1 < 1$, and so $P1$ has to choose between playing B and getting a payoff of 1, or playing T and getting a payoff of 2, knowing that $P3$ will always play L. Hence, a sequential equilibrium is: $\{(TL, R, LC); (\mu_1 = 0, \mu_3 = 0)\}$ with payoff $(2, 4, 0)$, which is sequentially rational and consistent.

- Now suppose that $\mu_1 = 1$ (and $\mu_3 = 1$). $P2$ knows that $P1$ is indifferent between playing L and R. So let p denote the probability that $P1$ will play L, and $1 - p$ be the probability that $P1$ will play R. So, moving up the tree, $P2$ has to choose between playing L and getting payoff $2(1 - p)$; or choosing R, then $P1$ plays L (since $\mu_1 = 1$) with probability p, then $P3$ picks C, and hence $P2$'s payoff is just p. Thus, $P2$ chooses L when the payoff from choosing L is greater than or equal to the payoff of playing R, which happens if:

$$
\begin{aligned}
2(1 - p) &\geq p \\
2 - 2p &\geq p \\
\Leftrightarrow p &\leq \frac{2}{3}.
\end{aligned}
$$

  Moving up the tree, $P1$ has to choose between playing B, where $P2$ plays L if $p \leq \frac{2}{3}$, and she gets a payoff of 3; or, choosing T, knowing that $P3$ will always play L, and getting a payoff of 2. Thus, the sequential equilibrium is: $\{(B \text{ and } L \text{ wp } p, L, LC); (\mu_1 = 1, \mu_3 = 1); p \leq \frac{2}{3}\}$ with payoff $(3, 2(1-p), p)$. In particular, the pure sequential equilibrium $(p = 0)$ is: $\{BR, L, LC); (\mu_1 = 1, \mu_3 = 1)\}$ with payoff $(3, 2, 0)$.

6 Cournot competition

Consider an n firm homogeneous product industry where firm i produces output q_i at cost cq_i. Price is:

$$p = \alpha - \beta Q, \ Q = \sum_{i=1}^{n} q_i.$$

What are the firms' outputs, prices, and profits in the Cournot equilibrium? What happens as $n \to \infty$.

Consider the behaviour of a firm i. Its profits are:

$$\begin{aligned}\pi_i &= pq_i - cq_i \\ &= (\alpha - \beta Q - c)q_i.\end{aligned}$$

Also denote the quantity of all the other firms' output:

$$Q_{-i} = \sum_{j=1}^{n} q_j, \ j \neq i,$$

and so we can write:

$$\pi_i = (\alpha - \beta Q_{-i} - \beta q_i - c)q_i. \tag{7}$$

Two things about (7): 1) Firm i's profit only depends on the aggregate output of the firms, and not on the distribution of output amongst the firms. 2) The profit function only applies for $Q < \alpha/\beta$. If this were not the case, price would be negative.

Let us calculate the best response of firm i in response to the quantities of the other firms. Fixing Q_{-i}, notice that (7) is strictly concave and continuous in q_i. It follows that the first order condition is both necessary and sufficient for a global maximum. Take the derivative to obtain:

$$\begin{aligned}\frac{\partial \pi_i}{\partial q_i} &= \alpha - \beta Q_{-i} - 2\beta q_i - c = 0 \\ \implies q_i^* &= \frac{\alpha - \beta Q_{-i} - c}{2\beta}.\end{aligned} \tag{8}$$

Alternatively, we could write:

$$\begin{aligned}\frac{\partial \pi_i}{\partial q_i} &= \alpha - \beta Q_{-i} - \beta q_i - c = \beta q_i \\ \implies q_i^* &= \frac{\alpha - \beta Q - c}{\beta},\end{aligned} \tag{9}$$

and note that each firm i's optimal output depends on the same common aggregate output, Q. This means that if all firms' FOCs are satisfied – i.e., we have a pure strategy Nash equilibrium – then all firms must be choosing the same quantity. Thus, this equilibrium must be symmetric, where $q_i = q^* \forall i$. Of course, this then means that

$$Q = nq^*.$$

The FOC then reduces to

$$\begin{aligned}
q^* &= \frac{\alpha - \beta n q^* - c}{\beta} \\
\beta q^* &= \alpha - n\beta q^* - c \\
q^*(\beta + n\beta) &= \alpha - c \\
\implies q^* &= \frac{\alpha - c}{\beta(1+n)}. \qquad (10)
\end{aligned}$$

Equilibrium price is:

$$\begin{aligned}
p &= \alpha - \beta Q = \alpha - \beta n q^* \\
&= \alpha - \beta n \left[\frac{\alpha - c}{\beta(1+n)}\right] \\
&= \alpha - n\frac{\alpha - c}{1+n} \\
&= \frac{\alpha(1+n) - n\alpha + nc}{1+n} \\
\implies p^* &= \frac{\alpha + nc}{1+n} \qquad (11)
\end{aligned}$$

and so we then get π_i^*:

$$\begin{aligned}
\pi_i^* &= pq_i - cq_i = q^*(p^* - c) \\
&= \left(\frac{\alpha - c}{\beta(1+n)}\right)\left(\frac{\alpha + nc}{1+n} - c\right) \\
&= \left(\frac{\alpha - c}{\beta(1+n)}\right)\left(\frac{\alpha + nc - c(1+n)}{1+n}\right) \\
&= \left(\frac{\alpha - c}{\beta(1+n)}\right)\left(\frac{\alpha - c}{1+n}\right) \\
&= \frac{1}{\beta}\left(\frac{\alpha - c}{1+n}\right)^2. \qquad (12)
\end{aligned}$$

Thus, we have found the relevant outputs, prices, and profits in a Cournot equilibrium. In other words, a pure strategy Nash equilibrium. Notice that price is never negative or zero. To finish off this part of the question, we consider the limiting behaviour as $n \to \infty$:

$$q^* = \frac{\alpha - c}{\beta(1+n)} \to 0, \qquad (13)$$

$$p^* = \frac{\alpha + nc}{1+n} \to c, \qquad (14)$$

$$Q^* = nq^* \to \frac{a - c}{\beta}, \qquad (15)$$

$$\pi^* = \frac{1}{\beta}\left(\frac{\alpha - c}{1+n}\right)^2 \to 0. \qquad (16)$$

6.1

Two firms merge. The merged firm has marginal costs of c, just as before. What happens to the merged firms' profits? What happens to the remaining firms' profits? Comment.

Prior to a merger, each firm enjoys a profit of:

$$\pi_i = \frac{1}{\beta}\left(\frac{\alpha - c}{1+n}\right)^2,$$

where we could say that π_i is a function of n, $\pi_i(n)$. So, clearly, as the number of firms increase, profits decrease (as shown above). Now, suppose then that $\pi_i(n-1)$ is the profit of the merged firm:

$$\begin{aligned} \pi_i(n-1) &= \frac{1}{\beta}\left(\frac{\alpha - c}{n}\right)^2 \\ \implies \pi_i(n-1) &> \pi_i(n). \end{aligned}$$

The combined profits of two firms prior to a merger is $2\pi_i(n)$, so it must be that the merge is profitable if and only if:

$$\begin{aligned} \pi_i(n-1) &> 2\pi_i(n) \\ \Leftrightarrow \frac{1}{\beta}\left(\frac{\alpha - c}{n}\right)^2 &> 2\frac{1}{\beta}\left(\frac{\alpha - c}{1+n}\right)^2 \\ \frac{1}{n^2} &> \frac{2}{(1+n)^2} \\ \frac{n(1+n)}{n^2} &> \frac{2n(1+n)}{(1+n)^2} \\ (1+n)^2 &> 2n^2 \\ \frac{1+n}{\sqrt{2}} &> n \\ 1 + n - n\sqrt{2} &> 0 \\ n &< \frac{1}{\sqrt{2}-1} \approx 2.414. \end{aligned} \tag{17}$$

Hence, a merger is only worthwhile if the two merging firms were originally duopolists. Why is this? When firms compete in a Cournot fashion, an expansion of production depresses the price and hence the profits of both them and their competitors. Of course, they care only about their own profits. Thus they over-expand their output relative to a monopolist. When firms merge, then internalise this affect, they hold back their production. But remaining firms now face less competition in terms of quantity, and hence there's a higher price. They follow by expanding their outputs, reducing the profits of the merged firms. By merging, the merged firms are less threatening to the remaining firms. Sometimes, therefore, it may better for them to remain separate.

6.2

Now suppose that any firm producing positive output incurs a fixed cost of F:

$$c_i(q_i) = F + cq_i, \text{ if } q_i > 0,$$

and $c_i(0) = 0$. Let $n = 4$, and suppose F satisfies:

$$F = \frac{1}{\beta}\left(\frac{2(\alpha - c)}{9}\right)^2.$$

What are the pure strategy equilibria?[4] *Without calculations, do you think there may be any mixed equilibria?*

We now search for pure strategy equilibria. In such an equilibrium, there may be some active firms (with $q_i > 0$) and some inactive firms (with $q_i = 0$). Any active firms face the standard Cournot problem. If they are active, then the fixed cost F is being paid anyway, and hence they must choose q_i optimally. This means that they choose the Cournot quantities from the previous part of the question. Suppose, then, that there is an equilibrium with n firms satisfying $q_i > 0$. For each of these firms, we must have what we found in (10)

$$q^* = \frac{\alpha - c}{\beta(1+n)}.$$

Of course, to be willing to remain active their variable profits must cover the fixed cost F. We thus require $\pi_i(n) \geq F$. This means that:

$$\begin{aligned} \pi_i(n) &\geq F \\ \Leftrightarrow \frac{1}{\beta}\left(\frac{\alpha - c}{1+n}\right)^2 &\geq \frac{1}{\beta}\left(\frac{2(\alpha - c)}{9}\right)^2 \\ \frac{1}{1+n} &\geq \left(\frac{2}{9}\right)^2 \\ \implies n &\leq \frac{7}{2}. \end{aligned} \tag{18}$$

Thus, if there are $n = 4$ firms active, they will not cover their fixed costs. This means that it would be a better response to set $q_i = 0$. Thus, we cannot have an equilibrium with $n = 4$ firms active. There remains the possibility, however, for equilibria with $n \leq 3$ firms active, since they will be able to cover the fixed cost F.

Suppose now that there are indeed $n \leq 3$ firms active. All active firms choose Cournot quantities and make positive profits. We must ask: is there an incentive for a inactive firm (there must be at least one) to start up and produce strictly positive output? Using

[4] Hint: There may be equilibria with only $m \leq n$ active firms, so try each case $m = 1, ..., 4$. Start by looking at the case where one firm is producing the monopoly output and the other firms are producing nothing. Can this be a Nash equilibrium? Does a firm not producing in this situation have an incentive to deviate? Then, look at the case when two firms are producing, and so on.

our previous analysis, the active firms each produce:

$$q^* = \frac{\alpha - c}{\beta(1+n)},$$
$$\implies Q = nq^* = \frac{n}{\beta}\left(\frac{\alpha - c}{1+n}\right).$$

Consider the best response quantity of a firm deciding to produce. This firm (say, i) will produce:

$$\begin{aligned} q_i &= \frac{\alpha - \beta Q(n) - c}{2\beta} \\ &= \frac{\alpha - c}{2\beta} - \frac{n}{2\beta}\left(\frac{\alpha - c}{1+n}\right) \\ &= \frac{\alpha - c}{2\beta}\left(1 - \frac{n}{1+n}\right) \\ &= \frac{\alpha - c}{2\beta(1+n)}, \end{aligned} \tag{19}$$

where we use (8) instead of (9) since firm i in this case is not producing yet. This leads to:

$$\begin{aligned} q_i + Q(n) &= \frac{\alpha - c}{2\beta(1+n)} + \frac{n}{\beta}\left(\frac{\alpha - c}{1+n}\right) \\ &= \left(\frac{1+2n}{2+2n}\right)\left(\frac{\alpha - c}{\beta}\right). \end{aligned} \tag{20}$$

Price is thus:

$$\begin{aligned} p &= \alpha - \beta(q_i + Q(n)) \\ &= \alpha - \beta\left(\frac{1+2n}{2+2n}\right)\left(\frac{\alpha - c}{\beta}\right) \\ &= \alpha - \frac{\alpha - c + 2n\alpha - 2nc}{2+2n} \\ &= \frac{\alpha + c(1+2n)}{2(1+n)}. \end{aligned} \tag{21}$$

Thus, if firm i begins producing at an optimal response level, it will obtain variable profits of:

$$\begin{aligned} \pi_i &= (p - c)q_i \\ &= \left[\frac{\alpha + c(1+2n)}{2(1+n)} - c\right]\frac{\alpha - c}{2\beta(1+n)} \\ &= \left[\frac{\alpha - c}{2(1+n)}\right]\frac{\alpha - c}{2\beta(1+n)} \\ &= \frac{1}{\beta}\left[\frac{a - c}{2(1+n)}\right]^2 \end{aligned} \tag{22}$$

Will this cover fixed costs F? For it to do so, we must have:

$$\begin{aligned} \pi_i &> F \\ \Leftrightarrow \frac{1}{\beta}\left[\frac{a-c}{2(1+n)}\right]^2 &> \frac{1}{\beta}\left(\frac{2(\alpha-c)}{9}\right)^2 \\ \frac{1}{2(1+n)} &> \frac{2}{9} \\ \frac{9}{2} &> 2(1+n) \\ \frac{9}{4} &> 1+n \\ \implies n &< \frac{5}{4}. \end{aligned}$$

Thus, it will only be profitable to start producing if $n = 1$ or $n = 0$. It follows that, if $n = 2$ or $n = 3$, it will not be profitable for an inactive firm to begin producing. we can conclude from this that there are two possible pure strategy equilibria. They involve either $n = 2$ or $n = 3$ active firms, producing the appropriate Cournot quantities. Since we have two equilibria – an even number – and generically games will only have an odd number of equilibria, we can expect there to be at least one mixed strategy Nash equilibrium as well.

7 An entry game

Consider the following two-period game with no discounting:

- *In period 1, four firms simultaneously and independently decide whether or not to pay 1 to enter an industry.*
- *In period 2, all firms that chose to enter now simultaneously and independently choose production levels, with fixed cost $F = 5$ and zero marginal cost $c = 0$. That is, a firm in the industry can either produce no output and incur no costs in period 2, or can produce any positive output and incur a total cost of 5 in period 2. All production is sold at price $10 - Q$ where Q is total industry output.*

Consider the five possible post-period 1 outcomes: n firms enter, where $n = \{n_i\}_{i=0}^4$.

Consider the Nash equilibria of the five different possible period 2 subgames corresponding to $\{n\}_{i=0}^4$ entrants. You should have a good understanding of these from the previous questions. Which period 1 outcomes are consistent with all firms choosing pure strategies in the Nash equilibrium of the whole game that are subgame perfect (i.e., consistent with backwards induction logic)? Explain.

Recall that we considered a Cournot oligopoly with n firms, each with marginal cost Q and an inverse demand function satisfying:

$$p(Q) = \alpha - \beta Q.$$

In this question, we have the following parameter values: $c = 0$, $\alpha = 10$, and $\beta = 1$. We begin by considering the stage 2 subgames. For now, let us not worry about the number of firms that enter, but instead consider the number of active firms in the subgame. Suppose that there are n active firms. From a previous question, we found that the profit of each active firm in a Nash equilibrium of the subgame was $\pi(n)$, where:

$$\pi(n) = \frac{1}{\beta}\left(\frac{\alpha - c}{1+n}\right)^2 = \frac{100}{(1+n)^2},$$

after accounting for the parameter values. If this is an equilibrium, we must have each active firm covering its fixed costs F – this is different than the entry cost, which is "sunk" at this point. For this question, we have $F = 5$. Hence, we require:

$$\begin{aligned} \pi(n) &\geq 5 \\ \Leftrightarrow \frac{100}{(1+n)^2} &\geq 5 \\ \implies n &\leq \sqrt{20} \approx 3.472. \end{aligned} \tag{23}$$

This means that we can only have a pure strategy Nash equilibrium of the subgame with $n \leq 3$ active firms.

This is not the end, however. Suppose that there are n active firms, but there are also inactive firms who have entered the industry but are (by assumption) setting $q_i = 0$. We have to check that such firms do not do better by becoming active. Again, refer to the solutions to the previous question. If there are n active firms, and inactive firm i becomes active in an optimal way, then it obtains variable profits of:

$$\pi_i = \frac{1}{\beta}\left[\frac{a-c}{2(1+n)}\right]^2 = \frac{25}{(1+n)^2}.$$

For a deviation to activity not to be profitable, we require:

$$\begin{aligned} \pi_i &\leq 5 \\ \Leftrightarrow \frac{25}{(1+n)^2} &\leq 5 \\ \implies n &\geq \sqrt{5} - 1 \approx 1.236. \end{aligned}$$

In words, this means that $n = 0$ and $n = 1$ cannot be equilibria when there is at least one inactive firm. It follows that with an unlimited number of firms present in the subgame, there are two pure strategy equilibria, corresponding to $n = 2$ and $n = 3$. Let us now consider the equilibria of the subgames, according to the number of firms entered:

None Nothing to do here. If there is no entry, then the game ends with $Q = 0$.

1 The entering firm is a monopolist, and produces the monopoly output.

2 There is a unique pure strategy Nash equilibrium of this subgame, a duopoly of $n = 2$ active firms. If only $n = 1$ were active, then the remaining firm would have an incentive to start producing.

3 There are two pure strategy Nash equilibria of this subgame, with either $n = 2$ or $n = 3$ firms active. If $n = 3$ are active, then variable profits exceed the fixed cost F. If there are $n = 2$ active firms, then the inactive firm has insufficient incentive to start producing, from the analysis above. There cannot be an equilibrium with $n = 1$ or $n = 0$ active firms, since an inactive firm would choose to produce.

4 There are two pure strategy Nash equilibria of this subgame, again with $n = 2$ or $n = 3$ active firms. With $n = 4$ active firms, fixed costs are not covered, and an active firm would shutdown. $n = 2, 3$ are equilibria by the argument given about.

This provides a first step to answering the question. Our next task is to check which options satisfy the subgame perfection criterion. We can do this by going through the different possibilities above.

None This cannot be a SPNE. There is a profitable deviation for one firm to enter. If it does, then $n = 1$. The unique equilibrium in the subgame is for this firm to supply the monopoly output. If it does, then its profits are, for $n = 1$:

$$\pi(n = 1) = \frac{1}{\beta}\left(\frac{\alpha - c}{1+n}\right)^2 = \frac{100}{2^2} = 25 > 5 + 1.$$

These profits exceeds the fixed costs and entry profits. Hence there is a profitable deviation, and this cannot be an equilibrium.

1 Once again, this cannot be an equilibrium. Suppose it is: One other (non-entering firm) could deviate. If it did, then there would be $n = 2$ firms in the second stage. There is a unique equilibrium of such a subgame with two active firms. Each would earn profits:

$$\pi(n = 2) = \frac{1}{\beta}\left(\frac{\alpha - c}{1 + n}\right)^2 = \frac{100}{3^2} \approx 11.11 > 5 + 1.$$

Hence, the deviation firm earns profits that exceed $F + 1$. This is a profitable deviation, and hence entry by 1 firm cannot be an equilibrium.

2 This can be an equilibrium. On the equilibrium path, there are $n = 2$ firms in the second stage, each earning variable profits that exceed the fixed and entry costs. Thus they have incentive to deviate. There remains, however, the possibility that a third firm may deviate and choose to enter. If this happens, then we have an equilibrium at stage 2. One such equilibrium is for the deviating firm to remain inactive. In this case, it would earn no variable profits, and incur no fixed cost, but this would mean that the entry cost would be wasted. Hence, given the equilibrium specified in the subgame, this is not a profitable deviation. Thus, the entry of $n = 2$ firms can be a SPNE. It requires, however, these two firms to ignore any off the equilibrium path entry by other firms, and to play the $n = 2$ equilibrium in the subgame.

3 This can be an equilibrium. On the equilibrium path, there are $n = 3$ firms in the second stage. We need to select an equilibrium in the second stage, however. Let us select the equilibrium in which $n = 3$ firms are active. We cannot select the SPNE with $n = 2$, since the inactive firm would choose not to enter. Now, the variable profits are:

$$\pi(n = 3) = \frac{1}{\beta}\left(\frac{\alpha - c}{1 + n}\right)^2 = \frac{100}{4^2} = 6.25 > 5 + 1.$$

Hence, these exceed fixed and entry costs. This is indeed a SPNE. No entering firm has an incentive to deviate.

4 This cannot be an equilibrium. It would only be an equlibrium if all four firms earned positive profits in the subgame. But there is no pure strategy equilibrium with $n = 4$ active firms in the subgame. Hence, at least one firm would have incentive to deviate and not enter.

In conclusion, there are two categories of pure strategy SPNE, in which either two or three firms enter the market.

7.1

One of the outcomes you found in the previous part is inconsistent with forwards induction logic. Which is it? Explain.

The SPNE in which $n = 2$ firms enter does not survive forward induction logic. Why? Suppose that a non-entering firm deviates, and enters the industry. The equilibrium strategy profile calls for firms to play a SPNE where the deviating firm is inactive, with only 2 active firms. But of course, observing the entry, the active firms would realise that the deviating firm is expecting the $n = 3$ active subgame equilibrium to be played. This is the only expectation that is consistent with optimising behaviour. Realising this, they would mutually expect the $n = 3$ active subgame equilibrium to be played. Of course, this leads to positive profits (net of fixed and entry costs) to the deviant. This argument eliminates the $n = 2$ equilibrium. Hence the only SPNE out come to satisfy forward induction logic is one in which 3 firms enter and are active. Notice that this removes our equilibrium selection problem that we previously had.

7.2

In addition to the subgame perfect outcomes ,there is one outcome consistent with all firms choosing pure strategies in a Nash equilibrium of the whole game that is imperfect. Which is it? Explain.

There is a Nash equilibrium in pure strategies with $n = 1$ firm entering. The strategy profile is as follows:

- Only one firm enters, call it $i = 1$, and produces the monopoly output.
- If, in stage 1, any of the remaining firms $i > 1$ enters, then firm $i = 1$ produces $Q = 10$ and floods the market in stage 2.

Consider firm $i = 1$. It has no incentive to deviate – it earns the maximum possible profit. Consider firms $i > 1$. If they enter, the market is flooded and they cannot recover fixed and entry costs. Hence this is a Nash equilibrium. It is not, however, subgame perfect. Why not? It requires an *incredible threat.* Firm 1 is threatening to flood the market. In the subgame (off the equilibrium path, of course) where it is called to do so, this is not equilibrium behaviour. This neatly illustrates the chief power of the SPNE and related concepts – the elimination of incredible threats.

7.3

Of the remaining period 1 outcomes, which are consistent with Nash equilibrium behaviour (perhaps including mixed strategies).

Outcomes $n = 0$ and $n = 4$ are both consistent with a mixed strategy SPNE. To see why, construct a strategy profile as follows. Each of the four firms enters with probability p. If there is entry by a single firm, they choose the monopoly quantity, yielding profit:

$$\pi(n = 1) - F,$$

where F is the fixed cost. If there is entry by two firms, they choose duopoly quantities, yielding profits:

$$\pi(n = 2) - F.$$

With three firms, they choose triopoly quantities, yielding:

$$\pi(n = 3) - F.$$

Finally, if four firms enter, they play a symmetric mixed strategy equilibrium of the subgame. I omit the details, but this requires indifference between activity and inactivity, since each firm mixes between a fixed positive quantity ($q_i > 0$) and no quantity ($q_i = 0$). Since they are indifferent to not producing, this yields profits in the subgame of 0, net of fixed costs. Now, by choosing to enter, the firm gains profits:

$$\tilde{\pi}(p) = (1-p)^3(\pi(n=1)-F) + 3p(1-p)^2(\pi(n=2)-F) + 3p^2(1-p)(\pi(n=3)-F) - 1.$$

The first term is the probability of no entry by others, multiplied by the payoff for being a monopolist. Similar interpretations are attached to the second and third terms. The final term is the entry cost. To be willing to mix, a firm must be indifferent between entering and not. This requires a p such that $\tilde{\pi}(p) = 0$. Does such a p exist? The answer is yes. Notice that $\tilde{\pi}(p)$ is continuous in p. Furthermore, we have:

$$\tilde{\pi}(p=0) = \pi(n=1) - F - 1 > 0,$$
$$\tilde{\pi}(p=1) = -1 < 0.$$

Hence, by the Mean Value Theorem[5], there exists a p such that $\tilde{\pi}(p) = 0$. All firms randomise over entry. This means that entry by either 0 or all firms may occur.

[5]**Mean Value Theorem**: If f is a continuous function on the closed interval $[a, b]$, and differentiable on the open interval (a, b), then there exists a point c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$